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A Computer-Oriented Method for Reducing Linearized Multibody System Equations by Incorporating Constraints

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Summary

Consider a spatial multibody system with rigid and elastic bodies. The bodies are linked by rigid interconnections (e.g. revolute joints) causing constraints, as well as by flexible interconnections (e.g. springs) causing applied forces. Small motions of the system with respect to a given nominal configuration can be described by linearized dynamic equations and kinematic constraint equations. We present a computer oriented procedure which allows to develop a minimum number of these equations. There are three problems. First: algorithmic selection of position coordinates; second: condensation of the dynamic equations; third: evaluation of the constraint forces. To demonstrate the procedure, a closed loop multibody system is used as an example.

1. INTRODUCTION

The linearized equations of motion can be written in the following form:

$$(1) \qquad \mathbf{M}\,\ddot{\mathbf{p}} + \mathbf{K}\,\mathbf{p} + \mathbf{D}\,\dot{\mathbf{p}} = \mathbf{g}$$

Here $\mathbf{M}$, $\mathbf{K}$, $\mathbf{D}$, are system matrices developed in [1, 2, 3]; $\mathbf{p}$ is the vector of the n_p position coordinates of the unconstraint system and $\mathbf{g}$ the vector of the internal and external forces. Any n_z holonomic constraints on the motions of the system can be represented in linearized form as

$$(2) \qquad \mathbf{C}_z{}^t\,\mathbf{p}(t) = z(t) \;;$$

where $\mathbf{C}_z$ is a constant $n_p x\, n_z$- matrix.

Equation (2) restricts the solution space of (1) to the set $\{\mathbf{a}(t)\} + \ker(\mathbf{C}_z{}^t)$ where $\mathbf{a}(t)$ is a particular solution of (2) and where $\ker(\mathbf{C}_z{}^t)$ is the null-space of $\mathbf{C}_z{}^t$. Without loss of generality it can be assumed that $\mathbf{a}(t)$ is an element of the orthogonal complement of $\ker(\mathbf{C}_z{}^t)$ which will be denoted by $\ker(\mathbf{C}_z{}^t)^+$. The dimension of $\ker(\mathbf{C}_z{}^t)$ corresponds to the number n_y of degrees of freedom. In the case of a system with a full rank constraint matrix we have $n_y = n_p - n_z$. Otherwise a reduction of $\mathbf{C}_z$ to a matrix with full rank must be performed to evaluate n_y. This can be done in a numerically stable manner by the singular value decomposition [4]. We assume for the following, that this has been previously done.

To describe a solution of the system (1),(2) with n_y (independent) reduced postion coordinates $\mathbf{y}$ it is necessary to choose a basis of the solution space $\ker(\mathbf{C}_z{}^t)$. The basis vectors will be denoted by the matrix $\mathbf{J}_y$. Equivalently a basis $\mathbf{J}_z$ of $\ker(\mathbf{C}_z{}^t)^+$ is chosen with the feature

$$(3) \qquad \mathbf{a}(t) = \mathbf{J}_z\, z(t)$$

These definitions are equivalent to the following matrix products:

(4) $\mathbf{C_z}^t \mathbf{J_y} = 0$

(5) $\mathbf{C_z}^t \mathbf{J_z} = \mathbf{I}$

(6) $\mathbf{J_y}^t \mathbf{J_z} = 0$

where $\mathbf{I}$ denotes the identity matrix. $\mathbf{p}$ can be splitted into a direct sum of two vectors with the following representation:

(7) $\mathbf{p} = \mathbf{J_y}\, \mathbf{y} + \mathbf{J_z}\, \mathbf{z}$.

For simple constrained systems one may have so much insight in the structure that it is possible to establish $\mathbf{J_y}$ by hand.

For more complex systems an automated approach will be necessary. A method to select the basis $\mathbf{J_y}$ has been proposed in [5,6] using the zero - eigenvalue theorem, which is equivalent to the singular value decompostion [4]. The coordinates obtained by this method form an orthogonal basis of $\ker(\mathbf{C_z}^t)$, which is from the numerical point of view optimal. But the reduced position vector $\mathbf{y}$ represented in an orthogonal basis has in general no more physically interpretable components and a backtransformation of the vector $\mathbf{y}$ to the vector $\mathbf{p}$ will be necessary, where the numerical advantage can be lost again.

This disadvantage is avoided by the method presented here. This method also automatically selects a basis $\mathbf{J_y}$ of the solution space, but the basis vectors are chosen so that the components of $\mathbf{y}$ are identical to some components of $\mathbf{p}$. This choice keeps the physical interpretability of $\mathbf{y}$. In addition one can prescribe some linear combination of the canonical basis vectors by giving a relation

(8) $\mathbf{C}_{\boldsymbol{\eta}}^t\, \mathbf{p} = \boldsymbol{\eta}$

to be taken as basis vectors of $\ker(C_z^t)$. η stands for some components of the reduced position vector $\mathbf{y}$.

2. NUMERICAL METHOD

For simplicity it is required that $\mathbf{C}_\eta$ has full rank. To avoid contradictions it is necessary to assume that the columns of $\mathbf{C}_z$ are linearly independent from those of $\mathbf{C}_\eta$. Otherwise linear dependency between columns of $\mathbf{C}_z$ and $\mathbf{C}_\eta$ would indicate, that coordinates η have been chosen, which cannot be used to describe the motion of the multibody system (c.f. example).

There are two steps to perform in order to obtain the basis vectors: Evaluation of $\mathbf{J}_y$ by solving equation (4) with the restriction (8) and evaluation of $\mathbf{J}_z$ by solving equation (5) and (6). The last step can be omitted if $\mathbf{z} = \mathbf{0}$ and if the computation of the constraint forces is not required.

If we choose $\mathbf{J}_z$ so, that

$$(9) \qquad \mathbf{C}_\eta^t \mathbf{J}_z = 0$$

holds,

$$(10) \qquad \mathbf{C}_\eta^t \mathbf{J}_y = [\mathbf{I},\mathbf{0}]$$

follows out of equation (7). Equation (4) and (10) can be combined to

$$(11) \qquad \mathbf{C}^t \mathbf{J}_y = \begin{bmatrix} 0 & 0 \\ \mathbf{I} & 0 \end{bmatrix} \qquad ; \quad \mathbf{C} = [\mathbf{C}_z \ , \ \mathbf{C}_\eta]$$

Using orthogonal Householder transformations and column pivoting [4], $\mathbf{C}^t$ can be decomposed into a product of an orthogonal matrix $\mathbf{Q}$, an upper triangular matrix $\mathbf{R}$, with its diagonal elements ordered in a sequence of decreasing absolute values, and a permutation matrix $\mathbf{P}$:

$$(12) \qquad \mathbf{C}^t = \mathbf{Q}\ \mathbf{R}\ \mathbf{P} \quad .$$

As $\mathbf{C}^t$ has full rank, $\mathbf{R}$ can be partitioned as

$$(13) \qquad \mathbf{R} = [\mathbf{R}_1, \mathbf{S}]$$

where $\mathbf{R}_1$ is quadratic with no zero diagonal elements. Thus the dimension of $\mathbf{R}_1$ equals the rank of $\mathbf{C}$. $\mathbf{J}_y$ is considered to be of the form

$$(14) \qquad \mathbf{J}_y = \mathbf{P} \begin{bmatrix} J_1 & J_2 \\ 0 & I \end{bmatrix}$$

where $\mathbf{J}_1$ is a $(n_z + n_\eta) \times n_\eta$ - matrix, $\mathbf{J}_2$ is a $(n_z + n_\eta) \times (n_p - n_\eta)$ - matrix. With these definitions equation (11) reads

$$(15) \qquad [\mathbf{Q}\ \mathbf{R}_1\ \mathbf{J}_1 \ , \ \mathbf{Q}\ \mathbf{R}_1\ \mathbf{J}_2 + \mathbf{Q}\ \mathbf{S}] = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

Denoting the first n_z rows of $\mathbf{Q}$ by $\mathbf{Q}_z$ and the remaining n_η by $\mathbf{Q}_\eta$ we obtain the following solution of equation (15)

$$(16) \qquad \mathbf{J}_y = \mathbf{P} \left[\begin{array}{c|c} R_1^{-1}\ Q_\eta & -R_1^{-1}\ S \\ 0 & I \end{array}\right]$$

Herein the orthogonality of $\mathbf{Q}$ has been used. The column vectors of $\mathbf{J_y}$ are indeed the basis vectors of $\ker(\mathbf{C_z}^t)$ and as a consequence of the identity matrix in the lower right corner of $\mathbf{J_y}$ the components of $\mathbf{y}$ generated by this method are really components of the unreduced position vector $\mathbf{p}$.

The step establishing $\mathbf{J_z}$ is easier. Equation (5) and (6) can be combined to the following system of linear equations

$$\begin{bmatrix} C_z{}^t \\ \\ J_y{}^t \end{bmatrix} J_z = \begin{bmatrix} I \\ \\ 0 \end{bmatrix} \tag{17}$$

which can be solved by standard methods.

This algorithm has been integrated and tested in a FORTRAN - program which generates the linearized equations of motion for general multibody systems [7,8]. For the orthogonal decomposition, subroutines from the LINPACK package [9] have been used. Methods using orthogonal matrix decomposition are numerically stable; they are preferable to those using Gauss related algorithms for determining the rank of a matrix.

3. CONDENSATION OF THE DYNAMIC EQUATIONS AND ELIMINATION OF THE CONSTRAINT FORCES

With the matrices $\mathbf{J_y}$ and $\mathbf{J_z}$ we can define the generalized applied forces $\mathbf{f_y}$ and the generalized constraint forces $\mathbf{f_z}$ as the projection of the total force $\mathbf{g}$ in the spaces $\ker(\mathbf{C_z}^t)$ and $\ker(\mathbf{C_z}^t)^+$:

$$\begin{aligned} \mathbf{f_y} &= J_y{}^t g \\ \mathbf{f_z} &= J_z{}^t g \; . \end{aligned} \tag{18}$$

Correspondingly $\mathbf{g}$ can be splitted into a direct sum of two forces $\mathbf{g} = \mathbf{g_y} + \mathbf{g_z}$; with the applied force $\mathbf{g_y}$, defined by

$$(19) \quad \mathbf{J_y}^t \mathbf{g} = \mathbf{J_y}^t \mathbf{g_y} \;.$$

and the constraint force $\mathbf{g_z}$, defined by

$$(20) \quad \mathbf{J_z}^t \mathbf{g} = \mathbf{J_z}^t \mathbf{g_z}$$

For simplicity we assume for the following the case $\mathbf{z} = \mathbf{0}$. The more general equations can be easily obtained from this case. Introducing (7) in (1) and premultiplying (1) with $\mathbf{J_y}^t$ gives

$$(21) \quad \mathbf{J_y}^t\ \mathrm{M}\ \mathbf{J_y}\ \ddot{\mathbf{y}} + \mathbf{J_y}^t\ \mathrm{K}\ \mathbf{J_y}\ \mathbf{y} + \mathbf{J_y}^t\ \mathrm{D}\ \mathbf{J_y}\ \dot{\mathbf{y}} = \mathbf{J_y}^t\ \mathbf{g_y}$$

By this procedure the number of equations of motion has been reduced to the minimum and the constraint forces have been eliminated, which follows from equation (19). Note that in order to obtain these results, the principle of d'Alembert was not needed; the elimination of the constraint forces was a consequence of geometrical considerations only.

4. EVALUATION OF THE CONSTRAINT FORCES

It follows from (20) that

$$(22) \quad \mathbf{J_z}^t \mathbf{g_y} = \mathbf{0}$$

holds. Premultiplying (1) with $\mathbf{J_z}^t$ instead of $\mathbf{J_y}^t$ gives an expression for the generation of the generalized constraint forces:

(23) $$\mathbf{f}_z = -\mathbf{J}_z^{\,t}(\mathrm{M}\ \mathbf{J}_y\ \ddot{\mathbf{y}} + \mathrm{K}\ \mathbf{J}_y\ \mathbf{y} + \mathrm{D}\ \mathbf{J}_y\ \dot{\mathbf{y}})$$

From these generalized constraint forces the forces $\mathbf{g}_z$ can be easily obtained by using the relation

(24) $$\mathbf{g}_z = \mathbf{C}_z\ \mathbf{f}_z\ .$$

This relation can be verified by premultiplying it with $\mathbf{J}_z^{\,t}$ and by using (4),(5) and (18):

(25) $$\mathbf{J}_z^{\,t}\mathbf{g}_z = \mathbf{J}_z^{\,t}\mathbf{C}_z\ \mathbf{f}_z = \mathbf{f}_z = \mathbf{J}_z^{\,t}\mathbf{g}$$

(26) $$\mathbf{J}_y^{\,t}\mathbf{g}_z = \mathbf{J}_y^{\,t}\mathbf{C}_z\ \mathbf{f}_z = \mathbf{0}$$

5. EXAMPLE

The kinematically closed chain treated in [10, page 182] has been used as an example to demonstrate the method. Small motions about the nominal triangular configuration, shown in figure 1, have been studied. There are $n_p = 30$ position coordinates and $n_z = 30$ constraint equations. The latter are linearly dependent and the system has one degree of freedom. The computational results of this example are shown in table 1. As one can see, the formalism chooses "y_1 = rotation of body 5 about axis 2" as reduced coordinate of the system in triangular - shape configuration. This choice and in consequence the matrix $\mathbf{J}_y$ are in general dependent on the nominal configuration of the system. The method applied to the cube - shape configuration of the chain does not lead to the same coordinate as found in the present case. The coordinate chosen by the algorithm in this case cannot be used to describe the motion of the chain in the cube shape configuration.

5.1.1 References

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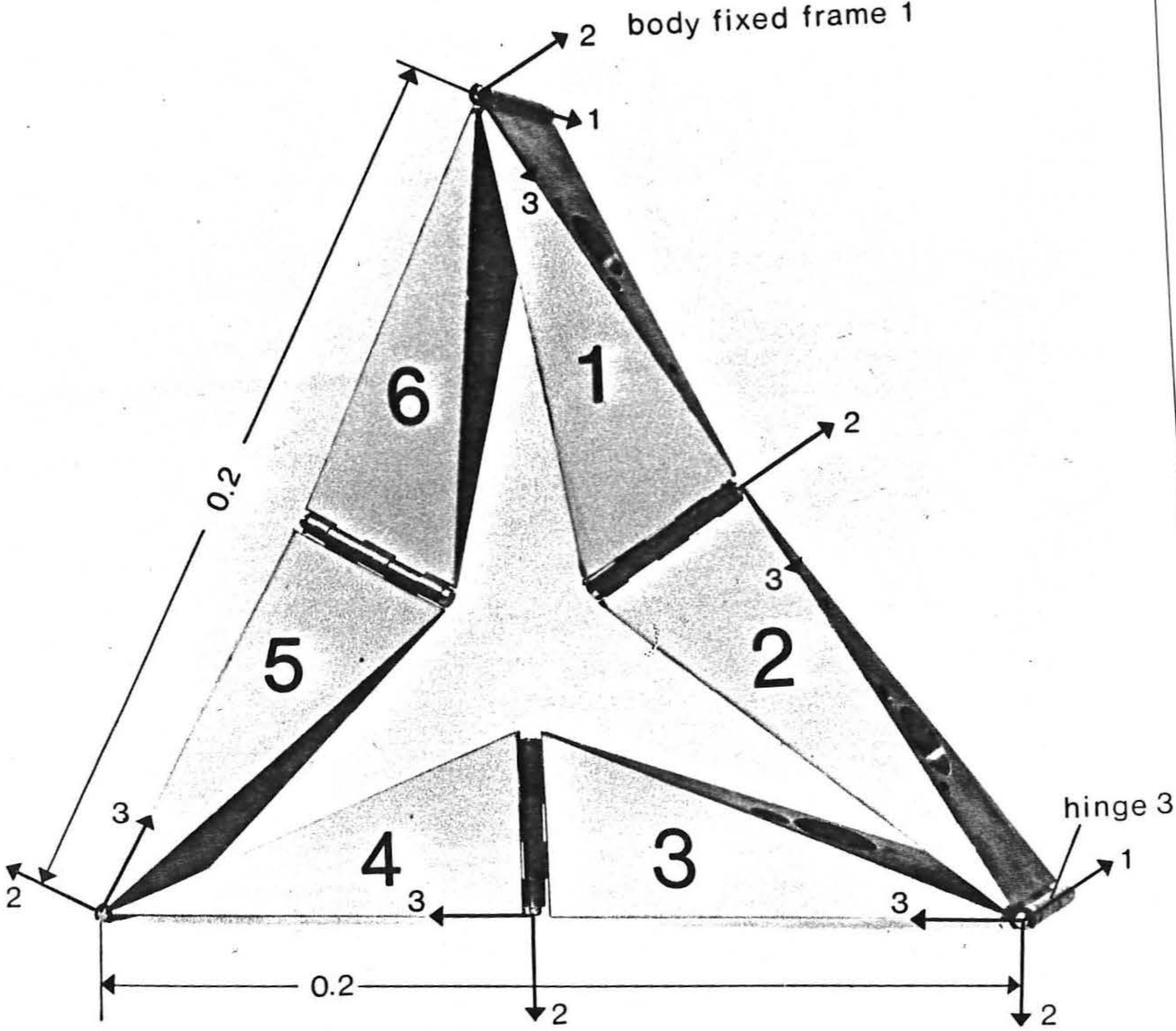

Figure 1. The kinematically closed chain in triangular configuration. Body 1 to 5 are free bodies, body 6 is fixed.
n_p = 30 postion coordinates, n_z = 30 constraints.

Number of degrees of freedom: $n_y = 1$

Definition of the reduced position variable : y_1 = rotation of body 5 about axis 2

Matrix $\mathbf{J_y}$ (30,1)	Position variables (defined in the body fixed frame) type	axis	body
0.0	translation in	1	1
0.0	" "	2	1
0.0	" "	3	1
0.0	rotation about	1	1
0.0	" "	2	1
0.0	" "	3	1
0.0	translation in	1	2
0.0	" "	2	2
0.0	" "	3	2
0.0	rotation about	1	2
- 1.0	" "	2	2
0.0	" "	3	2
- 0.1	translation in	1	3
0.0	" "	2	3
0.0	" "	3	3
0.0	rotation about	1	3
0.5	" "	2	3
0.866	" "	3	3
- 0.05	translation in	1	4
0.0	" "	2	4
0.0	" "	3	4
0.0	rotation about	1	4
- 0.5	" "	2	4
0.0	" "	3	4
- 0.1	translation in	1	5
0.0	" "	2	5
0.0	" "	3	5
0.0	rotation about	1	5
1.0	" "	2	5
0.0	" "	3	5

Table 1: Numerical results for the kinematically closed chain.