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THE GROUP OF LOCAL AUTOMORPHISMS OF THE MINKOFSKY SPACE

by

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Summary. We discuss the local automorphisms of the Minkosky space, and find that they form a simple Lie group, which is a subgroup of index two of $O(2,4)$.

I. - Introduction.

The Minkosky space M is the four-dimensional space time with the structure given by the light cones through each space time point. We will also use the term "event" for a space time point. Let a be a space time point. The forward light cone at a, is physicaly, the set of events that will be reached by a light signal given at a, and the backward light cone at a is the set of events from which a light signal will reach a.

We may identify the points of the Minkofsky space with the points in R^4. Let us introduse the Lorentz form $x \cdot y = x_0 y_0 - x_1 y_2 - x_2 y_2 - x_3 y_3$. The light cone at a is the set of points $x \in M$ such that $(x-a)^2 = 0$, where we use the notation $x^2 = x \cdot x$. The forward light cone is the subset such that $x_0 \geqslant a_0$, and the backward light cone is the subset such that $x_0 \leq a_0$. By an automorphism of the Minkofsky space M , we understand a one to one map of M onto M , such that light cones are mapped onto light cones, and forward light cones are mapped onto forward light cones. In [1] Zeeman has determined the group of automorphism of the Minkofsky space. An other way of looking on the group of automorphisms of the Minkofsky space is in terms of frames. The identification of M with R^4 considered above is of course in no way natural or unique. Such an identification may be called a coordinatization of M . We will say that a coordinatization is a frame iff the velocity of light is 1 in all directions and at any space time point. Written out, this is just that if a point in M has coordinates $a \in R^4$, then the events on the light cone at a has coordinates $(x-a)^2 = 0$, and the forward light cone at a , $x_0 \geqslant a_0$. The group of automorphisms of M is then nothing but the group of coordinate transformations between frames. Zeeman has proved in 1 that this group is the group G_0 , generated by the Poincaré-group P and the expansions.

By a local automorphism of M , we understand a one to one map of an open conected set of M onto an open conected set of M , such that the part of the light cones where it is defined goes into light cones, and the forward light cones into forward light cones. The set of local automorphisms of M , does not of course form a group under composition, but only a local group, since one may only compose two local automorphisms if the range of the one has something in common with the domain of the other. What we prove in this paper is that this local group comes from a group by restricting the operation of multiplication of two elements; in the sense that any local automorphism may be exdended in a unique way to an automorphism of a certain projectiv space, here called the projectiv Minkofsky space PM ; and that any automorphism of PM is also a local automorphism of M . The local automorphisms of M may also be looked upon in terms of local frame. A local frame is a local coordinatization of M , such that the velocity of light is 1 in all directions, at all events covered by the coordinate neighbourhood. The local automorphisms of M is then nothing but the coordinate transformations between local frames.

We see that the automorphisms of M , is dependent of the large scale structure of the physical space time, which in this case is take to be flat. While the main use of the group of automorphisms of M , as the group of coordinate transformations between frames, is in relativistic high energy physics in connection with scattering of elementary particles. These scattering experiments in high energy physics are of a very local nature as well in space as in time, and one should not expect the results to be dependent on the large scale structure of space time. Moreover the frames used as frames of reference for these experiments, are always local frames, that can not be extended to global frames, due to the rotation of the earth.

This indicates that the group of local automorphisms of M , may be better suited for the analysis of the experiments in high energy physics. One of the things the author has specially in mind is the group theoretical classification of the elementary particles.

In a forth coming paper, in this journal, the author will treat the question of classification of the elementary particles by the group of local automorphisms of the Minkofsky space.

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2. - The inversion in Minkofsky space.

In R^4 we introduce the inner product $x \cdot y = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$, and the notation $x^2 = x \cdot x$. The light cone at a point a R^4 , is the set of points $(x-a)^2 = 0$, the forward light cone at a is the subset of the light cone at a such that $x_0 \geqslant a_0$, and the backward light cone at a is the subset of the light cone at a such that $x_0 \leq a_0$.

Definition 1. The Minkofsky space M is R^4 with the structure given by all forward and backward light cones.

For two points a and b in M we will use the following notations. If b is inside the forward light cone at a , we say that b is in the future of a and that a is in the past of b . If b is neighter in the future nor in the past nor on the light come of a , we say that b is space like to a . We see that if b is space like to a then a is space like to b .

Definition 2. An automorphism of M is a point map of M that is one to one and onto and preserve the structure of M . That is, it maps forward light cones onto forward light cones, and backward light cones onto backward light cones.

We shall say that a set of points in M is open, iff it is open in the usual topology of R^4 . A line in R^4 that is contained in a light cone is called a light line in M . We see that any light line in M has a natural ordering, namely such that the part of it that is in the forward light cone comes after the part in the backward light cone. This ordering is obviously independent of the which light cone we choose. We shall say that a set of points in M is light convex, iff the intersection of the set with any light line is connected.

Definition 3. A local automorphism of M , is a one to one map of an open light convex set A onto an open light convex set B , such that it preserves the structure of M . That is, it maps the intersection of the forward light cone of a point in A onto the intersection of the forward light cone of the image point, with B , and correspondingly for the backward light cone.

We should remark that we do not require an automorphism nor a local automorphism to be continuous in the topology of R^4 .

The Lorentz group L is a subgroup of index two in O(1,3) , and L acts as a group of automorphisms of M , by its natural action as a group of linear transformations in R^4 leaving the form $x \cdot y$ of index (1,3) invariant. L is the group generated by the component of the identity in O(1,3) and the space reflection, so that L has two connected components. Let T be the abelian group of translations in R^4 . T acts then in a natural way as an abelian group of automorphisms of M . The semi direct product $P = L \cdot T$, is the Poincaré group. Let G_o be the group generated by P and the expansions in R^4 . G_o acts then as a group of automorphisms of M , and it is proved in [1], that this is the group of all automorphisms of M .

We will now exhibit a local automorphism of M , that is not an automorphism of M . It is what will be called the "inversion" in M . The inversion in M is most easely exhibit in the representation of M as two by two Hermitian matrices. Consider therefor the linear map h of R^4 onto the set of two by two Hermitian matrices, given by

$$h(x) = \begin{bmatrix} x_o+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_o-x_3 \end{bmatrix} \quad , \quad x = \{x_o, x_1, x_2, x_3\} .$$

We observe that the determinant of h(x) is equal to $x_o^2 - x_1^2 - x_2^2 - x_3^2$, so we get

$$|h(x)| = x^2$$

So we get that b is on the light cone at a iff $|h(b-a)| = 0$, and that b is on the forward light cone at a iff $|h(b-a)| = 0$ and h(b-a) is a positve matrix. The inversion I in M is then defined by

$$Ix = h^{-1}(-h(x))^{-1}$$

where $(-h(x))^{-1}$ is the invers matrix to $-h(x)$. Ix is then defined for all x such that $|h(x)| \neq 0$. That is for all $x \in M$, such that x is not on the light cone at origo. The complement of the light cone at origo is the union of three open light convex sets S , F and P . S is the set of points space like to origo, F is the set of points future to origo, and P is the set of points past to origo.

For the points in S , $h(x)$ is indefinite; for the points in F, $h(x)$ is positive definite; and for the points in P, $h(x)$ is negative definite. Hence we find that I maps S onto S , F onto P and P onto F . Moreover I^2 = identety.

Lemma 1. The restriction of I ot S , F or P , is a local automorphism of M .

Proof: Let a and b be in the complement of the light cone at origo, and b on the light cone at a , such that a and b are in the same component S , F or P of the complement of the light cone at origo. We will prove that Ib is on the light cone of Ia . Set $b = a + x$ where $|h(x)| = 0$. Then

$$h(Ib) - h(Ia) = (-h(b))^{-1} - (-h(a))^{-1}$$

$$= (h(a))^{-1} - (h(a) + h(x))^{-1}$$

$$= (h(a))^{-1} \cdot h(x) \cdot (h(b))^{-1}$$

Since $|h(x)| = 0$, we see that $|h(Ib) - h(Ia)| = 0$, hence Ib is on the light cone of Ia , and I maps light cones into light cones. We will now prove that the forward light cones are mapped into forward light cones. Let therefore b be on the forward light cone at a . b = a + x , where h(x) is a singular positive Hermitian matrix. Consider the continuous function from $[0,1]$ into the singular Hermitian two by two matrices, defined by

$$f(t) = h(a)^{-1}\, h(x)(h(a) + t h(x))^{-1} \;, \quad t \in [0,1]$$

Since h(x) is singular f(t) is singular. By the formula above, we have that

$$t \cdot f(t) = h(I(a+t\cdot x)) - h(Ia) \;.$$

This gives us that f(t) is Hermitian and non zero for $t \neq 0$. For t = 0 we find

$$f(0) = h(a)^{-1}\, h(x)\, h(a)^{-1}$$

which is obviously Hermitian and non zero. Moreover we see that f(0) is positive. Since it is impossible to come from the positive singular two by two matrices, by a continuous path of singular Hermitian two by two matrics, to a negative singular two by two matrix; with out ever crossing zero; we find that f(1) is a positive matrix. But

$$f(1) = h(Ib) - h(Ia) \;, \quad \text{hence}$$

Ib is on the forward light cone of Ia . This proves the lemma.

We may now generate local automorphisms of M by composing I with elements in G_0 . Let g_1 and g_2 be two elements in G_0 . $g_1 I g_2$ is then a local automorphism of M , that maps the complement of a light cone onto the complement of a light cone. Since the range of $g_1 I g_2$ is the complement of a light cone, it intersect the domain of definition of I in an open set, hence the composition Ig_1Ig_2 is again a local automorphism of M, and again we may compose by an element in G_o. In this way we generate a group of local automorphisms. We will denote, this group of local automorphisms generated by G_o and I, by G. In the next section we will study this group. and prove that G is a Lie group.

3. - The projective Minkofsky space.

Let $(,)$ be the bilinear form in R^6 with index $(2,4)$, given by

$$(u,v) = u_0v_0 - u_1v_1 - u_2v_2 - u_3v_3 - \tfrac{1}{2}u_4v_5 - \tfrac{1}{2}u_5v_4$$

Let C be the cone in R^6 given by $u \in C$ iff $(u,u) = 0$

Definition 4. The points of the projective Minkofsky space PM, is the set of lines in C. Let a be a point in PM, given by a line in C with direction $u \in R^6$. The light cone at a is the set of points, given by lines with direction v such that $(u,v) = 0$.

An automorphism of PM is a one to one map of the set of points of PM onto it self, such that the light cone at a point is mapped onto the light cone at the image point. We see that $O(2,4)$, the group of linear transformations of R^6 leaving the form $(,)$ invariant, acts as a group of automorphisms of PM.

There is a natural imbedding of M in PM, such that light cones are mapped into light cones. Let $x \in R^4$, $x = \{x_0, x_1, x_2, x_3\}$. The image of x in PM is then the line on C given by the direction $u = \{x_0, x_1, x_2, x_3, x^2, 1\}$. We see that $(u,u) = 0$, so that this line is in C. Let now y be on the light cone at x, $(x-y)^2 = 0$ or $x^2 + y^2 = 2x \cdot y$. Let $v = \{y_0, y_1, y_2, y_3, y^2, 1\}$ be the direction of line that represent y in PM. $(u,v) = x \cdot y - \frac{1}{2}x^2 - \frac{1}{2}y^2 = 0$. Hence the image of a light cone in M is contained in a light cone in PM.

We see that the image of M in PM is the set of lines in C with directions given by $u = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ with $u_5 \neq 0$. This set of lines in C will be called the set of finite points in PM, and its complement in PM will be called the lightcone at infinity. The line with direction $u = \{0, 0, 0, 0, 1, 0\}$ will be called the point at infinity. This notation is consistent since the light cone at infinity is the light cone at the point at infinity.

Lemma 2. If we identify M with the set of finite points in PM; then I, the inversion in M, is the restriction to the set of finite points in PM of an automorphism of PM that is in the connected component of the identity in $O(2,4)$.

Proof: We see that on its domain of definition I coinsides with the transformation induced on PM by the following linear transformation.

$$\{u_0, u_1, u_2, u_3, u_4, u_5\} \to \{-u_0, u_1, u_2, u_3, u_5, u_4\}$$

It is an immediate verification that this linear transformation is in the connected component of the identity in O(2,4) . This proves the lemma.

The subgroup of O(2,4) of index two generatied by the component of the identity in O(2,4) and the space reflection

$$S: \{u_o, u_1, u_2, u_3, u_4, u_5\} \rightarrow \{u_o, -u_1, -u_2, -u_3, -u_4, -u_5\}$$

will be denoted by G_1 .

Lemma 3. The subgroup of G_1 leaving the point at infinity fixed, is isomorphic to G_o ; and its action on the set of finite points in PM is identified with the action of G_o in M .

Proof: The subgroup of O(2,4) leaving the point at infinity fixed, i.e. leaving the direction $u = \{0,0,0,0,1,0\}$ invariant, will also leave the orthogonal subspace invariant, The orthogonal subspace is the subspace of all $v \in R^6$ such that $v_5 = 0$. The marix of a transformation leaving the direction u invariant is therefore given by

$$\begin{bmatrix} A & 0 & a \\ b' & \alpha & \beta \\ 0 & 0 & \gamma \end{bmatrix}$$

Where A is a 4 x 4 matrix, a a 4 vector and b' a transposed 4 vector and α, β and γ real numbers. Let $u = \{x_o, x_1, x_2, x_3, s, t\}$, then $(u,u) = x^2 - st$. Since the transformation should leave invariant the form (,) , we get

$$(Ax + ta)^2 - \gamma t(bx + s\alpha + t\beta) = x^2 - st .$$

By equating terms in the quadratic forms we get

$$(Ax)^2 = x^2 \ , \ 2\ Ax \cdot a = \gamma b \cdot x \ , \ a^2 = \gamma\beta \ , \ \alpha\gamma = -1$$

This gives us that A leaves the form x^2 invariant, i.e. $A \in O(1,3)$. Since G_1 is generated by its component of the identity and the space reflection S , we find that A must be in the subgroup of O(1,3) , generated by its component of the identity and the space reflection. It is A is in the Lorentz group L . The next equations gives us

$$a = \gamma/2\ Ab \ , \ \beta = \gamma/4\ b^2 \ \text{and} \ \alpha = -\gamma^{-1}$$

Hence we get the matrix representation of the subgroup of G_1 leaving the point at infinity fixed as

$$\begin{bmatrix} A & o & \gamma/2\ Ab \\ b' & -\gamma^{-1} & \gamma/4\ b^2 \\ 0 & 0 & \gamma \end{bmatrix}$$

where $A \in L$. A direct verification shows that this group operates on the set of finite points in PM as does G_o on M . This proves the lemma.

The component of the identity in O(2,4) acts transitively on the set of lines in C . To see this let $R^6 = V^+ \oplus V^-$ be a direct decomposition of R^6 in a subspace V^+ where (,) is positive definite and V^- where (,) is negative definite. The subgroup of O(2,4) respecting this decomposition is canonicaly isomorphic to O(2) x O(4) . It is easily seen that SO(2) x SO(4) acts transitively on pairs of linear subspaces, one from V^+ and one from V^- . Since any $u \in R^6$ such that (u,u) = 0 has a unique decomposition $u = u^+ + u^-$, $u^+ \in V^+$, $u^- \in V^-$, this gives us that

SO(2) x SO(4) acts transitively on the lines in C . If we combine this with lemma 3, we get

Lemma 4. G_1 acts transitively on PM , and the subgroup leaving a point fixed is isomorphic with G_0 .

Moreover the maximal compact subgroup of the component of the identity in G_1 , SO(2) x SO(4) , acts transitively on PM , and the subgroup leaving a point fixed is isomorphic with SO(3) , the maximal compact subgroup of the component of the identity in G_0 .

A direct verification utilizing the matrix representation exhibit above for the subgroup G_0 leaving the point at infinity fixed gives us that G_1 is generated by G_0 and the invertion I , where as pointed out earlier I in PM is represented by an element in the component of the identity in G_1 . This gives us then that the group of local automorphisms of M , generated by I and G_0 , which we have denoted by G is the same group as the subgroup of index 2 in O(2,4) that we have called G_1 . This gives us the following theorem.

Theorem 1. The group G of local automorphisms of M , generated by I and G_0 ; is a subgroup of O(2,4) of index two, generated by the component of the identity in O(2,4) and the space reflection S . Any of the local automorphisms of M in G , is induced by an automorphism of PM , and hence is defined in M outside a light cone. Namely the light cone that by the corresponding automorphism in PM is mapped into the light cone at infinity.

We will now state the main theorem without proof:

Theorem 2.

Let f be a local automorphism of M. Then f is a restriction of a unique transformation $g \in G$.

Reference

1. E.C. Zeemann: Causality implies the Lorentz group, J. Math. Phys. 5(1964) 490 - 493.