A non-standard treatment of the equation $y' = f(y,t)$

by

Bent Birkeland   Dag Normann

1. Introduction

In this note we will discuss the non-standard version of Peano's existence proof for solutions of the equation $y' = f(y,t)$ with initial condition $y(0) = a$. Without loss of much generality we will assume that $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous and that $\|f(y,t)\| \leq 1$ for all $(y,t) \in \mathbb{R}^n \times \mathbb{R}$. We will mainly be interested in solutions over the interval $t \in [0,1]$.

We will deal with the Euler-Cauchy-method for obtaining solutions to these equations, but our approach will clearly work for alternative methods too.

Bebbouchi [2] and [3] discuss problems of the same nature as we do.

We will assume familiarity with the elementary parts of non-standard analysis, see e.g. Robinson [8], Keisler [5] or Albeverio, Fenstad and Høegh-Krohn [1]. Throughout the note we will let N be a fixed non-standard natural number.

Definitions

a) $Y: \{k \in {}^*\mathbb{N}; k \leq N\} \to {}^*\mathbb{R}^n$ is an Euler-Cauchy-vector for f if Y is internal and Y satisfies the equation

$$Y(k+1) = Y(k) + \frac{1}{N}\,{}^*f\left(Y(k), \frac{k}{N}\right) \qquad \text{for } k < N$$

We write E-C-vector for Euler-Cauchy-vector.

b) An E-C--vector Y is near-standard if $Y(0)$ is near-standard.

c) If Y is a near-standard E-C-vector for f, let $y = y_Y$ be defined by

$$y(st(\tfrac{k}{N})) = st(Y(k)) \qquad \text{for} \quad 0 \leq k \leq N .$$

d) A solution for f is a (standard or internal) function $y:[0,1] \to \mathbb{R}^n$ $(y:{}^*[0,1] \to {}^*\mathbb{R}^n)$ satisfying the equation

$$y'(t) = f(y(t),t)$$

for all $t \in [0,1]$ $({}^*[0,1])$.

Remark

As in definition d) we will from now on omit some *'s when there may be no confusion.

By a standard non-standard argument we may show

Theorem 1

If Y is a near-standard E-C-vector for f then $y = y_Y$ is a standard solution for f with $y(0) = st(Y(0))$.

Theorem 1 gives us Peano's existence theorem.

The solutions are not always unique. The equation

$$y' = 3y^{2/3}$$
$$y(0) = 0$$

has the solutions

$$y(t) = \begin{cases} 0 & \text{if} \quad t \leq t_0 \\ (t-t_0)^3 & \text{if} \quad t \geq t_0 \end{cases}$$

for all t_0 in $[0,1]$.

In Hartman [4] pp. 18-20 there is an example where the set of solutions is very complex.

Remark

In examples we will sometimes violate the assumptions on f, but only those assumptions that do not restrict the generality of our arguments.

Our problem will be how to represent the set of solutions via E-C-vectors. The motivation is that a nice representation gives an easy access to the properties of this set.

P. Montel [7] showed that the Euler-Cauchy method gives all solutions if one solves the difference equation over intervals of non-constant lengths. Here we will only deal with intervals of length $\frac{1}{N}$. First we will show under a special assumption on f that all solutions may be represented by E-C-vectors for f. The example above, together with all equations

$$y' = K|y|^{\alpha}$$
$$y(0) = 0$$

for $0 < \alpha < 1$, will satisfy this special assumption.

Then we will show in general that all solutions for f may be represented by a E-C-vector for some g infinitesimally close to f. Finally we will use this to give a simple proof of a theorem of Kneser [6].

2. Backward Solutions

An E-C-vector Y for f is obviously uniquely determined by the initial value $Y(0)$.

Let $F: \mathbb{R}^n \to {}^*\mathbb{R}^n$ be defined by

$$F(Y(0)) = Y(N) .$$

Lemma 1

F is $*$-continuous and surjective.

Proof

Continuity is evident.

In order to show that F is surjective it is sufficient to show that $F_k(Y(k)) \overset{d}{=} Y(k+1)$ is surjective. For this we use Brouwer's fixed-point theorem: Let k and $Y(k+1)$ be given. The function $a \rightsquigarrow Y(k+1) - \frac{1}{N} f(a, \frac{k}{N})$ is continuous and maps the ball with center $Y(k+1)$ and radius $\frac{1}{N}$ into itself. So for some a in that ball

$$a = Y(k+1) - \frac{1}{N} f(a, \frac{k}{N}) .$$

Let $Y(k)$ be one such a. Then $Y(k+1) = F_k(Y(k))$.

Lemma 2

For given $a \in \mathbb{R}^n$, the set

$$C_a = \{b \in \mathbb{R} ; \exists Y \text{ (} Y \text{ is an E-C-vector for } f, \; Y(0) \simeq a \text{ and } Y(N) \simeq b)\}$$

is compact and connected in $\mathbb{R}^n$.

Proof

Let Y range over the E-C-vectors for f.

For every $n \in \mathbb{N}$ the set

$$B_{a,m} = \{Y(N) ; ||Y(0) - a|| < \frac{1}{m}\}$$

is connected in ${}^*\mathbb{R}^n$.

Let $C_{a,m} = \{st(b) ; b \in B_{a,m}\}$

$C_{a,m}$ is closed since $B_{a,m}$ is internal. $C_{a,m}$ is clearly bounded. If O_1 and O_2 are two disjoint sets separating $C_{a,m}$ into two disjoint nonempty sets it is easily seen that ${}^*O_1$ and ${}^*O_2$ will separate $B_{a,m}$.

Moreover $C_{a,m+1} \subseteq C_{a,m}$. It follows that $C_a = \bigcap_{m=1}^{\infty} C_{a,m}$ is compact and connected.

Now we know that the set of E-C-solutions y, i.e. those solutions obtained from an E-C-vector, with $y(0) = a$, cut out a compact connected set of each hyperplane "$t = \text{const}$". But we are unable to prove without further restrictions on f that we get all solutions this way.

Theorem 2

Assume that the solutions of f are unique to the left; i.e. that $y(t)$ for $t < 1$ is determined by $y(1)$. Then all standard solutions are of the form $y = y_Y$.

Remark

If the standard solutions are unique to the left, then by transfer all solutions are unique to the left. The equations $y' = ky^{\alpha}$, $y(0) \geqq 0$ for $0 > \alpha > 1$ are all covered by this theorem.

Proof

Let y be a standard solution. Then $y(1) = Y(N)$ for some E-C-vector Y for f. By uniqueness to the left we have $y = y_Y$.

Definition

Let $S(a,t) = \{y(t);\ y(0) = a$ and y is a solution for $f\}$.

Theorem 3

Assume that the solutions for f are unique to the left. Then $S(a,1)$ is compact, connceted and $\mathbb{R}^n \setminus S(a,1)$ is path-connected.

Proof

In this case $S(a,1) = C_a$ from Lemma 2, so we are left with showing that $\mathbb{R}^n \setminus S(a,1)$ is path-connected.

Let $b_1, b_2 \in \mathbb{R}^n \setminus S(a,1)$. We will show that there is a non-standard path from b_1 to b_2 avoiding ${}^*S(a,1)$ and then use the transfer principle "backwards". Let y_1 and y_2 be solutions of f with $y_1(1) = b_1$, $y_2(1) = b_2$. Let $a_1 = y_1(0)$ and $a_2 = y_2(0)$. Let σ be a standard path from a_1 to a_2 avoiding a .

Let Y_u be the E-C-vector for f with $Y_u(0) = \sigma(u)$ and let $\tilde{\sigma}(u) = Y_u(N)$. Then $\tilde{\sigma}$ is a non-standard path from $c_1 = Y_0(N)$ to $c_2 = Y_1(N)$ and $c_1 \simeq b_1$, $c_2 \simeq b_2$. If for some u $\tilde{\sigma}(u) \in {}^*S(a,1)$ then the unique solution y with $y(1) = st(\sigma'(u))$ will satisfy $y(0) = a$, so $\sigma(u) \simeq a$ contradicting the choice of σ . Also any infinitesimal ball around b_1 or b_2 will be disjoint from ${}^*S(a,1)$ So the path $\tilde{\sigma}$ may be extended to a path from b_1 to b_2 avoiding ${}^*S(a,1)$.

3. A special case

We will now consider the one-dimensional example

$$y' = 3y^{2/3}$$

over $[-1,1]$ with initial condition $y(-1) = -1$. This case is not covered by theorems 2 and 3 but the conclusions are still valid.

The E-C-vectors will here map $[-N,N] \cap {}^*\mathbb{N}$ into ${}^*\mathbb{R}$ with $Y(-N) \simeq -1$. Through a sequence of claims we will see that every solution may be obtained from one of these vectors. We only give hints of the proofs.

Claim 1

$\{Y(0): Y$ is an E-C-vector and $Y(-N) \simeq -1\}$ contains all non--positive infinitesimals.

Proof

If $Y(-N) \geq -1$ then $Y(0) \geq 0$

If $Y(-N) < -1-\varepsilon$ for some $\varepsilon > 0$ in $\mathbb{R}$ then $Y(0)$ is negative and not an infinitesimal. As $Y(0)$ is a *-continuous map of $Y(-N)$ the claim follows by standard analysis.

Claim 2

$Y(1)$ can take any value $\leqq \frac{4}{N^3}$ with $Y(0) \leqq 0$.

Proof

Standard calculus.

Claim 3

If $Y(1) = \frac{1}{N^3}$ then $Y(N) \simeq 1$.

Proof

Let $a_1 = 1$, $a_{n+1}^3 = a_n^3 + 3a_n^2$. Then $Y(n) = \frac{a_n^3}{N^3}$ and $Y(N) = \frac{a_N^3}{N^3}$ so $Y(N) \simeq \lim_{n\to\infty} \frac{a_n^3}{n^3}$ as a standard limit.

Clearly

$$a_n \to \infty \quad \text{as} \quad n \to \infty$$

and $a_{n+1} \leqq a_n + 1$.

We then have

$$(a_{n+1}-a_n)(a_{n+1}^2+a_{n+1}a_n+a_n^2) = a_{n+1}^3 - a_n^3 = 3a_n^2$$

so

$$a_{n+1} - a_n = \frac{3a_n^2}{a_{n+1}^2+a_{n+1}a_n+a_n^2} \leqq 1 .$$

But $a_{n+1} - a_n \geqq \frac{3a_n^2}{(a_n+1)^2 + (a_n+1)a_n + a_n^2} \to 1$ as $n \to \infty$.

So $\lim_{n\to\infty} (a_{n+1}-a_n) = 1$ and $\lim_{n\to\infty} \frac{a_n}{n} = 1$.

4. Perturbations of f

Let us now again work in the generality of the introduction.

Definition

Let $\delta > 0$ be in ${}^*\mathbb{R}$. Let $X_\delta = \{ g; g$ is internal, *-continuous and $\|f-g\|_\infty < \delta \}$, $(\| \;\|_\infty$ is the sup-norm).

Lemma 3

Let $\delta > 0$ be infinitesimal, $g \in X_\delta$ and Y a near-standard E-C-vector for g. Then y_Y is a solution for f.

The proof is like the proof of Theorem 1.

Lemma 4

Let $a \in \mathbb{R}^n$ be given. Let y be an internal solution for f with $y(0) = a$. Then there is an infinitesimal $\delta > 0$ and a $g \in X_\delta$ such that the E-C-vector Y for g with $Y(0) = a$ will satisfy

$$\forall k \leq N \; (Y(k) = y(\tfrac{k}{N})) \, .$$

Proof

Let $g(y(\frac{k}{N}), \frac{k}{N}) = N(y(\frac{k+1}{N}) - y(\frac{k}{N}))$.
Then $Y(k) = y(k)$ is an E-C-vector for g.

So far we have defined g at a hyperfinite set of points. We will show that for each of these points the distance to f is infinitesimal. Then the maximal distance will be infinitesimal and g may be extended to an element of some X_δ. We prove this for each coordinate $i \leq n$, f_i, g_i and y_i denote the i'th coordinate of f, g, y resp.

$$\begin{aligned} & |f_i(y(\tfrac{k}{N}), \tfrac{k}{N}) - g_i(y(\tfrac{k}{N}), \tfrac{k}{N})| \\ = \; & |f_i(y(\tfrac{k}{N}), \tfrac{k}{N}) - N(y_i(\tfrac{k+1}{N}) - y_i(\tfrac{k}{N}))| \\ = \; & |f_i(y(\tfrac{k}{N}), \tfrac{k}{N}) - y_i'(x)| \quad \text{for some} \quad x \in [\tfrac{k}{N}, \tfrac{k+1}{N}] \\ = \; & |f_i(y(\tfrac{k}{N}), \tfrac{k}{N}) - f_i(y(x), x)| \simeq 0 \end{aligned}$$

since f is uniformly continuous within the interesting area and y is Lipschitz-continuous with constant 1.

Theorem 4

There is an infinitesimal $\delta > 0$ such that for all standard solutions y for f there is a $g \in X_\delta$ and an E-C-vector Y for g such that

i) $y = y_Y$

ii) $y(0) = Y(0)$

iii) $y(1) = Y(N)$

Proof

For each near-standard a, let δ_a be the supremum of those infinitesimals needed in Lemma 3 for internal solutions y with $y(0) = a$. The set of internal solutions is an internal set so this supremum exists and is infinitesimal. δ_a will exist for a's that are not near-standard too, but will not necessarily be infinitesimal.

But $M = \{a; \delta_a < \frac{1}{\|a\|}\}$ is internal, contains all near-standard points and the δ_a's are all infinitesimals. Let $\delta = \sup\{\delta_a; a \in M\}$. Then δ will satisfy Theorem 4.

Remark

The δ we have constructed is dependent of f. In the standard ultraproduct model for non-standard analysis there is no infinitesimal δ that works for all f. Using the saturation-principle we may find a $\delta > 0$ that works uniformly for all standard f.

5. Applications

As the proof of Theorem 4 is rather simple it may be justified to use the theorem to obtain classically known results.

Corollary (Kneser [6])

Let $S(a,t)$ be as above, where $a \in \mathbb{R}^n$. $S(a,t)$ is a compact, connected set.

Proof

Let $t = 1$ without loosing generality. In Lemma 2 we showed that if an internal set A is bounded and connected in the non--standard sense, then $\{st(b); b \in A\}$ is compact and connected. Let $Z = \{Y_g(N): g \in X_\delta\}$ where Y_g is the E-C-vector g with $Y_g(0) = a$. X_δ is a connected space in the $\|\ \|_\infty$-topology and $g \rightsquigarrow Y_g(N)$ is continuous. By Theorem 4 $S(a,1) = \{st(b); b \in Z\}$. Then $S(a,1)$ is compact since Z is bounded and $S(a,1)$ is connected since Z is connected in the non-standard sense.

Remark

In general, $S(a,t)$ will not be simply-connected, see Hartmann [4, p.17, exerc. 4,3]. But under the additional assumption of left uniqueness, it is, by Theorem 3.

References

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