Simulation of nonlinear systems subject to modulated chirp signals

Abstract

- **Purpose**

  The purpose of the paper is to apply a novel technique for the simulation of nonlinear systems subject to modulated chirp signals.

- **Design/methodology/approach**

  The simulation technique is first described and its salient features are presented. Two examples are given to confirm the merits of the method.

- **Findings**

  The results indicate that the method is appropriate for simulating nonlinear systems subject to modulated chirp signals. In particular, the efficiency and accuracy of the method is seen to improve as the chirp frequency increases. In addition, error bounds are given for the method.

- **Originality/value**

  Chirp signals are employed in several important applications such as representing biological signals and in spread spectrum communications. Analysis of systems involving such signals requires accurate, appropriate and effective simulation techniques.

Introduction

Chirp signals are important in several applications such as sonar, radar applications (Capus 2000), (Flandrin 2001), (Leong 2013) and for system identification (Xiang 1997). They are also employed for signal approximation (Cui 2006) and (Qian 1998) and for analysis of nonlinear superconducting circuits (Murch 2012). Furthermore, modulated chirp signals are employed in spread spectrum communications for indoor wireless systems (Springer 2000). Modulated chirp signals are signals involving the product of a low-frequency signal and a high-frequency chirp signal. This paper applies a novel method for the simulation of nonlinear dynamical systems that are excited by such signals. Standard numerical integration techniques are inefficient for signals with widely-varying frequency content as the step size is governed by the highest frequency present in the signal. Several approaches have been proposed to deal with such cases including envelope simulation tools such as (Pedro 2002), multirate partial differential equation methods such as (Roychowdhury 2001) and multiscale methods such as (Ariel 2009). Envelope simulation methods have to be adapted for frequency modulation e.g. (Pulch 2008) or (Narayan 1999) and this process is not trivial. The technique also differs from recent works such as those proposed by Bitter and Brachtendorf (Bittner 2014a and Bittner 2014b). The method proposed in this paper is advantageous in that its efficiency increases with rising frequency and the method is not limited by a small step size as is the case in many quadrature methods that are based on local Taylor expansions.

Nonlinear system

Consider a nonlinear system of the form:

$$y'(t) = h(y(t)) + e(t) \tag{1}$$

where

$$e(t) = \sum_{m=1}^{N} a_m(t)\vartheta_m(t, f(t)) \tag{2}$$

and $h(y(t))$ is a nonlinear function governing the behavior of the system in question. $a_m(t)$ are slowly-varying functions that modulate the highly oscillatory chirp function $\vartheta_m(t)$. $f(t)$ is the time-varying frequency of the chirp signal. There are several types of chirps in existence. For example, a linear chirp is an oscillatory signal with a linearly increasing frequency, $f(t) = f_0 + \alpha t$. f_0 is the starting frequency and α is the rate at which the frequency increases. In contrast, an exponential chirp has $f(t) = f_0\alpha^t$.

For a signal composed of N modulated linear chirp-type signals with zero initial phase

$$e(t) = \sum_{m=1}^{N} a_m(t) e^{i2\pi\left(f_0 t + \alpha_m t^2/2\right)} \tag{3}$$

and for the exponential chirp-type

$$e(t) = \sum_{m=1}^{N} a_m(t) e^{i2\pi f_0\left(\frac{\alpha_m{}^t - 1}{\ln(\alpha_m)}\right)} \tag{4}$$

Asymptotic-Numerical Method

The proposed method was introduced in (Condon 2014). In the current paper, it is applied to the simulation of systems involving modulated chirp signals. Chirps fall into the category of signals for which the asymptotic-numerical method is applicable. Consider the case of a single modulated linear chirp function (A similar analysis applies for other chirp types).

$$e(t) = a(t) e^{i2\pi\left(f_0 t + \alpha t^2/2\right)} \tag{5}$$

α is a real constant.

It is assumed that $h(t)$ in (1) is smooth. It is assumed that for $t>0$ and some $\sigma>0$ and when $\omega = 2\pi f_0 >> 0$

$$\int_0^t h(\tau) e^{i2\pi\left(f_0\tau + \alpha\tau^2/2\right)} = O(\omega^{-\sigma})$$

For example, when $h(t)=1$

$$\int_0^t e^{i2\pi\left(f_0\tau + \alpha\tau^2/2\right)} = O(\omega^{-1})$$

The method involves expressing the solution to (1) as

$$y(t) \approx \sum_{r=0}^{\infty} p_r(t, f_0) \tag{6}$$

where each $p_r(t, f_0) = O(f_0^{-r\sigma})$ and therefore $p_r(t, f_0) = O(f_0^{-r})$ for a linear chirp.

It is stipulated that $p_0(t)$ is non-oscillatory and independent of f_0.
The series in (6) can be obtained via a recursive procedure as follows:

$$p_0(0) = y_0 \qquad p_r(0, f_0) \equiv 0$$

$$p_1'(t) = \frac{\partial h(p_0(t))}{\partial y} p_1(t) + a(t)e^{i2\pi\left(f_0 t + \alpha t^2/2\right)}, \quad t \geq 0, \quad p_1(0) = 0 \tag{7}$$

The solution of (7) is highly oscillatory but can be written down explicitly

$$p_1(t, f_0) = \phi(t)\int_0^t \phi^{-1}(\tau)a(\tau)e^{i2\pi(f_0\tau + \alpha\tau^2/2)}, \qquad t \geq 0 \tag{8}$$

where

$$\phi' = \frac{\partial h(p_0(t))}{\partial y}\phi, \quad t \geq 0, \quad \phi(0) = I$$

and I is the identity matrix.

In general, for $r \geq 2$,

$$p_r(t, f_0) = \phi(t)\int_0^t\int_0^t \cdots \int_0^t s_r(\zeta_1, \cdots \zeta_r)\prod_{j=1}^{r} e^{i2\pi(f_0\zeta_j + \alpha\zeta_j^2/2)} d\zeta_1 \cdots d\zeta_r, \qquad t \geq 0 \tag{9}$$

Let

$$\mathrm{I}_{m,r} = \left\{(k_1, k_2 \ldots k_m) : 1 \leq k_1 \leq k_2 \cdots k_m, \sum_{i=1}^{m} k_i = r\right\}, \ldots\ldots m, r \in N$$

and let $v_{m,r}$ be the number of m-tuples $\boldsymbol{k} \in N^m$ such that $\sum_{i=1}^{m} k_i = r$ and h_m is the m^{th} derivative tensor of the analytic function h in (1).

Then

$$s_r(\zeta) = \sum_{m=2}^{r} \frac{1}{m!} \sum_{\boldsymbol{k} \in I_{m,r}} v_{m,r} \int_{\max\{\zeta_1, \ldots \zeta_r\}} \phi^{-1}(\tau) h_m(p_0(\tau)) \left[\phi(\tau)s_{k_1}, \cdots \phi(\tau)s_{k_m}\right] d\tau \tag{10}$$

This is a non-oscillatory function and it is independent of f_0.

While the integral in (9) is highly oscillatory, it is linear and may be solved readily by quadrature e.g. (Huybrechs 2009) and (Deano 2009).

Error bounds and solution structure

In what follows, the Euclidean norm is denoted by $\| \ \|$.

Theorem

Suppose that $\|a(t)\|$ and $\|\vartheta(t)\|$ are uniformly bounded for $t \geq 0$ and suppose that derivatives of $h(t)$ of any order with respect to y or t are uniformly bounded. In addition, suppose that $\frac{\partial h(p_0(t))}{\partial y} = A + C(t)$ where $C(t) \to 0$ as $t \to \infty$, then if the eigenvalues of A have negative or zero real parts for $t \geq 0$, then $\|p_r(t)\|$ are uniformly bounded for $t \geq 0$.

Proof

From (Jordan and Smith 2003), the solutions of a differential equation

$\dot{x}(t) = \{A + C(t)\}x(t)$

are bounded if the solutions of $\dot{x}(t) = Ax(t)$ are bounded and $C(t)$ is continuous for $t \geq 0$ and $\int_0^t \|C(t)\| dt$ is bounded for $t > 0$.

If the equation is $\dot{x}(t) = \{A + C(t)\}x(t) + \zeta(t)$ where $\|\zeta(t)\|$ is bounded for $t \geq 0$ then as proven in (Wang and Li 2014), the solution $\|x(t)\|$ is uniformly bounded. □

Consider the series in (6). Each $p_r(t, f_0) = O(f_0^{-r\sigma})$ so if N terms are included to form the solution for $y(t)$, then the error is $O(f_0^{-(N+1)\sigma})$. This structure means that as the frequency f_0 rises, fewer terms in the series in (6) are required to attain a given level of accuracy and in practical situations, r=1 terms is likely to suffice.

Note that there is no traditional time-stepping involved in the method as described. In contrast, the method will involve time steps that are not small but are limited in size because the asymptotics tend to deteriorate if the time intervals are too long.

Numerical results

Consider the following example selected for its simplicity to emphasise the salient features of the proposed method.

$$y'(t) = -y^2 + \cos(0.2\pi t)e^{i2\pi(f_0 t + \alpha t^2/2)} \qquad y(0) = 1 \qquad (11)$$

In (11), α=0.5 and f_0 is varied to highlight the increasing accuracy of the method as frequency increases. Fig. 1a shows the absolute value of the error in the approximation of $y(t)$ if only one term is included in the series in (6) when $f_0 = 10$. Fig. 1b shows the error in the approximation $y(t)$ if two terms are included in the series in (6) when $f_0 = 10$. The reduction in the absolute error is clearly obvious.

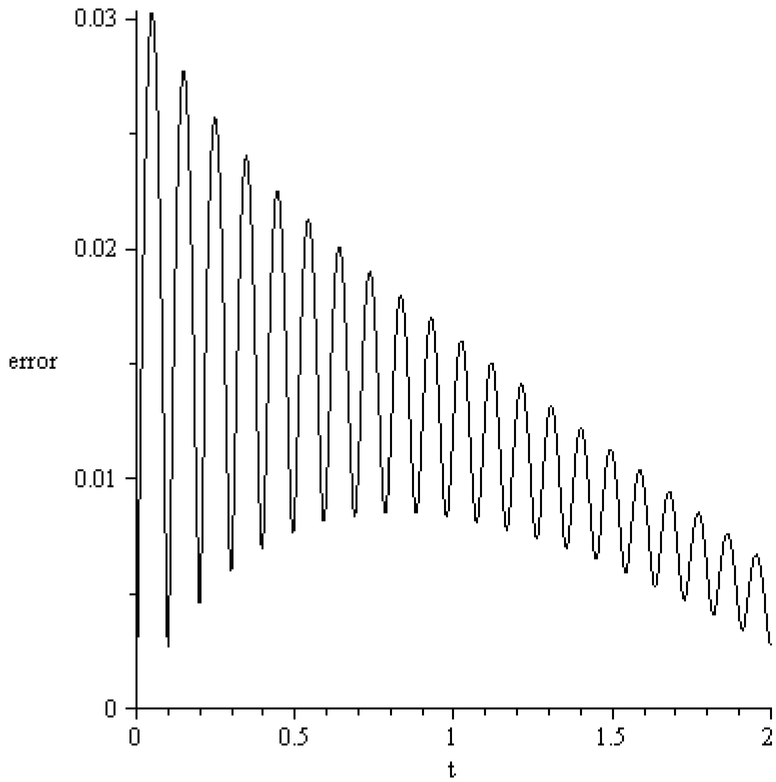

Fig. 1a Error when $f_0 = 10$ and only one term is included in the series in (6).

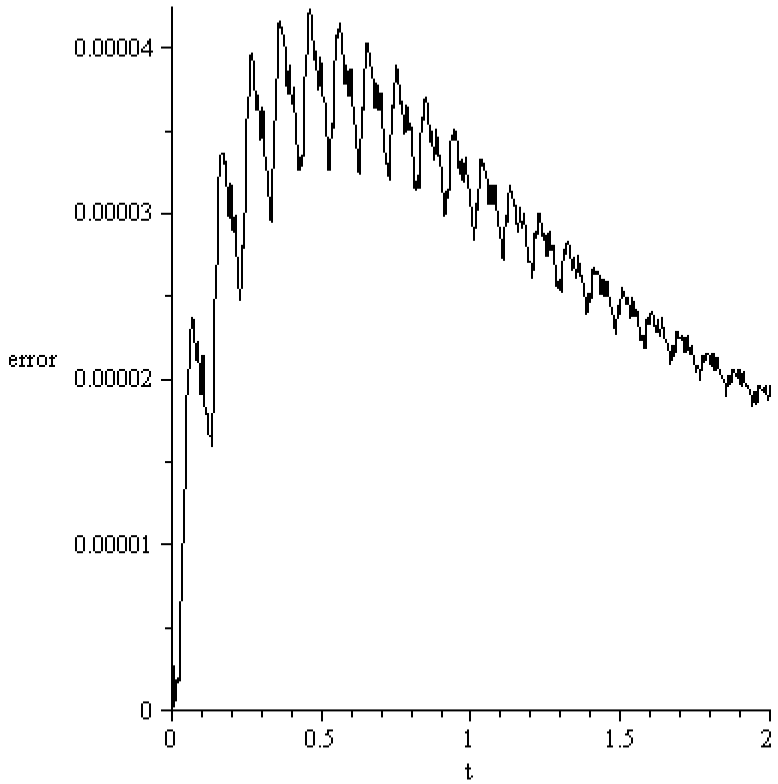

Fig. 1b Error when $f_0 = 10$ and two terms are included in the series in (6).

Now Fig. 1c shows the error when $f_0 = 100$ and only term is included in the series while Fig. 1d shows the error when two terms are included. From the foregoing theory, the error with one term is expected to be

$O(f_0^{-1})$ and with two terms $O(f_0^{-2})$. If further terms were included in the series, there would be a concomitant increase in accuracy.
The results confirm the merit of the method in that the error is reduced with increasing frequency.

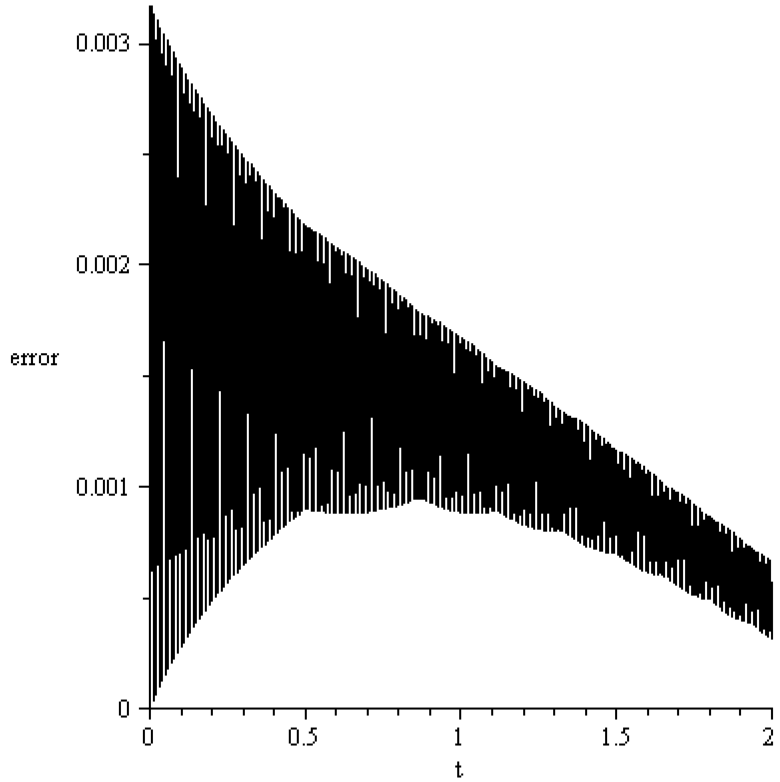

Fig. 1c Error when $f_0 = 100$ and only one term is included in the series in (6).

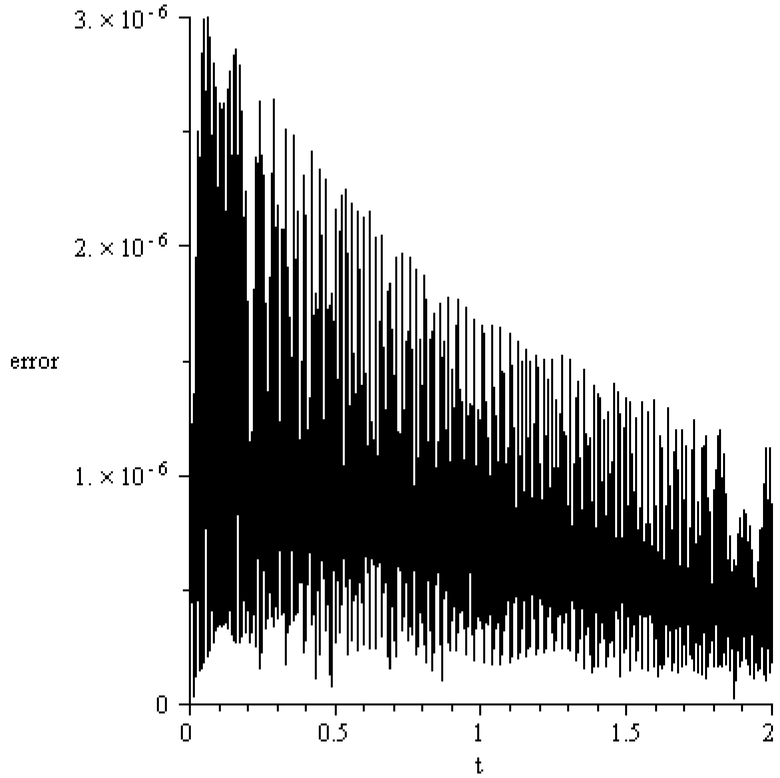

Fig. 1d Error when $f_0 = 100$ and two terms are included in the series in (6).

The second example involves a sum of chirps and a cubic nonlinearity

$$y'(t) = -y^3 + e^{-\frac{t^2}{2}} e^{j2\pi(f_0 t + \alpha t^2/4)} + e^{-0.02\pi t} e^{j2\pi(f_0 t + \alpha t^2/2)} \qquad y(0) = 1 \qquad (10)$$

Again as evident from Fig. 2, the error decays as expected with increasing frequency and with increasing the number of terms in the series. Importantly, as frequency increases, less terms are required in the series approximation to obtain a required level of accuracy.

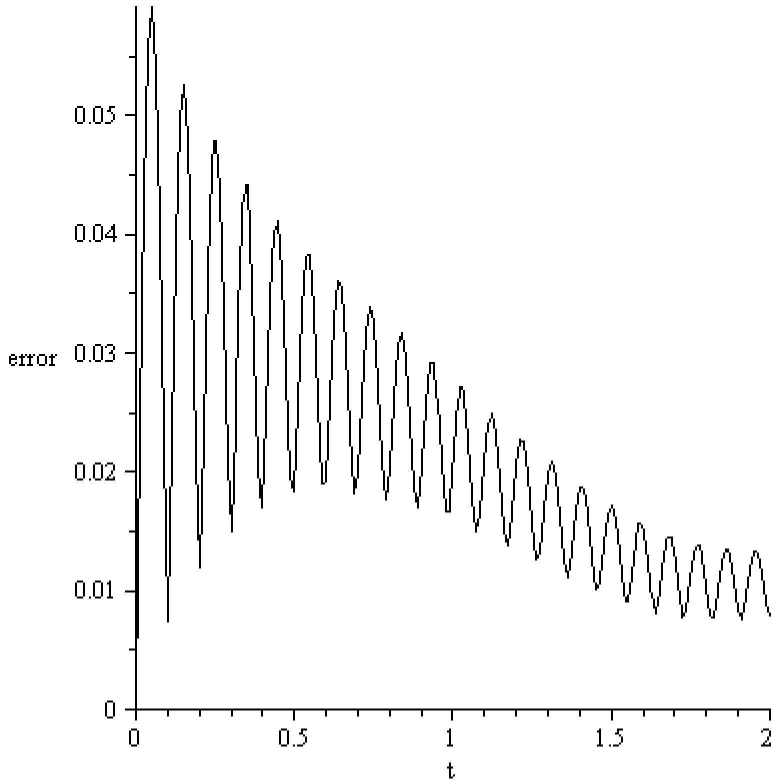

Fig. 2a Error when $f_0 = 10$ and only one term is included in the series in (6).

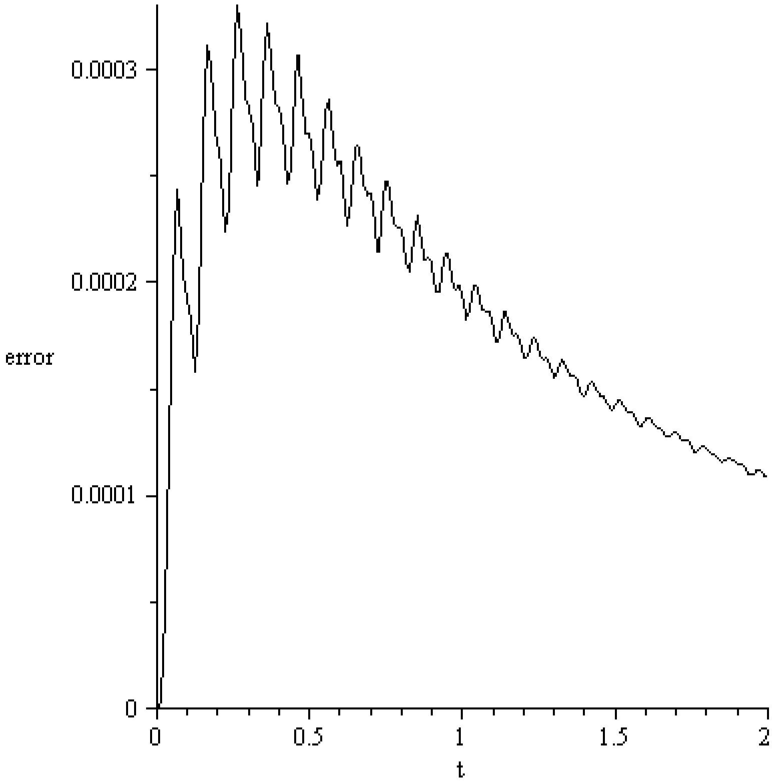

Fig. 2b Error when $f_0 = 10$ and two terms are included in the series in (6).

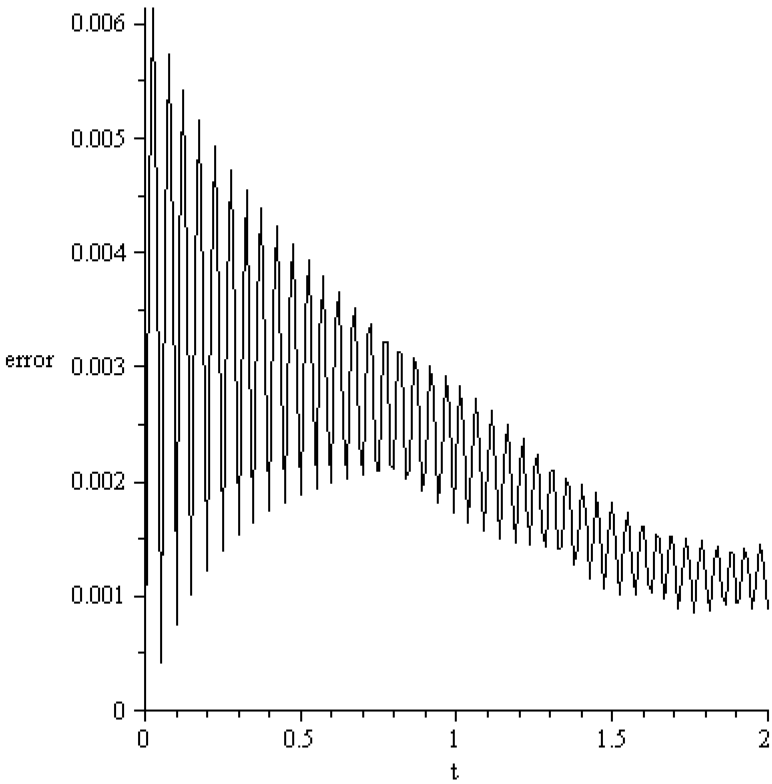

Fig. 2c Error when $f_0 = 100$ and only one term is included in the series in (6).

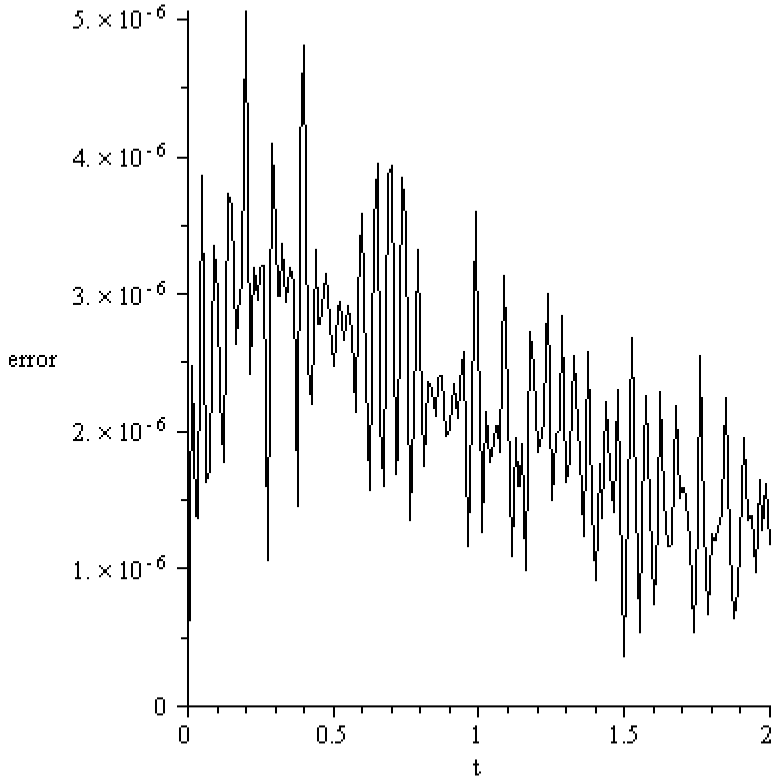

Fig. 2d Error when $f_0 = 100$ and two terms are included in the series in (6).

Conclusions

The paper has presented an asymptotic numerical method for simulating nonlinear systems subject to modulated chirp signals. Chirp signals arise in numerous applications such as spread spectrum techniques and their simulation is crucial for accurate analysis and design work. The efficiency of the proposed method increases with rising frequency as less terms are required in a truncated expansion in (6) to achieve a required degree of accuracy. In addition, the step size is not governed by the frequencies present in the signal.

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