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DEFORMATIONS OF REFLEXIVE SHEAVES OF RANK 2 ON $\mathbb{P}^3$

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DEFORMATIONS OF REFLEXIVE SHEAVES OF RANK 2 ON $\mathbb{P}^3_k$

In this paper we study deformations of reflexive sheaves of rank 2 on $\mathbb{P} = \mathbb{P}^3_k$ where k is an algebraically closed field of any characteristic. Let $\underline{F}$ be a reflexive sheaf with a section $s \in H^0(\underline{F}) = H^0(\mathbb{P},\underline{F})$ whose corresponding scheme of zeros is a curve C in $\mathbb{P}$. Moreover let $M = M(c_1,c_2,c_3)$ be the (coarse) moduli space of stable reflexive sheaves with Chern classes c_1, c_2 and c_3. The study of how the deformations of $C \subseteq \mathbb{P}$ correspond to the deformations of the reflexive sheaf $\underline{F}$ leads to a nice relationship between the local ring $O_{H,C}$ of the Hilbert scheme $H = H(d,g)$ of curves of degree d and arithmetic genus g at $C \subseteq \mathbb{P}$ and the corresponding local ring $O_{M,\underline{F}}$ of M at $\underline{F}$. In this paper we consider some examples where we use this relationship. In particular we prove that the moduli spaces $M(0,13,74)$ and $M(-1,14,88)$ contain generically non-reduced components.

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1. Deformations of a reflexive sheaf with a section.

If $\mathrm{Def}_{\underline{F}}$ is the local deformation functor of $\underline{F}$ defined on the category $\underline{l}$ of local artinian k-algebras with residue field k, then it is well known that $\mathrm{Ext}^1_{O_{\mathbb{P}}}(\underline{F},\underline{F})$ is the tangent space of $\mathrm{Def}_{\underline{F}}$ and that $\mathrm{Ext}^2_{O_{\mathbb{P}}}(\underline{F},\underline{F})$ contains the obstructions of deformation. See [H 3]. To deform the pair $(\underline{F},s)$ we consider the functor

$$\mathrm{Def}_{\underline{F},s} : \underline{l} \to \underline{\mathrm{Sets}}$$

defined by

$$\mathrm{Def}_{\underline{F},s}(R) = \{ O_{\mathbb{P}_R} \xrightarrow{s_R} \underline{F}_R \mid \underline{F}_R \in \mathrm{Def}_{\underline{F}}(R) \text{ and } s_R \otimes_R 1_k = s \}/\sim$$

where $\mathbb{P}_R = \mathbb{P} \times \mathrm{Spec}(R)$ and where $1_k : k \to k$ is the identity. Two deformations $(\underline{F}_R, s_R)$ and $(\underline{F}'_R, s'_R)$ are equivalent if there exist isomorphisms $O_{\mathbb{P}_R} \xrightarrow{\simeq} O_{\mathbb{P}_R}$, $\underline{F}_R \xrightarrow{\simeq} \underline{F}'_R$ and a commutative diagram

$$\begin{tikzcd} O_{\mathbb{P}_R} \arrow[r, "s_R"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circ"] & \underline{F}_R \arrow[d, "\simeq"] \\ O_{\mathbb{P}_R} \arrow[r, "s'_R"] & \underline{F}'_R \end{tikzcd}$$

such that $s_R \otimes_R 1_k = s'_R \otimes_R 1_k$. In fact we also identify the given pair $(\underline{F}, s)$ with any $(\underline{F}', s')$ where $s' \in H^0(\mathbb{P}, \underline{F}')$ if they fit together into such a commutative diagram.

<u>Proposition 1.1.</u> (i) The tangent space of $\mathrm{Def}_{\underline{F},s}$ is $\mathrm{Ext}^1_{O_{\mathbb{P}}}(\underline{I}_C(c_1), \underline{F})$ where $\underline{I}_C = \ker(O_{\mathbb{P}} \to O_C)$, and $\mathrm{Ext}^2_{O_{\mathbb{P}}}(\underline{I}_C(c_1), \underline{F})$ contains the obstructions of deformations.

(ii) The natural

$$\varphi : \mathrm{Def}_{\underline{F},s} \to \mathrm{Def}_{\underline{F}}$$

is a smooth morphism of functors on $\underline{l}$ provided

$$H^1(\underline{F}) = 0$$

By the correspondence [H 3, 4.1] there is a curve $C = (s)_0 \subseteq \mathbb{P}$ and an exact sequence

$$\xi : 0 \to O_{\mathbb{P}} \xrightarrow{s} \underline{F} \to \underline{I}_C(c_1) \to 0$$

associated to $(\underline{F}, s)$. The condition $H^1(\underline{F}) = 0$ is therefore equivalent to

$$H^1(\underline{I}_C(c_1)) = 0$$

<u>Proof</u> of (i). Using [L 2, § 2] or [Kl, 1.2] we know that there is a spectral sequence

$$E_2^{p,q} = \varprojlim^{(p)} \left\{ \begin{tikzcd}[ampersand replacement=\&] \mathrm{Ext}^q(\underline{F},\underline{F}) \arrow[dr, "\alpha^q"'] \& \& \mathrm{Ext}^q(O_{\mathbb{P}}, O_{\mathbb{P}}) \arrow[dl] \\ \& \mathrm{Ext}^q(O_{\mathbb{P}},\underline{F}) \& \end{tikzcd} \right\}$$

converging to some group $A^{(\cdot)}$ where A^1 is the tangent space of $\mathrm{Def}_{\underline{F},s}$ and A^2 contains the obstructions of deformation. Since $E_2^{p,q} = 0$ for $p \geq 2$, we have an exact sequence

$$0 \to E_2^{1,q-1} \to A^q \to E_2^{o,q} \to 0$$

Moreover

$$\mathrm{Ext}^q(O_{\mathbb{P}}, O_{\mathbb{P}}) = 0 \text{ for } q > 0 \text{ and } \mathrm{Ext}^q(O_{\mathbb{P}}, \underline{F}) = H^q(\underline{F}) \text{ for any } q,$$

and this gives

$$E_2^{o,q} = \ker \alpha^q \text{ and } E_2^{1,q} = \operatorname{coker} \alpha^q \text{ for } q > 0 .$$

Observe also that

$$E_2^{1,o} = \varprojlim^{(1)} \left\{ \begin{tikzcd}[ampersand replacement=\&] \mathrm{Hom}(\underline{F},\underline{F}) \arrow[dr, "\alpha_o^o"'] \& \& \mathrm{Hom}(O_{\mathbb{P}}, O_{\mathbb{P}}) \arrow[dl] \\ \& \mathrm{Hom}(O_{\mathbb{P}},\underline{F}) \& \end{tikzcd} \right\} = \operatorname{coker} \alpha^o$$

because $\mathrm{Hom}(O_{\mathbb{P}}, O_{\mathbb{P}}) \subseteq \mathrm{Hom}(\underline{F},\underline{F})$. We therefore have an exact sequence

$$0 \to \operatorname{coker} \alpha^{q-1} \to A^q \to \ker \alpha^q \to 0$$

for <u>any</u> $q > 0$. Combining with the long exact sequence

$$(*) \quad \begin{array}{l} \longrightarrow \operatorname{Hom}(\underline{F},\underline{F}) \xrightarrow{\alpha^0} H^0(\underline{F}) \to \operatorname{Ext}^1(\underline{I}_C(c_1),\underline{F}) \xrightarrow{\varphi^1} \operatorname{Ext}^1(\underline{F},\underline{F}) \\ \xrightarrow{\alpha^1} H^1(\underline{F}) \to \operatorname{Ext}^2(\underline{I}_C(c_1),\underline{F}) \xrightarrow{\varphi^2} \operatorname{Ext}^2(\underline{F},\underline{F}) \xrightarrow{\alpha^2} H^2(\underline{F}) \to \end{array}$$

deduced from the short exact sequence

$$0 \to O_{\mathbb{P}} \xrightarrow{s} \underline{F} \to \underline{I}_C(c_1) \to 0 ,$$

we find isomorphisms

$$A^q \simeq \operatorname{Ext}^q(\underline{I}_C(c_1),\underline{F}) \quad \text{for} \quad q > 0 .$$

(ii) Let $S \to R$ be a morphism in $\underline{l}$ whose kernel $\mathfrak{A}$ is a k-module via $R \twoheadrightarrow k$, let $s_R : O_{\mathbb{P}_R} \to \underline{F}_R$ be a deformation of $s : O_{\mathbb{P}} \to \underline{F}$ to R, and let $\underline{F}_S$ be a deformation of $\underline{F}_R$ to S. To prove the smoothness of φ, we must find a morphism s_S,

$$s_S : O_{\mathbb{P}_S} \to \underline{F}_S$$

such that $s_S \otimes_S 1_R = s_R$, i.e. we must prove that $s_R \in H^0(\underline{F}_R)$ is contained in the image of $H^0(\underline{F}_S) \to H^0(\underline{F}_R)$. Since

$$0 \to \underline{F} \otimes_k \mathfrak{A} \to F_S \to \underline{F}_R \to 0$$

is exact and since $H^1(\underline{F}) = 0$ by assumption, we see that $H^0(\underline{F}_S) \to H^0(\underline{F}_R)$ is surjective and we are done.

<u>Remark 1.2.</u> In the exact sequence (*) of this proof, φ^1 is the tangent map of $\varphi : \operatorname{Def}_{\underline{F},s} \to \operatorname{Def}_{\underline{F}}$ and φ^2 maps "obstructions to obstructions". In fact φ is a morphism of principal homogeneous spaces via φ^1. Using this it is in general rather easy to prove the smoothness of φ directly from the surjectivity of φ^1 and the injectivity of φ^2. This gives another proof of (1.1. ii) .

2. The relationship between the deformations of a reflexive sheaf with a section and the deformations of the corresponding curve.

Let $\underline{F}$, $s \in H^0(\underline{F})$ and $\underline{I} = \underline{I}_C = \ker(O_{\mathbb{P}} \to O_C)$ be as in the preceding section, and let $\mathrm{Def}_{\underline{I}} : \underline{l} \to \mathrm{Sets}$ be the deformation functor of the $O_{\mathbb{P}}$-Module $\underline{I}$. Then there is a natural map

$$\psi : \mathrm{Def}_{\underline{F},s} \to \mathrm{Def}_{\underline{I}}$$

defined by

$$\psi(\underline{F}_R, s_R) = \underline{M}_R \otimes (O_{\mathbb{P}}(-c_1) \otimes_k R)$$

where $\underline{M}_R = \operatorname{coker} s_R$. If $\mathrm{Hilb}_C : \underline{l} \to \underline{\mathrm{Sets}}$ is the local Hilbert functor at $C \subseteq \mathbb{P}$, we have also a natural map

$$\mathrm{Hilb}_C \to \mathrm{Def}_{\underline{I}}$$

of functors on $\underline{l}$. Recall that C is locally Cohen Macaulay and equidimensional [H 3, 4.1].

Proposition 2.1. (i) The natural morphism

$$\mathrm{Hilb}_C \to \mathrm{Def}_{\underline{I}}$$

is an isomorphism of functors.

(ii) If $H^1(\underline{F}(-4)) = 0$, then

$$\psi : \mathrm{Def}_{\underline{F},s} \to \mathrm{Def}_{\underline{I}}$$

is a smooth morphism of functors on l.

Observe also that

$$H^1(\underline{F}(-4)) \simeq H^1(\underline{I}_C(c_1-4))$$

and moreover by duality that

$$\mathrm{Ext}^2_{\mathcal{O}_{\mathbb{P}}}(\underline{I}_C(c_1),\mathcal{O}_{\mathbb{P}}) = H^1(\underline{I}_C(c_1-4))^\vee .$$

Proof of (i) If $\underline{N}_C = \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}}}(\underline{I},\mathcal{O}_C)$ is the normal bundle of C in $\mathbb{P}$, we proved in [K1,2.2] that

$$H^i(\underline{N}_C) \simeq \mathrm{Ext}^{i+1}_{\mathcal{O}_{\mathbb{P}}}(\underline{I},\underline{I}) \qquad \text{for } i = 0,1$$

as a consequence of the fact that the projective dimension of the $\mathcal{O}_{\mathbb{P}}$-Module $\underline{I}$ is 1, from which the conclusion of (i) is easy to understand. We will, however, give a direct proof.

To construct the inverse of $\mathrm{Hilb}_C(R) \to \mathrm{Def}_{\underline{I}}(R)$, let $\underline{M}_R$ be a deformation of $\underline{I}$ to R. Observe that there is an exact sequence

$$(*) \qquad 0 \to \underline{E} \to \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}}(-n_i) \xrightarrow{f} \underline{I} \to 0$$

where $\underline{E}$ is a vector bundle on $\mathbb{P}$ of rank r. $\overset{r}{\wedge}\underline{E}$ is therefore invertible, and we can identify it with $\mathcal{O}_{\mathbb{P}}(d_1)$ where $d_1 = -\Sigma n_i$. If $\underline{P} = \oplus\mathcal{O}_{\mathbb{P}}(-n_i)$, then there is a complex

$$(**) \qquad \underline{E} \to \underline{P} \simeq (\overset{r}{\wedge}\underline{P})^\vee(d_1) \to (\overset{r}{\wedge}\underline{E})^\vee(d_1) = \mathcal{O}_{\mathbb{P}}$$

and it is well known that the maps $\underline{P} \xrightarrow{f} \underline{I} \subseteq \mathcal{O}_{\mathbb{P}}$ and $\underline{P} \to \mathcal{O}_{\mathbb{P}}$ deduced from (*) and (**) respectively are equal up to a unit of k. We can assume equality. Now since $\underline{M}_R$ is a lifting of $\underline{I}$ to R, there is a map

$$f_R : \underline{P}_R = \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}_R}(-n_i) \to \underline{M}_R$$

such that $f_R \otimes_R 1_k = f : \underline{P} \to \underline{I}$. By Nakayama's lemma, f_R is surjective. Moreover if $\underline{E}_R = \ker f_R$, we easily see that $\underline{E}_R \otimes_R k = \underline{E}$

and $\underline{E}_R$ is R-flat. It follows that $\underline{E}_R$ is a locally free $O_{\mathbb{P}_R}$-Module of rank r satisfying

$$\overset{r}{\wedge}\underline{E}_R = O_{\mathbb{P}_R}(d_1) .$$

Furthermore there is a complex

$$\underline{E}_R \to \underline{P}_R \simeq (\overset{r}{\wedge}\underline{P}_R)^\vee(d_1) \to (\overset{r}{\wedge}\underline{E}_R)^\vee(d_1) = O_{\mathbb{P}_R}$$

which proves the existence of an $O_{\mathbb{P}_R}$-linear map

$$\alpha : \underline{M}_R \to O_{\mathbb{P}_R}$$

which reduces to the natural inclusion $\underline{I} \subseteq O_{\mathbb{P}}$ via $(-)\otimes_R k$. It is easy to see that α is injective, that $\operatorname{coker}\alpha$ is R-flat and that $\operatorname{coker}\alpha \otimes_R k = O_C$. We therefore have a deformation $C_R \subseteq \mathbb{P}_R$ of $C \subseteq \mathbb{P}$. Finally to see that the inverse of $\mathrm{Hilb}_C(R) \to \mathrm{Def}_{\underline{I}}(R)$ is well-defined, let $\beta : \underline{M}_R \xrightarrow{\sim} \underline{M}_R'$ and $\alpha' : \underline{M}_R' \to O_{\mathbb{P}_R}$ be $O_{\mathbb{P}_R}$-linear maps such that $\beta \otimes_R 1_k$ is the identity on $\underline{I}$ and $\alpha' \otimes_R 1_k$ is the natural inclusion $\underline{I} \subseteq R$. (We do not assume $\alpha'\beta = \alpha$). We claim that $\operatorname{Im}\alpha' = \operatorname{Im}\alpha$. In fact since

$$\operatorname{Ext}^i_{O_{\mathbb{P}}}(O_C, O_{\mathbb{P}}) = 0 \qquad \text{for } i = 0,1,$$

we have

$$k = \operatorname{Hom}_{O_{\mathbb{P}}}(O_{\mathbb{P}}, O_{\mathbb{P}}) \xrightarrow{\sim} \operatorname{Hom}_{O_{\mathbb{P}}}(\underline{I}, O_{\mathbb{P}}) .$$

We deduce that the map

$$R = \operatorname{Hom}_{O_{\mathbb{P}_R}}(O_{\mathbb{P}_R}, O_{\mathbb{P}_R}) \to \operatorname{Hom}_{O_{\mathbb{P}_R}}(\underline{M}_R, O_{\mathbb{P}_R})$$

induced by α, is surjective. Hence

$$\alpha'\beta = r\alpha$$

for some $r \in R$, and since $\alpha' \beta \otimes 1_k = \alpha \otimes 1_k$ is the natural inclusion $\underline{I} \subseteq O_{\mathbb{P}}$, r is a unit and we are done.

(ii) Let $S \to R$, $\mathfrak{a}$ and $s_R : O_{\mathbb{P}_R} \to \underline{F}_R$ be as in the proof of (1.1 ii). Moreover let $\underline{M}_R = \operatorname{coker} s_R$, and let $\underline{M}_S$ be a deformation of $\underline{M}_R$ to S. To prove smoothness we must find a deformation

$$s_S : O_{\mathbb{P}_S} \to \underline{F}_S$$

with cokernel $\underline{M}_S$ such that $s_S \otimes_S 1_R = s_R$. By theory of extensions it is sufficient to prove that the map

$$\operatorname{Ext}^1_{O_{\mathbb{P}_S}}(\underline{M}_S, C_{\mathbb{P}_S}) \to \operatorname{Ext}^1_{O_{\mathbb{P}_R}}(\underline{M}_R, O_{\mathbb{P}_R})$$

induced by $(-) \otimes_S R$ is surjective. Modulo isomorphisms we refind this map in the long exact sequence

$$\to \operatorname{Ext}^1(\underline{M}_S, O_{\mathbb{P}_S} \otimes \mathfrak{a}) \to \operatorname{Ext}^1(\underline{M}_S, O_{\mathbb{P}_S}) \to \operatorname{Ext}^1(\underline{M}_S, O_{\mathbb{P}_R}) \to \operatorname{Ext}^2(\underline{M}_S, O_{\mathbb{P}_S} \otimes \mathfrak{a}).$$

Since $\operatorname{Ext}^2_{O_{\mathbb{P}_S}}(\underline{M}_S,\ O_{\mathbb{P}_S} \otimes_S \mathfrak{a}) \simeq \operatorname{Ext}^2_{O_{\mathbb{P}}}(\underline{I}_C(c_1), O_{\mathbb{P}}) \otimes \mathfrak{a} = 0$ by assumption, we are done.

<u>Remark 2.2.</u> The short exact sequence

$$\xi : 0 \to O_{\mathbb{P}} \xrightarrow{s} \underline{F} \to \underline{I}_C(c_1) \to 0$$

induces a long exact sequence

$$\to \operatorname{Ext}^1_{O_{\mathbb{P}}}(\underline{I}_C(c_1), O_{\mathbb{P}}) \to \operatorname{Ext}^1_{O_{\mathbb{P}}}(\underline{I}_C(c_1), \underline{F}) \xrightarrow{\psi^1} \operatorname{Ext}^1_{O_{\mathbb{P}}}(\underline{I}_C, \underline{I}_C) \to$$

$$\operatorname{Ext}^2_{O_{\mathbb{P}}}(\underline{I}_C(c_1), O_{\mathbb{P}}) \to \operatorname{Ext}^2_{O_{\mathbb{P}}}(\underline{I}_C(c_1), \underline{F}) \xrightarrow{\psi^2} \operatorname{Ext}^2_{O_{\mathbb{P}}}(\underline{I}_C, \underline{I}_C) \to$$

where ψ^1 is the tangent map of ψ or more generally, ψ is a map of principal homogeneous spaces via ψ^1 and ψ^2 maps "obstructions to obstructions". As remarked in (1.2), the smoothness of ψ follows therefore from the surjectivity of ψ^1 and the injectivity of ψ^2.

Remark 2.3. Let ξ be the extension

$$0 \to O_{\mathbb{P}} \xrightarrow{s} \underline{F} \to \underline{I}_C(c_1) \to 0$$

and let $\mathrm{Def}_{C,\xi} : \underline{l} \to \underline{\mathrm{Sets}}$ be the functor defined by

$$\mathrm{Def}_{C,\xi}(R) = \left\{ (C_R, \xi_R) \;\middle|\; \begin{array}{l} (C_R \subseteq \mathbb{P}_R) \in \mathrm{Hilb}_C(R) \text{ and } \xi_R \in \\ \mathrm{Ext}^1(\underline{I}_{C_R}(c_1), O_{\mathbb{P}_R}) \text{ satisfies} \\ \xi_R \otimes_R k = \xi \end{array} \right\} / \sim$$

Two deformations (C_R, ξ_R) and (C'_R, ξ'_R) are equivalent if $C_R = C'_R \subseteq \mathbb{P}_R$ and if there is a commutative diagram

$$\begin{array}{ccccccccccc} \xi'_R : & 0 & \to & O_{\mathbb{P}_R} & \to & \underline{F}_R & \to & \underline{I}_{C_R}(c_1) & \to & 0 \\ & & & \downarrow & \circ & \downarrow & \circ & \| \; 1 & & \\ \xi'_R : & 0 & \to & O_{\mathbb{P}_R} & \to & \underline{F}'_R & \to & \underline{I}_{C_R}(c_1) & \to & 0 , \end{array}$$

both reducing to the extension ξ via $(-)\otimes_R k$. In the same way we identify the given (C, ξ) with any (C', ξ') provided $C = C'$ and $\xi' = u\xi$ for some unit $u \in k^*$. Note that we may in this definition of equivalence replace the identity 1 on $\underline{I}_{C_R}(c_1)$ by any $O_{\mathbb{P}_R}$ linear map. See [Ma 2, 6.1] and recall $\mathrm{Hom}(\underline{I}_C, \underline{I}_C) = k$. Now there is a forgetful map

$$\alpha : \mathrm{Def}_{C,\xi} \to \mathrm{Def}_{\underline{F},s} ,$$

and using (2.1i) we immediately have an inverse of α. Hence α is an isomorphism. Observe that we might construct the inverse of $\alpha(R)$ for $R \in \mathrm{ob}\,\underline{l}$ by considering the invertible sheaf $\det \underline{F}_R$ on $\mathbb{P}_R$. See [Ma 1, 4.2] or [G,4.1]. In fact if $(\underline{F}_R, s_R)$ is given, there is an $\mathbb{P}_R$ a morphism

$$i : \overset{2}{\wedge} \underline{F}_R \to \det \underline{F}_R \simeq O_{\mathbb{P}_R}(c_1)$$

and a complex

$$0 \to O_{\mathbb{P}_R} \xrightarrow{s_R} \underline{F}_R \xrightarrow{i[(-)\wedge s_R]} O_{\mathbb{P}_R}(c_1)$$

which after the tensorization $(-)\otimes_R k$ is exact. Hence

$$0 \to O_{\mathbb{P}_R} \xrightarrow{s_R} \underline{F}_R \to \operatorname{coker} s_R \to 0$$

is exact, $\operatorname{coker} s_R$ is R-flat and $\operatorname{coker} s_R \hookrightarrow O_{\mathbb{P}_R}(c_1)$, and putting this together, we can find an inverse of $\alpha(R)$. One should compare the isomorphism of α with [H 3, 4.1] which implies that there is a bijection between the set of pairs $(\underline{F}, s)$ and the set of (C, ξ) moduls equivalence under certain conditions on the pairs. Thinking of these families of pairs as moduli spaces, [H 3, 4.1] establishes a bijection on the k-points of these spaces while the isomorphism of α takes care of the scheme structure as well.

To be more precise we claim that there is a quasiprojective scheme D parametrizing equivalent pairs (C, ξ) where

1) C is an equidimensional Cohen Macaulay curve and where

2) the extension $\xi : 0 \to O_{\mathbb{P}} \to \underline{F} \to \underline{I}_C(c_1) \to 0$ is such that $\underline{F}$ is a stable reflexive sheaf.

Moreover there are projection morphisms

$$(*) \qquad \begin{array}{ccc} D & \xrightarrow[q]{} & H(d,g) \\ {\scriptstyle p}\downarrow & & \\ M(c_1,c_2,c_3) & & \end{array}$$

defined by $p(\underline{F}_K, s_K) = \underline{F}_K$ and $q(C_K, \xi_K) = C_K$ for a geometric K-point (C_K, ξ_K) corresponding to $(\underline{F}_K, s_K)$, such that the fibers of p and q are smooth connected schemes. Furthermore, p is smooth at $(\underline{F}_K, s_K)$ provided $H^1(\underline{F}_K) = 0$, and q is smooth at (C_K, ξ_K) provided $H^1(\underline{I}_{C_K}(c_1-4)) = 0$.

To indicate why [1)] let $\underline{Sch}/k$ be the category of locally noetherian k-schemes and let $\underset{\sim}{D} : \underline{Sch}/k \to \underline{Sets}$ be the functor defined by

$$\underset{\sim}{D}(S) = \left\{ (C_S, \underline{L}_S, \xi_S) \;\middle|\; \begin{array}{l} C_S \in \underset{\sim}{H(d,g)}(S),\ \underline{L}_S \text{ is invertible on } S \text{ and} \\ \xi_S \in \operatorname{Ext}^1(\underline{I}_{C_S}(c_1),\ O_{\mathbb{P}\times S} \otimes \underline{L}_S) \text{ such that} \\ C_S \times_S \operatorname{Spec}(K) \text{ satisfies (1) and } \xi_S \otimes K \neq 0 \\ \text{for any geometric K-point of } S \end{array} \right\} / \sim$$

Two deformations $(C_S, \underline{L}_S, \xi_S)$ and $(C_S', \underline{L}_S', \xi_S')$ are equivalent if $C_S = C_S'$ and if there is an isomorphism $\tau : \underline{L}_S \to \underline{L}_S'$ whose induced morphism $\operatorname{Ext}^1(\underline{I}_{C_S}(c_1), \tau)$ maps ξ_S onto ξ_S'. Now if $U \subseteq H(d,g)$ is the open set of equidimensional Cohen Macaulay curves and if $C_U \subseteq \mathbb{P} \times U \xrightarrow{\pi} U$ is the restricting of the universal curve to U, one may prove that $\underline{E} = \underline{\operatorname{Ext}}^1(\underline{I}_{C_U}(c_1), O_{\mathbb{P}\times U})$ is a coherent $O_{\mathbb{P}\times U}$-Module, flat over U. By [EGA,III,7.7.6] there is a unique coherent O_U-Module $\underline{Q}$ such that

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1) For good ideas of this construction, see the appendix [E,S], some of which appears in [S,M,S].

$$\underline{\mathrm{Hom}}_{O_U}(\underline{Q},\underline{R}) \simeq \pi_*(\underline{E}\otimes\underline{R})$$

for any quasicoherent O_U-Module $\underline{R}$. If $\mathbb{P}(\underline{Q}) = \mathrm{Proj}(\mathrm{Sym}(Q))$ is the projective fiber over U defined by $\underline{Q}$, we can use [EGA II,4.2.3] to prove that

$$\underset{\sim}{\underline{D}}(-) \simeq \mathrm{Mor}_k(-,\mathbb{P}(\underline{Q})) .$$

Now let $D \subseteq \mathbb{P}(\underline{Q})$ be the open set whose k-points are (C,ξ), $\xi : 0 \to O_{\mathbb{P}} \to \underline{F} \to \underline{I}_C(c_1) \to 0$, where $\underline{F}$ is a stable reflexive sheaf. Then we have a diagram (*) where the existence of the morphism p follows from the definition [Ma 1, 5.5] of the moduli space $M = M(c_1,c_2,c_3)$. Moreover since $\mathbb{P}(\underline{Q})$ represents the functor $\underset{\sim}{\underline{D}}$, the fiber of $q : D \to H(d,g)$ at a K-point $C_K \subseteq \mathbb{P}_K$ of $H(d,g)$ is just $D \cap \mathbb{P}(\mathrm{Ext}^1(\underline{I}_{C_K}(c_1),O_{\mathbb{P}_K})^\vee)$ where $(-)^\vee = \mathrm{Hom}_K(-,K)$. Moreover if we think of the fiber of p at a geometric K-point $\underline{F}_K$ of M as those sections $s \in H^0(\underline{F}_K)$ where $(s)_0$ is a curve, we understand that the fiber is an open subscheme of the linear space $\mathbb{P}(H^0(\underline{F}_K)^\vee)$. In particular the geometric fibers of p and q are smooth and connected.

Finally the smoothness of p and q at (C,ξ) follows from (1.1 ii) and (2.1 ii) provided we know that the morphism $p^* : O_{M,\underline{F}} \to O_{D,(\underline{F},s)}$ induced by $p : D \to M$ makes a commutative diagram

$$\begin{array}{ccc}
\mathrm{Def}_{\underline{F},s} & \simeq & \mathrm{Mor}(\hat{O}_{D,(\underline{F},s)},-) \\
\varphi\downarrow & \circ & \downarrow \mathrm{Mor}(p^*,-) \\
\mathrm{Def}_{\underline{F}} & \simeq & \mathrm{Mor}(\hat{O}_{M,\underline{F}},-)
\end{array}$$

of horisontal isomorphisms on $\underline{l}$. In fact the commutativity from the definition of a moduli space [Ma 1, 5.5] while the construction of M implies the lower horizontal isomorphism. See [Ma 2, 6.4] from which we immediately have that the morphism $\mathrm{Def}_{\underline{F}} \to \mathrm{Mor}(\hat{O}_{M,\underline{F}},-)$ is smooth, and since the morphism induces an isomorphism of tangent spaces, both isomorphic to $\mathrm{Ext}^1(\underline{F},\underline{F})$, it must be an isomorphism.

Remark 2.4. In particular the smoothness of $\mathrm{Def}_{\underline{F}} \to \mathrm{Mor}(\hat{O}_{M,\underline{F}},-)$ which is a consequence of the smoothness of the morphism treated in [Ma 2, 6.4], implies that $O_{M,\underline{F}}$ is a regular local ring if and only if $\mathrm{Def}_{\underline{F}}$ is a smooth functor on $\underline{l}$.

3. Non-reduced components of the moduli scheme $M(c_1,c_2,c_3)$.

One knows that the Hilbert scheme $H(d,g)$ is not always reduced. In fact if g is the largest number satisfying $g \leq \frac{d^2-4}{8}$, we proved in [Kl, 3.2.10] that $H(d,g)$ is non-reduced for every $d \geq 14$, and we explicitely described a non-reduced component in terms of the Picard group of a smooth general cubic surface.

Example 3.1. (Mumford [M 1]). For $d = 14$, we have $g = \frac{d^2-4}{8} = 24$, and there is an open irreducible subscheme $U \subseteq H(14,24)$ of smooth connected curves whose closure $\overline{U} = W$ makes a non-reduced component, such that for any $(C \subseteq \mathbb{P}) \in U$,

$$h^0(\underline{I}_C(\nu)) = \begin{cases} 0 & \text{for } \nu \leq 2 \\ 1 & \text{for } \nu = 3 \end{cases}$$

$$h^1(\underline{I}_C(\nu)) = 0 \qquad \text{for } \nu \notin \{3,4,5\},$$

$$h^1(O_C(\nu)) = \begin{cases} 0 & \text{for } \nu \geq 4 \\ 1 & \text{for } \nu = 3. \end{cases}$$

See [Kl,(3.2.4) and (3.1.3)]. In fact with $C \subseteq \mathbb{P}$ in U, there is a global complete intersection of two surfaces of degree 3 and 6 whose corresponding linked curve is a disjoint union of two coniques.

Now let $C \subseteq \mathbb{P}$ be a smooth connected curve satisfying

$$(*) \quad H^1(\underline{I}_C(c_1)) = 0, \quad H^1(\underline{I}_C(c_1-4)) = 0 \quad \text{and} \quad H^1(O_C(c_1-4)) \neq 0$$

for some integer c_1, let $\xi \in H^0(\omega_C(4-c_1)) = \mathrm{Ext}^1(\underline{I}_C(c_1), O_{\mathbb{P}})$ be non-trivial, and let $(\underline{F}, s)$, $s \in H^0(\underline{F})$, correspond to (C, ξ) via the usual correspondence. Then $\underline{F}$ is reflexive, and it is stable (resp. semistable) if and only if $c_1 > 0$ (resp. $c_1 \geq 0$) <u>and</u> C is not contained in any surface of degree $\leq \frac{1}{2} c_1$ (resp. $< \frac{1}{2} c_1$). See [H 3, 4.2]. Combining (1.1) and (2.1) with (2.4) in case $\underline{F}$ is stable, we find that $O_{M,\underline{F}}$ is non-reduced iff $O_{H,C}$ is non-reduced.

<u>Example 3.2.</u> Let $(C \subseteq \mathbb{P}) \in H(14,24)$ belong to the set U of (3.1) and let c_1 be an integer satisfying (*), i.e. $c_1 \leq 2$ or $c_1 = 6$.

(i) Let $c_1 = 6$. By virtue of (1.1) and (2.1) the hull of $\mathrm{Def}_{\underline{F}}$ is non-reduced. Moreover $\underline{F}$ is semistable with Chern classes $(c_1, c_2, c_3) = (6,14,18)$, and the normalized sheaf $\underline{F}(-3)$ has Chern classes $(c_1', c_2', c_3') = (0,5,18)$.

(ii) Let $c_1 = 2$. The corresponding reflexive sheaf is stable and must belong to at least one non-reduced component of $M(2,14,74)$, i.e. of $M(0,13,74)$.

(iii) With $c_1 = 1$ we find at least one non-reduced component of $M(1,14,88) \simeq M(-1,14,88)$.

Combining the discussion after (2.3) and in particular the irreducibility of the morphism q with the irreducibility of the set U of (3.1), we see that we obtain precisely one non-reduced component of $M(0,13,74)$ and $M(-1,14,88)$ in this way.

We will give one more example of a non-reduced component and include a discussion to better understand (1.1) and (2.1). In fact recall [Kl,2.3.6] that if an equidimensional Cohen Macaulay curve $(C \subseteq \mathbb{P}) \in H(d,g)$ is contained in a complete intersection $V(\underline{F}_1,\underline{F}_2)$ of two surfaces of degree $f_1 = \deg F_1$ and $f_2 = \deg F_2$ with

$$H^1(\underline{I}_C(f_i)) = 0 \quad \text{and} \quad H^1(\underline{I}_C(f_i-4)) = 0$$

for $i = 1,2$, and if $(C' \subseteq \mathbb{P}) \in H' = H(d',g')$ is the linked curve, then $O_{H,C}$ is reduced iff $O_{H',C'}$ is reduced. Since any curve $(C \subseteq \mathbb{P}) \in U$ of (3.1) is contained in a complete intersection $V(\underline{F}_1,\underline{F}_2)$ of two surfaces of degree $f_1 = f_2 = 6$, the linked curves $C' \subseteq \mathbb{P}$ must belong to at least one (and one may prove to exactly one) <u>non-reduced</u> <u>component</u>[1] $W \subseteq H(22,56)$ of dimension 88. See [Kl,2.3.9]. One may see that W contains smooth connected curves. Moreover using the fact that $\omega_C(4-f_1-f_2)$ and $\omega_{C'}(4-f_1-f_2)$ are the sheaves of ideals which define the closed subschemes $C' \subseteq V(\underline{F}_1,F_2)$ and $C \subseteq V(\underline{F}_1,\underline{F}_2)$ respectively, one proves easily that

$H^0(\underline{I}_{C'}(4)) = 0$, $H^1(\underline{I}_{C'}(\nu)) = 0$ for $\nu \notin \{3,4,5\}$ and $H^1(O_{C'}(5)) \neq 0$. See [S,P] and [Kl,2.3.3].

1) The condition $H^1(\underline{I}_C(f_i-4)) = 0$ implies also that the linked curves $C' \subseteq \mathbb{P}$ form an <u>open</u> subset of H'.

Example 3.3. Let $(C' \subseteq \mathbb{P}) \in W \subseteq H(22,56)$ be as above with C' smooth and connected. If c_1 is chosen among $1 \leq c_1 \leq 9$, then $C' \subseteq \mathbb{P}$ defines a stable reflexive sheaf $\underline{F}'$ and in fact a vector bundle if $c_1 = 9$ by the usual correspondence. Using (1.1) and (2.1) we find that $\underline{F}'$ belongs to a non-reduced component of $M(c_1,c_2,c_3)$ for the choices $1 \leq c_1 \leq 2$ or $c_1 = 6$. In particular there exists a non-reduced component of $M(6,22,66) \simeq M(0,13,66)$. Moreover we obtain precisely one non-reduced component in this way if we make use of the discussion after (2.3). If $c_1 = 9$, we find a reflexive sheaf $\underline{F}' \in M(9,22,0)$, and the normalized one is $\underline{F}'(-5) \in M(-1,2,0)$, but we can not conclude that $M(-1,2,0)$ is non-reduced, even though $H(22,56)$ is, because the condition $H^1(\underline{I}_C(c_1-4)) = 0$ of (2.1. ii) is not satisfied. In fact one knows that $M(-1,2,0)$ is a smooth scheme. See [H,S] or [S,M,S].

As a starting point of these final considerations, we will suppose as known that there is an open smooth connected subscheme $U_M \subseteq M(-1,2,0)$ of stable reflexive sheaves $\underline{F}$ for which there exists a global section $s \in H^0(\underline{F}(2))$ whose corresponding scheme of zero's $C' = (s)_0$ is a disjoint union of two coniques. Moreover $\dim U_M = 11$. In fact [H,S] proves even more. We then have an exact sequence

$$0 \to O_{\mathbb{P}} \to \underline{F}(2) \to \underline{I}_{C'}(3) \to 0$$

for $\underline{F} \in U_M$, and since the dimension of the cohomology groups $H^i(\underline{I}_{C'}(v))$ is easily found in case C' consists of two disjoint coniques, we get

$$h^0(\underline{F}(1)) = h^0(\underline{I}_{C'}(2)) = 1$$

and

$$h^1(\underline{F}(\nu)) = h^1(\underline{I}_{C'}(\nu+1)) = \begin{cases} 1 & \text{for } \nu = -1,1 \\ 2 & \text{for } \nu = 0 \\ 0 & \text{for } \nu \notin \{-1,0,1\}. \end{cases}$$

By $\dim U_M = 11$, $\mathrm{Ext}^2_{O_{\mathbb{P}}}(\underline{F},\underline{F}) = 0$. (The reader who is more familier with the Hilbert scheme may prove our assumptions on U_M by first proving that there is an open smooth connected subscheme $U \subseteq H(4,-1)$ of disjoint coniques C' and that $\dim U = 16$. This is in fact a very special case of [Kl, (3.1.10 i)]. See also [Kl, (3.1.4) and (2.3.18)]. With $c_1 = 3$, we have $H^1(\underline{I}_{C'}(c_1)) = H^1(\underline{I}_{C'}(c_1-4)) = 0$, and by the discussion after (2.3), there exists an open smooth connected subscheme of $M(3,4,0) \xrightarrow[i]{\sim} M(-1,2,0)$ defined by $U_M = i(p(q^{-1}(U)))$. Moreover $\dim U_M = 11$ because $\dim U_M + h^0(\underline{F}(2)) = \dim U + h^0(\omega_{C'}(4-c_1))$).

Fix an integer $\nu \geq 1$, and let $U(\nu)$ be the subset of $H(d,g)$ obtained by varying $\underline{F} \in U_M \subseteq M(-1,2,0)$ and by varying the sections $s \in H^0(\underline{F}(\nu))$ so that $C = (s)_0$ is a curve, i.e. let $U(\nu) = q(p^{-1}(U_M))$ and regard U_M as a subscheme of $M(c_1,c_2,0)$ with

$$c_1 = 2\nu-1, \quad c_2 = 2-\nu+\nu^2, \quad d = c_2 \quad \text{and} \quad g = 1 + \tfrac{1}{2} c_2 (c_1-4).$$

Recall that p and q are projection morphisms

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"] & H(d,g) \\ M(c_1,c_2,0) & \end{tikzcd}$$

For $(C \subseteq \mathbb{P}) \in U(\nu)$, there is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \underline{F}(\nu) \to \underline{I}_C(2\nu-1) \to 0$$

some $\underline{F}(\nu) \in U_M$. Now (1.1. ii) and (2.1 ii) apply for $\nu = 2$ and all $\nu \geq 6$, and it follows that $H(d,g)$ is <u>smooth</u> at any $(C \subseteq \mathbb{P})$ in the <u>open</u> subset $U(\nu) \subseteq H(d,g)$. Moreover by the irreducibility of p, $U(\nu)$ is an open smooth connected subscheme of $H(d,g)$. Furthermore

$$\dim U(\nu) = 4d + \frac{1}{6}\nu(\nu-5)(2\nu-5) \qquad \text{for} \quad \nu \geq 6$$

(resp $= 4d$ for $\nu = 2$) which asymptotically is $\sim 4d + \frac{1}{3}d^{3/2}$ for $\nu >> 0$. To find the dimension of $U(\nu)$, we use the fact that p and q are smooth morphisms of relative dimension $h^0(\underline{F}(\nu)) - 1$ and $h^0(\omega_C(4-c_1)) - 1$ respectively. This gives

$$\dim U_M + h^0(\underline{F}(\nu)) = \dim U(\nu) + h^0(\omega_C(4-c_1))$$

for $\nu = 2$ and $\nu \geq 6$, and since $h^0(\omega_C(4-c_1)) = h^1(\mathcal{O}_C(c_1-4)) = 1$ for $\nu \geq 6$ (resp. $= 2$ for $\nu = 2$), we get

$$\dim U(\nu) = 10 + h^0(\underline{F}(\nu)) \qquad \text{for} \quad \nu \geq 6$$

(resp. $= 9 + h^0(\underline{F}(\nu))$ for $\nu = 2$). The reader may verify that $h^0(\underline{F}(\nu)) = \chi(\underline{F}(\nu)) = \frac{1}{6}(\nu-1)(2\nu+3)(\nu+4) = 4d + \frac{1}{6}(\nu-5)(2\nu-5)\nu - 10$ for any $\nu \geq 2$, and the conclusion follows.

We will now discuss the cases $3 \leq \nu \leq 5$ where we can not guarantee the smoothness of q since (2.1. ii) does not apply. If $\nu = 5$, then the closure of $U(5)$ in $H(22,56)$ makes a non-reduced component by (3.3). For $\nu = 3$ or 4, we claim that $H(d,g)$ is smooth along $U(\nu)$ and the codimension

$$\dim W - \dim U(\nu) = h^1(\underline{I}_C(c_1-4)) = h^1(\underline{F}(-4))$$

where W is the irreducible component of $H(d,g)$ which contains $U(\nu)$. To see this it suffices to prove $H^1(\underline{N}_C) = 0$ and $\mathrm{Ext}^2(\underline{I}_C(c_1),\underline{F}(\nu)) = 0$ for any $(C \subseteq \mathbb{P}) \in U(\nu)$ because these conditions imply that the scheme D and $H(d,g)$ are non-singular at any (C,ξ) with $\xi \in H^0(\omega_C(4-c_1))$ and $(C \subseteq \mathbb{P}) \in H(d,g)$ respectively. See (1.1 i). Moreover if these "obstruction groups" vanish, we find

$$\dim W - \dim U(\nu) = \dim W - \dim q^{-1}(U(\nu)) = h^0(\underline{N}_C) - \dim \mathrm{Ext}^1(\underline{I}_C(c_1), \underline{F}(\nu))$$
$$= h^1(\underline{I}_C(c_1-4))$$

where $\dim U(\nu) = \dim q^{-1}(U(\nu))$ because of $h^0(\omega_C(4-c_1)) = 1$, and where the equality to the right follows from the long exact sequence of (2.2). Now to prove $\mathrm{Ext}^2(\underline{I}_C(c_1),\underline{F}(\nu)) = 0$ we use the long exact sequence (*) in the proof of (1.1. i) combined with $H^1(\underline{F}(\nu)) = 0$ and $\mathrm{Ext}^2(\underline{F},\underline{F}) = 0$, and to prove $H^1(\underline{N}_C) = 0$ we use the long exact sequence of (2.2) combined with $\mathrm{Ext}^2(\underline{I}_C(c_1),\underline{F}(\nu)) = 0$ and $\mathrm{Ext}^3(\underline{I}_C(c_1),O_{\mathbb{P}}) \simeq H^0(\underline{I}_C(c_1-4))^{\vee} = H^0(\underline{F}(\nu-4))^{\vee} = 0$ for $\nu = 3$ or $\nu = 4$, and we are done.

Computing numbers, we find for $\nu = 3$ that $U(3)$ is a locally closed subset of $H(8,5)$ of codimension 1, and any smooth connected curve $(C \subseteq \mathbb{P}) \in U(3)$ is a canonical curve, i.e. $\omega_C \simeq O_C(1)$. For $\nu = 4$, $U(4)$ is of codimension 2 in $H(14,22)$ and $\omega_C \simeq O_C(2)$ for any $(C \subseteq \mathbb{P}) \in U(4)$.

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