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RECURSIVE ORDINALS AND PROVABILITY IN FIRST ORDER NUMBER THEORY.

By

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A relation R on a set E is called a <u>well ordering</u> of E if R is a linear order of E such that every non-empty subset X of E has a "first" element with respect to R, i.e. there is an element $x_0 \in X$ such that $x_0 R x$ for all $x \in X$. An equivalent form of the last condition is that every strictly descending chain in E with respect to R is finite.

It is a basic fact of set theory that any two well ordered systems $\langle E,R \rangle$ and $\langle F,S \rangle$ either are order isomorphic or one is order isomorphic to an initial segment of the other. The set of natural numbers in the usual ordering is order isomorphic to an initial segment of any infinite well ordered set.

<u>Ordinals numbers</u> are introduced as invariants of well ordered sets. (One would like to define an ordinal number as an equivalence class of well ordered systems, but this leads to the usual difficulties in an axiomatic approach, so that within most formal systems ordinals are defined through a certain choice of representatives, always in such a manner that the natural numbers form an initial segment of the ordinals. The method most often adopted is due to von Neumann.)

Well orderings and thus ordinals occur in several connections within foundational studies. Gödel's incompleteness theorem for elementary formalized arithmetic shows that the consistency of this system is not provable within the system itself. On the other hand it was shown by G. Gentzen that the consistency of elementary arithmetic is provable by means of the method of transfinite induction up to the least epsilon-number ε_0 (i.e. the least solution of the ordinal equation $\omega^\alpha = \alpha$). Transfinite induction up to any ordinal $\alpha < \varepsilon_0$ is provable within arithmetic, and it follows from the result mention above that induction up to ε_0 is not so provable (see Schütte [15] for an exposition of this theory). (If α is a countable ordinal, then there exists a well order R of the natural numbers N of type α. Hence the scheme of transfinite induction up to α can be expressed as an elementary number theoretic scheme and may as such be provable.)

Gödel's incompleteness theorem asserts that any recursively enumerable extension of Peano arithmetic is incomplete. In fact, we know from some work of Tarski that number-theoretic truth is not arithmetically definable. It is, however, hyperarithmetic.

In order to bridge the gap between Gödel and Tarski the idea of using "ordinal logics" was introduced by Turing [19]. This idea was later elaborated by Feferman [2] who (extending greatly the results of Turing) developed a theory of "transfinite recursive progressions of axiomatic theories". The basic idea is simple. Let A_0 be the system of Peano arithmetic. From Gödel we know that Con_{A_0} (i.e. the number-theoretic sentence which asserts the consistency of A_0) is not provable within A_o. Add Con_{A_o} to the axioms of A_o to obtain a new system A_1. In this system Con_{A_o} is trivially provable, but, again from Gödel, Con_{A_1} is not provable within A_1. Thus by iterating the process we obtain stronger and stronger systems. We arrive at the following scheme of definition. Letting A_o denote the Peano axioms we define inductively for all ordinals less than some fixed λ_o:

$$A_{\alpha+1} = A_\alpha \cup \{\mathrm{Con}_{A_\alpha}\}, \quad \alpha < \lambda_o$$

and

$$A_K = \bigcup_{\alpha < K} A_\alpha, \quad K < \lambda_o, \quad K \text{ a limit number.}$$

There are at least two difficulties in this approach, one rather immediate, one somewhat more subtle. To mention the last problem first, it is not at all obvious how to choose the statements Con_A. In fact, Feferman has shown that there are seemingly plausible sentences Con_A such that $\vdash_A \mathrm{Con}_A$. In [1] he has analysed the situation and singled out a class of statements suitable for expressing consistency. (In the literature there has been much ambiguity on this point.) We do not enter into details here, but advise the reader that the point raised has caused seriour difficulties and has necessitated a delicate analysis of the concepts involved.

The first and rather obvious problem in constructing the "progression" of theories concerns the use of ordinals. We would like each A_α to be a recursively enumerable set of axioms. What segment of ordinals to choose and how to proveed at limit numbers ? An analysis of the situation shows that the class of recursive ordinals will recommend itself as the proper segment, but in order to make the construction of the various A_α's precise, we need an effective scheme of notations for the recursive ordinals. Thus in the final analysis we shall have a set O of notations for the recursive ordinals and we shall have a recursive function $d \rightarrow A_d$, such that if $d \in O$, then A_d is a (recursively enumerable) axiomatic system. Any system of notations is non-unique, thus we do not obtain a well ordered sequence of theories. (It has been shown by Feferman that A_{d_1} may differ from A_{d_2} even if d_1 and d_2 name the same ordinal.) What we do obtain has been called by Feferman a <u>transfinite recursive progression of axiomatic theories</u>.

The <u>Riemann hypothesis</u> is equivalent to a number-theoretic statement of the form $(\forall x)R(x)$, where R is a primitive recursive predicate [11]. A basic completeness result on transfinite progressions due to Turing asserts that each true statement of this form is provable at stage $\omega+1$ in a progression starting from Peano arithmetic and based on adding consistency.

We shall give the basic idea of the proof of the Turing completeness result. Let P denote the system of Peano arithmetic. Implicit in the second Gödel incompleteness theorem is the fact that with any (sufficiently well behaved) recursively enumerable consistent axiom system A, containing Peano arithmetic, one can associate a Gödel-sentence ν_A

such that

$$\vdash_P \nu_A \longleftrightarrow \sim Pr_A(\bar{\nu}_A).$$

Here Pr_A is the formalized notion of provability, and $\bar{\nu}_A$ is the arithmetized version of ν_A. Thus, in a sense, ν_A expresses its own unprovability. Further one may show that

$$\vdash_P Con_A \longleftrightarrow \nu_A.$$

The second Gödel incompleteness theorem is now a consequence of the fact that ν_A is not provable in A.

Let $(\forall x)R(x)$ be a true statement, where R is a primitive recursive predicate. R belongs to a formal system of arithmetic; to this R there corresponds a relation R^* defined in the set of natural numbers such that

$$n \in R^* \quad \text{iff} \quad \vdash_P R(\bar{n});$$

$$n \notin R^* \quad \text{iff} \quad \vdash_P \neg R(\bar{n}).$$

We shall define a certain ordinal notation of type $\omega+1$ using the recursive fixed-point theorem. This fixed point theorem, which incorporates all kinds of recursion theorems, asserts that if $\underline{\underline{F}}$ is a recursive functional, then the equation

$$f = \underline{\underline{F}}(f)$$

has a recursive solution. Recursive functions can be replaced by their Gödel numbers, hence it follows from the fixed point theorem that there exists a number e such that

$$\{e\}(n_o) = \begin{cases} n_o & \text{if} \quad (i)_{i \leq n} R^*(i), \\ 2^{3 \cdot 5^e} & \text{if} \quad (Ei)_{i \leq n} \neg R^*(i). \end{cases}$$

Let $d = 3 \cdot 5^e$. We know that $(\forall x)R(x)$ is true, i.e. $(i)R^*(i)$. This implies that $\{e\}(n_o) = n_o$ for all n, which means that d is a ordinal notation and $|d| = \omega$.

On the other hand, if we did not know that $(i)R^*(i)$, then for some n, $\{e\}(n_o)$ could be equal to 2^d; let n' be the least such number. From the definition of the recursive progression we know that

$$A_d = \bigcup_{n \in \omega} A_{\{e\}(n_o)},$$

thus $A_{2^d} = A_{\{e\}(n_o')} \subsetneq A_d$. But A_{2^d} contains the sentence Con_{A_d}, therefore $\vdash_{A_d} \nu_{A_d}$. What we have shown is:

If $(Ei) \neg R^*(i)$, then $\vdash_{A_d} \nu_{A_d}$.

(Of course A_d would in this case be inconsistent.) The basic point is now that the formalized version is provable in Peano arithmetic, i.e. we may obtain

$$\vdash_P (\exists x) \sim R(x) \rightarrow Pr_{A_d}(\bar{\nu}_{A_d}).$$

Again using a property of the Gödel sentence and the fact that $\vdash_{A_{2^d}} \gamma_{A_d}$, we may conclude that

$$\vdash_{A_{2^d}} (\forall x)R(x),$$

which is the Turing completeness result as $|2^d| = \omega+1$.

Hence either the Riemann hypothesis or its negation is provable from some (recursively enumerable) axiomatic system in the collection $\{A_d;$ d denotes the ordinal $\omega+1\}$. The notion of proof is effective, may we now employ a battalion of morons to settle the Riemann hypothesis ? Unfortunately not : there is no effective procedure for deciding whether a number d is a notation for the ordinal ω+1 or not. (This follows from the original Gödel incompleteness theorem, the undecideable statement is a true statement of the form $(\forall x)R(x)$.)

The original completeness result has been extended to a general completeness result for elementary arithmetic by Feferman. (The progression is now based on a certain formalized rule of complete induction.) In [4] we have given a more direct and simpler derivation of Feferman's completeness result for elementary arithmetic. If d' denotes the successor of d, let $A_{d'}$ consist of all sentence in A_d together with all sentences of the form

$$(\forall x)Pr_{A_d}(\overline{\phi(\dot{x})}) \rightarrow (\forall x)\phi(x),$$

where Pr_{A_d} is the arithmetized provability predicate for

A_d. (We have to take the arithmetized version in order to express the provability notion within the theory itself.) The principle expresses that if $\phi(\bar{n})$ is provable in A_d for each numeral $\bar{n}$, then $(\forall x)\phi(x)$ is provable at the next stage.

Our result is that <u>every true statement of elementary number theory is provable in the progression based on the above generating principle at a stage whose ordinal is less than ω^ω. And the bound ω^ω is the best possible</u>.

Let ϕ be a sentence of arithmetic. It is immediate that ϕ is provable from $\bigcup_{d \in O;\, |d| < \alpha} A_d$, where α is a limit number, iff

$$\phi \in \bigcup_{d \in O;\, |d| < \alpha} A_d.$$

Hence if $\alpha < \omega^\omega$, and we can then assume that $\alpha = \omega^N$ for some natural number N, then ϕ is a true sentence of number theory iff $(Ed)[d \in O_{\omega^N}\ \&\ \phi \in A_d]$. It is shown in [13] that the assertion $d \in O_{\omega^N}$ is arithmetic, and as $\phi \in A_d$ is recursively enumerable, this implies by the above equivalence that number-theoretic truth is arithmetically definable. This contradicts Tarski's theorem and shows that the bound ω^ω is sharp.

Again, our result does not constitute any decision method for number theoretic truth. It is in a sense a "normal form" theorem, i.e. a reduction theorem for elementary arithmetic. It asserts that every problem within elementary number theory, say, any question about the solvability of diophantine equations within the field of rationals, is effectively reducible to a problem about notations for ordinals less than ω^{ω} . And this it seems to us is of some conceptual interest in itself.

This report is concluded with two rather unrelated notes on recursive ordinals.

Ordinals can either be introduced as invariants of well ordered sets or as systems of notations. In note A we show the equivalence of the two approaches to recursive ordinals. This has previously been proved by W. Markwald [14]. We present a new proof in the spirit of Kleene [9].

Well orderings have so far been considered wrt arbitrary descending chains. What happens if we require the chains to be hyperarithmetic, arithmetic or recursive ? In the last note we discuss this problem.

A. RECURSIVE WELL ORDERINGS AND SYSTEMS OF NOTATIONS.

Let L be the set of all (gödel numbers of) functions $f : N \times N \longrightarrow N$ such that $f(x,y) = 0$ is a linear order of all $x \in N$ such that $f(x,x) = 0$. ($f(x,x) = 0$ means that x belongs to the field of the relation f.)

f is a well ordering of N if $f \in L$ and if every descending chain in (the domain of) f is finite, where by a chain in f we understand a number-theoretic function $\alpha : N \longrightarrow N$ such that $f(\alpha(x+1),\alpha(x)) = 0$ for all $x \in N$. Let W denote the collection of all well orderings of N. It is easily seen that an $f \in L$ belongs to W iff every descending chain α in f has a repeating element, i.e. $\alpha(x+1) = \alpha(x)$ for some number x.

For any $f \in W$ we let $|f|$ denote the order type of f. An ordinal α is called recursive if $\alpha = |f|$ for some $f \in W$. There exists a least non-recursive ordinal, denote this ordinal by ω_1^*.

Following the exposition of Kleene [9] we introduce a system of (non-unique) notations for a segment of the ordinals. (This is the system S3 of Church-Kleene.) The sequence $\langle n_o \rangle$ defined by the recursion relations $0_o = 1$ and $(n+1)_o = 2^{n_o}$, will name the initial segment of final ordinals. According to the terminology of Kleene we say that y defines y_n recursively as a function of n_o if

$(\forall n)[\{y\}(n_o)$ is defined and $= y_n]$; where $\{e\}$ is the recursive function with gödel number e. The set of notations O and a partial order relation $a <_o b$ is introduced by the following inductive definition:

01. $1 \in O$.

02. If $y \in O$, then $2^y \in O$ and $y <_o 2^y$.

03. If each $y_n \in O$ and $y_n <_o y_{n+1}$ for all n, and if y defines y_n recursively as a function of n_o, then $3 \cdot 5^y \in O$ and $y_n <_o 3 \cdot 5^y$ for all n.

We now require that O and $<_o$ are the least sets such that 01-03 are satisfied and such that $<_o$ is transitive on O.

For a development of the theory of O we refer the reader to papers by Kleene and Spector. In this note we shall mention a few results for later reference. Quite basic is the existence of a primitive recursive predicate V such that if the set $C(b)$ is defined as $a \in C(b)$ iff $(\exists x)V(a,b,x)$, then for any $b \in O$, $a \in C(b)$ iff $a <_o b$. The set O itself is only partially ordered by $<_o$, but any segment determined by a $b \in O$ is linearly ordered. Many of the elementary properties of ordinal arithmetic carry over to O. An addition of ordinal notations is definable, and this operation may be extended by induction to a definition of finite sums.

Through the following inductive definition O will be mapped onto a segment of the (classical) ordinals:

$$|1| = 0;$$

$$|2^y| = |y|+1 \quad \text{for} \quad y \in O,$$

$$|3\cdot 5^y| = \lim |y_n| ,$$

where $3\cdot 5^y \in O$ and $y_n = \{y\}(n_o)$. Each $b \in O$ represents a countable ordinal $|b|$, and $a <_o b$ implies $|a| < |b|$. Further for each $\alpha < |b|$ there is an $a \in O$ such that $a <_o b$ and $\alpha = |a|$. Thus the set $\{a | a <_o b\}$ for $b \in O$ is of order type $|b|$. The least ordinal not representable by any $b \in O$ is called the Church-Kleene ω_1.

THEOREM. ω_1^* equals the Church-Kleene ω_1.

That the Church-Kleene ω_1 is $\leq \omega_1^*$ is rather easy to show (it follows from the result mentioned above concerning the predicate V, details can be found in Kleene [9].)

That every recursive well ordering of N has a notation in O has been shown by Markwald [14] and Spector (unpublished). We present a new and fairly simple proof which depends on an analysis of the finite descending chains belonging to an $f \in W$.

We give some prepatory definitions: A sequence number w is a number of the form $w = 2^{s_o+1}\cdot 3^{s_1+1}\cdot\ldots\cdot p_n^{s_n+1}$. By $(w)_t$ we understand s_t, the t'th member of the finite sequence w, and $lh(w) = n+1$ counts the total number of members of w. The notions introduced are recursive.

$w \in Seq(f)$ iff (i) w is a sequence number (possibly the empty sequence 1), (ii) each member $(w)_t$ of w belongs to the field of f, (iii) w is descending in f.

The notion of <u>securable</u> $_f$ will be introduced in two different ways:

A. w is secured$_f$ iff $w \in Seq(f)$ and $(w)_t = (w)_{t+1}$ for some $t < lh(w) \dot{-} 1$.

w is securable$_f$ iff either w is secured$_f$ or $w \in Seq(f)$ and every descending extension α of w in f has a repeating element.

We say that w is immediately secured$_f$ if w is secured$_f$ and no proper segment of w is secured$_f$. If $w \in Seq(f)$ and is unsecured$_f$ (i.e. not secured$_f$), then w is a proper descending chain in f. We are interested in definition A simply because of the following result.

PROPOSITION. <u>Let $f \in L$, then $f \in W$ iff all $w \in Seq(f)$ are securable$_f$.</u>

If $f \in W$, then either w is secured$_f$ and thus securable$_f$, or $w \in Seq(f)$ but is not secured$_f$, which means that it is a strictly descending chain in f. But any extension α of w in f is finite, which means that w is securable$_f$ according to definition A. Conversely, if f is not a well ordering, then the empty sequence 1 which surly belongs to $Seq(f)$ is not securable$_f$ according to A.

We recast definition A as an inductive definition (following an almost analogous proof of Kleene in [9]).

B. If w is secured$_f$, then w is securable$_f$. If $w \in Seq(f)$ and w is not secured$_f$, but for every s such that $w * 2^{s+1} \in Seq(f)$ (here $w * u$ denotes the extension of w by u), $w * 2^{s+1}$ is securable$_f$, then w is securable$_f$.

Plainly we need the following lemma.

PROPOSITION. <u>w is securable$_f$ with respect to definition A iff w is securable$_f$ with respect to definition B</u>.

If w is securable$_f$ wrt B, then either w is secured$_f$, hence securable$_f$ wrt A, or every $w * 2^{s+1}$ which belongs to $Seq(f)$ is securable$_f$ wrt B, hence by inductive assumption also wrt A, which means as every extension α of w in f has an initial segment of the form $w * 2^{s+1} \in Seq(f)$, that w itself is securable$_f$ wrt A.

Conversely, if $w \in Seq(f)$ but is not securable$_f$ wrt B, then w cannot be secured$_f$. Thus there is some non-repeating extension $w * 2^{s_o+1}$ in f which is not securable$_f$. Repeating the argument we may further extend w to an unsecured$_f$ of the form $w * 2^{s_o+1} * 2^{s_1+1}$. Thus, inductively, we are able (non-effectively) to find an extension $s_o, s_1, \ldots$ of w in f without repeating ..

elements.

The set $S(f)$ of all sequences $w \in Seq(f)$ which are either unsecured$_f$ or immediately secured$_f$ can be given a well ordering if $f \in W$. The well order is defined as follows:

> $u \lhd w$ iff either u extends w or $(u)_t < (w)_t$ for the first member $(u)_t$ of u which differs from the corresponding member of w.

PROPOSITION. <u>If $f \in W$, then $S(f)$ is well ordered by $\lhd$</u>.

If $w_1 \rhd w_2 \rhd w_3 \dots$ is an infinite descending chain in $S(f)$, then each w_n must be unsecured$_f$, i.e. a strictly descending chain in f. The "lower envelope" of $w_1, w_2, \dots$ produces an infinite descending chain in f. (The <u>lower envelope</u> is the commen extension of all w_{i_n} such that $w_{i_n} \rhd w_{i_{n+1}}$ (i.e. $i_{n+1} > i_n$) and such that every w_{i_n} as a sequence is an initial segment of all w_{i_m} for $i_m > i_n$. The definition is best explained through a diagram! And the existence of the "lower envelope" follows from the definition of $\lhd$ and from the fact that every initial segment of N in the usual ordering is finite.)

The remainder of the proof will now be divided into two parts. First the easy one, we show that the order type of f is less than the order type of $S(f)$. Next we will construct an orderpreserving map of $S(f)$ into O. In fact we shall be able to bound the ordinal of $S(f)$ by a notation in O. This will conclude the proof.

PROPOSITION. <u>If $f \in W$, then $|f| \leq |S(f)|$</u>.

The proof preceeds by induction on the order type of f. Let f_α be (the gödel number of) an initial segment of f, i.e. $f_\alpha(x,y) = 0$ iff $f(x,x_o) = 0 \wedge f(y,x_o) = 0 \wedge f(x,y) = 0$, where x_o is some fixed element in the field of f. Every element $w \in S(f_\alpha)$ is also an element of $S(f)$, and as f_α is a restriction of f, the inclusion map is order preserving. So, $|f| = \sup(|f_\alpha|+1) \leq \sup(|S(f_\alpha)|+1) = |S(f)|$.

In order to construct a map $\xi(f,w)$ with values in O we need a last definition:

$$S(f,w) = \{w\}, \quad \text{if} \quad w \in S(f) \quad \text{and} \quad w \quad \text{is secured}_f.$$

$$S(f,w) = \{w * u \in S(f)\}, \quad \text{if} \quad w \quad \text{is unsecured}_f.$$

Note that $S(f) = S(f,1)$, further that $S(f,w)$ is the sum of all sets $\{w\},\ldots,S(f,w * 2^{s+1}),\ldots$ where $w * 2^{s+1} \in Seq(f)$. We shall use the recursion theorem to define the map ξ so as to satisfy the following conditions

$$\xi(w,f) = \begin{cases} 0 & \text{if} \quad w \notin Seq(f), \\ 1 & \text{if} \quad w \quad \text{is secured}_f, \\ 3 \cdot 5^{d_{f,w}} & \text{if} \quad w \in Seq(f) \text{ but is not secured}_f, \end{cases}$$

where $d_{f,w}$ defines the finite sum $\sum_{s<n_o} \eta(f,w * 2^{s+1})$ as a function of n_o. Here

$$\eta(f,w) = \begin{cases} \xi_f(w) +_o 1_o & \text{if } w \in \mathrm{Seq}(f) \\ 1_o & \text{otherwise.} \end{cases}$$

The details of the definition is as follows. Let $\mathrm{nat}(b) = \mu n_{n<b}[b = n_o]$. Further define

$$\theta(z,f,w,b) \simeq \sum_{s<\mathrm{nat}(b)} ([\Phi_2(z,f,w * 2^{s+1}) \cdot \varphi(w * 2^{s+1})$$
$$+(1 \dot{-} \varphi(w * 2^{s+1}))] +_o 1_o),$$

where $\varphi(w) = 0$ if $w \notin \mathrm{Seq}(f)$ and $\varphi(w) = 1$ otherwise. Choose a gödel number p of θ. Then the following function is recursive

$$\psi(z,f,w) = \begin{cases} 0 & \text{if } w \notin \mathrm{Seq}(f), \\ 1 & \text{if } w \text{ is secured}_f, \\ 3 \cdot 5^{S_1^3(p,z,f,w)} & \text{if } w \in \mathrm{Seq}(f) \text{ but not secured}_f. \end{cases}$$

Note that $\{S_1^3(p,z,f,w)\}(n_o) = \{p\}(z,f,w,n_o)$ which equals $\sum_{s<n}(\Phi_2(z,f,w * 2^{s+1}) +_o 1_o)$ if $w * 2^{s+1} \in \mathrm{Seq}(f)$ and equals 1_o otherwise. Choose e by the recursion theorem so that e defines ψ recursively. Then define $\xi = \{e\}$, i.e. $\xi(f,w) = \psi(e,f,w)$, and one easily verifies that the conditions on ξ stated above obtain.

PROPOSITION. <u>If $f \in W$, then $\xi(f,w) \in O$ for all $w \in Seq(f)$ and $|S(f,w)| \leq |\xi(f,w)|+1$.</u>

If $w \in Seq(f)$, then as shown above each $w \in Seq(f)$ is securable$_f$. Using definition B we prove that $\xi(f,w) \in O$: In the first case, if w is secured$_f$, then $\xi(f,w) = 0_o \in O$. If w is securable$_f$ but not secured$_f$, then by induction hypothesis each $\xi(f,w * 2^{s+1})$ such that $w * 2^{s+1} \in Seq(f)$, belongs to O. Then each $\eta(f,w * 2^{s+1})$ belongs to O, and as the finite sums are strictly increasing in O, it follows that $\xi(w,f) = 3 \cdot 5^{d_{f,w}} \in O$.

The inequality is proved as follows: If w is secured$_f$, then $S(f,w) = \{w\}$ and $|\xi(f,w)| = 0$, hence the estimate is correct.

It w is securable$_f$ but not secured$_f$, then

(i) if $w * 2^{s+1} \in Seq(f)$, then $w * 2^{s+1} \in S(f)$ and by induction hypothesis $|S(f,w * 2^{s+1})| \leq |\xi(f,w * 2^{s+1})|+1 = |\eta(f,w * 2^{s+1})|$;

(ii) if $w * 2^{s+1} \in Seq(f)$, then $|\eta(f,w * 2^{s+1})| = 1$.

We noted above that $S(f,w)$ is the sum of all sets $\{w\},\ldots,S(f,w * 2^{s+1}),\ldots$, where $w * 2^{s+1} \in Seq(f)$. Thus

$$|S(f,w)| = \sum |S(f,w * 2^{s+1})|+1,$$

where s runs through the increasing sequence of natural numbers such that $w * 2^{s+1} \in Seq(f)$. But by expanding the sequence we obtain

$$|S(f,w)| \leq \sum_{s=1}^{\infty} |\eta(f, w * 2^{s+1})| + 1$$

$$= |\xi(f,w)| + 1,$$

as finite sums of notations are defined such as to be order-preserving under the map $b \rightarrow |b|$, for $b \in O$.

In particular $\xi(f,1) \in O$ and $|S(f)| = |\xi(f,1)| + 1$. This, as noted above, concludes the proof.

B. ON SOME SPECIAL TYPES OF RECURSIVE LINEAR ORDERINGS.

It is known from some work of Spector [16] that the least ordinal not representable as a hyperarithmetic well ordering of N is the Church-Kleene ω_1. Thus using hyperarithmetic relations instead of recursive ones do not give us any larger ordinals. What happens if we use recursive linear orderings which are well orderings with respect to hyperarithmetic descending chains ? It would have been nice if we got exactly the class of recursive well orderings, but (as may be expected from the proof of the equivalence between the definitions A and B of securable$_f$ presented in the previous section) this is not so. The class of recursive well orderings with respect to hyperarithmetic chains is strictly larger than the class of recursive well orderings with respect to arbitrary chains. However, an absoluteness property similar to the one mentioned above might still be true: Does it matter whether we use hyperarithmetic chains or use only recursive descending chains ? Again, the answer is that it does, then are recursive linear orderings of N which are well orderings with respect to arbitrary arithmetic descending chains (hence, with respect to recursive ones) which are not well orderings with respect to hyperarithmetic chains.

In this section we present a uniform method for answering the problems discussed above. Some of the results we present are known, but some seems to be new (in particular the existence of a well ordering with respect to arithmetic descending chains which is not a well ordering with respect to hyperarithmetic ones).

Let $a \prec b$ be the recursive enumerable relation

$$a \prec b \quad \text{iff} \quad (\exists x)V(a,b,x),$$

discussed in the previous section. As there let $C(b) = \{a \mid a \prec b\}$, and let $C^*(b) = C(2^b) = \{a \mid a \preceq b\}$, where $a \preceq b$ iff $a \prec b \vee a = b$. We recall a few properties: (i) $a \prec 1$ never holds; (ii) If $b \neq 0$, then $a \prec 2^b$ iff $a \preceq b$; (iii) $a \prec 3 \cdot 5^e$ iff $(\exists n)[\{e\}(n_o)$ is defined and $a \preceq \{e\}(n_o)]$; (iv) If $b \in O$, then $a \prec b$ iff $a <_o b$.

Following the exposition of Feferman and Spector [3], we introduce a certain set M by letting $d \in M$ iff (i) $C^*(d)$ is linearly ordered by $\prec$; (ii) any element a of $C^*(d)$ is either of the form $a = 1$ or $a = 2^b$, where $b \neq 0$, or $a = 3 \cdot 5^e$, where $\{e\}(n_o)$ is defined for all n and $\{e\}(n_o) \prec \{e\}((n+1)_o)$.

Let $D_{\alpha,d} = \{n \mid \alpha(n) \neq 0\} \cap C^*(d)$, and consider the predicate $A(\alpha,d)$ defined by

$$A(\alpha,d) \quad \text{iff} \quad d \in M \wedge [D_{\alpha,d} \neq \emptyset \rightarrow \exists \text{ least element of } D_{\alpha,d} \text{ in the ordering } \prec].$$

The predicate $A(\alpha,d)$ will be used to introduce three sets of notations and the evaluations of the resulting predicates will give us all the results mentioned above.

(1) $d \in O$ iff $(\forall\alpha)A(\alpha,d)$.

(2) $d \in O^{*}$ iff $(\forall\alpha)[\alpha \in HA \to A(\alpha,d)]$.

(3) $d \in O^{**}$ iff $(\forall\alpha)[\alpha \in ARITH \to A(\alpha,d)]$.

Here HA is the collection of hyperarithmetic functions and ARITH denotes the class of functions whose graph is arithmetically definable. Obviously, $O \subseteq O^{*} \subseteq O^{**}$. Our first purpose is to prove that all inclusions are proper.

Before doing this there is one detail to take care of. O, the set of notations for recursive well orderings, was introduced in section A. That the two versions of O are the same needs a proof. (That $d \in O$ according to (1) above implies that d is a Church-Kleene O is proved by induction on $C^{*}(d)$ (note that $d \in C^{*}(d)$), using the fact that $d \in M$ and that $(\forall\alpha)A(\alpha,d)$ implies that $C^{*}(d)$ is well ordered by $\prec$. The converse is rather immediate on account of the minimality of O.)

PROPOSITION. <u>(i) O is a complete Π^1_1 set; (ii) O^{*} is a complete Σ^1_1 set; (iii) O^{**} is hyper-arithmetic</u>.

This proposition, whose proof will occupy the reminder of this section, answers our questions. A complete Π^1_1 set is not expressible in the dual form, hence $O \neq O^{*}$, and as $O \subseteq O^{*}$, there is a $d_0 \in O^{*}-O$. Using the techniques of Kleene [9], we may from this d_0 produce a recursive linear ordering of

N which is a well ordering with respect to hyperarithmetic chains, but not with respect to arbitrary chains. In the same manner it follows that O^* is not hyperarithmetic, hence any $d_1 \in O^{**}-O^*$ determines a recursive linear ordering of N which is a well ordering with respect to arithmetic (hence, recursive) chains, but not with respect to hyperarithmetic ones.

The proof of (i) was given by Kleene [9], and the proof that O^* is $\sum_1^1$ follows from results in Kleene [10]. The completeness of O^* as a $\sum_1^1$ predicate is a little more involved. Gandy and Spector (independently) have shown that every $\sum_1^1$ predicate can be expressed in the form $(\forall\alpha)[\alpha \in HA \rightarrow (\exists x)S(d,\bar{\alpha}(x))]$ for a suitable (primitive) recursive S. Every predicate of this form is a $\prod_1^1$ predicate relativized to HA, and O^* is O relativized to HA. Hence the obvious thing to do is to try to relativize the proof of (i) as given in Kleene [9].

This we have been informed, was done som years ago by Feferman, but his proof has remained unpublished. In 1966, without knowing the previous work of Feferman, I indicated to him a proof of the $\sum_1^1$-completeness of O^*. I then learned that J. Harrison in his thesis [6] also had given a proof of this very same result (as part of a more comprehensive investigation into the set O^*). I realized that a detailed working out of my proposal would produce a proof quite similar to the one given by Harrison. Hence I did not pursue the matter any further.

The success of the relativization depends upon the

following "uniformization lemma" for Π^1_1-predicates, due in substance to Kreisel [12]. For every $P \in \Pi^1_1$ we can define a $Q \in \Pi^1_1$ such that $Q(x,y) \rightarrow P(x,y)$,
$Q(x,y) \wedge Q(x,z) \rightarrow y = z$ and $(\exists y)P(x,y) \rightarrow (\exists y)Q(x,y)$.
In short, every Π^1_1 subset of $N \times N$ has a Π^1_1 cross-section.

Using this lemma we can prove the following induction principle: Let $X \subseteq N$ and $\lhd$ a recursive relation in X which is a w.o. wrt hyperarithmetic descending chains. Then every non-empty Π^1_1 set of X has a least element wrt. $\lhd$. - For the proof assume the converse, i.e. let $\emptyset \neq Y \subseteq X$, $Y \in \Pi^1_1$. As Y has no least element in the ordering $\lhd$, we have $(\forall x)[x \in Y \rightarrow (\exists y)[y \in Y \wedge y \lhd x]]$. Consider the predicate $P(x,y) \equiv. x \in Y \wedge y \in Y \wedge y \lhd x$. It is Π^1_1, hence it has a Π^1_1 cross-section $Q \in \Pi^1_1$. Q produces an infinite <u>hyperarithmetic</u> descending chain in X, viz. let d_o be some fixed element in Y and define $b_o = d_o$, b_{n+1} = the unique y such that $Q(b_n, y)$.

With this induction principle the adaphon of Kleene's proof is fairly straight forward, but a complete version of the proof would be rather long. The general induction principle was formulated explicitly by Harrison [6], whereas in my proposed version it was implicitly involved at several points.

It remains to show that O^{**} is hyperarithmetic. Our proof is a Kleene-type evaluation: Consider for convenience the predicate

$$(*) \qquad (\exists \alpha)[\alpha \in ARITH \wedge A(\alpha, d)],$$

where $A(\alpha,d)$ is any arithmetic predicate. Let H_y, $y \in O$, be the following hierarchy of predicates: $H_1(a)$ iff $a = a$; $H_{2^y}(a)$ iff $(\exists x)T_1^{H_y}(a,a,x)$; $H_{3\cdot 5^y}(a)$ iff $H_{y_{(a)_1}}((a)_0)$, if $3\cdot 5^y \in O$ and $y_n = \{y\}(n_0)$. It is known from work of Kleene [8] that any arithmetic predicate is recursive in some H_{n_0}. Using this we may rewrite (*) as follows:

$$(\exists\alpha)\left[\alpha \in \text{ARITH} \wedge A(\alpha,d)\right] \text{. iff.}$$

$$(\exists\alpha)(\exists n)\left[(\alpha \text{ is rec. in } H_{n_0}) \wedge A(\alpha,d)\right] \text{. iff.}$$

$$(\exists n)(\exists e)\left[(e \text{ is gnr. from } H_{n_0} \text{ of a total function } \alpha_e) \wedge A(\alpha_e,d)\right].$$

From the construction of the T-predicate it follows that if $(\exists z)T_1^{\Psi}(e,x,z)$, then z is unique. Using this (*) can be rewritten in the two following equivalent forms.

$$\text{(i)} \quad (\exists n)(\exists e)\left[(\forall i)(\exists t)T_1^{H_{n_0}}(e,i,t) \wedge (\forall\beta)\{(\forall i)T_1^{H_{n_0}}(e,i,\beta(i)) \rightarrow A(\lambda i\, U(\beta(i)),d)\}\right],$$

and

$$\text{(ii)} \quad (\exists n)(\exists e)\left[(\forall i)(\exists t)T_1^{H_{n_0}}(e,i,t) \wedge (\exists\beta)\{(\forall i)T_1^{H_{n_0}}(e,i,\beta(i)) \wedge A(\lambda i\, U(\beta(i)),d)\}\right].$$

(i) and (ii) together show that (*) is hyperarithmetic.

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