Matematisk Seminar
Universitetet i Oslo

Nr. 4
Mars 1965

GOOD STRATEGIES IN INFINITE GAMES

By

Jens Erik Fenstad

I. GENERAL NOTIONS FROM THE THEORY OF GAMES

The purpose of this report is to suggest a general approach to the study of "good" strategies in arbitrary two-person zero-sum games based upon fixed point theorems. Our contribution is to the "existence" part of the theory. We do not treat matters concerning calculation or characterization of optimal strategies, but are interested in an as general existence theorem for "good" strategies as possible, even if this theorem be highly non-effective. While the theorem we actually state is known, we do not believe that this result gives the limit of what is obtainable by this approach.

In this paper we shall always assume that a game is given in normal form, i.e. given as a triple

$$G = \langle A, B, k \rangle ,$$

where A and B are non-empty sets called sets of *pure* strategies and k is a real-valued function defined on the set $A \times B$ and called the *pay-off* function.

In brief outline, the game is played as follows. There are two participants, P_A and P_B. Each selects a strategy, i.e. P_A selects a point $a \in A$ and P_B selects a point $b \in B$. The outcome of the game is evaluated by calculating $k(a,b)$. "Pay-off" then consists in P_A receiving the amount $k(a,b)$ from P_B. (If $k(a,b) < 0$, P_A receives a negative amount which means that P_B gets the amount $-k(a,b)$ from P_A.)

Somewhat imprecisely we may say that the purpose of the game considered from the view-point of P_A is to select a strategy $a_o \in A$ so as to maximize his expected pay-off. Conversely P_B wants to minimize the return to P_A by selecting a "good" $b_o \in B$.

This preliminary description can be made precise in the following way.

For each $a \in A$ define

$$v_A(a) = \min_b k(a,b) .$$

(For the moment we assume that all entities entering into our calculations exist.) $v_A(a)$ represents the security level for P_A using strategy a, i.e. playing against an intelligent (or rational) opponent $v_A(a)$ is the maximum pay-off P_A may expect. A "good" strategy for P_A in the game G is then to select an $a_o \in A$ so as to maximize $v_A(a)$, i.e. try to obtain the amount

$$v_A = \max_a v_A(a) = \max_a \min_b k(a,b) .$$

Similarly we define

$$v_B(b) = \max_a k(a,b) .$$

Then $v_B(b)$ represents the maximum loss P_B may expect by choosing the strategy $b \in B$. The "good" thing for P_B is then to choose a $b_o \in B$ so as to minimize the maximum loss, i.e. try to hold P_A down to the amount

$$v_B = \min_b v_B(b) = \min_b \max_a k(a,b) .$$

It is easily seen that $v_A \leq v_B$. In an arbitrary game G, even if v_A and v_B exist, there may not exist strategies a_o and b_o such that $v_A = v_A(a_o)$ and $v_B = v_B(b_o)$, neither do we know whether $v_A < v_B$ or $v_A = v_B$. An investigation of these problems form the substance of this report.

It is necessary even for the simplest types of games to extend the concept of strategy. A "mixed" or "randomized" strategy is a probability distribution over the set of pure strategies. A randomized extension of a game G is obtained in the following way.

Let Σ_A and Σ_B be σ-algebras of subsets in A and B, respectively. Then the randomized extension of $G = \langle A,B,k \rangle$ with respect to the σ-algebras Σ_A and Σ_B is the triple

$$\Gamma = \langle \mathcal{A}, \mathcal{B}, K \rangle ,$$

where $\mathcal{A}$ and $\mathcal{B}$ are the sets of all probability distributions with respect to the σ-algebras Σ_A and Σ_B and $K(\mu,\lambda)$, where $\mu \in \mathcal{A}$ and $\lambda \in \mathcal{B}$, is the extended pay-off function defined by

$$K(\mu,\lambda) = \iint k(a,b)\,d\mu\,d\lambda ,$$

where the assumption is made that the integral exists (and can be evaluated either as a double integral or as an iterated integral).

An *equilibrium pair* $\langle \mu_o, \lambda_o \rangle$ in the game Γ is a pair of mixed strategies $\mu_o \in \mathcal{A}$ and $\lambda_o \in \mathcal{B}$ such that μ_o is good against λ_o, i.e. $K(\mu_o,\lambda_o) \geqslant K(\mu,\lambda_o)$ for all $\mu \in \mathcal{A}$, and λ_o is good against μ_o, i.e. $K(\mu_o,\lambda_o) \leqslant K(\mu_o,\lambda)$ for all $\lambda \in \mathcal{B}$. Then μ_o and λ_o satisfy the equations

$$\max_{\mu} K(\mu,\lambda_o) = K(\mu_o,\lambda_o) = \min_{\lambda} K(\mu_o,\lambda) .$$

The existence of an equilibrium pair $\langle \mu_o, \lambda_o \rangle$ implies the validity of the equation $v_A = v_B$ (where the notions are suitably extended to Γ). In fact, one has that $v_A = v_A(\mu_o) = v_B(\lambda_o) = v_B$ which follows from the inequalities

$$v_B \leqslant v_B(\lambda_o) = \max_{\mu} K(\mu, \lambda_o) = K(\mu_o, \lambda_o) = \min_{\lambda} K(\mu_o, \lambda) = v_A(\mu_o) \leqslant v_A$$

and the general inequality $v_A \leqslant v_B$. We note that if $K(\mu, \lambda)$ is defined for all $\mu \in \mathfrak{A}$ and $\lambda \in \mathfrak{B}$, then the existence of an equilibrium pair implies that all entities involved in our so far formal calculations are well defined.

Thus if $\langle \mu_o, \lambda_o \rangle$ is an equilibrium pair in the extended game Γ, then P_A by selecting the strategy μ_o can guarantee himself at least a pay-off equal to v_A, whereas P_B by selecting λ_o can hold P_A down to v_B, and the game is "fair" or in equilibrium as $v_A = v_B$.

This concludes our introduction to some general notions from game theory. We refer the reader to basic treatises such as Karlin ((2)), Luce and Raiffa ((4)) and von Neumann and Morgenstern ((6)) for further informations. Our problem is to determine whether any game G has some randomized extension Γ possessing an equilibrium point. In the next section we shall treat this problem by extending the fixed point technique given by Nash ((5)) for the case that both A and B are finite.

II. THE FIXED POINT THEOREM

We now consider some fixed extension $\Gamma = \langle \mathfrak{A}, \mathfrak{B}, K \rangle$ of the game $G = \langle A, B, k \rangle$ and assume that the pay-off function k is bounded, i.e. there shall exist an M_G such that

$$|k(a,b)| < M_G, \quad \text{for all } a \in A \text{ and } b \in B .$$

The restriction is further imposed that $k(a,b)$ is measurable with respect to the σ-algebra $\Sigma = \Sigma_A \times \Sigma_B$ on $A \times B$. Then $k(a,b)$ is integrable with respect to the product measure $\mu_1 \times \mu_2$ for all

$\mu_1 \in \mathcal{A}$ and $\mu_2 \in \mathcal{B}$, and we have unlimited access to the Fubini theorem.

In order to identify the pure strategies a and b with mixed ones, μ_a and μ_b, we shall require that each one point set in A and B belongs to the σ-algebras Σ_A and Σ_B. We may then define μ_a and μ_b by setting $\mu_a(X) = 1$ if $a \in X$ and 0 otherwise, where $X \in \Sigma_A$, and $\mu_b(Y) = 1$ if $b \in Y$ and 0 otherwise, for $Y \in \Sigma_B$. With these assumptions we observe that the real-valued function

$$a \longrightarrow K(\mu_a, \mu_2) ,$$

where μ_2 is considered as a parameter, is integrable on A which follows from the equalities

$$K(\mu_a, \mu_2) = \iint k(a',b)\,d\mu_a d\mu_2 = \int k(a,b)\,d\mu_2 ,$$

applying the Fubini theorem. In the sequel we denote μ_a and μ_b simply by a and b, respectively.

Next define for each $\mu = \mu_1 \times \mu_2 \in \mathcal{A} \times \mathcal{B}$

$$c_\mu(a) = [K(a, \mu_2) - K(\mu_1, \mu_2)] \vee 0 ,$$
$$d_\mu(b) = [K(\mu_1, \mu_2) - K(\mu_1, b)] \vee 0 .$$

Both c_μ and d_μ are integrable and it is seen that they jointly measure how far $\langle \mu_1, \mu_2 \rangle$ is from being a good strategy pair against the pure strategies a and b.

<u>Regularity assumption on $k: A \times B \longrightarrow R$</u>: For all $\varepsilon > 0$ there shall exist a finite covering of $A \times B$ of the form $U_i \times V_i$, $i = 1, \ldots, n$, such that $|k(a,b) - k(a',b')| < \varepsilon$ if both $\langle a,b \rangle$ and $\langle a',b' \rangle$ belong to the same set $U_i \times V_i$.

This regularity condition is in particular satisfied if both A and B are compact spaces and k continuous.

Using this regularity condition we shall construct two measures λ_A and λ_B suitable for measuring the average value of c_μ and d_μ over sets in Σ_A and Σ_B, respectively. (The regularity condition is only sufficient. Below we shall make some remarks on other conditions which could equally well serve and which apply in some cases where the above condition fails.)

For each $\varepsilon > 0$ let $U_1^\varepsilon, \ldots, U_m^\varepsilon$ be a refinement of the covering $U_1, \ldots, U_n$ of A such that if $U_j^\varepsilon \cap U_i \neq \emptyset$, then $U_j^\varepsilon \subseteq U_i$, and let a_j^ε be some point in U_j^ε. Let $a \in U_j^\varepsilon$, for any $b \in B$ there is some i such that $\langle a,b\rangle \in U_i \times V_i$. Hence $U_j^\varepsilon \cap U_i \neq \emptyset$. As then $U_j^\varepsilon \subseteq U_i$, this implies that $a_j^\varepsilon \in U_i$, therefore $\langle a_j^\varepsilon, b\rangle \in U_i \times V_i$. From the regularity condition we then may conclude that $|k(a,b) - k(a_j^\varepsilon,b)| < \varepsilon$. And if $a \in U_j^\varepsilon$, this inequality holds for all $b \in B$.

Next observe that $|c_\mu(a) - c_\mu(a')| \leq |K(a,\mu_2) - K(a',\mu_2)|$ which entails that

$$|c_\mu(a) - c_\mu(a')| \leq \int |k(a,b) - k(a',b)|\, d\mu_2 \leq \max_b |k(a,b) - k(a',b)| .$$

Thus for each $n \geq 1$ there exist points $a_1^n, \ldots, a_{m_n}^n$ with the property that given any $a \in A$ there is some a_i^n such that $|c_\mu(a) - c_\mu(a_i^n)| < \frac{1}{n}$ for all $\mu = \mu_1 \times \mu_2$.

We may now define λ_A for each $X \in \Sigma_A$ by

$$\lambda_A(X) = \sum_n \sum_{1 \leq i \leq m_n} \frac{(n;i)}{2^n \cdot m_n} ,$$

where $(n;i) = 1$ if $a_i^n \in X$ and 0 otherwise. In the same way we define a measure λ_B on $\langle B, \Sigma_B\rangle$.

We may establish the crucial property of the measure $_A$: c_μ must be identically 0 on A if $\int c_\mu(a)d\lambda_A = 0$. Suppose that $c_\mu(a') > 0$ for some $a' \in A$, then there would exist some n and i such that $c_\mu(a_i^n) > 0$. By construction c_μ is non-negative and $\lambda_A(\{a_i^n\}) \geq 1/2^n \cdot m_n$. Hence $\int c_\mu(a)d\lambda_A \geq \int_{\{a_i^n\}} c_\mu(a)d\lambda_A \geq c_\mu(a_i^n)/2^n \cdot m_n > 0$.

Similarly we may conclude that d_μ is identically 0 on B if $\int d_\mu(b)d\lambda_B = 0$.

The extended Nash transformation may now be defined for $X \in \Sigma_A$ and $Y \in \Sigma_B$ by

$$\mu_1'(X) = \frac{\mu_1(X) + \int_X c_\mu(a)d\lambda_A}{1 + \int_A c_\mu(a)d\lambda_A}$$

and

$$\mu_2'(Y) = \frac{\mu_2(Y) + \int_Y d_\mu(b)d\lambda_B}{1 + \int_B d_\mu(b)d\lambda_B}$$

It is immediate that $\mu_1' \in \mathcal{A}$ and $\mu_2' \in \mathcal{B}$. The transformation $T : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A} \times \mathcal{B}$ is obtainded by setting

$$T(\mu_1 \times \mu_2) = \mu_1' \times \mu_2'.$$

P r o p o s i t i o n . Let $G = \langle A,B,k \rangle$ be any game such that k satisfies the above stated regularity assumption and let Γ be any mixed extension. Then $\langle \mu_1, \mu_2 \rangle$ is an equilibrium pair for Γ if and only if $\mu_1 \times \mu_2$ is a fixed point for T.

P r o o f : I. Let $\langle \mu_1, \mu_2 \rangle$ be an equilibrium point for Γ. This means that

$$\max_{\lambda_1} K(\lambda_1, \mu_2) = K(\mu_1, \mu_2) = \min_{\lambda_2} K(\mu_1, \lambda_2) .$$

It follows at once from the definitions of c_μ and d_μ that $c_\mu(a) = 0$ for all $a \in A$ and $d_\mu(b) = 0$ for all $b \in B$. But then $\mu_1' = \mu_1$ and $\mu_2' = \mu_2$, i.e. $\mu_1 \times \mu_2$ is a fixed point for T.

II. To prove the converse we first observe that there are sets $X \in \sum_A$ and $Y \in \sum_B$ such that $\mu_1(X) > 0$, $\mu_2(Y) > 0$ and $K(\mu_1, \mu_2) \geqslant K(a, \mu_2)$ for all $a \in X$ and $K(\mu_1, \mu_2) \leqslant K(\mu_1, b)$ for all $b \in Y$. If this were not the case we would e.g. have $K(\mu_1, \mu_2) < K(a, \mu_2)$ for almost all $a \in A$ (with respect to μ_1), hence

$$K(\mu_1, \mu_2) = \int K(\mu_1, \mu_2) d\mu_1 < \int K(a, \mu_2) d\mu_1 = K(\mu_1, \mu_2) ,$$

a contradiction.

Using now the fact that $\mu_1 \times \mu_2$ is a fixed point for T we have

$$\mu_1(X) = \mu_1(X)/(1 + \int c_\mu(a) d\lambda_A)$$

and

$$\mu_2(Y) = \mu_2(Y)/(1 + \int d_\mu(b) d\lambda_B) .$$

But $\mu_1(X) > 0$ and $\mu_2(Y) > 0$, thus $\int c_\mu(a) d\lambda_A = 0$ and $\int d_\mu(b) d\lambda_B = 0$, from which we conclude that both c_μ and d_μ are identically 0. Hence from the definitions of c_μ and d_μ we obtain the inequalities, valid for all $a \in A$ and $b \in B$,

$$K(\mu_1, \mu_2) \geqslant K(a, \mu_2)$$

and

$$K(\mu_1, \mu_2) \leqslant K(\mu_1, b) .$$

Integrating with respect to arbitrary measures $\lambda_1 \in \mathfrak{A}$ and $\lambda_2 \in \mathfrak{B}$ this gives

$$K(\lambda_1, \mu_2) \leq K(\mu_1, \mu_2) \leq K(\mu_1, \lambda_2) ,$$

i.e. $\langle \mu_1, \mu_2 \rangle$ is an equilibrium point for Γ .

This completes the proof.

To conclude this section we shall make some remarks on the pay-off function k .

In the topological case there is another reasonable candidate for the averaging measure λ_A , viz. a measure which is strictly positive on non-empty open sets. (The Lebesgue measure is one such example.) Such measures exist in every locally compact and separable space: If $\{a_i\}$ is a countable dense subset define $\lambda_A(X) = \sum \frac{1}{2^i}$, $X \in \Sigma_A$, where the sum is taken over those i such that $a_i \in X$. As c_μ is non-negative, the continuity of c_μ implies that c_μ is identically 0 on A if $\int c_\mu(a)d\lambda_A = 0$ (see Halmos ((3)) , Ch. X).

However, it may be of interest to remark that <u>not every compact space admits a measure</u> λ (on the Borel sets) <u>which is strictly positive on non-empty open sets</u>. To prove this let A_o be a non-denumerable discrete space and A the one-point compactification. For each $a \in A_o$, $\{a\}$ is open, hence we assume that $x_a = \lambda(\{a\}) > 0$. ($x_\omega = \lambda(\{\omega\})$, $\omega \in A - A_o$, need not be positive.) If there is no countable subset K of A such that $\sum_{a \in K} x_a = 1$, then there is a least positive number $\varepsilon_o \in [0,1]$ such that $\sum_{a \in K} x_a \leq \varepsilon_o$, for all countable $K \subseteq A$. If $\sum_{a \in K} x_a = \varepsilon_o$ for some K , we pick an a' in $A_o - K$, hence $x_{a'} > 0$, and adding $x_{a'}$ to the sum the result will be $> \varepsilon_o$, - a contradiction. Thus $\sum_{a \in K} x_a < \varepsilon_o$ for all K .

But this is also impossible: Pick a sequence of positive reals $\varepsilon_n \uparrow \varepsilon_o$. For each n let K_n be a countable subset of A such that $\sum_{a \in K_n} \chi_a > \varepsilon_o - \varepsilon_n$. Then the sum over $K = UK_n$ will be $\geq \varepsilon_o$. From this we may conclude that $\lambda(\{a\})$ can be strictly positive for at most a countable subset of A_o, i.e. no Borel measure exists on the compact space A giving each non-empty open set positive measure.

One could also try to approach this problem by way of product measures as each compact space is embeddable in a product of intervals $[0,1]$. And, in fact, if A is sufficiently "thick" in the product space, i.e. if $O \cap A$ contains an open set from the base for each open O in the product space, then a suitable λ_A exists. But this is a somewhat restrictive condition, and as the problem of relativizing measures is rather complicated, we leave the matter here and in the sequel stick to our non-topological regularity condition on the pay-off function k.

III. TOPOLOGIES ON THE SPACES OF STRATEGIES

The set $\mathcal{D}$ of all bounded σ-measures on $\langle A \times B, \Sigma \rangle$ is a linear space. The subset $\mathcal{A} \times \mathcal{B}$ is not convex, but it is easy to extend T to the set of all probability measures, $\mathcal{C}$, on $\langle A \times B, \Sigma \rangle$ and $\mathcal{C}$ is a convex subset of $\mathcal{D}$. The extension $T : \mathcal{C} \to \mathcal{A} \times \mathcal{B}$ is obtained by setting for each $\mu \in \mathcal{C}$

$$T(\mu) = \mu_1' \times \mu_2' ,$$

where μ_1 and μ_2 are the projections on $\mathcal{A}$ and $\mathcal{B}$, respectively, i.e. $\mu_1(X) = \mu(X \times B)$, $X \in \Sigma_A$, and $\mu_2(Y) = \mu(A \times Y)$, $Y \in \Sigma_B$, and then compose with the map $\mu_1 \times \mu_2 \to \mu_1' \times \mu_2'$ as defined in the previous section.

Our task is now to set the stage for an application of the Schauder-Tychonoff fixed point theorem, ((1)) , p. 456, by searching for some topology on $\mathfrak{D}$ which (i) makes $\mathfrak{D}$ into a locally convex linear space such that (ii) $\mathcal{C}$ will be compact and (iii) T continuous. Then T will have a fixed point, necessarily in the set $\mathcal{A} \times \mathcal{B}$.

A most natural topology, taking into regard the definition of μ_1' and μ_2' , would be obtained by imbedding $\mathcal{C}$ (and $\mathfrak{D}$) into a product of real lines by using the set of linear maps $f : \mu \longrightarrow \mu(X)$, $X \in \Sigma$. It is immediate that $\Phi(\mu) = \langle f(\mu) \rangle$ imbeds $\mathcal{C}$ as a convex subset of a compact set in a locally convex linear topological space. Hence the first thing would be to show that $\mathcal{C}$ is closed.

However, $\mathcal{C}$ is not in general closed as the following example shows: Let $A = \{ a_1, a_2, \ldots, a_n, \ldots \}$ and $B = \{ b_1 \}$ and let Σ_A and Σ_B be the sets of all subsets of A and B , respectively. Then $\mathcal{C}$ essentially reduces to the set of all probability measures on A . We shall construct a λ in the closure of $\mathcal{C}$ (with respect to the above imbedding Φ) which is not σ-additive.

It is easy to show that each $\lambda \in \overline{\mathcal{C}}$ is a finitely additive measure on (A, Σ_A) . Let F be some ultrafilter refining the Fréchet filter on A . Define λ by $\lambda(X) = 1$ if $X \in F$ and 0 otherwise. λ is finitely additive but not σ-additive:

$$1 = \lambda(A) = \lambda(\cup \{a_i\}) > \sum \lambda(\{a_i\}) = 0 .$$

Let $X_1, \ldots, X_n \in \Sigma_A$ and assume that $X_1, \ldots, X_k \in F$, $X_{k+1}, \ldots, X_n \notin F$. As F has the finite intersection property, there is some point $a_o \in X_1 \cap \ldots \cap X_k \cap X'_{k+1} \cap \ldots \cap X'_n$. Let $\mu \in \mathcal{C}$ be defined by the condition $\mu(\{a_o\}) = 1$. Then μ approximates λ at $X_1, \ldots, X_n$ for any $\varepsilon > 0$; thus $\lambda \in \overline{\mathcal{C}}$.

It is therefore necessary to impose restrictions on our games G and Γ in order to obtain equilibrium points. Thus we now assume that A and B are compact spaces and k continuous. For Σ_A and Σ_B we take the Baire sets in A and B, respectively. (In this case not every one-point set need belong to the σ-algebras, but the reasoning above remains valid: e.g. to show that $K(\mu_a, \mu_2) = \int k(a,b)d\mu_2$ one extends μ_a to its uniquely associated regular Borel measure and evaluate the integral with respect to this measure. And as we have an extension, we obtain the correct value.)

It is well known that the space of all finite signed Baire measures is the dual of $C(A \times B)$, the set of continuous functions on $A \times B$. Further $\mathcal{C}$ is a convex, compact subset of this space in the topology induced by the maps $\mu \longrightarrow \mu(f) = \int f d\mu$, $f \in C(A \times B)$. Hence in order to apply the Schauder-Tychonoff fixed point theorem, it now remains to verify that $T : \mathcal{C} \longrightarrow \mathcal{A} \times \mathcal{B}$ is continuous in the "vague" topology.

In order to carry out this verification it will prove convenient to modify the definition of T somewhat by setting for each $f \in C(A)$ and $g \in C(B)$

$$\mu_1'(f) = (\mu_1(f) + \int f(a)c_\mu(a)d\lambda_A)/(1 + \int c_\mu(a)d\lambda_A) ,$$

$$\mu_2'(g) = (\mu_2(g) + \int g(b)d_\mu(b)d\lambda_B)/(1 + \int d_\mu(b)d\lambda_B) .$$

Here μ_1' and μ_2' must initially be conceived to be positive linear functionals, but by the well known duality already referred to, they correspond to uniquely defined measures μ_1' and μ_2' such that $\mu_1'(f) = \int f(a)d\mu_1'$ and $\mu_2'(g) = \int g(b)d\mu_2'$. As $\mu_1'(1) = 1$ and $\mu_2'(1) = 1$ we have $\mu_1' \in \mathcal{A}$ and $\mu_2' \in \mathcal{B}$.

The proof of the proposition of section II goes through with small modifications. In proving the sufficiency we now e.g. obtain a compact Baire set X such that $\mu_1(X) > 0$ and $c_\mu(a) = 0$ for all $a \in X$. Let then f_n be a decreasing sequence of continuous functions converging pointwise to the characteristic function of X (see Halmos ((3)), Ch. X). Observe that $f_n \circ c_\mu \longrightarrow 0$, hence going to the limit we have

$$\mu_1(X) = \mu_1(X)/(1 + \int c_\mu(a)d\lambda_A) ,$$

and the proof is completed as above.

We may now state the following result on the existence of good strategies.

T h e o r e m . <u>Let A and B be compact spaces and k a continuous real-valued function on $A \times B$. Let $\Gamma = \langle \mathfrak{A}, \mathfrak{B}, K \rangle$ be the mixed extension of the game $G = \langle A, B, k \rangle$ obtained by letting $\mathfrak{A}$ and $\mathfrak{B}$ be the sets of probability measures on the Baire set in A and B, respectively. Then Γ has an equilibrium pair.</u>

It remains to show that T is continuous, i.e. we must show that $\mu \longrightarrow \mu_1' \times \mu_2'(f)$ is continuous for all $f \in C(A \times B)$. It suffices to show that $\mu \longrightarrow \mu_1'(f)$, $f \in C(A)$, and $\mu \longrightarrow \mu_2'(g)$, $g \in C(B)$, are continuous: If this is proved, then $\mu \longrightarrow (\mu_1' \times \mu_2')(f \cdot g) = \mu_1'(f) \cdot \mu_2'(g)$ is continuous, hence also all the maps $\mu \longrightarrow (\mu_1' \times \mu_2')(\sum_{i=1}^{n} f_i g_i)$. But the set of maps $\sum_{i=1}^{n} f_i g_i$, $f_i \in C(A)$ and $g_i \in C(B)$, is uniformly dense in $C(A \times B)$ by virtue of the Stone-Weierstrass theorem. Thus the continuity of $\mu \longrightarrow \mu_1' \times \mu_2'(f)$ for an arbitrary $f \in C(A \times B)$ follows by the inequality:

$$|\mu_1' \times \mu_2'(f) - \lambda_1' \times \lambda_2'(f)| \leq 2 \cdot \| f - \sum f_i g_i \| + |\mu_1' \times \mu_2'(\sum f_i g_i) - \lambda_1' \times \lambda_2'(\sum f_i g_i)| .$$

The continuity of $\mu \longrightarrow \mu_1'(f)$ and $\mu \longrightarrow \mu_2'(g)$ is proved in exactly the same way, hence we treat only the first map. From the definition of $\mu_1'(f)$ it follows that we must verify that the maps $\mu \to \mu_1(f)$, $\mu \longrightarrow \int f(a)c_\mu(a)d\lambda_A$ and $\mu \longrightarrow \int c_\mu(a)d\lambda_A$ are continuous. This readily reduced to show that the map $\mu \longrightarrow \mu_1 \times \mu_2$ is continuous, - which is straight forward - , and that the family of maps $\mu \longrightarrow c_\mu(a)$, $a \in A$, is equicontinuous.

To verify this last assertion we need the compactness of $A \times B$ and the continuity of k: We first note that for all $\varepsilon > 0$ there exists a finite covering $U_1, \dots, U_m$ of A and points $a_i \in U_i$ such that $|k(a,b) - k(a_i,b)| < \varepsilon$ for all $b \in B$, provided $a \in U_i$. We also note that the map $\mu \longrightarrow K(a,\mu_2) = \int k(a,b)d\mu_2$ is continuous, a is here a fixed parameter. Making then use of the inequality

$$|K(a,\mu_2)-K(a,\lambda_2)| \leq 2\cdot\max_b |k(a,b)-k(a_i,b)| + |K(a_i,\mu_2)-K(a_i,\lambda_2)|$$

we may conclude that the family of maps $\mu \longrightarrow K(a,\mu_2)$, $a \in A$, is equicontinuous, because for each $\varepsilon > 0$ there is a f i n i t e number of points a_i for which we need to have the continuity of the map $\mu \longrightarrow K(a_i,\mu_2)$. By considering the inequality

$$|c_\mu(a) - c_\lambda(a)| \leq |K(a,\mu_2) - K(a,\lambda_2)| + |K(\mu_1,\mu_2) - K(\lambda_1,\lambda_2)|$$

we easily obtain that the maps $\mu \longrightarrow c_\mu(a)$, $a \in A$, are equicontinuous. This completes the proof.

References

((1)) N. Dunford and J.T. Schwartz: Linear operators. Part I. Interscience, New York, 1958.

((2)) S. Karlin: Mathematical methods and theory in games, programming and economics. Volume II: The theory of infinite games. Addison-Wesley, 1959.

((3)) P.R. Halmos: Measure theory. Van Nostrand, Princeton 1950.

((4)) R.D. Luce and H. Raiffa: Games and decisions. J. Wiley, New York, 1957.

((5)) J.F. Nash: Equilibrium points in n-person games. Proceedings of the National Academy of Sciences, U.S.A., vol. 36 (1950), pp. 48-49.

((6)) J. von Neumann and O. Morgenstern: Theory of games and economic behavior, 2. ed., Princeton University Press, 1947.