Homogeneous spaces
B. Komrakov seminar

THREE-DIMENSIONAL ISOTROPICALLY–FAITHFUL HOMOGENEOUS SPACES

VOLUME II

B. Komrakov A. Tchourioumov

E. Belin
Z. Belyankina
O. Chizh
B. Doubrov
I. Goncharov
T. Gur'yanova
S. Kimbar
B. Komrakov, Jnr.
L. Levin
N. Mozhei
A. Palvinsky
D. Shabes

International Sophus Lie Centre

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Foreword

We consider classification of lower-dimensional homogeneous spaces an immediate continuation and global version of classification results obtained by Sophus Lie. Two-dimensional homogeneous spaces were classified locally by Sophus Lie [L1] and globally by G.D. Mostow [M]. (See also our preprint [KTD], where the complete classification of two-dimensional homogeneous spaces, both locally and globally, is presented.) S. Lie also obtained some results in classification of three-dimensional homogeneous spaces and described all subalgebras in the Lie algebra $\mathfrak{gl}(3, \mathbb{C})$. A detailed account of these classifications can be found in [L2].

The problem of finding the complete description of three- and four-dimensional homogeneous spaces as pairs, (group, subgroup) or even (algebra, subalgebra), is extremely important and rich in applications, but it is a very difficult one: "The description of arbitrary transitive actions on manifolds M, where $\dim M \geqslant 3$, presently seems to be unattainable."([GO], p. 232)

Minimal transitive actions, that is, those that have no proper transitive subgroups, on three-dimensional manifolds were classified in [G]. The problem of local classification of three- and four-dimensional homogeneous spaces was chosen by one of the authors, B. Komrakov, as the topic of Dr. Sci. thesis for A. Tchourioumov, the other author. (Some of the results can be found in [Tch].)

An important subclass in all homogeneous spaces is formed by isotropically-faithful spaces. In particular, it contains all homogeneous spaces that admit an invariant affine connection. The present preprint gives the local classification of three-dimensional isotropically-faithful homogeneous spaces.

In 1990, the International Sophus Lie Centre, jointly with the University of Belarus, organized an experimental group of 25 students majoring in mathematics and working in accordance with a special syllabus oriented to modern differential-geometric methods in the study of nonlinear differential equations. The following idea arose: to split up the classification problem mentioned above into smaller parts and give each part to a student; in the process of learning new material, the student will then try to apply his newly acquired knowledge to this problem as an illustration.

Suppose, for example, that the student is learning about differential equations; he then writes out trajectories of one-parameter subgroups on the specific manifold that he has been given. Studying differential geometry, he computes invariant affine connections, metrics, curvature tensors, geodesics, etc., with special emphasis on his example, and so on.

In their first year, the students all took an advanced course in Lie algebras and the main part of the work on all these "smaller parts" was completed by 12 students. We had no time to give our students an introductory course in cohomologies of Lie algebras, and although their computation constitutes a considerable part of the work, we do not use this language.

This work was started in Tartu University, Estonia (August 1991), continued at the Institute of Astrophysics and Atmosphere Physics in Tõravere, Estonia (December 1991 to March 1992), then at the "Bears' Lakes" Space Center of the Special Research Bureau of Moscow Power Engineering Institute (August 1993), and finished at the University of Oslo and the Center for Advanced Study (SHS) at the

Norwegian Academy of Science and Letters. (Naturally, most of the time from August 1991 to November 1993 was spent in Minsk, Belarus.) The story of this work was rich in experiences and events only indirectly connected with mathematics, something we will not here dwell on at length. We would, however, like to express our gratitude to those who directly or indirectly made it possible for us to complete this work.

In the future, we are going to proceed with the study of geometry of three-dimensional homogeneous spaces in the following directions:

- description of invariant affine connections on three-dimensional homogeneous spaces together with their curvature and torsion tensors, holonomy groups, geodesics, etc.;
- description of invariant tensor geometric structures and their properties;
- global classification of three-dimensional isotropically-faithful homogeneous spaces and description of inclusions among the corresponding transformation groups;
- description of differential invariants for the homogeneous spaces to be found and of the corresponding invariant differential equations;
- description of discrete subgroups in transformation groups together with description of the corresponding topological factor spaces.

Introduction

It is known that the problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$ such that $G \subset \overline{G}$. Two pairs $(\overline{G}_1, G_1)$ and $(\overline{G}_2, G_2)$ are said to be equivalent if there exists an isomorphism of Lie groups $\pi\colon \overline{G}_1 \to \overline{G}_2$ such that $\pi(G_1) = G_2$.

By linearization, the problem can be reduced to the problem of classification of pairs of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ viewed up to equivalence of pairs. The structure of all pairs of Lie groups $(\overline{G}, G)$ corresponding to a given pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ was described in [M]. In the study of homogeneous spaces it is important to consider not the group $\overline{G}$ itself, but its image in $\mathrm{Diff}(M)$. In other words, it is sufficient to consider only the effective action of the group $\overline{G}$ on the manifold M. In terms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, this condition is equivalent to the condition for $\mathfrak{g}$ to contain no proper ideals of $\bar{\mathfrak{g}}$. In this case we say that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective.*

In the present work we classify all isotropically–faithful pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 3.

Definition. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropically-faithful* if the natural $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ is faithful.

We say that a homogeneous space $(\overline{G}, M)$ is isotropically-faithful if so is the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. From geometrical point of view it means that the natural action of the stabilizer $\overline{G}_x$ of an arbitrary point $x \in M$ on T_xM has discrete kernel.

We divide the solution of our problem into the following parts:

(1) We classify (up to isomorphism) all faithful three–dimensional $\mathfrak{g}$-modules U. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation.

(2) For each $\mathfrak{g}$-module U obtained in (1) we classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic.

In Chapter I we give basic definitions and introduce the notation to be employed. Here we also solve part (1) of the problem by classifying subalgebras in $\mathfrak{gl}(3, \mathbb{R})$.

In Chapter II we develop methods for constructing pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ given a three-dimensional faithful $\mathfrak{g}$-module U. This involves computation of the first cohomology space of $\mathfrak{g}$ with values in the natural module $\mathcal{L}(U, \mathfrak{g})$. A series of techniques described in Chapter II allows, in some cases, to simplify the computation considerably.

Finally, Chapter III gives the classification of three-dimensional isotropically-faithful pairs itself.

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3. Three-dimensional case

Proposition 3.1. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 3.1 is trivial.*

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	0	u_1	0	0
e_2	0	0	0	0	u_2	0
e_3	0	0	0	0	0	u_3
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	0	0	$-u_3$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial. And this proves the Proposition.

Proposition 3.2. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 3.2 is trivial.*

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	0	$-u_2$	u_1	0
e_2	0	0	0	u_1	u_2	0
e_3	0	0	0	0	0	u_3
u_1	u_2	$-u_1$	0	0	0	0
u_2	$-u_1$	$-u_2$	0	0	0	0
u_3	0	0	$-u_3$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial. And this proves the Proposition.

Proposition 3.3. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 3.3 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	0
e_2	$-2e_2$	0	e_1	0	u_1	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0
u_1	$-u_1$	0	$-u_2$	0	0	0
u_2	u_2	$-u_1$	0	0	0	0
u_3	0	0	0	0	0	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	0
e_2	$-2e_2$	0	e_1	0	u_1	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0
u_1	$-u_1$	0	$-u_2$	0	u_3	0
u_2	u_2	$-u_1$	0	$-u_3$	0	0
u_3	0	0	0	0	0	0

3.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	0
e_2	$-2e_2$	0	e_1	0	u_1	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0
u_1	$-u_1$	0	$-u_2$	0	0	u_1
u_2	u_2	$-u_1$	0	0	0	u_2
u_3	0	0	0	$-u_1$	$-u_2$	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.3 is trivial.*

Proof. Note that $\mathfrak{g}$ is a semisimple Lie algebra. By Proposition 12, Chapter II, without loss of generality it can be assumed that $q(\mathfrak{g}) = \{0\}$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure. Then

$$\begin{aligned}
&[e_1, e_2] = 2e_2, \\
&[e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \\
&[e_1, u_1] = u_1, \quad [e_2, u_1] = 0, \quad [e_3, u_1] = u_2, \\
&[e_1, u_2] = -u_2, \quad [e_2, u_2] = u_1, \quad [e_3, u_2] = 0, \\
&[e_1, u_3] = 0, \quad [e_2, u_3] = 0, \quad [e_3, u_3] = 0.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),$$

where

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) = \mathbb{R}e_3, \quad &&\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}u_2, \\
&\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_3, \quad &&\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1, \\
&\bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{R}e_2.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
[u_1, u_2] &= a_1 e_1 + \alpha_3 u_3, \\
[u_1, u_3] &= \beta_1 u_1, \\
[u_2, u_3] &= \gamma_2 u_2.
\end{aligned}
$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$2e_2$	$-2e_3$	u_1	$-u_2$	0
e_2	$-2e_2$	0	e_1	0	u_1	0
e_3	$2e_3$	$-e_1$	0	u_2	0	0
u_1	$-u_1$	0	$-u_2$	0	$\alpha_3 u_3$	$\gamma_2 u_1$
u_2	u_2	$-u_1$	0	$-\alpha_3 u_3$	0	$\gamma_2 u_2$
u_3	0	0	0	$-\gamma_2 u_1$	$-\gamma_2 u_2$	0 ,

where the coefficients α_3 and γ_2 satisfy the equation $\alpha_3 \gamma_2 = 0$.

Consider the following cases:

1°. $\alpha_3 = \gamma_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\alpha_3 \neq 0$, $\gamma_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$
\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, 2, 3, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= \frac{1}{\alpha_3} u_3.
\end{aligned}
$$

3°. $\alpha_3 = 0$, $\gamma_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$
\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, 2, 3 \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2, \\
\pi(u_3) &= \gamma_2 u_3.
\end{aligned}
$$

Now it remains to show that the pairs determinited in the Proposition are not equivalent to each other.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_3 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_1$, we conclude that the pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

Since $\mathcal{Z}\bar{\mathfrak{g}}_2 = \mathbb{R}u_3$ and $\mathcal{Z}\bar{\mathfrak{g}}_3 = 0$, we conclude that the pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

The proof of the Proposition is complete.

Proposition 3.4. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.4 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	e_3	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	0	0
u_2	0	$-u_1$	$-u_3$	0	0	0
u_3	u_3	$-u_2$	0	0	0	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	e_3	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	e_2	$-e_1$
u_2	0	$-u_1$	$-u_3$	$-e_2$	0	$-e_3$
u_3	u_3	$-u_2$	0	e_1	e_3	0

3.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	e_3	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	$-e_2$	e_1
u_2	0	$-u_1$	$-u_3$	e_2	0	e_3
u_3	u_3	$-u_2$	0	$-e_1$	$-e_3$	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.4 is trivial.*

Proof. Note that $\mathfrak{g}$ is a semisimple Lie algebra. By Proposition 12, Charter II, without loss of generality it can be assumed that $q(\mathfrak{g}) = \{0\}$. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.4. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the trivial virtual structure.

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Charter II). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}u_1, \quad \bar{\mathfrak{g}}^{-1)}(\mathfrak{h}) = \mathbb{R}e_3 \oplus \mathbb{R}u_3,$$

$$[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$
$$[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}),$$
$$[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}),$$

and

$$[u_1, u_2] = a_2e_2 + \alpha_1 u_1,$$
$$[u_1, u_3] = b_1e_1 + \beta_2 u_2,$$
$$[u_2, u_3] = c_3e_3 + \gamma_3 u_3.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	$e3$	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	$a_2e_2 + \alpha_1u_1$	$-a_2e_1 + \alpha_1u_2$
u_2	0	$-u_1$	$-u_3$	$-a_2e_2 - \alpha_1u_1$	0	$-a_2e_3 + \alpha_1u_3$
u_3	u_3	$-u_2$	0	$a_2e_1 - \alpha_1u_2$	$a_2e_3 - \alpha_1u_3$	0

Put $p = a_2 + \frac{\alpha_1^2}{4}$. Consider the following cases:

1°. $p = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_1) = u_1 - \frac{\alpha_1}{2}e_2,$$
$$\pi(u_2) = u_2 + \frac{\alpha_1}{2}e_1,$$
$$\pi(u_3) = u_3 + \frac{\alpha_1}{2}e_3.$$

2°. $p > 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_1) = (u_1 - \frac{\alpha_1}{2}e_2)/\sqrt{p},$$
$$\pi(u_2) = (u_2 + \frac{\alpha_1}{2}e_1)/\sqrt{p},$$
$$\pi(u_3) = (u_3 + \frac{\alpha_1}{2}e_3)/\sqrt{p}.$$

3°. $p < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_i) &= e_i,\ i = 1,2,3,\\
\pi(u_1) &= -p(u_1 - \frac{\alpha_1}{2}e_2)/\sqrt{-p},\\
\pi(u_2) &= (u_2 + \frac{\alpha_1}{2}e_1)/\sqrt{-p},\\
\pi(u_3) &= (u_3 + \frac{\alpha_1}{2}e_3)/\sqrt{-p}.
\end{aligned}$$

It remains to show that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent to each other. Indeed, since $\dim \mathfrak{r}(\bar{\mathfrak{g}}_1) \neq \dim \mathfrak{r}(\bar{\mathfrak{g}}_2)$ and $\dim \mathfrak{r}(\bar{\mathfrak{g}}_1) \neq \dim \mathfrak{r}(\bar{\mathfrak{g}}_3)$, we see that the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ is not equivalent to the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

Since $\bar{\mathfrak{g}}_3$ is a simple Lie algebra ($\bar{\mathfrak{g}}_3 \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$) and the Lie algebra $\bar{\mathfrak{g}}_2$ is not simple ($\bar{\mathfrak{g}}_2 \cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$), we see that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

This completes the proof of the Proposition.

Proposition 3.5. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.5 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_2$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	0	0
u_2	0	$-u_1$	u_3	0	0	0
u_3	$-u_1$	0	$-u_2$	0	0	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_2$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	e_2	e_1
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3
u_3	$-u_1$	0	$-u_2$	$-e_1$	$-e_3$	0

3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_2$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	$-e_2$	$-e_1$
u_2	0	$-u_1$	u_3	e_2	0	$-e_3$
u_3	$-u_1$	0	$-u_2$	e_1	e_3	0

Proof.

Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Lemma. *Any virtual structure q on generalized module 3.5 is trivial.*

Proof. Note that $\mathfrak{g}$ is a semisimple Lie algebra. By Proposition 12, Chapter II, without loss of generality it can be assumed that $q(\mathfrak{g}) = \{0\}$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.5. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial. Then

$$\begin{aligned}
&[e_1, e_2] = e_3, \\
&[e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1, \\
&[e_1, u_1] = -u_3, \quad [e_1, u_2] = 0, \quad [e_1, u_3] = u_1, \\
&[e_2, u_1] = -u_2, \quad [e_2, u_2] = u_1, \quad [e_2, u_3] = 0, \\
&[e_3, u_1] = 0, \quad [e_3, u_2] = -u_3, \quad [e_3, u_3] = u_2.
\end{aligned}$$

Put

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1 + a_2 e_2 + a_3 e_3 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\
[u_1, u_3] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\
[u_2, u_3] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_2$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	$a_2e_2 + \alpha_3u_3$	$a_2e_1 - \alpha_3u_2$
u_2	0	$-u_1$	u_3	$-a_2e_2 - \alpha_3u_3$	0	$a_2e_3 + \alpha_3u_1$
u_3	$-u_1$	0	$-u_2$	$\alpha_3u_2 - a_2e_1$	$-a_2e_3 - \alpha_3u_1$	0

Put

$$p = \frac{1}{\sqrt{|a_2 - \frac{1}{4}\alpha_3^2|}}, \quad \text{whenever } a_2 \neq \frac{1}{4}\alpha_3^2.$$

Consider the following cases:

1°. $4a_2 = \alpha_3^2$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, 2, 3, \\
\pi(u_1) &= u_1 + \frac{1}{2}\alpha_3 e_3, \\
\pi(u_2) &= u_2 - \frac{1}{2}\alpha_3 e_1, \\
\pi(u_3) &= u_3 + \frac{1}{2}\alpha_3 e_2.
\end{aligned}$$

2°. $4a_2 > \alpha_3^2$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2, 3,\\
&\pi(u_1) = p(u_1 + \frac{1}{2}\alpha_3 e_3),\\
&\pi(u_2) = p(u_2 - \frac{1}{2}\alpha_3 e_1),\\
&\pi(u_3) = p(u_3 + \frac{1}{2}\alpha_3 e_2).
\end{aligned}$$

3°. $4a_2 < \alpha_3^2$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2, 3,\\
&\pi(u_1) = p(u_1 + \frac{1}{2}\alpha_3 e_3),\\
&\pi(u_2) = p(u_2 - \frac{1}{2}\alpha_3 e_1),\\
&\pi(u_3) = p(u_3 + \frac{1}{2}\alpha_3 e_2).
\end{aligned}$$

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since $\mathfrak{r}(\bar{\mathfrak{g}}_1) \neq \{0\}$ and $\mathfrak{r}(\bar{\mathfrak{g}}_2) = \mathfrak{r}(\bar{\mathfrak{g}}_3) = \{0\}$, we see that no one of the pairs 3.5.2 and 3.5.3 is equivalent to the pair 3.5.1.

Note that the Lie algebra $\bar{\mathfrak{g}}_2$ is simple ($\bar{\mathfrak{g}}_2 \cong \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$) and the Lie algebra $\bar{\mathfrak{g}}_3$ is not simple ($\bar{\mathfrak{g}}_3 \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$). It follows that the pairs 3.5.2 and 3.5.3 are not equivalent.

This completes the proof of the Proposition.

Proposition 3.6. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.6 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_1	0	0	u_1
e_2	0	0	0	0	u_2	0
e_3	$-e_1$	0	0	0	0	u_3
u_1	0	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	$-u_1$	0	$-u_3$	0	0	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_1	e_1	0	u_1
e_2	0	0	0	0	u_2	0
e_3	$-e_1$	0	0	0	0	u_3
u_1	$-e_1$	0	0	0	0	u_3
u_2	0	$-u_2$	0	0	0	0
u_3	$-u_1$	0	$-u_3$	$-u_3$	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$, spanned by e_2 and e_3.

Lemma. *Any virtual structure q on generalized module 3.6 is equivalent to one of the following:*

$$C_1(e_1) = \begin{pmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = C_1(e_3) = 0.$$

Proof. Let q be a virtual structure on the generalized module 3.6. Without loss of generality it can be assumed that q is primary. Since

$$\mathfrak{g}^{(0,-1)}(\mathfrak{h}) \supset \mathbb{R}e_1, \quad \mathfrak{g}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_2 \oplus \mathbb{R}e_3,$$

$$U^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}u_1, \quad U^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_2, \quad U^{(0,1)} \supset \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} c_{11}^1 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & 0 & c_{33}^1 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^2 & 0 & 0 \\ c_{31}^2 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^3 & 0 & 0 \\ c_{31}^3 & 0 & 0 \end{pmatrix}.$$

Let us check condition (6), Chapter II, for $x, y \in \mathcal{E}$.

$$C([e_1, e_2]) = A(e_1)C(e_2) - C(e_2)B(e_1) - A(e_2)C(e_1) + C(e_1)B(e_2)$$

We have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_{31}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{21}^2 \\ 0 & 0 & c_{31}^2 \end{pmatrix}$$

and $c_{31}^2 = c_{21}^2 = 0$.

$$C([e_1, e_3]) = A(e_1)C(e_3) - C(e_3)B(e_1) - A(e_3)C(e_1) + C(e_1)B(e_3)$$

$$\begin{pmatrix} c_{11}^1 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & 0 & c_{33}^1 \end{pmatrix} = \begin{pmatrix} c_{31}^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{21}^3 \\ 0 & 0 & c_{31}^3 \end{pmatrix} +$$

$$+\begin{pmatrix} c_{11}^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & 0 & c_{33}^1 \end{pmatrix}$$

and $c_{31}^3 = c_{21}^3 = 0$. So, any virtual structure q on generalized module 3.6 has the form:

$$C(e_1) = \begin{pmatrix} c_{11}^1 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & 0 & c_{33}^1 \end{pmatrix}, \quad C(e_2) = C(e_3) = 0.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ c_{23}^1 & 0 & 0 \\ c_{33}^1 & 0 & 0 \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = \begin{pmatrix} c_{11}^1 + c_{33}^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = C_1(e_3) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.6. Then it can be assumed that the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is determined by one of the virtual structures determined in the Lemma. Then

$$\begin{aligned}
&[e_1, e_2] = 0, \\
&[e_1, e_3] = e_1, \quad [e_2, e_3] = 0, \\
&[e_1, u_1] = pe_1, \quad [e_2, u_1] = 0, \quad [e_3, u_1] = 0, \\
&[e_1, u_2] = 0, \quad [e_2, u_2] = u_2, \quad [e_3, u_2] = 0, \\
&[e_1, u_3] = u_1, \quad [e_2, u_3] = 0, \quad [e_3, u_3] = u_3.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*.$$

Thus

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) \supset \mathbb{R}e_1, \quad \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}u_1, \\
&\bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(0,1)} \supset \mathbb{R}u_3,
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] = \alpha_2 u_2, \\
&[u_1, u_3] = \beta_3 u_3, \\
&[u_2, u_3] = 0.
\end{aligned}$$

Checking the Jacobi identity, we obtain $\alpha_2 = 0$ and $\beta_3 = p$. It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_1	pe_1	0	u_1
e_2	0	0	0	0	u_2	0
e_3	$-e_1$	0	0	0	0	u_3
u_1	$-pe_1$	0	0	0	0	pu_3
u_2	0	$-u_2$	0	0	0	0
u_3	$-u_1$	0	$-u_3$	$-pu_3$	0	0 .

Consider the following cases:

1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1, 2, 3,$$
$$\pi(u_1) = pu_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = pu_3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 = 1$ and $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_2 = 3$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

This proves the Proposition.

Proposition 3.7. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.7 is trivial.*

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	0	u_1	λu_2	0
e_2	$-e_2$	0	e_2	0	0	u_1
e_3	0	$-e_2$	0	0	0	u_3
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0	0
u_3	0	$-u_1$	$-u_3$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1, e_3.

Lemma. *Any virtual structure q on generalized module 3.7 is equivalent to one of the following:*
a) $\lambda = 0$

$$C_1(e_1) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_3) = \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & s & 0 \end{pmatrix};$$

b) $\lambda \neq 0$

$$C_2(e_i) = 0, \quad i = 1, 2, 3.$$

Proof. Let q be a virtual structure on generalized module 3.7. By Proposition 9, Chapter II, without loss of generality it can be assumed that q is primary. Then

$$\mathfrak{g}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_3, \quad \mathfrak{g}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_2,$$

$$U^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_1, \quad U^{(\lambda,0)}(\mathfrak{h}) \supset \mathbb{R}u_2, \quad U^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3.$$

Consider the following cases:
1°. $\lambda = 0$. Then

$$C(e_1) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & r & 0 \end{pmatrix}.$$

Checking condition (6), Chapter II, for (e_1, e_2) and (e_2, e_3), we obtain $p = q, \quad s = r$. Put

$$H = \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}, \quad C_1(e_2) = 0, \quad C_1(e_3) = \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & s & 0 \end{pmatrix}.$$

2°. Then $\lambda \neq 0$. $C(e_1) = C(e_2) = C(e_3) = 0$.
And this completes the proof of the Lemma.

Consider the following cases: 1°. $\lambda = 0$. Then

$$\begin{array}{lll} [e_1, e_2] = e_2, & & \\ [e_1, e_3] = 0, & [e_2, e_3] = e_2, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, \\ [e_1, u_2] = p(e_1 + e_3), & [e_2, u_2] = 0, & [e_3, u_2] = s(e_1 + e_3), \\ [e_1, u_3] = 0, & [e_2, u_3] = 0, & [e_3, u_3] = u_3. \end{array}$$

Note that

$$\mathfrak{g}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_3 \oplus \mathbb{R}u_2, \quad \mathfrak{g}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_2,$$
$$\mathfrak{g}^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_1, \qquad \mathfrak{g}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3,$$

and

$$[u_1, u_2] = \alpha u_1,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = \beta u_3.$$

Checking the Jacobi identity for the triples (e_i, u_j, u_k), $i = 1,2,3$, $1 \leqslant j < k \leqslant 3$, and (u_1, u_2, u_3), we obtain that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	0	u_1	0	0
e_2	$-e_2$	0	e_2	0	0	u_1
e_3	0	$-e_2$	0	0	0	u_3
u_1	$-u_1$	0	0	0	αu_1	0
u_2	0	0	0	$-\alpha u_1$	0	$-\alpha u_3$
u_3	0	$-u_1$	$-u_3$	0	αu_3	0 .

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1,2,3,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 + \alpha(e_1 + e_3),$$
$$\pi(u_3) = u_3.$$

2°. $\lambda \neq 0$. Then

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0,$$

and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

The proof of the Proposition is complete.

Proposition 3.8. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.8 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$(1-\mu)e_3$	u_1	0	μu_3
e_2	0	0	$-\lambda e_3$	0	u_2	λu_3
e_3	$(\mu - 1)e_3$	λe_3	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	$-\mu u_3$	$-\lambda u_3$	$-u_1$	0	0	0

2. $\lambda = \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	0
e_2	0	0	0	0	u_2	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	0	$-u_2$	0	0	0	0
u_3	0	0	$-u_1$	$-e_3$	0	0

3. $\lambda = \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	0
e_2	0	0	0	0	u_2	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	0	$-u_2$	0	0	0	0
u_3	0	0	$-u_1$	e_3	0	0

4. $\lambda = \mu = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	0	u_1	0	u_3
e_2	0	0	$-e_3$	0	u_2	u_3
e_3	0	e_3	0	0	$2e_2 - e_1$	u_1
u_1	$-u_1$	0	0	0	u_3	0
u_2	0	$-u_2$	$e_1 - 2e_2$	$-u_3$	0	0
u_3	$-u_3$	$-u_3$	$-u_1$	0	0	0

5. $\lambda = 0,\ \mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$\frac{1}{2}e_3$	u_1	0	$e_3 + \frac{1}{2}u_3$
e_2	0	0	0	0	u_2	βe_3
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	$-e_3 - \frac{1}{2}u_3$	$-\beta e_3$	$-u_1$	0	0	0

6. $\lambda = 0,\ \mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$\frac{1}{2}e_3$	u_1	0	$\frac{1}{2}u_3$
e_2	0	0	0	0	u_2	e_3
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	$-\frac{1}{2}u_3$	$-e_3$	$-u_1$	0	0	0

7. $\lambda = 1,\ \mu = -1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$2e_3$	u_1	0	$-u_3$
e_2	0	0	$-e_3$	0	u_2	u_3
e_3	$-2e_3$	e_3	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_2
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	$-u_3$	$-u_1$	$-u_2$	0	0

8. $\lambda = -1$, $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	0
e_2	0	0	e_3	0	u_2	$-u_3$
e_3	$-e_3$	$-e_3$	0	0	0	u_1
u_1	$-u_1$	0	0	0	e_3	0
u_2	0	$-u_2$	0	$-e_3$	0	$2e_2 - e_1$
u_3	0	u_3	$-u_1$	0	$e_1 - 2e_2$	0

9. $\lambda = -\frac{1}{2}$ $\mu = \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$\frac{1}{2}e_3$	u_1	0	$\frac{1}{2}u_3$
e_2	0	0	$\frac{1}{2}e_3$	0	u_2	$-\frac{1}{2}u_3$
e_3	$-\frac{1}{2}e_3$	$-\frac{1}{2}e_3$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	e_3
u_3	$-\frac{1}{2}u_3$	$\frac{1}{2}u_3$	$-u_1$	0	$-e_3$	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-\mu \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu-1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors e_1 and e_2.

Lemma. *Any virtual structure q on generalized module 3.8 is equivalent to one of the following:*

a) $\mu = 1$, $\lambda = -1$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C(e_3) = 0;$$

b) $\mu = 2$, $\lambda = 0$

$$C(e_1) = C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} p & 0 & 0 \\ q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

c) $\mu = \lambda = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & q \\ p & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & -r \\ 0 & 0 & s \\ r & 0 & 0 \end{pmatrix}, \quad C(e_3) = 0;$$

d) $\mu = \lambda = 1$

$$C(e_1) = C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & p & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

e) $\mu = \frac{1}{2}$, $\lambda = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad C(e_3) = 0;$$

f) $(\lambda, \mu) \notin \{(-1,1),(0,2),(0,0),(1,1),(0,\frac{1}{2})\}$

$$C(e_i) = 0, \quad i = 1,2,3.$$

Proof. Let q be a virtual structure on generalized module 3.8. Without loss of generality it can be assumed that q is primary.

For example, suppose $\mu = 1, \lambda = -1$. Since

$$\begin{aligned} &\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, && U^{(0,-1)}(\mathfrak{h}) = \mathbb{R}u_1, \\ &\mathfrak{g}^{(0,1)}(\mathfrak{h}) = \mathbb{R}e_3, && U^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_2, \\ & && U^{(1,-1)}(\mathfrak{h}) = \mathbb{R}u_3, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^1 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^2 & 0 \end{pmatrix}, \quad C(e_3) = 0.$$

For a virtual structure defined by matrices $C(e_i)$, $i = 1,2,3$, condition (6), Chapter II, is satisfied.

Similarly we obtain the other results of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.8. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Consider the following cases:

1°. $\mu = 1$, $\lambda = -1$. Then

$$\begin{aligned}
&[e_1, e_2] = 0, \\
&[e_1, e_3] = 0, \quad [e_2, e_3] = e_3, \\
&[e_1, u_1] = u_1, \quad [e_2, u_1] = 0, \quad [e_3, u_1] = 0, \\
&[e_1, u_2] = pe_3, \quad [e_2, u_2] = qe_3 + u_2, \quad [e_3, u_2] = 0, \\
&[e_1, u_3] = u_3, \quad [e_2, u_3] = -u_3, \quad [e_3, u_3] = u_1.
\end{aligned}$$

Since

$$\begin{aligned}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) &= \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_1, \\
\bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) &= \mathbb{R}e_3 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}u_3,
\end{aligned}$$

we have:

$$\begin{aligned}
&[u_1, u_2] = 0, \\
&[u_1, u_3] = 0, \\
&[u_2, u_3] = \gamma_1 u_1.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	0	u_1	0	u_3
e_2	0	0	e_3	0	u_2	$-u_3$
e_3	0	$-e_3$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	$\gamma_1 u_1$
u_3	$-u_3$	u_3	$-u_1$	0	$-\gamma_1 u_1$	0

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2, 3, \\
&\pi(u_1) = u_1, \\
&\pi(u_2) = u_2 - \gamma_1 e_3, \\
&\pi(u_3) = u_3.
\end{aligned}$$

2°. $\mu = 2$, $\lambda = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

3°. $\mu = \lambda = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	0
e_2	0	0	0	0	u_2	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$b_3 e_3 + \beta_1 u_1$
u_2	0	$-u_2$	0	0	0	$\gamma_2 u_2$
u_3	0	0	$-u_1$	$-b_3 e_3 - \beta_1 u_1$	$-\gamma_2 u_2$	0

Put

$$a = \frac{1}{\sqrt{|b_3 + \frac{1}{4}\beta_1^2|}} \quad \text{for} \quad b_3 \neq -\frac{1}{4}\beta_1^2.$$

Consider the following cases:

3.1°. $b_3 + \frac{1}{4}\beta_1^2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_1) = u_1 - \frac{1}{2}\beta_1 e_3,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = u_3 + \frac{1}{2}\beta_1 e_1 + \gamma_2 e_2.$$

3.2°. $b_3 + \frac{1}{4}\beta_1^2 > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_1) = au_1 - \frac{a}{2}\beta_1 e_3,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = au_3 + \frac{a}{2}\beta_1 e_1 + a\gamma_2 e_2.$$

3.3°. $b_3 + \frac{1}{4}\beta_1^2 < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_1) = au_1 - \frac{a}{2}\beta_1 e_3,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = au_3 + \frac{a}{2}\beta_1 e_1 + a\gamma_2 e_2.$$

4°. $\mu = \lambda = 1$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	0	u_1	0	u_3
e_2	0	0	$-e_3$	0	u_2	u_3
e_3	0	e_3	0	0	$\gamma_3(2e_2 - e_1)$	u_1
u_1	$-u_1$	0	0	0	$\gamma_3 u_3$	0
u_2	0	$-u_2$	$\gamma_3(e_1 - 2e_2)$	$-\gamma_3 u_3$	0	0
u_3	$-u_3$	$-u_3$	$-u_1$	0	0	0

Consider the following cases:

4.1°. $\gamma_3 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

4.2°. $\gamma_3 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = \frac{1}{\gamma_3} u_2,$$
$$\pi(u_3) = u_3.$$

5°. $\mu = \frac{1}{2}$, $\lambda = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$\frac{1}{2}e_3$	u_1	0	$pe_3+\frac{1}{2}u_3$
e_2	0	0	0	0	u_2	qe_3
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	$-pe_3-\frac{1}{2}u_3$	$-qe_3$	$-u_1$	0	0	0

Consider the following cases:

5.1°. $p = q = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

5.2°. $p = 0$, $q \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1,2,3,$$
$$\pi(u_1) = \frac{1}{q}u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \frac{1}{q}u_3.$$

5.3°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1,2,3,$$
$$\pi(u_1) = \frac{1}{p}u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \frac{1}{p}u_3.$$

6°. $(\lambda, \mu) \notin \{(-1,1), (0,2), (0,0), (1,1), (0,\frac{1}{2})\}$.

6.1°. $\lambda = 1$, $\mu = -1$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$2e_3$	u_1	0	$-u_3$
e_2	0	0	$-e_3$	0	u_2	u_3
e_3	$-2e_3$	e_3	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\beta_2 u_2$
u_2	0	$-u_2$	0	0	0	0
u_3	u_3	$-u_3$	$-u_1$	$-\beta_2 u_2$	0	0

Consider the following cases:

6.1.1°. $\beta_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

6.1.2°. $\beta_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \ i = 1,2,3, \\ \pi(u_1) &= \frac{1}{\beta_2} u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= \frac{1}{\beta_2} u_3. \end{aligned}$$

6.2°. $\lambda = -1$, $\mu = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	0
e_2	0	0	e_3	0	u_2	$-u_3$
e_3	$-e_3$	$-e_3$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$a_3 e_3$	0
u_2	0	$-u_2$	0	$-a_3 e_3$	0	$a_3(2e_2 - e_1)$
u_3	0	u_3	$-u_1$	0	$a_3(e_1 - 2e_2)$	0

Consider the following cases:

6.2.1°. $a_3 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

6.2.2°. $a_3 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_8 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \ i = 1,2,3, \\ \pi(u_1) &= \frac{1}{a_3} u_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= \frac{1}{a_3} u_3. \end{aligned}$$

6.3°. $\lambda = -\frac{1}{2}$, $\mu = \frac{1}{2}$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$\frac{1}{2}e_3$	u_1	0	$\frac{1}{2}u_3$
e_2	0	0	$\frac{1}{2}e_3$	0	u_2	$-\frac{1}{2}u_3$
e_3	$-\frac{1}{2}e_3$	$-\frac{1}{2}e_3$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	$c_3 e_3$
u_3	$-\frac{1}{2}u_3$	$\frac{1}{2}u_3$	$-u_1$	0	$-c_3 e_3$	0

Consider the following cases:

6.3.1°. $c_3 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

6.3.2°. $c_3 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_9 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2, 3,\\
&\pi(u_1) = \frac{1}{c_3} u_1,\\
&\pi(u_2) = u_2,\\
&\pi(u_3) = \frac{1}{c_3} u_3.
\end{aligned}$$

6.4°. $(\lambda, \mu) \notin \{(1, -1), (-1, 0), (-\frac{1}{2}, \frac{1}{2})\}$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Consider the homomorphisms

$$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R}) \quad (i = 1, 2, 3),$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{D\bar{\mathfrak{g}}_i} x$ in the basis $e_3, u_1, u_2, x \in \bar{\mathfrak{g}}_i$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$ $(i = 1, 2, 3)$, are not conjugate to each other, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent. Similarly we can prove that no one of the pairs 3.8.5 and 3.8.6 is equivalent to the pair 3.8.1.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_4 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_1$, we see that the pairs $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

Consider the algebras

$$\tilde{\mathfrak{g}}_5 = \bar{\mathfrak{g}}_5 / \mathcal{D}^2 \bar{\mathfrak{g}}_5, \quad \tilde{\mathfrak{g}}_6 = \bar{\mathfrak{g}}_6 / \mathcal{D}^2 \bar{\mathfrak{g}}_6,$$

and the homomorphisms

$$f_i : \tilde{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R}) \ (i = 5, 6),$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\mathcal{D}\tilde{\mathfrak{g}}_i} x$ in the basis $\{e_3 + \mathbb{R}u_1, u_2 + \mathbb{R}u_1, u_3 + \mathbb{R}u_1\}$, $x \in \tilde{\mathfrak{g}}_i$.

Since the subalgebras $f_5(\tilde{\mathfrak{g}}_5)$ and $f_6(\tilde{\mathfrak{g}}_6)$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ are not equivalent.

Consider the pairs $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ and $(\bar{\mathfrak{g}}'_5, \mathfrak{g}'_5)$ with parameters β and β' respectively. It is possible to show that these pairs are not equivalent, whenever $\beta \neq \beta'$.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_7 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_1$, we see that the pairs $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent. Since $\dim \mathcal{D}\bar{\mathfrak{g}}_8 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_1$, we see that the pairs $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent. Similarly, since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_9 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_1$, the pairs $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

This completes the proof of the Proposition.

Proposition 3.9. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.9 is trivial.*

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2	u_3
e_2	0	0	0	0	u_1	u_2
e_3	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	$-u_3$	$-u_2$	$-u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial. This proves the Proposition.

Proposition 3.10. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.10 is trivial.*

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	λu_2
e_2	0	0	$-e_3$	0	u_2	$u_2 + u_3$
e_3	$-e_3$	e_3	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	$-\lambda u_2$	$-u_2 - u_3$	$-u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Proposition 3.11. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.11 is trivial.*

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_1	u_1	0	u_2
e_2	0	0	0	0	u_2	u_3
e_3	$-e_1$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0
u_3	$-u_2$	$-u_3$	$-u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Proposition 3.12. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.12 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-e_3$	0	0	u_3
e_2	e_2	0	0	0	0	u_2
e_3	e_3	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	0	0	0	0	0	0
u_3	$-u_3$	$-u_2$	$-u_1$	0	0	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-e_3$	0	0	u_3
e_2	e_2	0	0	e_3	$2e_2$	u_2
e_3	e_3	0	0	0	e_3	u_1
u_1	0	$-e_3$	0	0	$-u_1$	0
u_2	0	$-2e_2$	$-e_3$	u_1	0	$2u_3$
u_3	$-u_3$	$-u_2$	$-u_1$	0	$-2u_3$	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.12 is equivalent to one of the following:*

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ p_1 & p_3 & 0 \\ p_2 & p_4 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ p_5 & p_1 & 0 \\ p_6 & p_2 & 0 \end{pmatrix}.$$

Proof. Let q be a virtual structure on generalized module 3.12. Without loss of generality it can be assumed that q is primary. Since

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

where

$$\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}e_3, \quad \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} c_{11}^1 & c_{12}^1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ c_{21}^2 & c_{22}^2 & 0 \\ c_{31}^2 & c_{32}^2 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c_{13}^3 \\ c_{21}^3 & c_{22}^3 & 0 \\ c_{31}^3 & c_{32}^3 & 0 \end{pmatrix}.$$

Since for any virtual structure q condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ c_{21}^2 & c_{22}^2 & 0 \\ c_{31}^2 & c_{32}^2 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c_{13}^3 \\ c_{21}^3 & c_{21}^2 + c_{13}^3 & 0 \\ c_{31}^3 & c_{31}^2 - c_{13}^2 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c_{13}^3 & -c_{13}^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $C_1(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^2 & c_{22}^2 & 0 \\ c_{31}^2 & c_{32}^2 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^3 & c_{21}^2 & 0 \\ c_{31}^3 & c_{31}^2 & 0 \end{pmatrix}.$$

By corollary 3, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.12. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{lll} [e_1, e_2] = -e_2 & & \\ [e_1, e_3] = -e_3 & [e_2, e_3] = 0 & \\ [e_1, u_1] = 0 & [e_2, u_1] = p_1 e_2 + p_2 e_3 & [e_3, u_1] = p_5 e_2 + p_6 e_3 \\ [e_1, u_2] = 0 & [e_2, u_2] = p_3 e_2 + p_4 e_3 & [e_3, u_2] = p_1 e_2 + p_2 e_3 \\ [e_1, u_3] = u_3 & [e_2, u_3] = u_2 & [e_3, u_3] = u_1 \end{array}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

where

$$\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}e_3, \quad \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_3.$$

Therefore

$$[u_1, u_2] = a_1 e_1 + \alpha_1 u_1 + \alpha_2 u_2,$$
$$[u_1, u_3] = \beta_3 u_3,$$
$$[u_2, u_3] = \gamma_3 u_3.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-e_3$	0	0	u_3
e_2	e_2	0	0	$p_1e_2+p_2e_3$	$2p_2e_2$	u_2
e_3	e_3	0	0	$2p_1e_3$	$p_1e_2+p_2e_3$	u_1
u_1	0	$-p_1e_2-p_2e_3$	$-2p_1e_3$	0	$-p_2u_1+p_1u_2$	$2p_1u_3$
u_2	0	$-2p_2e_2$	$-p_1e_2-p_2e_3$	$p_2u_1-p_1u_2$	0	$2p_2u_3$
u_3	$-u_3$	$-u_2$	$-u_1$	$-2p_1u_3$	$-2p_2u_3$	0

Consider the following cases:

1°. $p_1 = p_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $p_1 \neq 0$, $p_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_3,$$
$$\pi(e_3) = e_2,$$
$$\pi(u_1) = p_1 u_2,$$
$$\pi(u_2) = p_1 u_1,$$
$$\pi(u_3) = p_1 u_3.$$

3°. $p_1 = 0$, $p_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\pi(e_i) = e_i, \; i = 1, 2, 3,$$
$$\pi(u_j) = p_2 u_j, \; j = 1, 2, 3.$$

4°. $p_1 \neq 0$, $p_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(e_3) = p_1 e_2 + e_3,$$
$$\pi(u_1) = p_2 u_1 + p_1 p_2 u_2,$$
$$\pi(u_2) = p_2 u_2,$$
$$\pi(u_3) = p_2 u_3.$$

Since $\mathcal{Z}(\bar{\mathfrak{g}}_1) = \mathbb{R}u_1 + \mathbb{R}u_2$ and $\mathcal{Z}(\bar{\mathfrak{g}}_2) = 0$, we see that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

This completes the proof of the Proposition.

Proposition 3.13. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 3.13 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(\lambda-\mu)e_2$	$(1-\mu)e_3$	u_1	λu_2	μu_3
e_2	$(\mu-\lambda)e_2$	0	0	0	0	u_2
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0	0
u_3	$-\mu u_3$	$-u_2$	$-u_1$	0	0	0

2. $\lambda = 1-2\mu,\ 0 \leqslant \mu < 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-3\mu)e_2$	$(1-\mu)e_3$	u_1	$(1-2\mu)u_2$	μu_3
e_2	$(3\mu-1)e_2$	0	0	0	0	u_2
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$(2\mu-1)u_2$	0	0	0	0	e_3
u_3	$-\mu u_3$	$-u_2$	$-u_1$	0	$-e_3$	0

3. $\lambda = 1-\mu,\ 0 \leqslant \mu < 2$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-2\mu)e_2$	$(1-\mu)e_3$	u_1	$(1-\mu)u_2$	μu_3
e_2	$(2\mu-1)e_2$	0	0	0	0	u_2
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$(\mu-1)u_2$	0	0	0	0	u_1
u_3	$-\mu u_3$	$-u_2$	$-u_1$	0	$-u_1$	0

4. $\lambda = 1+2\mu,\ -1 < \mu < 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1+\mu)e_2$	$(1-\mu)e_3$	u_1	$(1+2\mu)u_2$	μu_3
e_2	$-(\mu+1)e_2$	0	0	0	0	u_2
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_2
u_2	$-(2\mu+1)u_2$	0	0	0	0	0
u_3	$-\mu u_3$	$-u_2$	$-u_1$	$-e_2$	0	0

5. $\lambda = 1+\mu,\ -2 < \mu < 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$(1-\mu)e_3$	u_1	$(1+\mu)u_2$	μu_3
e_2	$-e_2$	0	0	0	0	u_2
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_2
u_2	$-(\mu+1)u_2$	0	0	0	0	0
u_3	$-\mu u_3$	$-u_2$	$-u_1$	$-u_2$	0	0

6. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-\mu e_2$	$(1-\mu)e_3$	u_1	0	μu_3
e_2	μe_2	0	0	e_3	$2e_2$	u_2
e_3	$(\mu-1)e_3$	0	0	0	e_3	u_1
u_1	$-u_1$	$-e_3$	0	0	$-u_1$	0
u_2	0	$-2e_2$	$-e_3$	u_1	0	$2u_3$
u_3	$-\mu u_3$	$-u_2$	$-u_1$	0	$-2u_3$	0

7. $\lambda = 2\mu$, $-\frac{1}{2} \leqslant \mu \leqslant \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	μe_2	$(1-\mu)e_3$	u_1	$2\mu u_2$	$e_2 + \mu u_3$
e_2	$-\mu e_2$	0	0	0	0	u_2
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-2\mu u_2$	0	0	0	0	0
u_3	$-e_2 - \mu u_3$	$-u_2$	$-u_1$	0	0	0

8. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	λe_2	e_3	u_1	λu_2	0
e_2	$-\lambda e_2$	0	0	0	0	$\alpha u_2 + \beta e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	$-\lambda u_2$	0	0	0	0	$\beta u_2 - \alpha e_2$
u_3	0	$-\alpha u_2 - \beta e_2$	$-u_1$	$-e_3$	$\alpha e_2 - \beta u_2$	0

, $\alpha > 0$

9. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	λe_2	e_3	u_1	λu_2	0
e_2	$-\lambda e_2$	0	0	0	0	$\alpha u_2 + \beta e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	$-\lambda u_2$	0	0	0	0	$\beta u_2 - \alpha e_2$
u_3	0	$-\alpha u_2 - \beta e_2$	$-u_1$	e_3	$\alpha e_2 - \beta u_2$	0

, $\alpha > 0$

10. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	λe_2	e_3	u_1	λu_2	0
e_2	$-\lambda e_2$	0	0	0	0	$u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	$-\lambda u_2$	0	0	0	0	βu_2
u_3	0	$-u_2 - \alpha e_2$	$-u_1$	$-e_3$	$-\beta u_2$	0

11. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	λe_2	e_3	u_1	λu_2	0
e_2	$-\lambda e_2$	0	0	0	0	$u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	$-\lambda u_2$	0	0	0	0	βu_2
u_3	0	$-u_2 - \alpha e_2$	$-u_1$	e_3	$-\beta u_2$	0

12. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	λe_2	e_3	u_1	λu_2	0
e_2	$-\lambda e_2$	0	0	0	0	$u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0	$\alpha u_2 - e_2$
u_3	0	$-u_2 - \alpha e_2$	$-u_1$	0	$e_2 - \alpha u_2$	0

, $\alpha \geqslant 0$

13. $\mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	λe_2	e_3	u_1	λu_2	0
e_2	$-\lambda e_2$	0	0	0	0	$u_2 + e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0	αu_2
u_3	0	$-u_2 - e_2$	$-u_1$	0	$-\alpha u_2$	0

$, \quad -1 \leqslant \alpha \leqslant 1$

14. $\mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(\lambda - \frac{1}{2})e_2$	$\frac{1}{2}e_3$	u_1	λu_2	$e_3 + \frac{1}{2}u_3$
e_2	$(\frac{1}{2} - \lambda)e_2$	0	0	0	0	u_2
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	0	0	0	0	0
u_3	$-e_3 - \frac{1}{2}u_3$	$-u_2$	$-u_1$	0	0	0

15. $\lambda = \frac{2}{3}$, $\mu = \frac{1}{3}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$2e_3$	$3u_1$	$2u_2$	$e_2 + u_3$
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-2e_3$	0	0	0	0	u_1
u_1	$-3u_1$	0	0	0	0	0
u_2	$-2u_2$	0	0	0	0	u_1
u_3	$-e_2 - u_3$	$-u_2$	$-u_1$	0	$-u_1$	0

16. $\lambda = 0$, $\mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	$2u_1$	0	$e_3 + u_3$
e_2	e_2	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-2u_1$	0	0	0	0	0
u_2	0	0	0	0	0	e_3
u_3	$-e_3 - u_3$	$-u_2$	$-u_1$	0	$-e_3$	0

17. $\lambda = \frac{1}{2}$, $\mu = \frac{1}{4}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$3e_3$	$4u_1$	$2u_2$	$e_2 + u_3$
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-3e_3$	0	0	0	0	u_1
u_1	$-4u_1$	0	0	0	0	0
u_2	$-2u_2$	0	0	0	0	e_3
u_3	$-e_2 - u_3$	$-u_2$	$-u_1$	0	$-e_3$	0

18. $\lambda = 1$, $\mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	$2u_1$	$2u_2$	$e_2 + e_3 + u_3$
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-2u_1$	0	0	0	0	0
u_2	$-2u_2$	0	0	0	0	0
u_3	$-e_2 - e_3 - u_3$	$-u_2$	$-u_1$	0	0	0

19. $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	$2u_1$	u_2	$e_3 + u_3$
e_2	0	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-2u_1$	0	0	0	0	0
u_2	$-u_2$	0	0	0	0	u_1
u_3	$-e_3 - u_3$	$-u_2$	$-u_1$	0	$-u_1$	0

20. $\lambda = 0$, $\mu = 0$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	e_2
e_2	0	0	0	0	0	$\alpha u_2 + \beta e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	0	0	0	0	0	$\beta u_2 - \alpha e_2$
u_3	$-e_2$	$-\alpha u_2 - \beta e_2$	$-u_1$	$-e_3$	$\alpha e_2 - \beta u_2$	0

, $\alpha > 0$

21. $\lambda = 0$, $\mu = 0$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	e_2
e_2	0	0	0	0	0	$\alpha u_2 + \beta e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	0	0	0	0	0	$\beta u_2 - \alpha e_2$
u_3	$-e_2$	$-\alpha u_2 - \beta e_2$	$-u_1$	e_3	$\alpha e_2 - \beta u_2$	0

, $\alpha > 0$

22. $\lambda = 0$, $\mu = 0$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	e_2
e_2	0	0	0	0	0	$u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	0	0	0	0	0	βu_2
u_3	$-e_2$	$-u_2 - \alpha e_2$	$-u_1$	$-e_3$	$-\beta u_2$	0

23. $\lambda = 0$, $\mu = 0$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	e_2
e_2	0	0	0	0	0	$u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	0	0	0	0	0	βu_2
u_3	$-e_2$	$-u_2 - \alpha e_2$	$-u_1$	e_3	$-\beta u_2$	0

24. $\lambda = 0$, $\mu = 0$

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	e_2
e_2	0	0	0	0	0	$u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	0	0	0	0	$\alpha u_2 - e_2$
u_3	$-e_2$	$-\alpha e_2 - u_2$	$-u_1$	0	$e_2 - \alpha u_2$	0

, $\alpha \geqslant 0$

25. $\lambda = 0,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	e_2
e_2	0	0	0	0	0	$u_2 + e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	0	0	0	0	αu_2
u_3	$-e_2$	$-u_2 - e_2$	$-u_1$	0	$-\alpha u_2$	0

$, \quad -1 \leqslant \alpha \leqslant 1$

26. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$e_2 + \alpha e_3$
u_2	$-u_2$	0	0	0	0	$\beta e_3 + u_1$
u_3	0	$-u_2$	$-u_1$	$-e_2 - \alpha e_3$	$-u_1 - \beta e_3$	0

27. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	αe_3
u_2	$-u_2$	0	0	0	0	$e_2 + u_1$
u_3	0	$-u_2$	$-u_1$	$-\alpha e_3$	$-e_2 - u_1$	0

28. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	αe_3
u_2	$-u_2$	0	0	0	0	$-e_2 + u_1$
u_3	0	$-u_2$	$-u_1$	$-\alpha e_3$	$e_2 - u_1$	0

29. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	$-u_2$	0	0	0	0	u_1
u_3	0	$-u_2$	$-u_1$	$-e_3$	$-u_1$	0

30. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	$-u_2$	0	0	0	0	u_1
u_3	0	$-u_2$	$-u_1$	e_3	$-u_1$	0

31. $\lambda = 1$, $\mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	αe_3
u_2	$-u_2$	0	0	0	0	$\alpha e_2 + e_3 + u_1$
u_3	0	$-u_2$	$-u_1$	$-\alpha e_3$	$-\alpha e_2 - e_3 - u_1$	0

32. $\lambda = 1$, $\mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$u_1 + \alpha e_3$
u_2	$-u_2$	0	0	0	0	$e_3 + \beta e_2$
u_3	0	$-u_2$	$-u_1$	$-\alpha e_3 - u_1$	$-e_3 - \beta e_2$	0

33. $\lambda = 1$, $\mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$u_1 + e_2 + \alpha e_3$
u_2	$-u_2$	0	0	0	0	$\gamma e_3 + \beta e_2$
u_3	0	$-u_2$	$-u_1$	$-e_2 - \alpha e_3 - u_1$	$-\gamma e_3 - \beta e_2$	0

34. $\lambda = 1$, $\mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\gamma e_2 + \alpha e_3 - u_2$
u_2	$-u_2$	0	0	0	0	$\alpha e_2 + \beta e_3 + u_1$
u_3	0	$-u_2$	$-u_1$	$-\gamma e_2 - \alpha e_3 + u_2$	$-\alpha e_2 - \beta e_3 - u_1$	0

, $|\beta| \leqslant |\gamma|$

35. $\lambda = 1$, $\mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	$-u_2$	0	0	0	0	$e_2 + e_3$
u_3	0	$-u_2$	$-u_1$	$-e_3$	$-e_2 - e_3$	0

36. $\lambda = 1$, $\mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	$-u_2$	0	0	0	0	$-e_2 + e_3$
u_3	0	$-u_2$	$-u_1$	e_3	$e_2 - e_3$	0

37. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$e_2 + \alpha e_3$
u_2	$-u_2$	0	0	0	0	$\alpha e_2 - e_3$
u_3	0	$-u_2$	$-u_1$	$-e_2 - \alpha e_3$	$-\alpha e_2 + e_3$	0

$, \quad \alpha \geqslant 0$

38. $\lambda = 1,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_2$	0	0	0	0	e_3
u_3	0	$-u_2$	$-u_1$	0	$-e_3$	0

Proof. Let $\mathcal{E} = \{e_1,\ e_2,\ e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda-\mu & 0 \\ 0 & 0 & 1-\mu \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \mu-\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu-1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma 1. *Any virtual structure q on generalized module 3.13 is equivalent to one of the following:*

a) $\lambda = \mu = 0$

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ -p & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ r & 0 & 0 \end{pmatrix}, \quad C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r & 0 \end{pmatrix};$$

b) $\lambda = 0, \mu = \frac{1}{2}$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ q & 0 & 0 \end{pmatrix}, \quad C_2(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix};$$

c) $\lambda = 0, \mu =\notin \{0, \frac{1}{2}\}$

$$C_3(e_1) = 0, \quad C_3(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ p & 0 & 0 \end{pmatrix}, \quad C_3(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix};$$

d) $\lambda = \frac{1}{2}, \mu = 0$

$$C_4(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & -\frac{1}{2}p & 0 \\ -p & 0 & 0 \end{pmatrix}, \quad C_4(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_4(e_3) = 0;$$

e) $\mu = 0, \lambda \notin \{0, \frac{1}{2}\}$

$$C_5(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & -\lambda p & 0 \\ -p & 0 & 0 \end{pmatrix}, \quad C_5(e_2) = C_5(e_3) = 0;$$

f) $\lambda = \frac{1}{2}, \mu = \frac{1}{2}$

$$C_6(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_6(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_6(e_3) = 0;$$

g) $\lambda = \frac{1}{2}, \mu = \frac{1}{4}$

$$C_7(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad C_7(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_7(e_3) = 0;$$

h) $\lambda = \frac{1}{2}, \mu = 1$

$$C_8(e_1) = C_8(e_3) = 0, \quad C_8(e_2) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix};$$

i) $\lambda = \frac{1}{2}, \mu \notin \{0, \frac{1}{4}, \frac{1}{2}, 1\}$

$$C_9(e_1) = C_9(e_3) = 0, \quad C_9(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix};$$

j) $\lambda = 1, \mu = \frac{1}{2}$

$$C_{10}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & q \end{pmatrix}, \quad C_{10}(e_2) = C_{10}(e_3) = 0;$$

k) $\mu = 1, \lambda \notin \{0, \frac{1}{2}, 1\}$

$$C_{11}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_{11}(e_2) = C_{11}(e_3) = 0;$$

l) $\lambda = 2\mu, \mu \notin \{0, \frac{1}{4}, \frac{1}{2}\}$

$$C_{12}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad C_{12}(e_2) = C_{12}(e_3) = 0;$$

m) $\lambda, \mu \notin \{0, \frac{1}{2}\}, \lambda \neq 2\mu$

$$C_{13}(e_i) = 0, \quad i = 1, 2, 3.$$

Proof. Put

$$C(e_i) = \left(c^i_{jk}\right)_{1 \leqslant j,k \leqslant 3}, \quad i = 1, 2, 3.$$

Now put

$$H = \begin{pmatrix} c^1_{11} & c^1_{13} & 0 \\ c^3_{23} & c^2_{23} & 0 \\ c^3_{33} & c^2_{33} & 0 \end{pmatrix},$$

and $C'(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C'(e_1) = \begin{pmatrix} 0 & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & 0 \\ c^2_{21} & c^2_{22} & 0 \\ c^2_{31} & c^2_{32} & 0 \end{pmatrix},$$

$$C'(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & c^3_{13} \\ c^3_{21} & c^3_{22} & 0 \\ c^3_{31} & c^3_{32} & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C' are equivalent.

Let us check condition (6), Chapter II. Direct calculation shows that

$$C'(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & (\mu - \lambda)c^1_{13} & c^1_{23} \\ (\mu - 1)c^1_{13} & 0 & c^1_{33} \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} 0 & c^2_{12} & 0 \\ 0 & c^2_{22} & 0 \\ c^2_{31} & c^2_{32} & 0 \end{pmatrix},$$

$$C'(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c^2_{31} & 0 \end{pmatrix},$$

where

$$\begin{cases} (2\lambda - \mu)c^2_{12} = 0, \\ (\mu - 1)c^2_{12} = 0, \\ \lambda c^2_{31} = 0, \\ \lambda c^2_{22} = 0, \\ (1 - 2\lambda)c^2_{32} = 0. \end{cases} \qquad (*)$$

a) Suppose $\lambda = \mu = 0$. From (*) it follows that

$$c^2_{12} = c^2_{32} = 0.$$

Put

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c_{33}^1 \end{pmatrix},$$

and $C_1(x) = C'(x) + A(x)H_1 - H_1B(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & c_{13}^1 \\ 0 & 0 & c_{23}^1 \\ -c_{13}^1 & 0 & 0 \end{pmatrix}, \ C_1(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^2 & 0 \\ c_{31}^2 & 0 & 0 \end{pmatrix}, \ C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{31}^2 & 0 \end{pmatrix}.$$

b) Suppose $\lambda \notin \{0, \frac{1}{2},\}$, $\mu = 0$. From (*) it follows that

$$c_{12}^2 = c_{32}^2 = c_{22}^2 = c_{31}^2 = 0.$$

Put

$$H_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda}c_{23}^1 \\ 0 & 0 & -c_{33}^1 \end{pmatrix},$$

and $C_5(x) = C'(x) + A(x)H_5 - H_5B(x)$ for $x \in \mathfrak{g}$. Then

$$C_5(e_1) = \begin{pmatrix} 0 & 0 & c_{13}^1 \\ 0 & -\lambda c_{13}^1 & 0 \\ -c_{13}^1 & 0 & 0 \end{pmatrix}, \quad C_5(e_2) = C_5(e_3) = 0.$$

In a similar way we obtain the other results of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.13. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form determined in Lemma 1. Put

$$\begin{aligned}
[u_1, u_2] &= a_1e_1 + a_2e_2 + a_3e_3 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3, \\
[u_1, u_3] &= b_1e_1 + b_2e_2 + b_3e_3 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3, \\
[u_2, u_3] &= c_1e_1 + c_2e_2 + c_3e_3 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.
\end{aligned}$$

Consider the following cases:

1°. Suppose $\mu = 0, \lambda \notin \{0, \frac{1}{2}\}$. Then

$$\begin{aligned}
&[e_1, e_2] = \lambda e_2, \\
&[e_1, e_3] = e_3, && [e_2, e_3] = 0, \\
&[e_1, u_1] = -pe_3 + u_1, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = -\lambda pe_2 + \lambda u_2, && [e_2, u_2] = 0, && [e_3, u_2] = 0, \\
&[e_1, u_3] = pe_1, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1.
\end{aligned}$$

1.1°. $\lambda \neq 1$.

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	λe_2	e_3	u_1	λu_2	0
e_2	$-\lambda e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$b_3e_3 + \beta_1u_1$
u_2	$-\lambda u_2$	0	0	0	0	$c_2e_2 + \gamma_2u_2$
u_3	0	$-u_2$	$-u_1$	$-b_3e_3 - \beta_1u_1$	$-c_2e_2 - \gamma_2u_2$	0

Let $W = \mathcal{D}(\bar{\mathfrak{g}})$. Note that W is a commutative subalgebra of the Lie algebra $\bar{\mathfrak{g}}$ and the matrices of the endomorphisms $\mathrm{ad}_W e_1$ and $\mathrm{ad}_W(-u_3)$ in the basis $\{e_3, u_1, e_2, u_2\}$ have the form:

$$\mathrm{ad}_W e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \mathrm{ad}_W(-u_3) = \begin{pmatrix} 0 & b_3 & 0 & 0 \\ 1 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 1 & \gamma_2 \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

It is clear that there is a one-to-one correspondence between the set of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set of pairs of matrices (X, Y), where $X, Y \in \mathfrak{gl}(2, \mathbb{R})$.

Lemma 2. *Suppose $(\bar{\mathfrak{g}}^1, \mathfrak{g}^1)$ and $(\bar{\mathfrak{g}}^2, \mathfrak{g}^2)$ are the pairs corresponding to pairs of matrices (X_1, Y_1) and (X_2, Y_2) respectively. The pairs $(\bar{\mathfrak{g}}^1, \mathfrak{g}^1)$ and $(\bar{\mathfrak{g}}^2, \mathfrak{g}^2)$ are equivalent if and only if there exist*

$$\mathbf{A}, \mathbf{B} \in \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, ac \neq 0 \right\}, \quad \theta \in \mathbb{R}^*, \ \tau \in \mathbb{R},$$

such that

$$X_1 = \theta \mathbf{A} X_2 \mathbf{A}^{-1} + \tau E,$$
$$Y_1 = \theta \mathbf{B} Y_2 \mathbf{B}^{-1} + \lambda \tau E.$$

Proof. Suppose $f : \bar{\mathfrak{g}}^2 \to \bar{\mathfrak{g}}^1$ is a mapping and the matrix of f in the basis $\{e_1, e_3, u_1, e_2, u_2, u_3\}$ has the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \tau \\ 0 & & & 0 & 0 & 0 \\ 0 & & \mathbf{A} & 0 & 0 & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & \mathbf{B} & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta \end{pmatrix},$$

where the matrices $\mathbf{A}, \mathbf{B}$ and the numbers τ, θ satisfy the conditions of the Lemma. Then the mapping f establishes the equivalence of the pairs $(\bar{\mathfrak{g}}^1, \mathfrak{g}^1)$ and $(\bar{\mathfrak{g}}^2, \mathfrak{g}^2)$.

Conversely, suppose the pairs $(\bar{\mathfrak{g}}^1, \mathfrak{g}^1)$ and $(\bar{\mathfrak{g}}^2, \mathfrak{g}^2)$ are equivalent, and $f : \bar{\mathfrak{g}}^2 \to \bar{\mathfrak{g}}^1$ is the mapping establishing the equivalence. Let $W_1 = \mathcal{D}(\bar{\mathfrak{g}}^1)$ and $W_2 = \mathcal{D}(\bar{\mathfrak{g}}^2)$. It is clear that $f(W_2) = W_1$ and $f(\bar{\mathfrak{g}}^2) = \bar{\mathfrak{g}}^1$. Then the matrix of f in the basis $\{e_1, e_3, u_1, e_2, u_2, u_3\}$ has the form:

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} \\ a_{21} & & & a_{24} & a_{25} & a_{26} \\ 0 & & \mathbf{A} & 0 & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & & & a_{46} \\ 0 & 0 & a_{53} & & \mathbf{B} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{pmatrix},$$

where

$$\mathbf{A}, \mathbf{B} \in \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| ac \neq 0 \right\}, \quad a_{11}, a_{66} \in \mathbb{R}^*.$$

It is clear that

$$\operatorname{ad}_{W_1} f(e_1) = f|_{W_2} \circ \operatorname{ad}_{W_2} e_1 \circ f^{-1}|_{W_2} = \operatorname{ad}_{W_2} e_1.$$

This implies $f|_{W_2} \circ \operatorname{ad}_{W_2} e_1 = \operatorname{ad}_{W_2} e_1 \circ f|_{W_2}$. Therefore the matrix of f has the form:

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} \\ a_{21} & & & 0 & 0 & a_{26} \\ 0 & & \mathbf{A} & 0 & 0 & a_{36} \\ a_{41} & 0 & 0 & & & a_{46} \\ 0 & 0 & 0 & & \mathbf{B} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{pmatrix}.$$

Then

$$\operatorname{ad}_{W_1} f(-u_3) = a_{66} f|_{W_2} \circ \operatorname{ad}_{W_2}(-u_3) \circ f^{-1}|W_2 - a_{16} f|_{W_2} \circ \operatorname{ad}_{W_2} e_1 \circ f^1|_{W_2}.$$

This implies

$$X_1 = a_{66} \mathbf{A} X_2 \mathbf{A}^{-1} - a_{16} E,$$
$$Y_1 = a_{66} \mathbf{B} Y_2 \mathbf{B}^{-1} - a_{16} \lambda E.$$

The proof of Lemma 2 is complete.

Up to transformations determined in Lemma 2, any pair

$$X = \begin{pmatrix} 0 & b_3 \\ 1 & \beta_1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & c_2 \\ 1 & \gamma_2 \end{pmatrix}$$

is equivalent to one and only one of the following pairs:

1.1.1°.

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix} \right\}, \quad \alpha > 0.$$

The corresponding pair is $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$.

1.1.2°.

$$\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \beta & -\alpha \\ \alpha & \beta \end{pmatrix} \right\}, \quad \alpha > 0.$$

The corresponding pair is $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$.

1.1.3°.

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 1 & \beta \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$.

1.1.4°.

$$\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 1 & \beta \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$.

1.1.5°.

$$\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}\right\}, \quad \alpha \geqslant 0.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$.

1.1.6°.

$$\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & \alpha \end{pmatrix}\right\}, \quad |\alpha| \leqslant 1.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$.

1.1.7°.

$$\left\{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{1}, \mathfrak{g}_{1})$.

1.2°. $\lambda = 1$.

Using the Jacoby identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$$\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & u_1 & u_2 & u_3 \\
\hline
e_1 & 0 & e_2 & e_3 & u_1 & u_2 & 0 \\
e_2 & -e_2 & 0 & 0 & 0 & 0 & u_2 \\
e_3 & -e_3 & 0 & 0 & 0 & 0 & u_1 \\
u_1 & -u_1 & 0 & 0 & 0 & 0 & B \\
u_2 & -u_2 & 0 & 0 & 0 & 0 & C \\
u_3 & 0 & -u_2 & -u_1 & -B & -C & 0
\end{array},$$

where

$$B = b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2,$$
$$C = c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2.$$

Let $W = \mathcal{D}(\bar{\mathfrak{g}})$. Note that W is a commutative subalgebra of the Lie algebra $\bar{\mathfrak{g}}$ and the matrices of the endomorphisms $\mathrm{ad}_W\, e_1$ and $\mathrm{ad}_W(-u_3)$ in the basis $\{e_3, u_1, e_2, u_2\}$ have the form:

$$\mathrm{ad}_W\, e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{ad}_W(-u_3) = \begin{pmatrix} 0 & 0 & b_3 & c_3 \\ 0 & 0 & b_2 & c_2 \\ 1 & 0 & \beta_1 & \gamma_1 \\ 0 & 1 & \beta_2 & \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 & X \\ E & Y \end{pmatrix}.$$

It is clear that there is a one-to-one correspondence between the set of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set of pairs of matrices (X, Y), where $X, Y \in \mathfrak{gl}(2, \mathbb{R})$.

Lemma 3. *Suppose $(\bar{\mathfrak{g}}^1, \mathfrak{g}^1)$ and $(\bar{\mathfrak{g}}^2, \mathfrak{g}^2)$ are the pairs corresponding to pairs of matrices (X_1, Y_1) and (X_2, Y_2) respectively. The pairs $(\bar{\mathfrak{g}}^1, \mathfrak{g}^1)$ and $(\bar{\mathfrak{g}}^2, \mathfrak{g}^2)$ are equivalent if and only if there exist*

$$\mathbf{A} \in \mathrm{GL}(2, \mathbb{R}), \quad \theta \in \mathbb{R}^*, \ \tau \in \mathbb{R},$$

such that

$$X_1 = \mathbf{A}(-\tau^2 E + \theta^2 X_2 + \theta\tau Y_2)\mathbf{A}^{-1},$$
$$Y_1 = \theta \mathbf{A} Y_2 \mathbf{A}^{-1} - \mu E.$$

Proof. The proof is similar to that of Lemma 2.

Up to transformations determined in Lemma 3, any pair of matrices (X, Y) is equivalent to one and only one of the following pairs:

1.2.1°.

$$\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

1.2.2°.

$$\left\{\begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{26}, \mathfrak{g}_{26})$.

1.2.3°.

$$\left\{\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{27}, \mathfrak{g}_{27})$.

1.2.4 o .

$$\left\{\begin{pmatrix} \alpha & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{28}, \mathfrak{g}_{28})$.

1.2.5°.

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{29}, \mathfrak{g}_{29})$.

1.2.6°.

$$\left\{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{30}, \mathfrak{g}_{30})$.

1.2.7°.

$$\left\{\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{31}, \mathfrak{g}_{31})$.

1.2.8°.

$$\left\{\begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{32}, \mathfrak{g}_{32})$.

1.2.9°.

$$\left\{\begin{pmatrix} \alpha & \gamma \\ 1 & \beta \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{33}, \mathfrak{g}_{33})$.

1.2.10°.

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{34}, \mathfrak{g}_{34})$.

1.2.11°.

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{35}, \mathfrak{g}_{35})$.

1.2.12°.

$$\left\{ \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{36}, \mathfrak{g}_{36})$.

1.2.13°.

$$\left\{ \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{37}, \mathfrak{g}_{37})$.

1.2.14°.

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{2}, \mathfrak{g}_{2})$.

1.2.15°.

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

The corresponding pair is $(\bar{\mathfrak{g}}_{1}, \mathfrak{g}_{1})$.

2°. Suppose $\lambda = \mu = \frac{1}{2}$. Then

$$\begin{aligned}
&[e_1, e_2] = 0, \\
&[e_1, e_3] = \frac{1}{2} e_3, && [e_2, e_3] = 0, \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = \frac{1}{2} u_2, && [e_2, u_2] = q e_3, && [e_3, u_2] = 0, \\
&[e_1, u_3] = p e_3 + \frac{1}{2} u_3, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	$\frac{1}{2}e_3$	u_1	$\frac{1}{2}u_2$	$pe_3+\frac{1}{2}u_3$
e_2	0	0	0	0	0	u_2
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{1}{2}u_2$	0	0	0	0	$\gamma_1 u_1$
u_3	$-pe_3-\frac{1}{2}u_3$	$-u_2$	$-u_1$	0	$-\gamma_1 u_1$	0

2.1°. $\gamma_1 = p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2.2°. $\gamma_1 p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{19}, \mathfrak{g}_{19})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{19} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}\pi(e_1) &= 2e_1,\\ \pi(e_2) &= \frac{2p}{\gamma_1}e_2,\\ \pi(e_3) &= e_3,\\ \pi(u_1) &= 2pu_1,\\ \pi(u_2) &= \frac{2p}{\gamma_1}u_2,\\ \pi(u_3) &= u_3.\end{aligned}$$

2.3°. $\gamma_1 = 0,\ p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{14}, \mathfrak{g}_{14})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{14} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}\pi(e_i) &= e_i,\ i = 1,2,3,\\ \pi(u_j) &= \frac{1}{p}u_j,\ j = 1,2,3.\end{aligned}$$

2.4°. $\gamma_1 \neq 0,\ p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}\pi(e_i) &= e_i,\ i = 1,2,3,\\ \pi(u_j) &= \frac{1}{\gamma_1}u_j,\ j = 1,2,3.\end{aligned}$$

It is possible to show that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$, $(\bar{\mathfrak{g}}_{14}, \mathfrak{g}_{14})$, and $(\bar{\mathfrak{g}}_{19}, \mathfrak{g}_{19})$ are not equivalent to each other.

In a similar way we obtain the other results of the Proposition.

Proposition 3.14. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.14 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-(1+\mu)e_2$	$(1-\mu)e_3$	u_1	$-u_2$	μu_3
e_2	$(1+\mu)e_2$	0	0	0	0	u_2
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	0
u_3	$-\mu u_3$	$-u_2$	$-u_1$	0	0	0

2. $\mu = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-2e_2$	0	u_1	$-u_2$	u_3
e_2	$2e_2$	0	0	0	0	u_2
e_3	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	e_3
u_3	$-u_3$	$-u_2$	$-u_1$	0	$-e_3$	0

3. $\mu = 2$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-3e_2$	$-e_3$	u_1	$-u_2$	$2u_3$
e_2	$3e_2$	0	0	0	0	u_2
e_3	e_3	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	u_1
u_3	$-2u_3$	$-u_2$	$-u_1$	0	$-u_1$	0

4. $\mu = \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-\frac{3}{2}e_2$	$\frac{1}{2}e_3$	u_1	$-u_2$	$\frac{1}{2}u_3 + e_3$
e_2	$\frac{3}{2}e_2$	0	0	0	0	u_2
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	0
u_3	$-\frac{1}{2}u_3 - e_3$	$-u_2$	$-u_1$	0	0	0

5. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	u_1	$-u_2$	0
e_2	e_2	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	u_2
u_3	0	$-u_2$	$-u_1$	0	$-u_2$	0

6. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	u_1	$-u_2$	0
e_2	e_2	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	$e_2 + \alpha u_2$
u_3	0	$-u_2$	$-u_1$	0	$-e_2 - \alpha u_2$	0

, $\alpha \geqslant 0$

7. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	u_1	$-u_2$	0
e_2	e_2	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	$-e_2 + \alpha u_2$
u_3	0	$-u_2$	$-u_1$	0	$e_2 - \alpha u_2$	0

, $\alpha \geqslant 0$

8. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	u_1	$-u_2$	0
e_2	e_2	0	0	0	0	$u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	u_2	0	0	0	0	βu_2
u_3	0	$-u_2$	$-u_1$	0	$e_2 - \alpha u_2$	0

, $0 < \beta - \alpha \leqslant 2$

9. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	u_1	$-u_2$	0
e_2	e_2	0	0	0	0	$\beta u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	u_2	0	0	0	0	$\alpha u_2 - \beta e_2$
u_3	0	$-\alpha e_2 - \beta u_2$	$-u_1$	$-e_3$	$\beta e_2 - \alpha u_2$	0

$,\alpha \geqslant 0,\ \beta > 0$

10. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	u_1	$-u_2$	0
e_2	e_2	0	0	0	0	$\beta u_2 + \alpha e_2$
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	u_2	0	0	0	0	$\alpha u_2 - \beta e_2$
u_3	0	$-\alpha e_2 - \beta u_2$	$-u_1$	e_3	$\beta e_2 - \alpha u_2$	0

$,\alpha \geqslant 0,\ 0 < \beta \leqslant 1$

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1-\mu & 0 \\ 0 & 0 & 1-\mu \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1+\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu-1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.14 is equivalent to one of the following:*

a) $\mu = 0$

$$C(e_2) = C(e_3) = 0, \quad C(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & p & 0 \\ -p & 0 & 0 \end{pmatrix}, \quad p \in \mathbb{R};$$

b) $\mu = \frac{1}{2}$

$$C(e_2) = C(e_3) = 0, \quad C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad p \in \mathbb{R};$$

c) $\mu \notin \{0, \frac{1}{2}\}$

$$C(e_i) = 0, \quad i = 1, 2, 3.$$

Proof. Let q be a virtual structure on the generalized module 2.19. Without loss of generality it can be assumed that q is primary. Since

$$\begin{aligned} \mathfrak{g}^{(0)}(\mathfrak{h}) &= \mathbb{R}e_1, & U^{(1)}(\mathfrak{h}) &\supseteq \mathbb{R}u_1, \\ \mathfrak{g}^{(-\mu-1)}(\mathfrak{h}) &\supseteq \mathbb{R}e_2, & U^{(\mu)}(\mathfrak{h}) &\supseteq \mathbb{R}u_3, \\ \mathfrak{g}^{(1-\mu)}(\mathfrak{h}) &\supseteq \mathbb{R}e_3, & U^{(-1)}(\mathfrak{h}) &= \mathbb{R}u_2, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & c^1_{22} & 0 \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & 0 & 0 \\ 0 & 0 & c^2_{23} \\ 0 & 0 & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3_{33} \end{pmatrix}.$$

If $\mu = 0$, then

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & c^1_{22} & 0 \\ c^1_{31} & 0 & 0 \end{pmatrix}, \; C(e_2) = \begin{pmatrix} c^2_{11} & 0 & 0 \\ 0 & 0 & c^2_{23} \\ 0 & 0 & 0 \end{pmatrix}, \; C(e_3) = \begin{pmatrix} 0 & c^3_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3_{33} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -c^2_{23} & 0 \\ -c^3_{33} & 0 & 0 \end{pmatrix}$$

and $C_1(x) = C(x) - A(x)H + HB(x)$. By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

Since for any virtual structure q condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_2) = C(e_3) = 0, \quad C(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & p & 0 \\ -p & 0 & 0 \end{pmatrix}, \quad p \in \mathbb{R}.$$

Similarly we obtain the other results of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.14. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*.$$

Thus

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1-\mu)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1-\mu)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(\mu)}(\mathfrak{h}),$$

where

$$\bar{\mathfrak{g}}^{(-1-\mu)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1,$$
$$\bar{\mathfrak{g}}^{(1-\mu)}(\mathfrak{h}) = \mathbb{R}e_3, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}u_2,$$
$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1, \quad \bar{\mathfrak{g}}^{\mu}(\mathfrak{h}) = \mathbb{R}u_3.$$

Therefore

$$[u_1, u_2] \in \bar{\mathfrak{g}}^{(0)},$$
$$[u_1, u_3] \in \bar{\mathfrak{g}}^{(1+\mu)},$$
$$[u_2, u_3] \in \bar{\mathfrak{g}}^{(\mu-1)}.$$

Consider the following cases:

1°. $\mu = 0$. Then

$$\begin{aligned}
&[e_1, e_2] = -(1+\mu)e_2, && && \\
&[e_1, e_3] = (1-\mu)e_3, && [e_2, e_3] = 0, && \\
&[e_1, u_1] = u_1 + pe_3, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = -u_2 + pe_2, && [e_2, u_2] = 0, && [e_3, u_2] = 0, \\
&[e_1, u_3] = pe_1, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1.
\end{aligned}$$

and

$$[u_1, u_2] = a_1e_1 + \alpha_3u_3,$$
$$[u_1, u_3] = b_3e_3 + \beta_1u_1,$$
$$[u_2, u_3] = c_2e_2 + \gamma_2u_2.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	u_1	$-u_2$	0
e_2	e_2	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$b_3e_3 + \beta_1u_1$
u_2	u_2	0	0	0	0	$\gamma_2u_2 + c_2e_2$
u_3	0	$-u_2$	$-u_1$	$-b_3e_3 - \beta_1u_1$	$-c_2e_2 - \gamma_2u_2$	0

Consider the following cases:

1.1°. $b_3 + \frac{\beta_1^2}{4} = 0$.

1.1.1°. $c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4} = 0$ and $\gamma_2 + \frac{\beta_1}{2} = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

1.1.2°. $c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4} = 0$ and $\gamma_2 + \frac{\beta_1}{2} \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi\colon \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= (\gamma_2 + \frac{\beta_1}{2})e_2,\\
\pi(e_3) &= (\gamma_2 + \frac{\beta_1}{2})e_3,\\
\pi(u_1) &= (\gamma_2 + \frac{\beta_1}{2})\frac{\beta_1}{2}e_3 + u_1,\\
\pi(u_2) &= -(\gamma_2 + \frac{\beta_1}{2})\frac{\beta_1}{2}e_2 + u_2,\\
\pi(u_3) &= (\gamma_2 + \frac{\beta_1}{2})u_3 - (\gamma_2 + \frac{\beta_1}{2})\frac{\beta_1}{2}e_1.
\end{aligned}$$

1.1.3°. $c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4} > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \bar{\mathfrak{g}}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \mathfrak{g}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= \sqrt{c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4}}e_2,\\
\pi(e_3) &= \sqrt{c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4}}e_3,\\
\pi(u_1) &= u_1 + \sqrt{c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4}}\frac{\beta_1}{2}e_3,\\
\pi(u_2) &= u_2 + \sqrt{c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4}}\frac{\beta_1}{2}e_2,\\
\pi(u_3) &= \sqrt{c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4}}u_3.
\end{aligned}$$

1.1.4°. $c_2 - \frac{\beta_1\gamma_2}{2} - \frac{\beta_1^2}{4} < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= \sqrt{-c_2 + \frac{\beta_1\gamma_2}{2} + \frac{\beta_1^2}{4}}e_2,\\
\pi(e_3) &= \sqrt{-c_2 + \frac{\beta_1\gamma_2}{2} + \frac{\beta_1^2}{4}}e_3,\\
\pi(u_1) &= u_1 + \sqrt{-c_2 + \frac{\beta_1\gamma_2}{2} + \frac{\beta_1^2}{4}}\frac{\beta_1}{2}e_3\\
\pi(u_2) &= u_2 + \sqrt{-c_2 + \frac{\beta_1\gamma_2}{2} + \frac{\beta_1^2}{4}}\frac{\beta_1}{2}e_2,\\
\pi(u_3) &= \sqrt{-c_2 + \frac{\beta_1\gamma_2}{2} + \frac{\beta_1^2}{4}}u_3.
\end{aligned}$$

1.2°. $\beta_3 + \beta_1^2/4 > 0$ and $\gamma_2^2 + 4c_2 > 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_8 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= \lambda e_2,\\
\pi(e_3) &= \lambda e_3,\\
\pi(u_1) &= \frac{\lambda\beta_1}{2}e_3 + u_1,\\
\pi(u_2) &= \lambda(\frac{\gamma_2}{2} + \sqrt{c_2 + \frac{\gamma_2^2}{4}}e_2 + u_2,\\
\pi(u_3) &= \lambda^{-1}(u_3 + \mu e_1),
\end{aligned}$$

where $\lambda = \sqrt{\beta_3 + \frac{\beta_1^2}{4}}$, $\mu = -\frac{\lambda\beta_1}{2}$.

1.3°. $\beta_3 + \frac{\beta_1^2}{4} > 0$ and $\gamma_2^2 + 4c_2 < 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_9 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= \frac{\lambda}{b}e_2,\\
\pi(e_3) &= \lambda e_3,\\
\pi(u_1) &= \frac{\lambda\beta_1}{2}e_3 + u_1,\\
\pi(u_2) &= \frac{\lambda\gamma_2}{2b}e_2 + u_2,\\
\pi(u_3) &= \lambda^{-1}(u_3 + \mu e_1),
\end{aligned}$$

where $\lambda = \sqrt{\beta_3 + \frac{\beta_1^2}{4}}$, $\mu = -\frac{\lambda\beta_1}{2}$, $b = \lambda\sqrt{-c_2 - \frac{\gamma_2^2}{4}}$.

1.4°. $\beta_3 + \frac{\beta_1^2}{4} < 0$ and $\gamma_2^2 + 4c_2 < 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{10} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= \frac{\lambda}{b}e_2,\\
\pi(e_3) &= \lambda e_3,\\
\pi(u_1) &= \frac{\lambda\beta_1}{2}e_3 + u_1,\\
\pi(u_2) &= \frac{\lambda\gamma_2}{2b}e_2 + u_2,\\
\pi(u_3) &= \lambda^{-1}(u_3 + \mu e_1),
\end{aligned}$$

where $\lambda = \sqrt{\beta_3 + \frac{\beta_1^2}{4}}$, $\mu = -\frac{\lambda\beta_1}{2}$, $b = \lambda\sqrt{-c_2 - \frac{\gamma_2^2}{4}}$.

Note that two pairs $(\bar{\mathfrak{g}}_6', \mathfrak{g}_6')$ and $(\bar{\mathfrak{g}}_6'', \mathfrak{g}_6'')$ corresponding, respectively, to the values α and $-\alpha$ of the parameter are equivalent by means of the following mapping

$\pi\colon \bar{\mathfrak{g}}'_6 \to \bar{\mathfrak{g}}''_6$:

$$\begin{aligned}\pi(e_1) &= e_1,\\ \pi(e_2) &= -e_2,\\ \pi(e_3) &= -e_3,\\ \pi(u_1) &= u_1\\ \pi(u_2) &= u_2\\ \pi(u_3) &= -u_3.\end{aligned}$$

So, it can be assumed that $\alpha \geqslant 0$. Similarly, we obtain additional conditions on parameters in the pairs 3.14.7, ... , 3.14.10.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4,\mathbb{R})$, $i = 1,5,\dots,10$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{e_2, e_3, u_1, u_2\}$ of $\mathcal{D}\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1,5,\dots,10$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1,5,\dots,10$ are not equivalent to each other.

2°. $\mu = 1$. Then

$$\begin{aligned}&[e_1,e_2] = -2e_2,\\ &[e_1,e_3] = 0, && [e_2,e_3] = 0,\\ &[e_1,u_1] = u_1, && [e_2,u_1] = 0, && [e_3,u_1] = 0,\\ &[e_1,u_2] = -u_2, && [e_2,u_2] = 0, && [e_3,u_2] = 0,\\ &[e_1,u_3] = u_3, && [e_2,u_3] = u_2, && [e_3,u_3] = u_1.\end{aligned}$$

and

$$\begin{aligned}[u_1,u_2] &= a_1e_1 + a_3e_3,\\ [u_1,u_3] &= 0,\\ [u_2,u_3] &= c_1e_1 + c_3e_3.\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-2e_2$	0	u_1	$-u_2$	u_3
e_2	$2e_2$	0	0	0	0	u_2
e_3	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	b_3e_3
u_3	$-u_3$	$-u_2$	$-u_1$	0	$-b_3e_3$	0

2.1°. $b_3 = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2°. $b_3 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}\pi(e_1) &= e_1,\\ \pi(e_2) &= b_3e_2,\\ \pi(e_3) &= e_3,\\ \pi(u_1) &= u_1,\\ \pi(u_2) &= b_3u_2,\\ \pi(u_3) &= u_3.\end{aligned}$$

3°. $\mu = 2$. Then

$$\begin{array}{lll}
[e_1, e_2] = -3e_2 & & \\
[e_1, e_3] = -e_3 & [e_2, e_3] = 0 & \\
[e_1, u_1] = u_1 & [e_2, u_1] = 0 & [e_3, u_1] = 0 \\
[e_1, u_2] = -u_2 & [e_2, u_2] = 0 & [e_3, u_2] = 0 \\
[e_1, u_3] = 2u_3 & [e_2, u_3] = u_2 & [e_3, u_3] = u_1
\end{array}$$

and

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1, \\
[u_1, u_3] &= 0, \\
[u_2, u_3] &= \alpha_1 u_1.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-3e_2$	$-e_3$	u_1	$-u_2$	$2u_3$
e_2	$3e_2$	0	0	0	0	u_2
e_3	e_3	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	$\alpha_1 u_1$
u_3	$-2u_3$	$-u_2$	$-u_1$	0	$-\alpha_1 u_1$	0 .

3.1°. $\alpha_1 = 0$.
Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.
3.2°. $\alpha_1 \neq 0$.
Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, 2, 3, \\
\pi(u_j) &= \alpha_1 u_j, \ j = 1, 2, 3.
\end{aligned}$$

4°. $\mu = \frac{1}{2}$. Then

$$\begin{array}{lll}
[e_1, e_2] = -\dfrac{3}{2}e_2, & & \\
[e_1, e_3] = \dfrac{1}{2}e_3, & [e_2, e_3] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, \\
[e_1, u_2] = -u_2, & [e_2, u_2] = 0, & [e_3, u_2] = 0, \\
[e_1, u_3] = \dfrac{1}{2}u_3 + pe_3, & [e_2, u_3] = u_2, & [e_3, u_3] = u_1,
\end{array}$$

and

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1, \\
[u_1, u_3] &= 0, \\
[u_2, u_3] &= 0.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-\frac{3}{2}e_2$	$\frac{1}{2}e_3$	u_1	$-u_2$	$\frac{1}{2}u_3 + pe_3$
e_2	$\frac{3}{2}e_2$	0	0	0	0	u_2
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	u_2	0	0	0	0	0
u_3	$-\frac{1}{2}u_3 - pe_3$	$-u_2$	$-u_1$	0	0	0

4.1°. $\alpha_1 = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

4.2°. $\alpha_1 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(e_3) = p^{-1}e_3,$$
$$\pi(u_1) = p^{-1}u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = u_3.$$

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R})$, $i = 1, 4$, where $f_i(x)$ is the matrix of the mapping $ad_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{e_2, e_3, u_1, u_2, u_3\}$ of $\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, 4$, are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, 4$, are not equivalent.

5°. $\mu \notin \{0, 1, 2, \frac{1}{2}\}$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

The proof of the Proposition is complete.

Proposition 3.15. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.15 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 + \lambda e_2$	$\lambda e_3 - e_2$	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
e_2	$-e_3 - \lambda e_2$	0	0	0	0	u_2
e_3	$e_2 - \lambda e_3$	0	0	0	0	u_1
u_1	$u_2 - \lambda u_1$	0	0	0	0	0
u_2	$-u_1 - \lambda u_2$	0	0	0	0	0
u_3	0	$-u_2$	$-u_1$	0	0	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 + \lambda e_2$	$\lambda e_3 - e_2$	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
e_2	$-e_3 - \lambda e_2$	0	0	0	0	u_2
e_3	$e_2 - \lambda e_3$	0	0	0	0	u_1
u_1	$u_2 - \lambda u_1$	0	0	0	0	e_3
u_2	$-u_1 - \lambda u_2$	0	0	0	0	e_2
u_3	0	$-u_2$	$-u_1$	$-e_3$	$-e_2$	0

3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3+\lambda e_2$	λe_3-e_2	λu_1-u_2	$u_1+\lambda u_2$	0
e_2	$-e_3-\lambda e_2$	0	0	0	0	u_2
e_3	$e_2-\lambda e_3$	0	0	0	0	u_1
u_1	$u_2-\lambda u_1$	0	0	0	0	$-e_3$
u_2	$-u_1-\lambda u_2$	0	0	0	0	$-e_2$
u_3	0	$-u_2$	$-u_1$	e_3	e_2	0

4.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3+\lambda e_2$	λe_3-e_2	λu_1-u_2	$u_1+\lambda u_2$	0
e_2	$-e_3-\lambda e_2$	0	0	0	0	u_2
e_3	$e_2-\lambda e_3$	0	0	0	0	u_1
u_1	$u_2-\lambda u_1$	0	0	0	0	$e_2+\alpha e_3$
u_2	$-u_1-\lambda u_2$	0	0	0	0	αe_2-e_3
u_3	0	$-u_2$	$-u_1$	$-e_2-\alpha e_3$	$-\alpha e_2+e_3$	0

5.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3+\lambda e_2$	λe_3-e_2	λu_1-u_2	$u_1+\lambda u_2$	0
e_2	$-e_3-\lambda e_2$	0	0	0	0	u_2
e_3	$e_2-\lambda e_3$	0	0	0	0	u_1
u_1	$u_2-\lambda u_1$	0	0	0	0	$-e_2+\alpha e_3$
u_2	$-u_1-\lambda u_2$	0	0	0	0	αe_2+e_3
u_3	0	$-u_2$	$-u_1$	$e_2-\alpha e_3$	$-\alpha e_2-e_3$	0

6.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3+\lambda e_2$	λe_3-e_2	λu_1-u_2	$u_1+\lambda u_2$	0
e_2	$-e_3-\lambda e_2$	0	0	0	0	u_2
e_3	$e_2-\lambda e_3$	0	0	0	0	u_1
u_1	$u_2-\lambda u_1$	0	0	0	0	$\alpha e_2+\beta e_3+u_1$
u_2	$-u_1-\lambda u_2$	0	0	0	0	$\beta e_2-\alpha e_3+u_2$
u_3	0	$-u_2$	$-u_1$	$-\alpha e_2-\beta e_3-u_1$	$-\beta e_2+\alpha e_3-u_2$	0

Proof. Let $\mathcal{E} = \{e_1,\, e_2,\, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 1 & \lambda \end{pmatrix}, \; A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Consider the complex generalized module $(\mathfrak{g}^{\mathbb{C}},\, U^{\mathbb{C}})$. Put

$$\tilde{e}_i = e_i \otimes 1, \; i = 1, 2, 3, \text{ and } \tilde{u}_j = u_j \otimes 1, \; j = 1, 2, 3.$$

Then $\tilde{\mathcal{E}} = \{\tilde{e}_1,\, \tilde{e}_2,\, \tilde{e}_3\}$ is a basis of $\mathfrak{g}^{\mathbb{C}}$. The vector space $U^{\mathbb{C}}$ can be identified with $\mathbb{C}^3$, and $\{\tilde{u}_1,\, \tilde{u}_2,\, \tilde{u}_3\}$ is a standard basis of $U^{\mathbb{C}}$.

Lemma. *Any virtual structure q on generalized module 3.15 is equivalent to one of the following:*

$$C(e_1) = \begin{pmatrix} 0 & 0 & p \\ p & -\lambda p & 0 \\ -\lambda p & -p & 0 \end{pmatrix}, \quad C(e_2) = C(e_3) = 0, \ p \in \mathbb{R}.$$

Proof. Suppose q is a virtual structure on the generalized module $(\mathfrak{g}, U)$. By Proposition 15, Chapter II, without loss of generality it can be assumed that $q^{\mathbb{C}}$ is a primary virtual structure on $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$ (with respect to $\mathfrak{h}^{\mathbb{C}}$). We have

$$\begin{aligned}
&(\mathfrak{g}^{\mathbb{C}})^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{e}_1, && (U^{\mathbb{C}})^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{u}_3,\\
&(\mathfrak{g}^{\mathbb{C}})^{(\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3), && (U^{\mathbb{C}})^{(\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 + i\tilde{u}_1),\\
&(\mathfrak{g}^{\mathbb{C}})^{(\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_3 + i\tilde{e}_2), && (U^{\mathbb{C}})^{(\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_1 + i\tilde{u}_2).
\end{aligned}$$

Then

$$\begin{aligned}
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_3) && \in \mathbb{C}\tilde{e}_1,\\
&q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_3) && \in \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3),\\
&q^{\mathbb{C}}(\tilde{e}_3 + i\tilde{e}_2)(\tilde{u}_3) && \in \mathbb{C}(\tilde{e}_3 + i\tilde{e}_2),\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2 + i\tilde{u}_1) && \in \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3),\\
&q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_2 + i\tilde{u}_1) && = 0,\\
&q^{\mathbb{C}}(\tilde{e}_3 + i\tilde{e}_2)(\tilde{u}_2 + i\tilde{u}_1) && \in \mathbb{C}(\tilde{e}_1),\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1 + i\tilde{u}_2) && \in \mathbb{C}(\tilde{e}_3 + i\tilde{e}_2),\\
&q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_1 + i\tilde{u}_2) && \in \mathbb{C}\tilde{e}_1,\\
&q^{\mathbb{C}}(\tilde{e}_3 + i\tilde{e}_2)(\tilde{u}_1 + i\tilde{u}_2) && = 0.
\end{aligned}$$

Therefore,

$$\begin{array}{lll}
q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1) \in \mathbb{C}\tilde{e}_2 + \mathbb{C}\tilde{e}_3 & q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_1) \in \mathbb{C}\tilde{e}_1 & q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_1) \in \mathbb{C}\tilde{e}_1\\
q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2) \in \mathbb{C}\tilde{e}_2 + \mathbb{C}\tilde{e}_3 & q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2) \in \mathbb{C}\tilde{e}_1 & q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_2) \in \mathbb{C}\tilde{e}_1\\
q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_3) \in \mathbb{C}\tilde{e}_1 & q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_3) \in \mathbb{C}\tilde{e}_2 + \mathbb{C}\tilde{e}_3 & q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_3) \in \mathbb{C}\tilde{e}_2 + \mathbb{C}\tilde{e}_3
\end{array}$$

Since the matrices of the mappings $q(eh_i)$ and $q^{\mathbb{C}}(e_i)$, $i = 1, 2, 3$, coincide, we obtain

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ c^1_{21} & c^1_{22} & 0 \\ c^1_{31} & c^1_{32} & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & 0 \\ 0 & 0 & c^2_{23} \\ 0 & 0 & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & 0 \\ 0 & 0 & c^3_{23} \\ 0 & 0 & c^3_{33} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ c_{23}^3 & c_{23}^2 & 0 \\ c_{33}^3 & c_{33}^2 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & c_{13}^1 \\ c_{21}^1 & c_{22}^1 & 0 \\ c_{31}^1 & c_{32}^1 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} c_{11}^2 & c_{12}^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} c_{11}^3 & c_{12}^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

Since for any virtual structure q condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & c_{13}^1 \\ c_{13}^1 & -\lambda c_{13}^1 & 0 \\ -\lambda c_{13}^1 & -c_{13}^1 & 0 \end{pmatrix}, \quad C_1(e_2) = C_1(e_3) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.15. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{lll}
[e_1, e_2] = e_3 + \lambda e_2, & & \\
[e_1, e_3] = \lambda e_3 - e_2, & [e_2, e_3] = 0, & \\
[e_1, u_1] = pe_1 - \lambda pe_3 + \lambda u_1 - u_2, & [e_2, u_1] = 0, & [e_3, u_1] = 0, \\
[e_1, u_2] = -\lambda pe_2 - pe_3 + u_1 + \lambda u_2, & [e_2, u_2] = 0, & [e_3, u_2] = 0, \\
[e_1, u_3] = pe_1, & [e_2, u_3] = u_2, & [e_3, u_3] = u_1.
\end{array}$$

Since $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$, we have

$$(\bar{\mathfrak{g}}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) = (\mathfrak{g}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \times (U^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \text{ for all } \alpha \in (\mathfrak{h}^{\mathbb{C}})^*$$

(Proposition 10, Chapter II). Thus

$$(\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_3 + i\tilde{e}_2) \oplus \mathbb{C}(\tilde{u}_1 + i\tilde{u}_2).$$

Therefore,

$$[\tilde{u}_2 + i\tilde{u}_1, \tilde{u}_3] \in \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3) \oplus \mathbb{C}(\tilde{u}_2 + i\tilde{u}_1),$$
$$[\tilde{u}_1 + i\tilde{u}_2, \tilde{u}_3] \in \mathbb{C}(\tilde{e}_3 + i\tilde{e}_2) \oplus \mathbb{C}(\tilde{u}_1 + i\tilde{u}_2),$$
$$[\tilde{u}_2 + i\tilde{u}_1, \tilde{u}_1 + i\tilde{u}_2] \in \mathbb{C}\tilde{e}_1 \oplus \mathbb{C}\tilde{u}_3,$$

and

$$[u_1, u_2] = a_1 e_1 + \alpha_3 u_3,$$
$$[u_1, u_3] = b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2,$$
$$[u_2, u_3] = c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 + \lambda e_2$	$\lambda e_3 - e_2$	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
e_2	$-e_3 - \lambda e_2$	0	0	0	0	u_2
e_3	$e_2 - \lambda e_3$	0	0	0	0	u_1
u_1	$u_2 - \lambda u_1$	0	0	0	0	A
u_2	$-u_1 - \lambda u_2$	0	0	0	0	B
u_3	0	$-u_2$	$-u_1$	$-A$	$-B$	0

where $A = b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2$, $B = b_3 e_2 - b_2 e_3 - \beta_2 u_1 + \beta_1 u_2$.

The mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_1) = \frac{\beta_2}{2} e_2 - \frac{\beta_2}{2} \lambda e_3 + u_1,$$
$$\pi(u_2) = -\frac{\beta_2}{2} \lambda e_2 - \frac{\beta_2}{2} e_3 + u_2,$$
$$\pi(u_3) = \frac{\beta_2}{2} e_1 + u_3,$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 + \lambda e_2$	$\lambda e_3 - e_2$	$\lambda u_1 - u_2$	$u_1 + \lambda u_2$	0
e_2	$-e_3 - \lambda e_2$	0	0	0	0	u_2
e_3	$e_2 - \lambda e_3$	0	0	0	0	u_1
u_1	$u_2 - \lambda u_1$	0	0	0	0	A
u_2	$-u_1 - \lambda u_2$	0	0	0	0	B
u_3	0	$-u_2$	$-u_1$	$-A$	$-B$	0 ,

where $A = k_1 e_2 + k_2 e_3 + k_3 u_1$, $B = k_2 e_2 - k_1 e_3 + k_3 u_2$,

$$k_1 = \frac{\beta_1 \beta_2}{2} + b_2$$
$$k_2 = b_3 - \frac{\beta_2^2(\lambda^2 + 1)}{4} - \frac{\beta_1 \beta_2 \lambda}{2}$$
$$k_3 = \beta_2 \lambda + \beta_1.$$

Consider the following cases:

1°. $k_3 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = k_3 u_j, \ j = 1, 2, 3.$$

2°. $k_3 = 0$.

2.1°. $k_1 < 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \sqrt{-k_1} u_j, \ j = 1, 2, 3.$$

2.2°. $k_1 > 0$. Then the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}'$ such that

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \sqrt{k_1} u_j, \ j = 1, 2, 3,$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}', \mathfrak{g}')$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$.

2.3°. $k_1 = 0$.

2.3.1.° $k_2 < 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \sqrt{-k_2} u_j, \ j = 1, 2, 3.$$

2.3.2°. $k_2 > 0$. Then the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}'$ such that

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \sqrt{k_2} u_j, \ j = 1, 2, 3,$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}', \mathfrak{g}')$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$.

2.3.3°. $k_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

Now it remains to show that the pairs determined in hte Proposition are not equivalent to each other.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R})$, $i = 1, \ldots, 6$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{e_2, e_3, u_1, u_2\}$ of $\mathcal{D}\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, \ldots, 6$, are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \ldots, 6$, are not equivalent.

This proves the Proposition.

Proposition 3.16. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.16 is trivial pair.*

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(\lambda-\mu)e_2 + e_3$	$(\lambda-\mu)e_3 - e_2$	$\lambda u_1 - u_2$	$\lambda u_2 + u_1$	μu_3
e_2	$(\mu - \lambda)e_2 - e_3$	0	0	0	0	u_2
e_3	$(\mu - \lambda)e_3 + e_2$	0	0	0	0	u_1
u_1	$u_2 - \lambda u_1$	0	0	0	0	0
u_2	$-\lambda u_2 - u_1$	0	0	0	0	0
u_3	$-\mu u_3$	$-u_2$	$-u_1$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda-\mu & -1 \\ 0 & 1 & \lambda-\mu \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda-\mu & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \mu-\lambda & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra spanned by the vector e_1.

Consider the complex generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$. Put

$$\tilde{e}_i = e_i \otimes 1,\ 1 \leqslant i \leqslant 3, \text{ and } \tilde{u}_j = u_j \otimes 1,\ 1 \leqslant j \leqslant 3.$$

Then $\tilde{\mathcal{E}} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is a basis of $\mathfrak{g}^{\mathbb{C}}$. The vector space $U^{\mathbb{C}}$ can be identified with $\mathbb{C}^3$, and $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ is the standard basis of $U^{\mathbb{C}}$.

Lemma. *Any virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.16 is trivial.*

Proof. Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a virtual pair defined by a virtual structure q. By Proposition 15, Charter II, without loss of generality it can be assumed that $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$ (with respect to $\mathfrak{h}^{\mathbb{C}}$). Since

$$\begin{aligned}
&(\mathfrak{g}^{\mathbb{C}})^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{e}_1,\\
&(\mathfrak{g}^{\mathbb{C}})^{(\lambda-\mu-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3),\\
&(\mathfrak{g}^{\mathbb{C}})^{(\lambda-\mu+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3),\\
&(U^{\mathbb{C}})^{(\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_1 + i\tilde{u}_2),\\
&(U^{\mathbb{C}})^{(\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_1 - i\tilde{u}_2),\\
&(U^{\mathbb{C}})^{(\mu)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{u}_3,
\end{aligned}$$

we have

$$\begin{aligned}
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1 + i\tilde{u}_2) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1 - i\tilde{u}_2) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_3) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_1 + i\tilde{u}_2) \in (\mathfrak{g}^{\mathbb{C}})^{(2\lambda-\mu)}(\mathfrak{h}^{\mathbb{C}}),\\
&q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_1 - i\tilde{u}_2) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_3) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_1 + i\tilde{u}_2) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_1 - i\tilde{u}_2) \in (\mathfrak{g}^{\mathbb{C}})^{(2\lambda-\mu)}(\mathfrak{h}^{\mathbb{C}}),\\
&q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_3) = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned} &q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1) = q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2) \\ =&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_3) = q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_3) \\ =&q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_3) = 0, \\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_1) = -q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_2), \\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2) = q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_1), \\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_1) \in \mathbb{C}e_1, \\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2) \in \mathbb{C}e_1. \end{aligned}$$

Since the matrices of the mappings $q(e_i)$ and $q^{\mathbb{C}}(\tilde{e}_i)$, $1 \leqslant i \leqslant 3$, coincide, we obtain

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c_1 & c_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_c) = \begin{pmatrix} c_2 & -c_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Checking condition (6), Charter II, for e_2 and e_3, we obtain $c_2 = c_1 = 0$. This completes the proof of the Lemma.

Thus it can be assumed that the virtual structure q determining the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the zero mapping of $\mathfrak{g}$ into $\mathcal{L}(U, \mathfrak{g})$. Then

$$\begin{aligned} &[e_1, e_2] = (\lambda - \mu)e_2 + e_3, && && \\ &[e_1, e_3] = (\lambda - \mu)e_3 - e_2, && [e_2, e_3] = 0, && \\ &[e_1, u_1] = \lambda u_1 - u_2, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\ &[e_1, u_2] = u_1 + \lambda u_2, && [e_2, u_2] = 0, && [e_3, u_2] = 0, \\ &[e_1, u_3] = \mu u_3, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1. \end{aligned}$$

Since $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$, we have

$$(\bar{\mathfrak{g}}^{\mathbb{C}})^{(\alpha)}(\mathfrak{h}^{\mathbb{C}}) = (\mathfrak{g}^{\mathbb{C}})^{(\alpha)}(\mathfrak{h}^{\mathbb{C}}) \times (U^{\mathbb{C}})^{(\alpha)}(\mathfrak{h}^{\mathbb{C}}), \text{ for all } \alpha \in (\mathfrak{h}^{\mathbb{C}})^*$$

(Proposition 10, Charter II). Thus,

$$\begin{aligned} &(\bar{\mathfrak{g}}^{\mathbb{C}})^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}e_1, \\ &(\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda-\mu-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3), \\ &(\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda-\mu+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3), \\ &(\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_1 + i\tilde{u}_2), \\ &(\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_1 - i\tilde{u}_2), \\ &(\bar{\mathfrak{g}}^{\mathbb{C}})^{(\mu)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_3), \end{aligned}$$

Therefore,

$$\begin{aligned} &[\tilde{u}_1 + i\tilde{u}_2, \tilde{u}_3] \in (\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda+\mu+i)}(\mathfrak{h}^{\mathbb{C}}), \\ &[\tilde{u}_1 - i\tilde{u}_2, \tilde{u}_3] \in (\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda+\mu-i)}(\mathfrak{h}^{\mathbb{C}}), \end{aligned}$$

and

$$[\tilde{u}_1 + i\tilde{u}_2, \tilde{u}_3] = [\tilde{u}_1, \tilde{u}_3] + i[\tilde{u}_2, \tilde{u}_3] = 0,$$
$$[\tilde{u}_1 - i\tilde{u}_2, \tilde{u}_3] = [\tilde{u}_1, \tilde{u}_3] - i[\tilde{u}_2, \tilde{u}_3] = 0,$$
$$[\tilde{u}_1 + i\tilde{u}_2, \tilde{u}_1 - i\tilde{u}_2] = -2i[\tilde{u}_1, \tilde{u}_2] \in \mathbb{C}\tilde{e}_1 + \mathbb{C}\tilde{u}_3.$$

Hence,

$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0,$$
$$[u_1, u_2] = a_1 e_1 + \alpha_3 u_3.$$

Using the Jacobi identity we obtain $a_1 = \alpha_3 = 0$. Thus the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Proposition 3.17. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.17 is trivial.*

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	$\lambda u_1 + u_2$	0
e_2	0	0	$-e_3$	0	u_1	u_3
e_3	$-e_3$	e_3	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_1 - u_2$	$-u_1$	0	0	0	0
u_3	0	$-u_3$	$-u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & \lambda + 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Proposition 3.18. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.18 is trivial.*

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	$u_1 + u_2$	0
e_2	0	0	$-e_3$	0	0	u_3
e_3	$-e_3$	e_3	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_1 - u_2$	0	0	0	0	0
u_3	0	$-u_3$	$-u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Proposition 3.19. *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 3.19 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-\lambda e_3$	0	u_2	λu_3
e_2	e_2	0	0	0	u_1	0
e_3	λe_3	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	$-\lambda u_3$	0	$-u_1$	0	0	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-\lambda e_3$	0	u_2	λu_3
e_2	e_2	0	0	e_2	u_1	0
e_3	λe_3	0	0	0	0	$u_1 - e_1$
u_1	0	$-e_2$	0	0	u_2	0
u_2	$-u_2$	$-u_1$	0	$-u_2$	0	0
u_3	$-\lambda u_3$	0	$e_1 - u_1$	0	0	0

3. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	e_3
e_2	e_2	0	0	0	u_1	0
e_3	0	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	$-e_3$	0	$-u_1$	0	0	0

4. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	0
e_2	e_2	0	0	0	u_1	0
e_3	0	0	0	0	0	$u_1 + e_3$
u_1	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	0	0	$-u_1 - e_3$	0	0	0

5. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	e_3
e_2	e_2	0	0	0	u_1	0
e_3	0	0	0	0	0	$u_1 + e_3$
u_1	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	$-e_3$	0	$-u_1 - e_3$	0	0	0

6. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	$e_1 - u_1$
e_2	e_2	0	0	e_2	u_1	0
e_3	0	0	0	0	0	$u_1 - e_1$
u_1	0	$-e_2$	0	0	u_2	0
u_2	$-u_2$	$-u_1$	0	$-u_2$	0	0
u_3	$u_1 - e_1$	0	$e_1 - u_1$	0	0	0

7. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	0
e_2	e_2	0	0	0	u_1	0
e_3	0	0	0	0	0	$u_1+\alpha e_3$
u_1	0	0	0	0	0	u_1
u_2	$-u_2$	$-u_1$	0	0	0	u_2
u_3	0	0	$-u_1-\alpha e_3$	$-u_1$	$-u_2$	0

8. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	e_3
e_2	e_2	0	0	0	u_1	0
e_3	0	0	0	0	0	$u_1+\alpha e_3$
u_1	0	0	0	0	0	u_1
u_2	$-u_2$	$-u_1$	0	0	0	u_2
u_3	$-e_3$	0	$-u_1-\alpha e_3$	$-u_1$	$-u_2$	0

9. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	e_3
e_2	e_2	0	0	e_2	u_1	0
e_3	0	0	0	0	0	$u_1-e_1+\alpha e_3$
u_1	0	$-e_2$	0	0	u_2	0
u_2	$-u_2$	$-u_1$	0	$-u_2$	0	0
u_3	$-e_3$	0	$e_1-u_1-\alpha e_3$	0	0	0

10. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	0	0	u_2	$-e_3$
e_2	e_2	0	0	e_2	u_1	0
e_3	0	0	0	0	0	$u_1-e_1+\alpha e_3$
u_1	0	$-e_2$	0	0	u_2	0
u_2	$-u_2$	$-u_1$	0	$-u_2$	0	0
u_3	e_3	0	$e_1-u_1-\alpha e_3$	0	0	0

11. $\lambda = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-e_3$	0	u_2	u_3
e_2	e_2	0	0	e_3	u_1	e_1
e_3	e_3	0	0	0	0	u_1
u_1	0	$-e_3$	0	0	0	u_2
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	$-u_3$	$-e_1$	$-u_1$	$-u_2$	0	0

12. $\lambda = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-e_3$	0	u_2	u_3
e_2	e_2	0	0	e_3	u_1	e_1
e_3	e_3	0	0	$-e_2$	$-e_1$	u_1
u_1	0	$-e_3$	e_2	0	$-u_3$	u_2
u_2	$-u_2$	$-u_1$	e_1	u_3	0	0
u_3	$-u_3$	$-e_1$	$-u_1$	$-u_2$	0	0

13. $\lambda = -1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	0	u_2+e_3	e_2-u_3
e_2	e_2	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-e_3-u_2$	$-u_1$	0	0	0	0
u_3	u_3-e_2	0	$-u_1$	0	0	0

14. $\lambda = -1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	0	u_2	$-u_3$
e_2	e_2	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	0	0	0	0	e_3	e_2
u_2	$-u_2$	$-u_1$	0	$-e_3$	0	e_1
u_3	u_3	0	$-u_1$	$-e_2$	$-e_1$	0

15. $\lambda = -1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	e_3	0	e_3+u_2	e_2-u_3
e_2	e_2	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	0	0	0	0	e_3	e_2
u_2	$-e_3-u_2$	$-u_1$	0	$-e_3$	0	e_1
u_3	$-e_2+u_3$	0	$-u_1$	$-e_2$	$-e_1$	0

16. $\lambda = -\frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-2e_2$	e_3	0	$2u_2$	$-u_3$
e_2	$2e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-2u_2$	$-u_1$	0	0	0	e_3
u_3	u_3	0	$-u_1$	0	$-e_3$	0

17. $\lambda = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-2e_2$	$-e_3$	0	$2u_2$	u_3
e_2	$2e_2$	0	0	0	u_1	e_3
e_3	e_3	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-2u_2$	$-u_1$	0	0	0	0
u_3	$-u_3$	$-e_3$	$-u_1$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma.
(i) Suppose $\lambda \neq 1$. Then any virtual structure q on generalized module 3.19 is equivalent to one of the following:
a) $\lambda \notin \{0, \frac{1}{2}, -1\}$

$$C_1(e_1) = 0, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_3) = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

b) $\lambda = 0$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2(e_3) = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix};$$

c) $\lambda = \frac{1}{2}$

$$C_3(e_1) = 0, \quad C_3(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad C_3(e_3) = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

d) $\lambda = -1$

$$C_4(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix}, \quad C_4(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C_4(e_3) = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

(ii) Suppose $\lambda = 1$. Then any virtual structure q on generalized module 3.19 is equivalent to one and only one of the following:
a)

$$C_5(e_i) = 0, \quad i = 1, 2, 3;$$

b)

$$C_6(e_1) = 0, \quad C_6(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_6(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

c)

$$C_7(e_1) = 0, \quad C_7(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_7(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

d)

$$C_8(e_1) = 0, \quad C_8(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_8(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Let q be a virtual structure on generalized module 3.19. Without loss of generality it can be assumed that q is primary. Since

$$\begin{aligned} \mathfrak{g}^{(0)}(\mathfrak{h}) &\supset \mathbb{R}e_1, & U^{(0)}(\mathfrak{h}) &\supset \mathbb{R}u_1, \\ \mathfrak{g}^{(-1)}(\mathfrak{h}) &\supset \mathbb{R}e_2, & U^{(1)}(\mathfrak{h}) &\supset \mathbb{R}u_2, \\ \mathfrak{g}^{(-\lambda)}(\mathfrak{h}) &\supset \mathbb{R}e_3, & U^{(\lambda)}(\mathfrak{h}) &\supset \mathbb{R}u_3, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} c_{11}^1 & 0 & c_{13}^1 \\ 0 & 0 & c_{23}^1 \\ c_{31}^1 & c_{32}^1 & c_{33}^1 \end{pmatrix}, \; C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & c_{13}^2 \\ c_{21}^2 & 0 & c_{23}^2 \\ c_{31}^2 & c_{32}^2 & c_{33}^2 \end{pmatrix}, \; C(e_3) = \begin{pmatrix} c_{11}^3 & c_{12}^3 & c_{13}^3 \\ c_{21}^3 & c_{22}^3 & 0 \\ c_{31}^3 & 0 & c_{33}^3 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c_{12}^2 & 0 & -c_{23}^2 \\ 0 & 0 & 0 \\ c_{32}^2 & 0 & 0 \end{pmatrix},$$

and $C'(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C'(e_1) = \begin{pmatrix} c_{11}^1 & 0 & c_{13}^1 \\ 0 & 0 & c_{23}^1 \\ c_{31}^1 & c_{32}^1 & c_{33}^1 \end{pmatrix}, \; C'(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ c_{21}^2 & 0 & 0 \\ c_{31}^2 & 0 & c_{33}^2 \end{pmatrix}, \; C'(e_3) = \begin{pmatrix} c_{11}^3 & c_{12}^3 & c_{13}^3 \\ c_{21}^3 & c_{22}^3 & 0 \\ c_{31}^3 & 0 & c_{33}^3 \end{pmatrix}.$$

Since for any virtual structure q condition (6), Charter II, must be satisfied, after direct calculation we obtain:

$$C'(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & c_{32}^1 & c_{33}^1 \end{pmatrix}, \; C'(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ c_{21}^2 & 0 & 0 \\ c_{31}^2 & 0 & c_{33}^2 \end{pmatrix}, \; C'(e_3) = \begin{pmatrix} 0 & c_{12}^3 & c_{13}^3 \\ c_{21}^3 & 0 & 0 \\ 0 & 0 & c_{33}^3 \end{pmatrix},$$

where

$$\begin{cases} (1-\lambda)c_{12}^3 = 0, \\ (1-\lambda)c_{21}^3 = 0, \\ \lambda c_{33}^3 = 0, \\ (1-\lambda)c_{13}^2 = 0, \\ (1-\lambda)c_{31}^2 = 0, \\ \lambda c_{13}^2 = c_{31}^2, \\ c_{21}^3 = c_{12}^3, \\ (2\lambda - 1)c_{33}^2 = 0, \\ c_{13}^2 = -c_{21}^2. \end{cases} \qquad (*)$$

Consider the following cases:

a) $\lambda \notin \{0, -1, \frac{1}{2}\}$. Since q is primary, we obtain

$$c_{32}^1 = c_{23}^1 = c_{33}^1 = c_{31}^2 = c_{13}^2 = c_{33}^2 = c_{21}^3 = c_{12}^3 = c_{33}^3 = 0.$$

From (*) it follows that $c_{13}^3 = -c_{21}^2$.

b) $\lambda = 0$. Since q is primary, we obtain

$$c_{32}^1 = c_{23}^1 = c_{31}^2 = c_{13}^2 = c_{33}^2 = c_{21}^3 = c_{12}^3 = 0.$$

From (*) it follows that $c_{13}^3 = -c_{21}^2$.

c) $\lambda = \frac{1}{2}$. Since q is primary, we obtain

$$c_{32}^1 = c_{23}^1 = c_{33}^1 = c_{31}^2 = c_{13}^2 = c_{33}^3 = c_{21}^3 = c_{12}^3 = 0.$$

From (*) it follows that $c_{13}^3 = -c_{21}^2$.

d) $\lambda = -1$. Since q is primary, we obtain

$$c_{33}^1 = c_{31}^2 = c_{13}^2 = c_{33}^2 = c_{21}^3 = c_{12}^3 = c_{33}^3 = 0.$$

From (*) it follows that $c_{13}^3 = -c_{21}^2$.

e) $\lambda = 1$. Since q is primary, we obtain

$$c_{32}^1 = c_{23}^1 = c_{33}^1 = c_{33}^2 = c_{33}^3 = 0.$$

From (*) it follows that $c_{13}^2 = c_{31}^2$, $c_{12}^3 = c_{21}^3$, and $c_{13}^3 = -c_{21}^2$. Thus

$$C'(e_1) = 0, \quad C'(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ c_{21}^2 & 0 & 0 \\ c_{13}^2 & 0 & 0 \end{pmatrix}, \quad C'(e_3) = \begin{pmatrix} 0 & c_{12}^3 & -c_{21}^2 \\ c_{12}^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Proposition 7, Charter II, virtual structures C' and C'' are equivalent if and only if there exist $P \in \operatorname{Aut}(\mathfrak{g})$ and $H \in \operatorname{Mat}_{3\times 3}(\mathbb{R})$ such that

$$C''(x) = FC'(\varphi^{-1}(x))P^{-1} - A(x)H + HB(x) \text{ for all } x \in \mathfrak{g},$$

where $\varphi = \varphi(P)$, and F is the matrix of φ.

Direct calculations show that the virtual structure q is equivalent to one and only one of the virtual structures C_5, C_6, C_7, C_8.

The proof of the Lemma is complete.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.19. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II).

Consider the following cases:

1°. $\lambda \notin \{0, \frac{1}{2}, -1, 1\}$. Then

$$\begin{aligned}
&[e_1, e_2] = -e_2, \\
&[e_1, e_3] = -\lambda e_3, \quad [e_2, e_3] = 0, \\
&[e_1, u_1] = 0, \qquad [e_2, u_1] = pe_2, \quad [e_3, u_1] = 0, \\
&[e_1, u_2] = u_2, \qquad [e_2, u_2] = u_1, \quad [e_3, u_2] = 0, \\
&[e_1, u_3] = \lambda u_3, \quad [e_2, u_3] = 0, \qquad [e_3, u_3] = u_1 - pe_1.
\end{aligned}$$

Since

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_1, \\
&\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2, \qquad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_2, \\
&\bar{\mathfrak{g}}^{(-\lambda)}(\mathfrak{h}) = \mathbb{R}e_3, \qquad \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) = \mathbb{R}u_3,
\end{aligned}$$

we have

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] = \alpha_2 u_2, \\
&[u_1, u_3] = \beta_3 u_3, \\
&[u_2, u_3] = c_3 e_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-\lambda e_3$	0	u_2	λu_3
e_2	e_2	0	0	pe_2	u_1	0
e_3	λe_3	0	0	0	0	$u_1 - pe_1$
u_1	0	$-pe_2$	0	0	pu_2	0
u_2	$-u_2$	$-u_1$	0	$-pu_2$	0	$c_3 e_3$
u_3	$-\lambda u_3$	0	$pe_1 - u_1$	0	$-c_3 e_3$	0

,

where $c_3(1 + 2\lambda) = 0$, and $pc_3 = 0$.

1.1°. $c_3 = p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.2°. $c_3 = 0$, $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2, 3, \\
&\pi(u_j) = \frac{1}{p} u_j, \ j = 1, 2, 3.
\end{aligned}$$

1.3°. $c_3 \neq 0$, $\lambda = -\frac{1}{2}$, $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{16}, \mathfrak{g}_{16})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{16} \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned}
&\pi(e_1) = 2e_1, \qquad \pi(u_1) = c_3 u_1, \\
&\pi(e_2) = c_3 e_2, \qquad \pi(u_2) = u_2, \\
&\pi(e_3) = c_3 e_3, \qquad \pi(u_3) = u_3.
\end{aligned}$$

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent. For $\lambda - \frac{1}{2}$ we have $\dim \mathcal{D}\bar{\mathfrak{g}}_2 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_{16}$, and $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_{16}$. This implies that the pair $(\bar{\mathfrak{g}}_{16}, \mathfrak{g}_{16})$ is not equivalent to the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ for $\lambda = -\frac{1}{2}$.

2°. $\lambda = \frac{1}{2}$. Then

$$\begin{aligned}
&[e_1, e_2] = -e_2, \\
&[e_1, e_3] = -\frac{1}{2}e_3, \quad [e_2, e_3] = 0, \\
&[e_1, u_1] = 0, \qquad [e_2, u_1] = pe_2, \quad [e_3, u_1] = 0, \\
&[e_1, u_2] = u_2, \qquad [e_2, u_2] = u_1, \quad [e_3, u_2] = 0, \\
&[e_1, u_3] = \frac{1}{2}u_3, \qquad [e_2, u_3] = qe_3, \quad [e_3, u_3] = u_1 - pe_1.
\end{aligned}$$

Since

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_1, \\
&\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2, \qquad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_2, \\
&\bar{\mathfrak{g}}^{(-1/2)}(\mathfrak{h}) = \mathbb{R}e_3, \qquad \bar{\mathfrak{g}}^{(1/2)}(\mathfrak{h}) = \mathbb{R}u_3,
\end{aligned}$$

we have

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(1/2)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(3/2)}(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] = \alpha_2 u_2, \\
&[u_1, u_3] = \beta_3 u_3, \\
&[u_2, u_3] = 0.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-\frac{1}{2}e_3$	0	u_2	$\frac{1}{2}u_3$
e_2	e_2	0	0	pe_2	u_1	qe_3
e_3	$\frac{1}{2}e_3$	0	0	0	0	u_1-pe_1
u_1	0	$-pe_2$	0	0	pu_2	0
u_2	$-u_2$	$-u_1$	0	$-pu_2$	0	0
u_3	$-\frac{1}{2}u_3$	$-qe_3$	pe_1-u_1	0	0	0 ,

where $pq = 0$.

2.1°. $q = p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2.2°. $p \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2, 3, \\
&\pi(u_j) = \frac{1}{p}u_j, j = 1, 2, 3.
\end{aligned}$$

2.3°. $q \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{17}, \mathfrak{g}_{17})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{17} \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i,\ i = 1, 2, 3,$$
$$\pi(u_j) = \frac{1}{q} u_j,\ j = 1, 2, 3.$$

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$, $\dim \mathcal{D}\bar{\mathfrak{g}}_2 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_{17}$, and $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_{17}$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_{17}, \mathfrak{g}_{17})$ are not equivalent to each other.

In a similar way we obtain the other results of the Proposition.

Proposition 3.20. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.20 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	$(1-\mu)e_3$	u_1	λu_2	μu_3
e_2	$(\lambda-1)e_2$	0	0	0	u_1	0
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0
u_3	$-\mu u_3$	0	$-u_1$	0	0	0

2. $\lambda = 0$, $\mu > 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$(1-\mu)e_3$	u_1	0	μu_3
e_2	$-e_2$	0	0	0	u_1	0
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_1	0
u_2	0	$-u_1$	0	$-u_1$	0	$-u_3$
u_3	$-\mu u_3$	0	$-u_1$	0	u_3	0

3. $\lambda \leqslant 0$, $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	e_3	u_1	λu_2	0
e_2	$(\lambda-1)e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_1
u_2	$-\lambda u_2$	$-u_1$	0	0	0	u_2
u_3	0	0	$-u_1$	$-u_1$	$-u_2$	0

4. $\lambda < \frac{1}{3}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	$2\lambda e_3$	u_1	λu_2	$(1-2\lambda)u_3$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	0
e_3	$-2\lambda e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	e_2
u_3	$(2\lambda-1)u_3$	0	$-u_1$	0	$-e_2$	0

5. $\lambda = 1 - 2\mu$, $\mu \geqslant \frac{1}{3}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$2\mu e_2$	$(1-\mu)e_3$	u_1	$(1-2\mu)u_2$	μu_3
e_2	$-2\mu e_2$	0	0	0	u_1	0
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$(2\mu-1)u_2$	$-u_1$	0	0	0	e_3
u_3	$-\mu u_3$	0	$-u_1$	0	$-e_3$	0

6. $\lambda = \frac{1}{2}$, $\mu = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	0	u_1	$e_2 + \frac{1}{2}u_2$	u_3
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1	0
e_3	0	0	0	0	e_2	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-e_2 - \frac{1}{2}u_2$	$-u_1$	$-e_2$	0	0	0
u_3	$-u_3$	0	$-u_1$	0	0	0

7. $\lambda = 0$, $\mu = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	0	u_1	e_3	$e_2 + u_3$
e_2	$-e_2$	0	0	0	u_1	0
e_3	0	0	0	0	e_3	u_1
u_1	$-u_1$	0	0	0	u_1	0
u_2	$-e_3$	$-u_1$	$-e_3$	$-u_1$	0	αe_2
u_3	$-e_2 - u_3$	0	$-u_1$	0	$-\alpha e_2$	0

8. $\lambda = \frac{1}{3}$, $\mu = \frac{2}{3}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{2}{3}e_2$	$\frac{1}{3}e_3$	u_1	$e_3 + \frac{1}{3}u_2$	$e_2 + \frac{2}{3}u_3$
e_2	$-\frac{2}{3}e_2$	0	0	0	u_1	0
e_3	$-\frac{1}{3}e_3$	0	0	0	e_2	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-e_3 - \frac{1}{3}u_2$	$-u_1$	$-e_2$	0	0	0
u_3	$-e_2 - \frac{2}{3}u_3$	0	$-u_1$	0	0	0

9. $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	u_1	$e_2 + \frac{1}{2}u_2$	$e_3 + \frac{1}{2}u_3$
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1	0
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-e_2 - \frac{1}{2}u_2$	$-u_1$	0	0	0	0
u_3	$-e_3 - \frac{1}{2}u_3$	0	$-u_1$	0	0	0

10. $\lambda = 0,\ \mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$\frac{1}{2}e_3$	u_1	0	$e_3 + \frac{1}{2}u_3$
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-\frac{1}{2}e_3$	0	0	0	e_3	u_1
u_1	$-u_1$	0	0	0	$2u_1$	0
u_2	0	$-u_1$	$-e_3$	$-2u_1$	0	$\alpha e_3 - u_3$
u_3	$-e_3 - \frac{1}{2}u_3$	0	$-u_1$	0	$-\alpha e_3 + u_3$	0

11. $\lambda = 0,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	e_2	$e_3 + u_1$
u_1	$-u_1$	0	0	0	$-e_2$	0
u_2	0	$-u_1$	$-e_2$	e_2	0	0
u_3	0	0	$-e_3 - u_1$	0	0	0

12. $\lambda = 0,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	e_2	$e_3 + u_1$
u_1	$-u_1$	0	0	0	0	$-u_1$
u_2	0	$-u_1$	$-e_2$	0	0	$-u_2$
u_3	0	0	$-e_3 - u_1$	u_1	u_2	0

13. $\lambda = 0,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$-e_2$	$e_3 + u_1$
u_1	$-u_1$	0	0	0	e_2	0
u_2	0	$-u_1$	e_2	$-e_2$	0	0
u_3	0	0	$-e_3 - u_1$	0	0	0

14. $\lambda = 0,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$-e_2$	$e_3 + u_1$
u_1	$-u_1$	0	0	0	0	$-u_1$
u_2	0	$-u_1$	e_2	0	0	$-u_2$
u_3	0	0	$-e_3 - u_1$	u_1	u_2	0

15. $\lambda = 0,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$-e_2$	$e_3 + u_1$
u_1	$-u_1$	0	0	0	$\frac{3}{2}e_2 - e_3 - \frac{3}{2}u_1$	$-\frac{1}{2}e_2 + \frac{1}{2}u_1$
u_2	0	$-u_1$	e_2	$-\frac{3}{2}e_2 + e_3 + \frac{3}{2}u_1$	0	$\frac{1}{2}e_1 + \frac{1}{2}u_2 + \frac{3}{2}u_3$
u_3	0	0	$-e_3 - u_1$	$\frac{1}{2}e_2 - \frac{1}{2}u_1$	$-\frac{1}{2}e_1 - \frac{1}{2}u_2 - \frac{3}{2}u_3$	0

16. $\lambda = \frac{1}{4}$, $\mu = \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{3}{4}e_2$	$\frac{1}{2}e_3$	u_1	$\frac{1}{4}u_2$	$e_3 + \frac{1}{2}u_3$
e_2	$-\frac{3}{4}e_2$	0	0	0	u_1	0
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{1}{4}u_2$	$-u_1$	0	0	0	e_2
u_3	$-e_3 - \frac{1}{2}u_3$	0	$-u_1$	0	$-e_2$	0

17. $\lambda = 0$, $\mu = \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$\frac{1}{2}e_3$	u_1	0	$e_3 + \frac{1}{2}u_3$
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	e_3
u_3	$-e_3 - \frac{1}{2}u_3$	0	$-u_1$	0	$-e_3$	0

18. $\lambda < \frac{1}{2}$, $\mu = \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	$\frac{1}{2}e_3$	u_1	λu_2	$e_3 + \frac{1}{2}u_3$
e_2	$(\lambda - 1)e_2$	0	0	0	u_1	0
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0
u_3	$-e_3 - \frac{1}{2}u_3$	0	$-u_1$	0	0	0

19. $\lambda = \frac{1}{2}$, $\mu \geqslant \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	$(1-\mu)e_3$	u_1	$e_2 + \frac{1}{2}u_2$	μu_3
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1	0
e_3	$(\mu - 1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-e_2 - \frac{1}{2}u_2$	$-u_1$	0	0	0	0
u_3	$-\mu u_3$	0	$-u_1$	0	0	0

20. $\lambda \leqslant 0$, $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	e_3	u_1	λu_2	0
e_2	$(\lambda - 1)e_2$	0	0	0	u_1	e_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	αu_1
u_2	$-\lambda u_2$	$-u_1$	0	0	0	$(\alpha - 1)u_2$
u_3	0	$-e_2$	$-u_1$	$-\alpha u_1$	$(1-\alpha)u_2$	0

21. $\lambda = 0,\ \mu > 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$(1-\mu)e_3$	u_1	0	μu_3
e_2	$-e_2$	0	0	0	u_1	0
e_3	$(\mu-1)e_3$	0	0	0	e_3	u_1
u_1	$-u_1$	0	0	0	$(1-\alpha)u_1$	0
u_2	0	$-u_1$	$-e_3$	$(\alpha-1)u_1$	0	αu_3
u_3	$-\mu u_3$	0	$-u_1$	0	$-\alpha u_3$	0

22. $\lambda = 0,\ \mu = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$\frac{1}{2}e_3$	u_1	0	$\frac{1}{2}u_3$
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-\frac{1}{2}e_3$	0	0	0	e_3	u_1
u_1	$-u_1$	0	0	0	$2u_1$	0
u_2	0	$-u_1$	$-e_3$	$-2u_1$	0	$e_3 - u_3$
u_3	$-\frac{1}{2}u_3$	0	$-u_1$	0	$-e_3 + u_3$	0

23. $\lambda = 0,\ \mu = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	0	u_1	0	u_3
e_2	$-e_2$	0	0	0	u_1	0
e_3	0	0	0	0	e_3	u_1
u_1	$-u_1$	0	0	0	u_1	0
u_2	0	$-u_1$	$-e_3$	$-u_1$	0	e_2
u_3	$-u_3$	0	$-u_1$	0	$-e_2$	0

24. $\lambda = 0,\ \mu = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	e_3
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	0	$-e_3$	$-u_1$	$-u_1$	$-u_2$	0

25. $\lambda = 2\mu,\ \mu \leqslant 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-2\mu)e_2$	$(1-\mu)e_3$	u_1	$2\mu u_2$	μu_3
e_2	$(2\mu-1)e_2$	0	0	0	u_1	e_3
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-2\mu u_2$	$-u_1$	0	0	0	0
u_3	$-\mu u_3$	$-e_3$	$-u_1$	0	0	0

26. $\lambda > 0,\ \mu = 2\lambda$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	$(1-2\lambda)e_3$	u_1	λu_2	$2\lambda u_3$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	0
e_3	$(2\lambda-1)e_3$	0	0	0	e_2	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	$-e_2$	0	0	0
u_3	$-2\lambda u_3$	0	$-u_1$	0	0	0

27. $\lambda = \frac{1}{5}$, $\mu = \frac{2}{5}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{4}{5}e_2$	$\frac{3}{5}e_3$	u_1	$\frac{1}{5}u_2$	$\frac{2}{5}u_3$
e_2	$-\frac{4}{5}e_2$	0	0	0	u_1	0
e_3	$-\frac{3}{5}e_3$	0	0	0	e_2	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{1}{5}u_2$	$-u_1$	$-e_2$	0	0	e_3
u_3	$-\frac{2}{5}u_3$	0	$-u_1$	0	$-e_3$	0

28. $\lambda = 1 - \mu$, $\mu \geqslant \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	μe_2	$(1-\mu)e_3$	u_1	$e_3 + (1-\mu)u_2$	$e_2 + \mu u_3$
e_2	$-\mu e_2$	0	0	0	u_1	0
e_3	$(\mu - 1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-e_3 + (\mu - 1)u_2$	$-u_1$	0	0	0	0
u_3	$-e_2 - \mu u_3$	0	$-u_1$	0	0	0

29. $\lambda = 0$, $\mu = 1$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	0	u_1	e_3	$e_2 + u_3$
e_2	$-e_2$	0	0	0	u_1	0
e_3	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-e_3$	$-u_1$	0	0	0	e_2
u_3	$-e_2 - u_3$	0	$-u_1$	0	$-e_2$	0

Proof.
Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\mu \end{pmatrix}, \; A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda-1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ \mu-1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *on generalized module 3.20 is equivalent to one of the following:*

a) $\mu \neq \frac{1}{2}$, $\lambda \neq \frac{1}{2}$, $\mu \neq 0$, $\lambda \neq 0$, $\lambda \neq 2\mu$, $\mu \neq 2\lambda$, $\mu + \lambda \neq 1$, $\mu \neq \lambda + 1$

$$C(e_i) = 0, \quad i = 1, 2, 3;$$

b) $\mu = \frac{1}{2},\ \lambda < \frac{1}{2},\ \lambda \notin \{0,\ \frac{1}{4}\}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = C(e_3) = 0;$$

c) $\lambda = \frac{1}{2},\ \mu > \frac{1}{2},\ \mu \neq 1$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = C(e_3) = 0;$$

d) $\mu = 0,\ \lambda < 0$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = 0;$$

e) $\lambda = 0,\ \mu > 0,\ \mu \notin \{\frac{1}{2},\ 1\}$

$$C(e_1) = C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix};$$

f) $\lambda = 2\mu,\ \mu < 0$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_3) = 0;$$

g) $\mu = 2\lambda,\ \lambda > 0,\ \lambda \notin \{\frac{1}{4},\ \frac{1}{3},\ \frac{1}{2}\}$

$$C(e_1) = C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

h) $\lambda + \mu = 1,\ \mu > \frac{1}{2},\ \mu \notin \{\frac{2}{3},\ 1\}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & q & 0 \end{pmatrix}, \quad C(e_2) = C(e_3) = 0;$$

i) $\mu = \lambda + 1,\ \lambda \notin \{-2,\ -1, -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1\}$

$$C(e_1) = C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & p & 0 \\ p(\lambda - 1) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

j) $\lambda = \frac{1}{2},\ \mu = 1$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

k) $\lambda = 0,\ \mu = 1$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & q & 0 \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & s & 0 \\ -s & 0 & 0 \\ 0 & r & 0 \end{pmatrix};$$

l) $\lambda = \frac{1}{3},\ \mu = \frac{2}{3}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & q & 0 \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

m) $\lambda = \frac{1}{2},\ \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & r \\ 0 & s & q \end{pmatrix}, \quad C(e_2) = C(e_3) = 0;$$

n) $\lambda = \frac{1}{4},\ \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

o) $\lambda = 0,\ \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix};$$

p) $\lambda = 0,\ \mu = 0$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & q \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{pmatrix};$$

q) $\lambda = -2,\ \mu = -1$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & q & 0 \\ -3q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$r)$ $\lambda = -1,\ \mu = 0$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & q & 0 \\ -2q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$s)$ $\lambda = -\frac{1}{2},\ \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & q & 0 \\ -\frac{3}{2}q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$t)$ $\lambda = \frac{1}{2},\ \mu = \frac{3}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & q & 0 \\ -\frac{1}{2}q & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$u)$ $\lambda = 1,\ \mu = 2$

$$C(e_1) = C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & q & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Put

$$C(e_i) = \left(c^i_{jk}\right)_{1\leqslant j,k\leqslant 3}, \quad i = 1, 2, 3.$$

Let q be a virtual structure on generalized module 3.20. Without loss of generality it can be assumed that q is primary.

Since

$$\begin{aligned} \mathfrak{g}^{(0)}(\mathfrak{h}) &\supseteq \mathbb{R}e_1, & U^{(1)}(\mathfrak{h}) &\supseteq \mathbb{R}u_1, \\ \mathfrak{g}^{(1-\lambda)}(\mathfrak{h}) &\supseteq \mathbb{R}e_2, & U^{(\lambda)}(\mathfrak{h}) &\supseteq \mathbb{R}u_2, \\ \mathfrak{g}^{(1-\mu)}(\mathfrak{h}) &\supseteq \mathbb{R}e_3, & U^{(\mu)}(\mathfrak{h}) &\supseteq \mathbb{R}u_3, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} 0 & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & 0 & c^2_{13} \\ 0 & c^2_{22} & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & 0 \\ c^3_{21} & c^3_{22} & c^3_{23} \\ 0 & c^3_{32} & c^3_{33} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ c^3_{23} & 0 & 0 \\ c^2_{32} & 0 & 0 \end{pmatrix},$$

and $C'(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C'(e_1) = \begin{pmatrix} 0 & c_{12}^1 & c_{13}^1 \\ c_{21}^1 & c_{22}^1 & c_{23}^1 \\ c_{31}^1 & c_{32}^1 & c_{33}^1 \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} c_{11}^2 & 0 & c_{13}^2 \\ 0 & c_{22}^2 & c_{23}^2 \\ c_{31}^2 & 0 & c_{33}^2 \end{pmatrix},$$

$$C'(e_3) = \begin{pmatrix} c_{11}^3 & c_{12}^3 & 0 \\ c_{21}^3 & c_{22}^3 & 0 \\ 0 & c_{32}^3 & c_{33}^3 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C' are equivalent.

Since for any virtual structure q condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C'(e_1) = \begin{pmatrix} 0 & c_{12}^1 & c_{13}^1 \\ 0 & c_{22}^1 & c_{23}^1 \\ 0 & c_{32}^1 & c_{33}^1 \end{pmatrix}, \; C'(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^2 & c_{23}^2 \\ 0 & 0 & c_{33}^2 \end{pmatrix}, \; C'(e_3) = \begin{pmatrix} 0 & c_{12}^3 & 0 \\ c_{21}^3 & c_{22}^3 & 0 \\ 0 & c_{32}^3 & c_{33}^3 \end{pmatrix},$$

where the set of coefficients c_{ij}^k satisfies the following system:

$$\begin{cases} (2\mu - \lambda)c_{33}^2 = 0, \\ (\mu - 2\lambda)c_{22}^3 = 0, \\ (1 - \mu + \lambda)c_{12}^3 = 0, \\ (1 - \mu + \lambda)c_{21}^3 = 0, \\ (\lambda - 1)c_{12}^3 = c_{21}^3, \\ \lambda c_{22}^2 = (1 - \lambda)c_{12}^1, \\ \mu c_{23}^2 = (1 - \lambda)c_{13}^1, \\ \lambda c_{32}^3 = (1 - \mu)c_{12}^1, \\ \mu c_{33}^3 = (1 - \mu)c_{13}^1. \end{cases}$$

For example, consider the following cases:

a) $\mu, \lambda \notin \{0, \frac{1}{2}\}$, $\lambda \neq 2\mu$, $\mu \neq 2\lambda$, $\mu + \lambda \neq 1$, $\mu \neq \lambda + 1$.

From the system it follows that $c_{33}^2 = c_{22}^3 = c_{12}^3 = c_{21}^3 = 0$. Put

$$H_1 = \begin{pmatrix} 0 & \frac{1}{\lambda}c_{12}^1 & \frac{1}{\mu}c_{13}^1 \\ 0 & \frac{1}{2\lambda-1}c_{22}^1 & \frac{1}{\lambda+\mu-1}c_{23}^1 \\ 0 & \frac{1}{\lambda+\mu-1}c_{32}^1 & \frac{1}{2\mu-1}c_{33}^1 \end{pmatrix},$$

and $C_1(x) = C'(x) + A(x)H_1 - H_1B(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_i) = 0, \quad i = 1, 2, 3.$$

m) $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$.

From the system it follows that $c_{33}^2 = c_{22}^3 = c_{12}^3 = c_{21}^3 = 0$. Put

$$H_{13} = \begin{pmatrix} 0 & 2c_{12}^1 & 2c_{13}^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_{13}(x) = C'(x) + A(x)H_{13} - H_{13}B(x)$ for $x \in \mathfrak{g}$. Then

$$C_{13}(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c^1_{22} & c^1_{23} \\ 0 & c^1_{32} & c^1_{33} \end{pmatrix}, \quad C_{13}(e_2) = C_{13}(e_3) = 0.$$

$p)$ $\lambda = 0,\ \mu = 0$.
From the system it follows that $c^1_{12} = c^1_{13} = c^3_{12} = c^3_{21} = 0$. Put

$$H_{16} = \begin{pmatrix} 0 & c^3_{32} & c^2_{23} \\ 0 & -c^1_{22} & -c^1_{23} \\ 0 & -c^1_{32} & -c^1_{33} \end{pmatrix},$$

and $C_{16}(x) = C'(x) + A(x)H_{16} - H_{16}B(x)$ for $x \in \mathfrak{g}$. Then

$$C_{16}(e_1) = 0, \quad C_{16}(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c^2_{22} & 0 \\ 0 & 0 & c^2_{33} \end{pmatrix}, \quad C_{16}(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c^3_{22} & 0 \\ 0 & 0 & c^3_{33} \end{pmatrix}.$$

In a similar way we obtain the other results of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.20. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. The virtual structure q is trivial. Then

$$\begin{aligned} &[e_1, e_2] = (1-\lambda)e_2, \\ &[e_1, e_3] = (1-\mu)e_3, \quad [e_2, e_3] = 0, \\ &[e_1, u_1] = u_1, \qquad [e_2, u_1] = 0, \quad [e_3, u_1] = 0, \\ &[e_1, u_2] = \lambda u_2, \qquad [e_2, u_2] = u_1, \quad [e_3, u_2] = 0, \\ &[e_1, u_3] = \mu u_3, \qquad [e_2, u_3] = 0, \quad [e_3, u_3] = u_1. \end{aligned}$$

Since the virtual structure q is primary, we have

$$\begin{aligned} &[u_1, u_2] = a_1 e_1 + a_2 e_2 + a_3 e_3 + \alpha_1 u_1 + \alpha_3 u_3, \\ &[u_1, u_3] = b_1 e_1 + b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2, \\ &[u_2, u_3] = c_1 e_1 + c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	$(1-\mu)e_3$	u_1	λu_2	μu_3
e_2	$(\lambda-1)e_2$	0	0	0	u_1	0
e_3	$(\mu-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	$\alpha_1 u_1$	$\beta_1 u_1$
u_2	$-\lambda u_2$	$-u_1$	0	$-\alpha_1 u_1$	0	A
u_3	$-\mu u_3$	0	$-u_1$	$-\beta_1 u_1$	$-A$	0

$$A = c_2e_2 + c_3e_3 + \beta_1u_2 - \alpha_1u_3,$$

where the set of coefficients satisfies the following system:

$$\begin{cases} \lambda\alpha_1 = 0, \\ \mu\beta_1 = 0, \\ (1 - 2\lambda - \mu)c_2 = 0, \\ (1 - \lambda - 2\mu)c_3 = 0. \end{cases}$$

1.1°. $c_2 = c_3 = \alpha_1 = \beta_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.2°. $\lambda = 0,\ c_2 = c_3 = \beta_1 = 0,\ \alpha_1 \neq 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi\colon \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, & \pi(u_1) &= u_1, \\ \pi(e_2) &= \frac{1}{\alpha_1}e_2, & \pi(u_2) &= \alpha_1u_2, \\ \pi(e_3) &= e_3, & \pi(u_3) &= u_3. \end{aligned}$$

1.3°. $\mu = 0,\ c_2 = c_3 = \alpha_1 = 0,\ \beta_1 \neq 0$.

Then the mapping $\pi\colon \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, & \pi(u_1) &= u_1, \\ \pi(e_2) &= e_2, & \pi(u_2) &= u_2, \\ \pi(e_3) &= \frac{1}{\beta_1}e_3, & \pi(u_3) &= \beta_1u_3, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

1.4°. $\mu = 1 - 2\lambda,\ c_3 = \alpha_1 = \beta_1 = 0,\ c_2 \neq 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi\colon \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, & \pi(u_1) &= u_1, \\ \pi(e_2) &= e_2, & \pi(u_2) &= u_2, \\ \pi(e_3) &= \frac{1}{c_2}e_3, & \pi(u_3) &= c_2u_3. \end{aligned}$$

1.5°. $\lambda = 1 - 2\mu,\ c_2 = \alpha_1 = \beta_1 = 0,\ c_3 \neq 0$.

Then the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, & \pi(u_1) &= u_1, \\ \pi(e_2) &= \frac{1}{c_3}e_2, & \pi(u_2) &= c_3u_2, \\ \pi(e_3) &= e_3, & \pi(u_3) &= u_3, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$.

1.6°. $\lambda = 0,\ \mu = 1,\ c_3 = \beta_1 = 0,\ \alpha_1 \neq 0.$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ $(\mu = 1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \pi(u_1) = u_1, \\ \pi(e_2) &= \frac{1}{\alpha_1} e_2, \pi(u_2) = \frac{c_2}{\alpha_1} e_3 + \alpha_1 u_2, \\ \pi(e_3) &= e_3, \pi(u_3) = \frac{c_2}{\alpha_1^2} e_2 + u_3. \end{aligned}$$

1.7°. $\lambda = 0,\ \mu = \frac{1}{2},\ c_2 = \beta_1 = 0,\ \alpha_1 \neq 0.$

Then the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, \pi(u_1) = u_1, \\ \pi(e_2) &= \frac{1}{\alpha_1} e_2, \pi(u_2) = \alpha_1 u_2, \\ \pi(e_3) &= e_3, \pi(u_3) = \frac{c_3}{\alpha_1} e_2 + u_3. \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ $(\mu = \frac{1}{2})$.

1.8°. $\lambda = 0,\ \mu = 0,\ c_2 = c_3 = 0,\ \alpha_1 \neq 0.$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ $(\lambda = 0)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, & \pi(u_1) &= u_1, \\ \pi(e_2) &= -\frac{\beta_1}{\alpha_1} e_2 + \frac{1}{\alpha_1} e_3, & \pi(u_2) &= \alpha_1 u_3, \\ \pi(e_3) &= e_2, & \pi(u_3) &= u_2 + \beta_1 u_3. \end{aligned}$$

1.9°. $\lambda = \frac{1}{3},\ \mu = \frac{1}{3},\ \alpha_1 = \beta_1 = 0,\ c_2 \neq 0.$

Then the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_1) &= e_1, & \pi(u_1) &= u_1, \\ \pi(e_2) &= c_3 e_2 + \frac{1}{c_2} e_3, & \pi(u_2) &= c_2 u_3, \\ \pi(e_3) &= -c_2 e_2, & \pi(u_3) &= -\frac{1}{c_2} u_2 + c_3 u_3, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ $(\mu = \frac{1}{3})$.

2°. $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$. Then

$$
\begin{aligned}
&[e_1, e_2] = \frac{1}{2} e_2, \\
&[e_1, e_3] = \frac{1}{2} e_3, && [e_2, e_3] = 0, \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = pe_2 + se_3 + \frac{1}{2} u_2, && [e_2, u_2] = u_1, && [e_3, u_2] = 0, \\
&[e_1, u_3] = re_2 + qe_3 + \frac{1}{2} u_3, && [e_2, u_3] = 0, && [e_3, u_3] = u_1.
\end{aligned}
$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1/2)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

where

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1, \qquad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1,$$
$$\bar{\mathfrak{g}}^{(1/2)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_3.$$

Therefore

$$
\begin{aligned}
&[u_1, u_2] = 0, \\
&[u_1, u_3] = 0, \\
&[u_2, u_3] = \gamma_1 u_1.
\end{aligned}
$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	u_1	$pe_2+re_3+\frac{1}{2}u_2$	$re_2+qe_3+\frac{1}{2}u_3$
e_2	$\frac{1}{2}e_2$	0	0	0	u_1	0
e_3	$\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-pe_2-re_3-\frac{1}{2}u_2$	$-u_1$	0	0	0	$\gamma_1 u_1$
u_3	$-re_2-qe_3-\frac{1}{2}u_3$	0	$-u_1$	0	$-\gamma_1 u_1$	0

Then the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$
\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, 2, 3, \\
&\pi(u_1) = u_1, \\
&\pi(u_2) = u_2, \\
&\pi(u_3) = -\gamma_1 e_2 + u_3,
\end{aligned}
$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	$\frac{1}{2}e_3$	u_1	$pe_2+re_3+\frac{1}{2}u_2$	$re_2+qe_3+\frac{1}{2}u_3$
e_2	$\frac{1}{2}e_2$	0	0	0	u_1	0
e_3	$\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-pe_2-re_3-\frac{1}{2}u_2$	$-u_1$	0	0	0	0
u_3	$-re_2-qe_3-\frac{1}{2}u_3$	0	$-u_1$	0	0	0

Now we determine the group of all transformations of mappings q. Let

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & r \\ 0 & r & q \end{pmatrix}, \quad C(e_2) = C(e_3) = 0,$$

and

$$C'(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p' & r' \\ 0 & r' & q' \end{pmatrix}, \quad C'(e_2) = C'(e_3) = 0.$$

Put

$$Q = \begin{pmatrix} p & r \\ r & q \end{pmatrix}, \quad \text{and} \quad Q' = \begin{pmatrix} p' & r' \\ r' & q' \end{pmatrix}.$$

The virtual structures C and C' are equivalent if and only if there exist matrices $P \in \mathcal{A}(\mathfrak{g})$ and $H \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ such that

$$C'(x) = FC(\varphi^{-1}(x))P^{-1} - A(x)H + HB(x) \quad \text{for all} \quad x \in \mathfrak{g},$$

where $\varphi(x) = PxP^{-1}$, and F is the matrix of the mapping φ. Direct calculation shows that the virtual structures C and C' are equivalent if and only if there exist a number $a \neq 0$ and a matrix $A \in \mathrm{GL}(2, \mathbb{R})$ such that the following condition holds:

$$Q' = \frac{a}{(\det A)^2} AQA^t.$$

Using this condition, we see that any virtual structure on generalized module 3.20 ($\lambda = \mu = \frac{1}{2}$) is equivalent to one and only one of the following:

a)

$$C^1(e_i) = 0, \quad i = 1, 2, 3;$$

b)

$$C^2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^2(e_2) = C^2(e_3) = 0;$$

c)

$$C^3(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C^3(e_2) = C^3(e_3) = 0;$$

d)

$$C^4(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^4(e_2) = C^4(e_3) = 0.$$

Note that the virtual structure C^1 was already considered in case 1°, and the virtual structures C^2 and C^3 are special cases of the virtual structures in the case $\lambda = \frac{1}{2}$ and $\lambda + \mu = 1$, respectively.

For the virtual structure C^4 we obtain the pair $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$.

3°. $\lambda = 0,\ \mu = 0$. Then

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & q \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{pmatrix}.$$

Put

$$C'(e_1) = 0, \quad C'(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p' & 0 \\ 0 & 0 & q' \end{pmatrix}, \quad C'(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r' & 0 \\ 0 & 0 & s' \end{pmatrix}.$$

The virtual structures C and C' are equivalent if and only if there exist matrices $P \in \mathcal{A}(\mathfrak{g})$ and $H \in \mathrm{Mat}_{3\times3}(\mathbb{R})$ such that

$$C'(x) = FC(\varphi^{-1}(x))P^{-1} - A(x)H + HB(x) \quad \text{for all} \quad x \in \mathfrak{g},$$

where $\varphi(x) = PxP^{-1}$, and F is the matrix of the mapping φ. Direct calculation shows that the virtual structures C and C' are equivalent if and only if there exist numbers a, b, c, d, g such that $\Delta = ad - bc \neq 0,\ g \neq 0$, and the following conditions hold:

$$\begin{aligned}
p' &= \frac{1}{g\Delta}[a^2(dp - cr) + b^2(dq - cs) - 2ac(-bp + ar) - 2bd(-bq + as)], \\
q' &= \frac{1}{g\Delta}[c^2(dp - cr) + d^2(dq - cs)], \\
r' &= \frac{1}{g\Delta}[a^2(-bp + ar) + b^2(-bq + as)], \\
s' &= \frac{1}{g\Delta}[c^2(-bp + ar) + d^2(-bq + as) - 2ac(dp - cr) - 2bd(dq - cs)].
\end{aligned}$$

Using these conditions, we see that any virtual structure on generalized module 3.20 ($\lambda = \mu = 0$) is equivalent to one and only one of the following:

a)

$$C^1(e_i) = 0, \quad i = 1, 2, 3;$$

b)

$$C^2(e_1) = C^2(e_2) = 0, \quad C^2(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

c)

$$C^3(e_1) = C^3(e_2) = 0, \quad C^3(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

d)

$$C^4(e_1) = C^4(e_2) = 0, \quad C^4(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

e)

$$C^5(e_1) = C^5(e_2) = 0,\ C^5(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the virtual structure C^1 was considered in case 1°, and the virtual structures C^2 and C^3 are special cases of the virtual structures in the cases $\lambda = 2\mu$ and $\mu = 0$, respectively.

For the virtual structures C^4 and C^5 we obtain the following nonequivalent pairs: $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$, $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$, $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$, $(\bar{\mathfrak{g}}_{14}, \mathfrak{g}_{14})$, $(\bar{\mathfrak{g}}_{15}, \mathfrak{g}_{15})$.

In a similar way we obtain the other results of the Proposition.

Proposition 3.21. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.21 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-\lambda e_2 - e_3$	$-\lambda e_3 + e_2$	0	$\lambda u_2 - u_3$	$u_2 + \lambda u_3$
e_2	$\lambda e_2 + e_3$	0	0	0	u_1	0
e_3	$\lambda e_3 - e_2$	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$u_3 - \lambda u_2$	$-u_1$	0	0	0	0
u_3	$-u_2 - \lambda u_3$	0	$-u_1$	0	0	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-\lambda e_2 - e_3$	$-\lambda e_3 + e_2$	0	$\lambda u_2 - u_3$	$u_2 + \lambda u_3$
e_2	$\lambda e_2 + e_3$	0	$e_2 - \lambda e_3$	0	u_1	$-e_1$
e_3	$\lambda e_3 - e_2$	0	0	$\lambda e_2 + e_3$	e_1	u_1
u_1	0	$-e_2 + \lambda e_3$	$-e_3 - \lambda e_2$	0	$u_2 + \lambda u_3$	$u_3 - \lambda u_2$
u_2	$u_3 - \lambda u_2$	$-u_1$	$-e_1$	$-u_2 - \lambda u_3$	0	0
u_3	$-u_2 - \lambda u_3$	e_1	$-u_1$	$\lambda u_2 - u_3$	0	0

3. $\lambda = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	$+e_2$	0	$-u_3$	$u_2 + e_3$
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	u_3	$-u_1$	0	0	0	0
u_3	$-e_3 - u_2$	0	$-u_1$	0	0	0

4. $\lambda = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	$+e_2$	0	$-u_3$	$u_2 + e_3$
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	e_2	e_3
u_2	u_3	$-u_1$	0	$-e_2$	0	e_1
u_3	$-e_3 - u_2$	0	$-u_1$	$-e_3$	$-e_1$	0

5. $\lambda = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	$+e_2$	0	$-u_3$	$u_2 + e_3$
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	$-e_2$	$-e_3$
u_2	u_3	$-u_1$	0	e_2	0	$-e_1$
u_3	$-e_3 - u_2$	0	$-u_1$	e_3	e_1	0

6. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	$+e_2$	0	$-u_3$	u_2
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	e_2	e_3
u_2	u_3	$-u_1$	0	$-e_2$	0	e_1
u_3	$-u_2$	0	$-u_1$	$-e_3$	$-e_1$	0

7. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	$+e_2$	0	$-u_3$	u_2
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	$-e_2$	$-e_3$
u_2	u_3	$-u_1$	0	e_2	0	$-e_1$
u_3	$-u_2$	0	$-u_1$	e_3	e_1	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & -1 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1. Consider the complex generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$. Put

$$\tilde{e}_i = e_i \otimes 1, \ i = 1, 2, 3, \quad \text{and} \quad \tilde{u}_j = u_j \otimes 1, \ j = 1, 2, 3.$$

Then $\tilde{\mathcal{E}} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is a basis of $\mathfrak{g}^{\mathbb{C}}$. The vector space $U^{\mathbb{C}}$ can be identified with $\mathbb{C}^3$, and $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ is a standard basis of $U^{\mathbb{C}}$.

Lemma. *Any virtual structure on the generalized module $(\mathfrak{g}, U)$ of type 3.21 is equivalent to one of the following virtual structure:*

a)

$$C_1(e_1) = 0, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & -p \\ p & 0 & 0 \\ -\lambda p & 0 & 0 \end{pmatrix}, \quad C_1(e_3) = \begin{pmatrix} 0 & p & 0 \\ \lambda p & 0 & 0 \\ p & 0 & 0 \end{pmatrix};$$

b) $\lambda = 0$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q & r \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & -p \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2(e_3) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix}.$$

Proof. Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a virtual pair defined by a virtual structure q. By Proposition 15, Chapter II, without loss of generality it can be assumed that $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$ (with respect to $\mathfrak{h}^{\mathbb{C}}$). Then

$$\begin{aligned} &\mathfrak{g}^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{e}_1, && (U^{\mathbb{C}})^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{u}_1,\\ &\mathfrak{g}^{(-\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3), && (U^{\mathbb{C}})^{(\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 + i\tilde{u}_3),\\ &\mathfrak{g}^{(-\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3), && (U^{\mathbb{C}})^{(\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 - i\tilde{u}_3). \end{aligned}$$

Suppose $\lambda \neq 0$. Then we have

$$\begin{aligned} &q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1) &&\in \mathbb{C},\\ &q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_1) &&\in \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3),\\ &q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2 + i\tilde{u}_3) &&= 0,\\ &q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_2 + i\tilde{u}_3) &&= 0,\\ &q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2 - i\tilde{u}_3) &&= 0,\\ &q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_2 - i\tilde{u}_3) &&\in \mathbb{C},\\ &q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_1) &&\in \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3),\\ &q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_2 + i\tilde{u}_3) &&\in \mathbb{C}\tilde{e}_1,\\ &q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_2 - i\tilde{u}_3) &&= 0. \end{aligned}$$

Therefore

$$\begin{aligned} &q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1) \in \mathbb{C}\tilde{e}_1,\\ &q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2) = q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_3) = 0,\\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_1) \in \mathbb{C}\tilde{e}_2 + \mathbb{C}\tilde{e}_3,\\ &q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_1) \in \mathbb{C}\tilde{e}_2 + \mathbb{C}\tilde{e}_3,\\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2) \in \mathbb{C}\tilde{e}_1,\\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_3) \in \mathbb{C}\tilde{e}_1,\\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2) = q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_3),\\ &q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_3) = -q^{\mathbb{C}}(\tilde{e}_3)(\tilde{u}_2). \end{aligned}$$

Since the matrices of the mappings $q(e_i)$ and $q^{\mathbb{C}}(\tilde{e}_i)$, $i = 1, 2, 3$, concide, we obtain

$$C(e_1) = \begin{pmatrix} c_{11}^1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & c_{13}^2\\ c_{21}^2 & 0 & 0\\ c_{31}^2 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & c_{13}^2 & c_{12}^2\\ c_{21}^3 & 0 & 0\\ c_{31}^3 & 0 & 0 \end{pmatrix}.$$

Checking condition (6), Chapter II, for $x, y \in \mathcal{E}$, we obtain

$$\begin{aligned} c_{11}^1 &= c_{12}^2 = 0,\\ c_{21}^2 &= c_{31}^3 = -c_{13}^2,\\ -c_{31}^2 &= c_{21}^3 = -\lambda c_{13}^2. \end{aligned}$$

Finally, put $C_1 = C$. In a similar way we obtain the result for $\lambda = 0$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.21. Then it can be assumed that the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is determined by one of the virtual pairs determined in the Lemma:

Consider the following cases.

1°. $\lambda \neq 0$. Then

$$\begin{aligned}
&[e_1, e_2] = -\lambda e_2 - e_3, && &&\\
&[e_1, e_3] = e_2 - \lambda e_3, && [e_2, e_3] = 0, &&\\
&[e_1, u_1] = 0, && [e_2, u_1] = pe_2 + p\lambda e_3, && [e_3, u_1] = p\lambda e_2 + pe_3,\\
&[e_1, u_2] = \lambda u_2 - u_3, && [e_2, u_2] = u_1, && [e_3, u_2] = pe_1,\\
&[e_1, u_3] = u_2 + \lambda u_3, && [e_2, u_3] = -pe_1, && [e_3, u_3] = u_1.
\end{aligned}$$

Since $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$, we have

$$(\bar{\mathfrak{g}}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) = (\mathfrak{g}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \times (U^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \text{ for all } \alpha \in (\mathfrak{h}^{\mathbb{C}})^*$$

(Proposition 10, Chapter II). Thus

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{e}_1, && (\bar{\mathfrak{g}}^{\mathbb{C}})^{(0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}\tilde{u}_1,\\
&\bar{\mathfrak{g}}^{(-\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3), && (\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 + i\tilde{u}_3),\\
&\bar{\mathfrak{g}}^{(-\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3), && (\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 - i\tilde{u}_3).
\end{aligned}$$

Therefore

$$\begin{aligned}
&[\tilde{u}_1, \tilde{u}_2 + i\tilde{u}_3] \in \mathbb{C}(\tilde{u}_2 + i\tilde{u}_3),\\
&[\tilde{u}_1, \tilde{u}_2 - i\tilde{u}_3] \in \mathbb{C}(\tilde{u}_2 - i\tilde{u}_3),\\
&[\tilde{u}_2 + i\tilde{u}_3, \tilde{u}_2 - i\tilde{u}_3] = -2i[\tilde{u}_2, \tilde{u}_3] = 0,
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] = \alpha_2 u_2 + \alpha_3 u_3,\\
&[u_1, u_3] = \beta_2 u_2 + \beta_3 u_3,\\
&[u_2, u_3] = 0.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-\lambda e_2 - e_3$	$-\lambda e_3 + e_2$	0	$\lambda u_2 - u_3$	$u_2 + \lambda u_3$
e_2	$\lambda e_2 + e_3$	0	0	$pe_2 - p\lambda e_3$	u_1	$-pe_1$
e_3	$\lambda e_3 - e_2$	0	0	$p\lambda e_2 + pe_3$	pe_1	u_1
u_1	0	$-pe_2 + p\lambda e_3$	$-pe_3 - p\lambda e_2$	0	$pu_2 + p\lambda u_3$	$pu_3 - p\lambda u_2$
u_2	$u_3 - \lambda u_2$	$-u_1$	$-pe_1$	$pu_2 + p\lambda u_3$	0	0
u_3	$-u_2 - \lambda u_3$	pe_1	$-u_1$	$p\lambda u_2 - pu_3$	0	0

1.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, such that

$$\begin{aligned}
&\pi(e_i) = e_i, \; i = 1, 2, 3,\\
&\pi(u_j) = \frac{1}{p} u_j, \; j = 1, 2, 3.
\end{aligned}$$

2°. Now suppose $\lambda = 0$. Then

$$\begin{aligned}
&[e_1, e_2] = -e_3, && && \\
&[e_1, e_3] = e_2, && [e_2, e_3] = 0, && \\
&[e_1, u_1] = 0, && [e_2, u_1] = pe_2, && [e_3, u_1] = pe_3, \\
&[e_1, u_2] = qe_3 - u_3, && [e_2, u_2] = u_1, && [e_3, u_2] = pe_1, \\
&[e_1, u_3] = u_2 + re_3, && [e_2, u_3] = -pe_1, && [e_3, u_3] = u_1.
\end{aligned}$$

Since $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$, we have

$$\begin{aligned}
(\bar{\mathfrak{g}}^{\mathbb{C}})^{(0)}(\mathfrak{h}^{\mathbb{C}}) &= \mathbb{C}\tilde{e}_1 + \mathbb{C}\tilde{u}_1, \\
(\bar{\mathfrak{g}}^{\mathbb{C}})^{(i)}(\mathfrak{h}^{\mathbb{C}}) &= \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3) + \mathbb{C}(\tilde{u}_2 + i\tilde{u}_3), \\
(\bar{\mathfrak{g}}^{\mathbb{C}})^{(-i)}(\mathfrak{h}^{\mathbb{C}}) &= \mathbb{C}(\tilde{u}_2 - i\tilde{u}_3) + \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3).
\end{aligned}$$

Therefore

$$\begin{aligned}
[u_1, u_2] &= a_2 e_2 + a_3 e_3 + \alpha_2 u_2 + \alpha_3 u_3, \\
[u_1, u_3] &= b_2 e_2 + b_3 e_3 + \beta_2 u_2 + \beta_3 u_3, \\
[u_2, u_3] &= c_1 e_1 + \gamma_1 u_1.
\end{aligned}$$

2.1°. $p \neq 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	e_2	0	$-u_3$	u_2
e_2	e_3	0	0	pe_2	u_1	$-pe_1$
e_3	$-e_2$	0	0	pe_3	pe_1	u_1
u_1	0	$-pe_2$	$-pe_3$	0	pu_2	pu_3
u_2	u_3	$-u_1$	$-pe_1$	pu_2	0	0
u_3	$-u_2$	pe_1	$-u_1$	$-pu_3$	0	0

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, such that

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, 2, 3, \\
\pi(u_j) &= \frac{1}{p} u_j, \ j = 1, 2, 3.
\end{aligned}$$

2.2°. $p = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	e_2	0	$-u_3$	u_2+re_3
e_2	e_3	0	0	0	u_1	0
e_3	$-e_2$	0	0	0	0	u_1
u_1	0	0	0	0	$a_2 e_2$	$a_2 e_3$
u_2	u_3	$-u_1$	0	$-a_2 e_2$	0	$a_2 e_1+\gamma_1 u_1$
u_3	$-re_3-u_2$	0	$-u_1$	$-a_2 e_3$	$-a_2 e_1-\gamma_1 u_1$	0

2.2.1°. $a_2 = r = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_1$, such that

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, 2, 3, \\
\pi(u_1) &= u_1, \\
\pi(u_2) &= u_2 - \frac{\gamma_1}{2} e_3, \\
\pi(u_3) &= u_3 - \frac{\gamma_1}{2} e_2.
\end{aligned}$$

2.2.2°. $a_2 = 0$, $r \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, such that

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1,2,3,\\
\pi(u_1) &= \frac{1}{r} u_1,\\
\pi(u_2) &= \frac{1}{r}(u_2 - \frac{\gamma_1}{2} e_3),\\
\pi(u_3) &= \frac{1}{r}(u_3 - \frac{\gamma_1}{2} e_2).
\end{aligned}$$

2.2.3°. $a_2 > 0$, $r \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, such that

$$\begin{aligned}
\pi(e_1) &= e_1,\\
\pi(e_2) &= \frac{r}{\sqrt{a_2}} e_2,\\
\pi(e_3) &= \frac{r}{\sqrt{a_2}} e_3,\\
\pi(u_1) &= \frac{r}{\sqrt{a_2}} u_1,\\
\pi(u_2) &= \frac{1}{\sqrt{a_2}}(u_2 - \frac{\gamma_1}{2} e_3),\\
\pi(u_3) &= \frac{1}{\sqrt{a_2}}(u_3 - \frac{\gamma_1}{2} e_2).
\end{aligned}$$

2.2.4°. $a_2 < 0$, $r \neq 0$. We see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$.

2.2.5°. $a_2 > 0$, $r = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, such that

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1,2,3,\\
\pi(u_1) &= \frac{1}{\sqrt{a_2}} u_1,\\
\pi(u_2) &= \frac{1}{\sqrt{a_2}}(u_2 - \frac{\gamma_1}{2} e_3),\\
\pi(u_3) &= \frac{1}{\sqrt{a_2}}(u_3 - \frac{\gamma_1}{2} e_2).
\end{aligned}$$

2.2.6°. $a_2 < 0$, $r = 0$. We see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$.

It remains to show that the pairs determined in the Proposition are not equivalent to each other. Note that

a) $\dim[\mathcal{D}\bar{\mathfrak{g}}_2, \bar{\mathfrak{g}}_2] \neq \dim[\mathcal{D}\bar{\mathfrak{g}}_i, \bar{\mathfrak{g}}_i]$, where $i \in \{1, 3, 4, 5, 6, 7\}$;

b) no one of the virtual pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$, $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ is isomorphic to any of the virtual pairs $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$, $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$, $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$;

c) $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_6$, $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_7$;

d) any Levi subalgebra of $\bar{\mathfrak{g}}_6$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and any Levi subalgebra of $\bar{\mathfrak{g}}_7$ is isomorphic to $\mathfrak{su}(2)$;

e) $\dim \mathcal{D}\bar{\mathfrak{g}}_3 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_4$, $\dim \mathcal{D}\bar{\mathfrak{g}}_3 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_5$;

f) any Levi subalgebra of $\bar{\mathfrak{g}}_4$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and any Levi subalgebra of $\bar{\mathfrak{g}}_5$ is isomorphic to $\mathfrak{su}(2)$.

It follows that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \dots, 7$, are not equivalent to each other.

Proposition 3.22. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.22 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(\lambda-\mu)e_2 - e_3$	$e_2 + (\lambda-\mu)e_3$	λu_1	$\mu u_2 - u_3$	$u_2 + \mu u_3$
e_2	$(\mu-\lambda)e_2 + e_3$	0	0	0	u_1	0
e_3	$(\mu-\lambda)e_3 - e_2$	0	0	0	0	u_1
u_1	$-\lambda u_1$	0	0	0	0	0
u_2	$u_3 - \mu u_2$	$-u_1$	0	0	0	0
u_3	$-u_2 - \mu u_3$	0	$-u_1$	0	0	0

2. $\lambda = 2\mu,\ \mu > 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\mu e_2 - e_3$	$e_2 + \mu e_3$	$2\mu u_1$	$\mu u_2 - u_3$	$u_2 + \mu u_3 + e_3$
e_2	$-\mu e_2 + e_3$	0	0	0	u_1	0
e_3	$-\mu e_3 - e_2$	0	0	0	0	u_1
u_1	$-2\mu u_1$	0	0	0	0	0
u_2	$u_3 - \mu u_2$	$-u_1$	0	0	0	0
u_3	$-u_2 - \mu u_3 - e_3$	0	$-u_1$	0	0	0

.

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & -1 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda-\mu & 1 \\ 0 & -1 & \lambda-\mu \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \mu-\lambda & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ \mu-\lambda & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on the generalized module 3.22 is equivalent to one of the following:*

a) $\lambda \neq 2\mu$

$$C_1(e_i) = 0, \quad i = 1, 2, 3;$$

b) $\lambda = 2\mu$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & q \end{pmatrix}, \quad C_2(e_2) = C_2(e_3) = 0.$$

Proof. Put

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix}.$$

Consider the following cases:

1°. $\lambda \neq 2\mu$. Put

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix},$$

where H satisfies the condition $A(e_1)H - HB(e_1) + C(e_1) = 0$,

$$\begin{pmatrix} -\lambda h_{11} & -\mu h_{12} + h_{13} & -h_{12} - \mu h_{13} \\ -\mu h_{21} + h_{31} & (\lambda - 2\mu)h_{22} + h_{32} + h_{23} & (\lambda - 2\mu)h_{23} + h_{22} - h_{33} \\ -h_{21} - \mu h_{31} & (\lambda - 2\mu)h_{32} - h_{22} + h_{33} & (\lambda - 2\mu)h_{33} - h_{23} - h_{32} \end{pmatrix} +$$

$$+ \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the solution of this equation exists since all elements of matrix $A(e_1)H - HB(e_1)$ are linearly independent.

Put $C_1(x) = C(x) + A(x)H - HB(x)$.

By corollary 3, Chapter II, the virtual structures defined by mappings C and C_1 are equivalent. Then $C_1(e_1) = 0$ and put

$$C_1(e_i) = (c^i_{jk})_{1 \leqslant j,k \leqslant 3}, \quad i = 2, 3.$$

Let us check condition (6), Chapter II, for $x, y \in \mathcal{E}$.

$$C_1([e_1, e_2]) = A(e_1)C_1(e_2) - C_1(e_2)B(e_1) - A(e_2)C_1(e_1) + C_1(e_1)B(e_2)$$

We have

$$\begin{pmatrix} 0 & 0 & 0 \\ (\lambda - \mu)c^2_{21} + c^2_{31} & (\lambda - \mu)c^2_{22} + c^2_{32} & (\lambda - \mu)c^2_{23} + c^2_{33} \\ -c^2_{21} + (\lambda - \mu)c^2_{31} & -c^2_{22} + (\lambda - \mu)c^2_{32} & -c^2_{23} + (\lambda - \mu)c^2_{33} \end{pmatrix} -$$

$$- \begin{pmatrix} \lambda c^2_{11} & \mu c^2_{12} - c^2_{13} & c^2_{12} + \mu c^2_{13} \\ \lambda c^2_{21} & \mu c^2_{22} - c^2_{23} & c^2_{22} + \mu c^2_{23} \\ \lambda c^2_{31} & \mu c^2_{32} - c^2_{33} & c^2_{32} + \mu c^2_{33} \end{pmatrix} =$$

$$= (\lambda - \mu)\begin{pmatrix} c_{11}^2 & c_{12}^2 & c_{13}^2 \\ c_{21}^2 & c_{22}^2 & c_{23}^2 \\ c_{31}^2 & c_{32}^2 & c_{33}^2 \end{pmatrix} - \begin{pmatrix} c_{11}^3 & c_{12}^3 & c_{13}^3 \\ c_{21}^3 & c_{22}^3 & c_{23}^3 \\ c_{31}^3 & c_{32}^3 & c_{33}^3 \end{pmatrix}.$$

We obtain the following equations:

$$\begin{cases} (\lambda - \mu)c_{11}^2 = c_{11}^3 & (1) \\ \lambda c_{12}^2 - c_{13}^2 = c_{12}^3 & (2) \\ \lambda c_{13}^2 + c_{12}^2 = c_{13}^3 & (2) \\ \lambda c_{21}^2 - c_{31}^2 = c_{21}^3 & (3) \\ \mu c_{22}^2 - c_{32}^2 - c_{23}^2 = c_{22}^3 & (4) \\ \mu c_{23}^2 + c_{22}^2 - c_{33}^2 = c_{23}^3 & (4) \\ \lambda c_{31}^2 + c_{21}^2 = c_{31}^3 & (3) \\ \mu c_{32}^2 + c_{22}^2 - c_{33}^2 = c_{32}^3 & (4) \\ \mu c_{33}^2 + c_{23}^2 + c_{32}^2 = c_{33}^3 & (4) \end{cases}$$

$$C_1([e_1, e_3]) = A(e_1)C_1(e_3) - C_1(e_3)B(e_1) - A(e_3)C_1(e_1) + C_1(e_1)B(e_3)$$

We have

$$\begin{pmatrix} 0 & 0 & 0 \\ (\lambda - \mu)c_{21}^3 + c_{31}^3 & (\lambda - \mu)c_{22}^3 + c_{32}^3 & (\lambda - \mu)c_{23}^3 + c_{33}^3 \\ -c_{21}^3 + (\lambda - \mu)c_{31}^3 & -c_{22}^3 + (\lambda - \mu)c_{32}^3 & -c_{23}^3 + (\lambda - \mu)c_{33}^3 \end{pmatrix} -$$

$$- \begin{pmatrix} \lambda c_{11}^3 & \mu c_{12}^3 - c_{13}^3 & c_{12}^3 + \mu c_{13}^3 \\ \lambda c_{21}^3 & \mu c_{22}^3 - c_{23}^3 & c_{22}^3 + \mu c_{23}^3 \\ \lambda c_{31}^3 & \mu c_{32}^3 - c_{33}^3 & c_{32}^3 + \mu c_{33}^3 \end{pmatrix} =$$

$$= (\lambda - \mu)\begin{pmatrix} c_{11}^3 & c_{12}^3 & c_{13}^3 \\ c_{21}^3 & c_{22}^3 & c_{23}^3 \\ c_{31}^3 & c_{32}^3 & c_{33}^3 \end{pmatrix} + \begin{pmatrix} c_{11}^2 & c_{12}^2 & c_{13}^2 \\ c_{21}^2 & c_{22}^2 & c_{23}^2 \\ c_{31}^2 & c_{32}^2 & c_{33}^2 \end{pmatrix}.$$

We obtain the following equations:

$$\begin{cases} (\lambda - \mu)c_{11}^3 = -c_{11}^2 & (1) \\ \lambda c_{12}^3 + c_{13}^3 = c_{12}^2 & (2) \\ -\lambda c_{13}^3 - c_{12}^3 = c_{13}^2 & (2) \\ -\lambda c_{21}^3 + c_{31}^3 = c_{21}^2 & (3) \\ -\mu c_{22}^3 + c_{32}^3 + c_{23}^3 = c_{22}^2 & (4) \\ -\mu c_{23}^3 - c_{22}^3 + c_{33}^3 = c_{23}^2 & (4) \\ -\lambda c_{31}^3 - c_{21}^3 = c_{31}^2 & (3) \\ -\mu c_{32}^3 - c_{22}^3 + c_{33}^3 = c_{32}^2 & (4) \\ -\mu c_{33}^3 - c_{23}^3 - c_{32}^3 = c_{33}^3 & (4) \end{cases}$$

The matrix of the system of the equations marked by (1) has the form

$$\begin{pmatrix} \lambda - \mu & -1 \\ 1 & \lambda - \mu \end{pmatrix};$$

therefore $c_{11}^2 = c_{11}^3 = 0$.

The matrix of the system of the equations marked by (2) and (3) has the form

$$\begin{pmatrix} \lambda & 1 & 1 & 0 \\ -1 & \lambda & 0 & 1 \\ -1 & 0 & \lambda & 1 \\ 0 & -1 & -1 & \lambda \end{pmatrix}.$$

If $\lambda \neq 0$, this matrix is non-degenerate. Therefore $c_{12}^2 = c_{12}^3 = c_{13}^2 = c_{13}^3 = 0$ and $c_{21}^2 = c_{21}^3 = c_{31}^2 = c_{31}^3 = 0$.

The matrix of the system of equations marked by (4) has the form

$$\begin{pmatrix} \mu & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & \mu & 0 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & \mu & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & \mu & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & \mu & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & \mu & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & \mu & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & \mu \end{pmatrix}.$$

This matrix is non-degenerate for all $\mu \in \mathbb{R}$. Therefore $c_{22}^2 = c_{22}^3 = c_{23}^2 = c_{23}^3 = c_{32}^2 = c_{32}^3 = c_{33}^2 = c_{33}^3 = 0$.

Thus, $C_1(e_1) = C_1(e_2) = C_1(e_3) = 0$.

2°. $\lambda = 2\mu$. Put

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix},$$

where H satisfies the condition

$$A(e_1)H - HB(e_1) + C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & q \end{pmatrix},$$

$$\begin{pmatrix} -2\mu h_{11} & -\mu h_{12} + h_{13} & -h_{12} - \mu h_{13} \\ -\mu h_{21} + h_{31} & h_{32} + h_{23} & h_{22} - h33 \\ -h_{21} - \mu h_{31} & -h_{22} + h_{33} & -h_{23} - h_{32} \end{pmatrix} + \begin{pmatrix} c_{11}^1 & c_{12}^1 & c_{13}^1 \\ c_{21}^1 & c_{22}^1 & c_{23}^1 \\ c_{31}^1 & c_{32}^1 & c_{33}^1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^1 - c_{23}^1 & c_{33}^1 - c_{22}^2 \end{pmatrix}.$$

Put $C_2(x) = C(x) + A(x)H - HB(x)$.

By corollary 3, Chapter II, the virtual structures defined by mappings C and C_2 are equivalent. Then

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & q \end{pmatrix}$$

and put

$$C_2(e_i) = (c^i_{jk})_{1\leqslant j,k\leqslant 3}, \quad i = 2,3.$$

Note that

$$A(e_i)C_2(e_1) = C_2(e_1)B(e_i) = 0, \quad i = 2,3.$$

Therefore, as in the previous case, $C_2(e_2) = C_2(e_3) = 0$.

This completes the proof of the Lemma.

Consider the following cases:

1°. $\lambda \neq 2\mu$. Then

$$\begin{array}{lll}
[e_1,e_2] = (\lambda-\mu)e_2 - e_3 & & \\
[e_1,e_3] = e_2 + (\lambda-\mu)e_3 & [e_2,e_3] = 0 & \\
[e_1,u_1] = \lambda u_1 & [e_2,u_1] = 0 & [e_3,u_1] = 0 \\
[e_1,u_2] = \mu u_2 - u_3 & [e_2,u_2] = u_1 & [e_3,u_2] = 0 \\
[e_1,u_3] = u_2 + \mu u_3 & [e_2,u_3] = 0 & [e_3,u_3] = u_1
\end{array}$$

Put

$$\begin{aligned}
[u_1,u_2] &= a_1e_1 + a_2e_2 + a_3e_3 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,\\
[u_1,u_3] &= b_1e_1 + b_2e_2 + b_3e_3 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,\\
[u_2,u_3] &= c_1e_1 + c_2e_2 + c_3e_3 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

2°. $\lambda = 2\mu$. Then

$$\begin{array}{lll}
[e_1,e_2] = \mu e_2 - e_3 & & \\
[e_1,e_3] = e_2 + \mu e_3 & [e_2,e_3] = 0 & \\
[e_1,u_1] = 2\mu u_1 & [e_2,u_1] = 0 & [e_3,u_1] = 0 \\
[e_1,u_2] = \mu u_2 - u_3 + pe_3 & [e_2,u_2] = u_1 & [e_3,u_2] = 0 \\
[e_1,u_3] = u_2 + \mu u_3 + qe_3 & [e_2,u_3] = 0 & [e_3,u_3] = u_1
\end{array}$$

Put

$$\begin{aligned}
[u_1,u_2] &= a_1e_1 + a_2e_2 + a_3e_3 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,\\
[u_1,u_3] &= b_1e_1 + b_2e_2 + b_3e_3 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,\\
[u_2,u_3] &= c_1e_1 + c_2e_2 + c_3e_3 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\mu e_2 - e_3$	$e_2 + \mu e_3$	$2\mu u_1$	$\mu u_2 - u_3$	$u_2 + \mu u_3 + qe_3$
e_2	$-\mu e_2 + e_3$	0	0	0	u_1	0
e_3	$-\mu e_3 - e_2$	0	0	0	0	u_1
u_1	$-2\mu u_1$	0	0	0	0	0
u_2	$u_3 - \mu u_2$	$-u_1$	0	0	0	0
u_3	$-u_2 - \mu u_3 - qe_3$	0	$-u_1$	0	0	0

Consider the following cases:

2.1°. $q = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2°. $q \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= e_1, \\ \pi(e_2) &= qe_2, \\ \pi(e_3) &= qe_3, \\ \pi(u_1) &= qu_1, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= u_3. \end{aligned}$$

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Consider the homomorphisms $f_i : \mathfrak{g}_i \to \mathfrak{gl}(5, \mathbb{R})$, $i = 1, 2$, where $f_i(x)$ is the matrix of the linear mapping $\operatorname{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i} x$. Since the subalgebras $f_1(\mathfrak{g}_1)$ and $f_2(\mathfrak{g}_2)$ of the Lie algebra $\mathfrak{gl}(5, \mathbb{R})$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Proposition 3.23. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.23 is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	$2(1-\lambda)e_3$	u_1	λu_2	$(2\lambda-1)u_3$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	u_2
e_3	$2(\lambda-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	0
u_3	$(1-2\lambda)u_3$	$-u_2$	$-u_1$	0	0	0

2. $\lambda = \frac{3}{5}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{2}{5}e_2$	$\frac{4}{5}e_3$	u_1	$\frac{3}{5}u_2$	$\frac{1}{5}u_3$
e_2	$-\frac{2}{5}e_2$	0	0	0	u_1	u_2
e_3	$-\frac{4}{5}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{3}{5}u_2$	$-u_1$	0	0	0	e_3
u_3	$-\frac{1}{5}u_3$	$-u_2$	$-u_1$	0	$-e_3$	0

3. $\lambda = \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	e_3	u_1	$\frac{1}{2}u_2$	0
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	0	e_2
u_3	0	$-u_2$	$-u_1$	0	$-e_2$	0

4. $\lambda = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	e_3	u_1	$\frac{1}{2}u_2$	0
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	0	$-e_2$
u_3	0	$-u_2$	$-u_1$	0	e_2	0

5. $\lambda = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	e_3	u_1	$\frac{1}{2}u_2$	0
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_1
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	0	$\alpha e_2 + u_2$
u_3	0	$-u_2$	$-u_1$	$-u_1$	$-\alpha e_2 - u_2$	0

6. $\lambda = \frac{3}{4}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{4}e_2$	$\frac{1}{2}e_3$	u_1	$\frac{3}{4}u_2$	$\frac{1}{2}u_3 + e_3$
e_2	$-\frac{1}{4}e_2$	0	0	0	u_1	u_2
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{3}{4}u_2$	$-u_1$	0	0	0	0
u_3	$-\frac{1}{2}u_3 - e_3$	$-u_2$	$-u_1$	0	0	0

7. $\lambda = 0$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$2e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	0	$-e_3$	$u_1 - 2e_2$	u_2
e_3	$-2e_3$	0	0	0	$-e_3$	u_1
u_1	$-u_1$	e_3	0	0	u_1	0
u_2	0	$-u_1 + 2e_2$	e_3	$-u_1$	0	$-2u_3$
u_3	u_3	$-u_2$	$-u_1$	0	$2u_3$	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 2-2\lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-2\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda-1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\lambda-2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector e_1.

Lemma. *Any virtual structure q on generalized module 3.23 is equivalent to one of the following:*

a) $\lambda \notin \{0, \frac{2}{3}, \frac{3}{4}\}$

$$C_1(e_i) = 0, \quad i = 1, 2, 3;$$

b) $\lambda = \frac{3}{4}$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_2(e_2) = C_2(e_3) = 0;$$

c) $\lambda = 0$

$$C_3(e_1) = 0, \quad C_3(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ p & 0 & 0 \end{pmatrix}, \quad C_3(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix};$$

d) $\lambda = \frac{2}{3}$

$$C_4(e_2) = C_4(e_3) = 0, \quad C_4(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. Let q be a virtual structure on generalized module 3.23. Without loss of generality it can be assumed that q is primary. Since

$$\begin{aligned} &\mathfrak{g}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_2, && U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1, \\ &\mathfrak{g}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1, && U^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_2, \\ &\mathfrak{g}^{(2-2\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_3, && U^{(2\lambda-1)}(\mathfrak{h}) \supset \mathbb{R}u_3, \end{aligned}$$

we have:

$$C(e_1) = \begin{pmatrix} 0 & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & 0 & c^2_{13} \\ 0 & c^2_{22} & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & 0 \\ c^3_{21} & 0 & c^3_{23} \\ 0 & c^3_{32} & c^3_{33} \end{pmatrix}.$$

Checking condition (6), Chapter II, obtain:

a) $\lambda \neq \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & \lambda c^2_{13} & c^1_{13} \\ \lambda c^3_{23} & c^1_{22} & c^1_{23} \\ (2\lambda - 1)c^2_{32} & (3\lambda - 2)c^3_{33} & c^1_{33} \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & (1-\lambda)c^2_{13} + c^3_{23} & \frac{\lambda-1}{1-2\lambda}c^1_{13} - \frac{1}{1-2\lambda}c^1_{22} \\ 0 & c^2_{32} & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^3 \\ 0 & (2\lambda-2)c_{13}^2 & \frac{2\lambda-2}{1-2\lambda}c_{13}^1 + c_{32}^2 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & c_{13}^2 & \frac{1}{2\lambda-1}c_{13}^1 \\ c_{23}^3 & \frac{1}{2\lambda-1}c_{22}^1 & \frac{1}{3\lambda-2}c_{23}^1 \\ c_{32}^2 & c_{33}^2 & \frac{1}{4\lambda-3}c_{33}^1 \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_i) = 0, \quad i = 1, 2, 3.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

b) $\lambda = \frac{1}{2}$. Then

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^2 \\ 0 & c_{32}^2 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^3 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & c_{33}^3 - c_{32}^2 \\ 0 & c_{23}^2 - \frac{1}{2}(c_{33}^3 - c_{32}^2) & 0 \\ c_{32}^2 & 0 & 0 \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_i) = 0, \quad i = 1, 2, 3.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.

c) $\lambda = \frac{3}{4}$. Then

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^1 \end{pmatrix}, \quad C(e_2) = C(e_3) = 0.$$

Now we can put $C_2 = C$.

d) $\lambda = 0$. Then

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ 0 & c_{22}^2 & 0 \\ c_{32}^3 - 2c_{13}^2 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^3 \\ 0 & c_{32}^3 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & c_{13}^2 & 0 \\ c_{23}^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $C_3(x) = C(x) + A(x)H - HB(x)$. Then

$$C_3(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^2 & 0 \\ c_{32}^3 - 2c_{13}^2 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^3 - 2c_{13}^2 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_3 are equivalent.

e) $\lambda = \frac{2}{3}$. Then

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c^1_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2_{33} \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c^2_{33} & 0 \end{pmatrix}$$

and $C_4(x) = C(x) + A(x)H - HB(x)$. Then

$$C_4(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c^1_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad C_4(e_2) = C_4(e_3) = 0.$$

By corollary 2, Chapter II, the virtual structures C and C_4 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.23. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda \notin \{0, \frac{2}{3}, \frac{3}{4}\}$. Then

$$\begin{aligned}
&[e_1, e_2] = (1-\lambda)e_2, && && \\
&[e_1, e_3] = (2-2\lambda)e_3, && [e_2, e_3] = 0, && \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = \lambda u_2, && [e_2, u_2] = u_1, && [e_3, u_2] = 0, \\
&[e_1, u_3] = (2\lambda - 1)u_3, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Thus

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_2, && \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supset \mathbb{R}e_1, \\
&\bar{\mathfrak{g}}^{(2-2\lambda)}(\mathfrak{h}) \supset \mathbb{R}e_3, && \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1, \\
&\bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) \supset \mathbb{R}u_2, && \bar{\mathfrak{g}}^{(2\lambda-1)}(\mathfrak{h}) \supset \mathbb{R}u_3,
\end{aligned}$$

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1+\lambda)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(2\lambda)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(3\lambda-1)}(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}[u_1, u_2] &= a_1 e_1 + a_3 e_3 + \alpha_3 u_3 \\ [u_1, u_3] &= b_2 e_2 + b_3 e_3 + \beta_1 u_1 \\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + \gamma_2 u_2.\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$\lambda \neq \frac{1}{2}$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\lambda)e_2$	$2(1-\lambda)e_3$	u_1	λu_2	$(2\lambda-1)u_3$
e_2	$(\lambda-1)e_2$	0	0	0	u_1	u_2
e_3	$2(\lambda-1)e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\lambda u_2$	$-u_1$	0	0	0	$c_3 e_3$
u_3	$(1-2\lambda)u_3$	$-u_2$	$-u_1$	0	$-c_3 e_3$	0

,

where $c_3(\lambda - \frac{3}{5}) = 0$, or

$\lambda = \frac{1}{2}$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	e_3	u_1	$\frac{1}{2}u_2$	0
e_2	$-\frac{1}{2}e_2$	0	0	0	u_1	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\beta_1 u_1$
u_2	$-\frac{1}{2}u_2$	$-u_1$	0	0	0	$c_2 e_2 + \beta_1 u_2$
u_3	0	$-u_2$	$-u_1$	$-\beta_1 u_1$	$-c_2 e_2 - \beta_1 u_2$	0

Consider the following cases:

1.1°. $\lambda \neq \frac{1}{2}$

1.1.1°. $c_3 = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

1.1.2°. $\lambda = \frac{3}{5}$, $c_3 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}\pi(e_1) &= e_1, \\ \pi(e_2) &= \frac{1}{\sqrt[3]{c_3}} e_2, \\ \pi(e_3) &= \frac{1}{\sqrt[3]{c_3^2}} e_3, \\ \pi(u_1) &= \frac{1}{\sqrt[3]{c_3^4}} u_1, \\ \pi(u_2) &= \frac{1}{c_3} u_2, \\ \pi(u_3) &= \frac{1}{\sqrt[3]{c_3^2}} u_3.\end{aligned}$$

1.2°. $\lambda = \frac{1}{2}$.

1.2.1°. $\beta_1 = 0$, $c_2 = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

1.2.2°. $\beta_1 = 0$, $c_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, where $i = 3$ or $i = 4$, by means of the mapping $\pi : \bar{\mathfrak{g}}_i \to \bar{\mathfrak{g}}$, where

$$\pi(e_j) = e_j, \ j = 1, 2, 3,$$
$$\pi(u_k) = \frac{1}{\sqrt{|c_2|}} u_k, \ k = 1, 2, 3.$$

(if $c_2 > 0$ then $i = 3$, if $c_2 < 0$ then $i = 4$).

1.2.3°. $\beta_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \frac{1}{\beta_1} u_j, \ j = 1, 2, 3.$$

2°. $\lambda = \frac{3}{4}$. Then

$$\begin{aligned}
&[e_1, e_2] = \frac{1}{4} e_2, && && \\
&[e_1, e_3] = \frac{1}{2} e_3, && [e_2, e_3] = 0, && \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = \frac{3}{4} u_2, && [e_2, u_2] = u_1, && [e_3, u_2] = 0, \\
&[e_1, u_3] = \frac{1}{2} u_3 + p e_3, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\bar{\mathfrak{g}}^{(1/4)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1,$$
$$\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1, \quad \bar{\mathfrak{g}}^{(3/4)}(\mathfrak{h}) = \mathbb{R}u_2,$$
$$\bar{\mathfrak{g}}^{(1/2)}(\mathfrak{h}) = \mathbb{R}u_3 \oplus \mathbb{R}e_3,$$

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(7/4)}(\mathfrak{h}), \quad \text{and} \quad && [u_1, u_2] = 0, \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(3/2)}(\mathfrak{h}), && [u_1, u_3] = 0, \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(5/4)}(\mathfrak{h}), && [u_2, u_3] = 0.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{4}e_2$	$\frac{1}{2}e_3$	u_1	$\frac{3}{4}u_2$	$\frac{1}{2}u_3 + pe_3$
e_2	$-\frac{1}{4}e_2$	0	0	0	u_1	u_2
e_3	$-\frac{1}{2}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{3}{4}u_2$	$-u_1$	0	0	0	0
u_3	$-\frac{1}{2}u_3 - pe_3$	$-u_2$	$-u_1$	0	0	0

2.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

2.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6,\mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1,2,3,$$
$$\pi(u_j) = \frac{1}{p}u_j, \ j = 1,2,3.$$

3°. $\lambda = 0$. Then

$$\begin{aligned}
&[e_1, e_2] = e_2, && && \\
&[e_1, e_3] = 2e_3, && [e_2, e_3] = 0, && \\
&[e_1, u_1] = u_1, && [e_2, u_1] = pe_3, && [e_3, u_1] = 0, \\
&[e_1, u_2] = 0, && [e_2, u_2] = u_1 + qe_2, && [e_3, u_2] = pe_3, \\
&[e_1, u_3] = -u_3, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1.
\end{aligned}$$

Since the virtual structure q is primary, we have

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}u_1, && \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_2, \\
&\bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{R}e_3, && \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}u_3,
\end{aligned}$$

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), && \text{and} && [u_1, u_2] = a_2e_2 + \alpha_1u_1, \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), && && [u_1, u_3] = b_1e_1 + \beta_2u_2, \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}), && && [u_2, u_3] = \gamma_3u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$2e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	0	pe_3	$u_1 + 2pe_2$	u_2
e_3	$-2e_3$	0	0	0	pe_3	u_1
u_1	$-u_1$	$-pe_3$	0	0	$-pu_1$	0
u_2	0	$-u_1 - 2pe_2$	$-pe_3$	pu_1	0	$2pu_3$
u_3	u_3	$-u_2$	$-u_1$	0	$-2pu_3$	0

3.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

3.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, && \pi(u_1) = u_1, \\
&\pi(e_2) = pe_2, && \pi(u_2) = \frac{1}{p}u_2, \\
&\pi(e_3) = p^2e_3, && \pi(u_3) = \frac{1}{p^2}u_3.
\end{aligned}$$

4°. $\lambda = \frac{2}{3}$. Then

$$
\begin{aligned}
&[e_1, e_2] = \frac{1}{3}e_2, && && \\
&[e_1, e_3] = \frac{2}{3}e_3, && [e_2, e_3] = 0, && \\
&[e_1, u_1] = u_1, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = \frac{2}{3}u_2, && [e_2, u_2] = u_1, && [e_3, u_2] = 0, \\
&[e_1, u_3] = \frac{1}{3}u_3 + pe_2, && [e_2, u_3] = u_2, && [e_3, u_3] = u_1.
\end{aligned}
$$

Since the virtual structure q is primary, we have

$$
\begin{aligned}
&\bar{\mathfrak{g}}^{(1/3)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}u_3, \quad \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1, \\
&\bar{\mathfrak{g}}^{(2/3)}(\mathfrak{h}) = \mathbb{R}e_3 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_1,
\end{aligned}
$$

$$
\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(5/3)}(\mathfrak{h}), && \text{and} && [u_1, u_2] = 0, \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(4/3)}(\mathfrak{h}), && && [u_1, u_3] = 0, \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), && && [u_2, u_3] = \gamma_1 u_1.
\end{aligned}
$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{3}e_2$	$\frac{2}{3}e_3$	u_1	$\frac{2}{3}u_2$	$\frac{1}{3}u_3$
e_2	$-\frac{1}{3}e_2$	0	0	0	u_1	u_2
e_3	$-\frac{2}{3}e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{2}{3}u_2$	$-u_1$	0	0	0	$\gamma_1 u_1$
u_3	$-\frac{1}{3}u_3$	$-u_2$	$-u_1$	0	$-\gamma_1 u_1$	0

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$
\begin{aligned}
&\pi(e_i) = e_i, \; i = 1, 2, 3, \\
&\pi(u_1) = u_1, \\
&\pi(u_2) = u_2, \\
&\pi(u_3) = u_3 + \gamma_1 e_2.
\end{aligned}
$$

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Let $\lambda = \frac{3}{4}$. Consider the homomorphisms

$$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(5, \mathbb{R}), \ i = 1, 6,$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}|_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in basis $\{e_2, e_3, u_1, u_2, u_3\}$ of $\mathcal{D}\bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, 6$, are not conjugated, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ are not equivalent.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_7$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ are not equivalent.

Let $\lambda = \frac{1}{2}$. Consider the homomorphisms

$$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R}), \ i = 1, 3, 4, 5,$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}|_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in basis $\{e_2, e_3, u_1, u_2\}$ of $\bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$ are not conjugated, we conclude that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$, $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$, $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ are not equivalent to each other.

Thus the proof of the Proposition is complete.

Proposition 3.24. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-2e_3$	0	u_2	$2u_3$
e_2	e_2	0	0	0	u_1	u_2
e_3	$2e_3$	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	$-2u_3$	$-u_2$	$-u_1$	0	0	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-2e_3$	0	u_2	$2u_3$
e_2	e_2	0	0	0	u_1	u_2
e_3	$2e_3$	0	0	0	e_2	$e_1 + u_1$
u_1	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	$-e_2$	0	0	0
u_3	$-2u_3$	$-u_2$	$-e_1 - u_1$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on the generalized module 3.24 is equivalent to one of the following:*

$$C(e_1)=0,\quad C(e_2)=\begin{pmatrix}0&0&0\\p&0&0\\0&0&0\end{pmatrix},\quad C(e_3)=\begin{pmatrix}0&0&r\\0&p+r&0\\0&0&0\end{pmatrix}.$$

Proof. Put $C(e_i)=(c^i_{jk})_{1\leqslant j,k\leqslant 3}$, $1\leqslant i\leqslant 3$. Let us check conditions (6), Chapter II. Direct calculation shows that

$$C(e_1)=\begin{pmatrix}0&c^1_{12}&c^1_{13}\\c^1_{21}&c^1_{22}&c^1_{23}\\c^1_{31}&c^1_{32}&c^1_{33}\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&c^2_{12}&c^1_{12}\\c^2_{21}&c^1_{21}-c^1_{12}&\frac12(c^1_{22}-c^1_{12})\\0&\frac12 c^1_{31}&\frac13 c^1_{32}\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix}0&0&c^3_{13}\\0&c^3_{13}+c^2_{21}&c^1_{21}\\-2c^2_{12}&-c^1_{12}&\frac12 c^1_{31}-c^1_{13}\end{pmatrix}.$$

Put

$$H=\begin{pmatrix}c^2_{12}&c^1_{12}&\frac12 c^1_{13}\\c^1_{21}&\frac12 c^1_{22}&\frac13 c^1_{23}\\\frac12 c^1_{31}&\frac13 c^1_{32}&\frac14 c^1_{33}\end{pmatrix}$$

and $C_1(x)=C(x)+A(x)H-HB(x)$ for $x\in\mathfrak{g}$. Then

$$C_1(e_1)=0,\quad C_1(e_2)=\begin{pmatrix}0&0&0\\c^2_{21}+c^2_{12}&0&0\\0&0&0\end{pmatrix},$$

$$C_1(e_3)=\begin{pmatrix}0&0&c^3_{13}-c^2_{12}\\0&c^3_{13}+c^2_{21}&0\\0&0&0\end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.
This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type 3.24. Then it can be assumed that corresponding virtual pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is defined by the virtual structure determined in the Lemma. Then

$$\begin{aligned}
&[e_1,e_2]=-e_2,\\
&[e_1,e_3]=-2e_3,\quad [e_2,e_3]=0,\\
&[e_1,u_1]=0,\quad [e_2,u_1]=pe_2,\quad [e_3,u_1]=0,\\
&[e_1,u_2]=u_2,\quad [e_2,u_2]=u_1,\quad [e_3,u_2]=(p+r)e_2,\\
&[e_1,u_3]=2u_3,\quad [e_2,u_3]=u_2,\quad [e_3,u_3]=re_1+u_1.
\end{aligned}$$

Put

$$[u_1, u_2] = a_1e_1 + a_2e_2 + a_3e_3 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,$$
$$[u_1, u_3] = b_1e_1 + b_2e_2 + b_3e_3 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,$$
$$[u_2, u_3] = c_1e_1 + c_2e_2 + c_3e_3 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.$$

Using the Jacobi identity we see that the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) has the form:

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_2$	$-2e_3$	0	u_2	$2u_3$
e_2	e_2	0	0	0	u_1	u_2
e_3	$2e_3$	0	0	0	re_2	$re_1 + u_1$
u_1	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	$-re_2$	0	0	0
u_3	$-2u_3$	$-u_2$	$-re_1 - u_1$	0	0	0

Consider the following cases:

1°. $r = 0$.

Then the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) is equivalent to the trivial pair ($\bar{\mathfrak{g}}_1$,$\mathfrak{g}_1$).

2°. $r \neq 0$.

Then the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) is equivalent to the pair ($\bar{\mathfrak{g}}_2$,$\mathfrak{g}_2$) by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_2$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_i) = \frac{1}{r}u_i, \ i = 1, 2, 3.$$

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$, we see that the pairs ($\bar{\mathfrak{g}}_1$,$\mathfrak{g}_1$) and ($\bar{\mathfrak{g}}_2$,$\mathfrak{g}_2$) are not equivalent.

This proves the Proposition.

Proposition 3.25. *Any pair ($\bar{\mathfrak{g}}$, $\mathfrak{g}$) of type is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	0	$-u_1$	$-u_2$	0	0	0

2.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	e_2
u_3	0	$-u_1$	$-u_2$	0	$-e_2$	0

3.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	e_2
u_2	$-u_1$	0	0	0	0	e_3
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_3$	0

4.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	e_2
u_2	$-u_1$	0	0	0	0	$e_2 + e_3$
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_2 - e_3$	0

5.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	$-e_2$
u_2	$-u_1$	0	0	0	0	$-e_3$
u_3	0	$-u_1$	$-u_2$	e_2	e_3	0

6.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	$-e_2$
u_2	$-u_1$	0	0	0	0	$e_2 - e_3$
u_3	0	$-u_1$	$-u_2$	e_2	$-e_2 + e_3$	0

7.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	$\alpha e_2 + u_1$
u_2	$-u_1$	0	0	0	0	$\alpha e_3 + u_2$
u_3	0	$-u_1$	$-u_2$	$-\alpha e_2 - u_1$	$-\alpha e_3 - u_2$	0

8.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	$\alpha e_2 + u_1$
u_2	$-u_1$	0	0	0	0	$e_2 + \alpha e_3 + u_2$
u_3	0	$-u_1$	$-u_2$	$-\alpha e_2 - u_1$	$-e_2 - \alpha e_3 - u_2$	0

9.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	$-e_3$	$-u_1$	$-u_2$	0	0	0

10.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	e_2
u_3	$-e_3$	$-u_1$	$-u_2$	0	$-e_2$	0

11.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	e_2
u_2	$-u_1$	0	0	0	0	e_3
u_3	$-e_3$	$-u_1$	$-u_2$	$-e_2$	$-e_3$	0

12.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	e_2
u_2	$-u_1$	0	0	0	0	$e_2 + e_3$
u_3	$-e_3$	$-u_1$	$-u_2$	$-e_2$	$-e_2 - e_3$	0

13.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	$-e_2$
u_2	$-u_1$	0	0	0	0	$-e_3$
u_3	$-e_3$	$-u_1$	$-u_2$	e_2	e_3	0

14.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	u_2
u_1	0	0	0	0	0	$-e_2$
u_2	$-u_1$	0	0	0	0	$e_2 - e_3$
u_3	$-e_3$	$-u_1$	$-u_2$	e_2	$-e_2 + e_3$	0

15.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	$-e_2$	0	0	0	0	u_2
u_1	0	0	0	0	0	$\alpha e_2 + u_1$
u_2	$-u_1$	0	0	0	0	$\alpha e_3 + u_2$
u_3	$-e_3$	$-u_1$	$-u_2$	$-\alpha e_2 - u_1$	$-\alpha e_3 - u_2$	0

16.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_3
e_2	0	0	0	0	0	u_1
e_3	$-e_2$	0	0	0	0	u_2
u_1	0	0	0	0	0	$\alpha e_2 + u_1$
u_2	$-u_1$	0	0	0	0	$e_2 + \alpha e_3 + u_2$
u_3	$-e_3$	$-u_1$	$-u_2$	$-\alpha e_2 - u_1$	$-e_2 - \alpha e_3 - u_2$	0

17.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	u_2
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	$-e_1$	0	0	0
u_3	0	$-u_1$	$-u_2$	0	0	0

18.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	u_2
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	$-e_1$	0	0	e_2
u_3	0	$-u_1$	$-u_2$	0	$-e_2$	0

19.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	u_2
u_1	0	0	0	0	0	e_2
u_2	$-u_1$	0	$-e_1$	0	0	e_3
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_3$	0

20.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	u_2
u_1	0	0	0	0	0	e_2
u_2	$-u_1$	0	$-e_1$	0	0	$e_2 + e_3$
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_2 - e_3$	0

21.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	u_2
u_1	0	0	0	0	0	$-e_2$
u_2	$-u_1$	0	$-e_1$	0	0	$-e_3$
u_3	0	$-u_1$	$-u_2$	e_2	e_3	0

22.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	u_2
u_1	0	0	0	0	0	$-e_2$
u_2	$-u_1$	0	$-e_1$	0	0	$e_2 - e_3$
u_3	0	$-u_1$	$-u_2$	e_2	$-e_2 + e_3$	0

23.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	$e_1 + e_3$
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	$-e_3 + u_2$
u_1	0	0	0	0	0	$\alpha e_2+(1+\beta)u_1$
u_2	$-u_1$	0	0	0	0	$\alpha e_3+\beta u_2$
u_3	$-e_1-e_3$	$-u_1$	e_3-u_2	$-\alpha e_2-(1+\beta)u_1$	$-\alpha e_3-\beta u_2$	0

24.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	$e_1 + e_3$
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	$-e_3 + u_2$
u_1	0	0	0	0	0	$\alpha e_2+(1+\beta)u_1$
u_2	$-u_1$	0	0	0	0	$e_2+\alpha e_3+\beta u_2$
u_3	$-e_1-e_3$	$-u_1$	e_3-u_2	$-\alpha e_2-(1+\beta)u_1$	$-e_2-\alpha e_3-\beta u_2$	0

25.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_1
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	$-e_3+u_2$
u_1	0	0	0	0	0	$be_2+(1+\beta)u_1$
u_2	$-u_1$	0	0	0	0	$\alpha e_3+\beta u_2$
u_3	$-e_3$	$-u_1$	e_3-u_2	$-\alpha e_2-(1+\beta)u_1$	$-\alpha e_3-\beta u_2$	0

26.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_1
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	0	$-e_3+u_2$
u_1	0	0	0	0	0	$be_2+(1+\beta)u_1$
u_2	$-u_1$	0	0	0	0	$e_2+\alpha e_3+\beta u_2$
u_3	$-e_3$	$-u_1$	e_3-u_2	$-\alpha e_2-(1+\beta)u_1$	$-e_2-\alpha e_3-\beta u_2$	0

27.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_1
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	$-e_3+u_2$
u_1	0	0	0	0	0	αe_2+3u_1
u_2	$-u_1$	0	$-e_1$	0	0	αe_3+2u_2
u_3	$-e_1$	$-u_1$	e_3-u_2	$-\alpha e_2-3u_1$	$-\alpha e_3-2u_2$	0

28.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_1
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	$-e_3+u_2$
u_1	0	0	0	0	0	αe_2+3u_1
u_2	$-u_1$	0	$-e_1$	0	0	$e_2+\alpha e_3+2u_2$
u_3	$-e_1$	$-u_1$	e_3-u_2	$-\alpha e_2-3u_1$	$-e_2-\alpha e_3-2u_2$	0

29.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	e_1
e_2	0	0	0	0	0	u_1
e_3	e_2	0	0	0	e_1	$-e_3+u_2$
u_1	0	0	0	0	0	αe_2+3u_1
u_2	$-u_1$	0	$-e_1$	0	0	$-e_2+\alpha e_3+2u_2$
u_3	$-e_1$	$-u_1$	e_3-u_2	$-\alpha e_2-3u_1$	$e_2-\alpha e_3-2u_2$	0

30.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	e_2	u_1
e_3	e_2	0	0	e_2	$2e_3$	u_2
u_1	0	0	$-e_2$	0	$-u_1$	0
u_2	$-u_1$	$-e_2$	$-2e_3$	u_1	0	$2u_3$
u_3	0	$-u_1$	$-u_2$	0	$-2u_3$	0

31.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	e_2	u_1
e_3	e_2	0	0	e_2	e_1+2e_3	u_2
u_1	0	0	$-e_2$	0	$-u_1$	0
u_2	$-u_1$	$-e_2$	$-e_1-2e_3$	u_1	0	$2u_3$
u_3	0	$-u_1$	$-u_2$	0	$-2u_3$	0

Proof. Let $\mathcal{E}=\{e_1,e_2,e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad e_3=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}.$$

Then

$$A(e_1)=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\quad A(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad A(e_3)=\begin{pmatrix}0&0&0\\-1&0&0\\0&0&0\end{pmatrix},$$

and for $x\in\mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 3.25 is equivalent to one of the following:*

$$C(e_1)=\begin{pmatrix}0&0&p\\0&0&0\\0&0&r\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&0\\0&s&0\\0&0&0\end{pmatrix},\quad C(e_3)=\begin{pmatrix}0&t&0\\s&k&0\\0&2s&-p\end{pmatrix}.$$

Proof. Put

$$C(e_i)=(c^i_{jk})_{1\leqslant j,k\leqslant 3},\quad i=1,2,3.$$

Since for any virtual structure q condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1)=\begin{pmatrix}0&c^1_{12}&c^1_{13}\\c^1_{21}&c^1_{22}&c^1_{23}\\0&-c^1_{21}&c^1_{33}\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&c^1_{12}\\0&c^2_{22}&c^2_{23}\\0&0&-c^1_{21}\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix}0&c^3_{12}&c^3_{13}\\c^2_{22}+c^1_{12}&c^3_{22}&c^3_{23}\\0&2c^2_{12}&c^2_{23}-c^1_{13}-c^1_{22}\end{pmatrix}.$$

Put

$$H=\begin{pmatrix}c^1_{12}&c^3_{13}&0\\c^2_{23}&c^3_{23}&0\\-c^1_{21}&c^2_{23}-c^1_{22}&-c^1_{23}\end{pmatrix},$$

and $C_1(x)=C(x)+A(x)H-HB(x)$ for $x\in\mathfrak{g}$. Then

$$C_1(e_1)=\begin{pmatrix}0&0&c^1_{13}\\0&0&0\\0&0&c^1_{33}\end{pmatrix},\quad C_1(e_2)=\begin{pmatrix}0&0&0\\0&c^2_{22}&0\\0&0&0\end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} 0 & c_{12}^3 & 0 \\ c_{22}^2 & c_{22}^3 - c_{13}^3 & 0 \\ 0 & 2c_{22}^2 & -c_{13}^1 \end{pmatrix}.$$

By corollary 3, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.25. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{lll} [e_1, e_2] = 0, & & \\ [e_1, e_3] = e_2, & [e_2, e_3] = 0, & \\ [e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = se_2, \\ [e_1, u_2] = u_1, & [e_2, u_2] = se_2, & [e_3, u_2] = te_1 + ke_2 + 2se_3, \\ [e_1, u_3] = pe_1 + re_3, & [e_2, u_3] = u_1, & [e_3, u_3] = -pe_3 + u_2. \end{array}$$

Put

$$\begin{aligned} [u_1, u_2] &= a_1 e_1 + a_2 e_2 + a_3 e_3 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\ [u_1, u_3] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	pe_1+re_3
e_2	0	0	0	0	se_2	u_1
e_3	$-e_2$	0	0	se_2	te_1+2se_3	$-pe_3+u_2$
u_1	0	0	$-se_2$	0	$-\gamma_2 se_2 - su_1$	A
u_2	$-u_1$	$-se_2$	$-te_1-2se_3$	$\gamma_2 se_2+su_1$	0	B
u_3	$-pe_1-re_3$	$-u_1$	pe_3-u_2	$-A$	$-B$	0

where

$$\begin{aligned} A &= (rt + 2sp)e_1 + b_2 e_2 + (\gamma_2 + p)u_1, \\ B &= \gamma_1 s e_1 + c_2 e_2 + b_2 e_3 + \gamma_1 u_1 + \gamma_2 u_2 + 2su_3, \end{aligned}$$

$$\begin{cases} \gamma_2 t - 2pt = 0, \\ 2\gamma_2 s - 2sp - tr = 0, \\ rs = 0, \\ s(tr + 2sp) = 0, \\ \gamma_2 sp + 2\gamma_2^2 s - 3sb_2 = 0, \\ rt + 2\gamma_2 s + 4sp = 0. \end{cases}$$

The rest of the proof is similar to that of Proposition 2.13.

Proposition 3.26. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.26 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_1+e_3	0	0	u_1	0
e_2	$-e_1-e_3$	0	$-e_3$	0	u_2	u_2+u_3
e_3	0	e_3	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	$-u_2$	0	0	0	0
u_3	0	$-u_2-u_3$	$-u_1$	0	0	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_1+e_3	0	e_3	u_1	e_2
e_2	$-e_1-e_3$	0	$-e_3$	0	u_2	u_2+u_3
e_3	0	e_3	0	0	0	u_1
u_1	$-e_3$	0	0	0	0	u_2
u_2	$-u_1$	$-u_2$	0	0	0	0
u_3	$-e_2$	$-u_2-u_3$	$-u_1$	$-u_2$	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \quad A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on the generalized module 3.24 is equivalent to one of the following:*

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ p & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = C_1(e_3) = 0.$$

Proof. Put $C(e_i) = (c^i_{jk})_{1 \leqslant j,k \leqslant 3}$, $1 \leqslant i \leqslant 3$. Let us check condition (2), Chapter II. Direct calculation shows that

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ 0 & -c^1_{11} & c^1_{23} \\ c^1_{11} + c^1_{23} & c^1_{32} & c^1_{13} \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & c^2_{13} \\ 0 & c^2_{11} - c^1_{12} & c^2_{11} - c^1_{12} - c^1_{13} \\ c^1_{32} + 2c^2_{11} - c^1_{12} & c^2_{32} & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c_{11}^2 \\ 0 & 0 & -c_{11}^1 \\ c_{11}^1 & c_{12}^1 - c_{11}^2 & c_{32}^1 - c_{12}^1 + c_{11}^2 + c_{13}^1 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c_{11}^2 & \frac{1}{2}c_{12}^2 & \frac{1}{2}(c_{13}^2 - \frac{1}{2}c_{12}^2) \\ -c11^1 & c_{11}^2 - c_{12}^1 & -c_{13}^1 \\ c_{11}^2 - c_{12}^1 + c_{32}^1 & \frac{1}{2}(c_{32}^2 - \frac{1}{2}c_{12}^2) & \frac{1}{4}(c_{12}^2 - c_{13}^2 - c_{32}^2) + \frac{1}{2}c_{33}^2 \end{pmatrix},$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ c_{23}^1 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = C_1(e_3) = 0.$$

By corollary 2, Chapter II, the virtual structure C and C_1 are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type 3.26. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is defined by the virtual structure determined in the Lemma. Then

$$\begin{aligned}
&[e_1, e_2] = e_1 + e_2, \\
&[e_1, e_3] = 0, && [e_2, e_3] = -e_3, \\
&[e_1, u_1] = pe_3, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = u_1, && [e_2, u_2] = u_2, && [e_3, u_2] = 0, \\
&[e_1, u_3] = pe_2, && [e_2, u_3] = u_2 + u_3, && [e_3, u_3] = u_1.
\end{aligned}$$

Put

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1 + a_2 e_2 + a_3 e_3 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\
[u_1, u_3] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\
[u_2, u_3] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form:

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_1 + e_3$	0	pe_3	u_1	pe_2
e_2	$-e_1 - e_3$	0	$-e_3$	0	u_2	$u_2 + u_3$
e_3	0	e_3	0	0	0	u_1
u_1	$-pe_3$	0	0	0	0	pu_2
u_2	$-u_1$	$-u_2$	0	0	0	0
u_3	$-pe_2$	$-u_2 - u_3$	$-u_1$	$-pu_2$	0	0

Consider the following cases:

1°. $p = 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the trivial pair$(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

2°. $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_2$, where

$$\pi(e_i) = e_i, \ i = 1,2,3,$$
$$\pi(u_i) = \frac{1}{p}u_i, \ i = 1,2,3.$$

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$ we see that the pairs $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ are not equivalent. This proves the Proposition.

Proposition 3.27. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type is equivalent to one and only one of the following pairs:*

1.

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3-(1-\lambda)e_1$	0	0	u_1	0
e_2	$(1-\lambda)e_1 - e_3$	0	$(1-\lambda)e_3$	u_1	λu_2	$u_2+\lambda u_3$
e_3	0	$(\lambda-1)e_3$	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	0
u_2	$-u_1$	$-\lambda u_2$	0	0	0	0
u_3	0	$-u_2-\lambda u_3$	$-u_1$	0	0	0

2. $\lambda = \frac{1}{3}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3-\frac{2}{3}e_1$	0	0	u_1	0
e_2	$\frac{2}{3}e_1-e_3$	0	$\frac{2}{3}e_3$	u_1	$\frac{1}{3}u_2$	$u_2+\frac{1}{3}u_3$
e_3	0	$-\frac{2}{3}e_3$	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	0
u_2	$-u_1$	$-\frac{1}{3}u_2$	0	0	0	e_3
u_3	0	$-u_2-\frac{1}{3}u_3$	$-u_1$	0	$-e_3$	0

3. $\lambda = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - e_1$	0	0	u_1	e_3
e_2	$e_1 - e_3$	0	e_3	u_1	0	u_2
e_3	0	$-e_3$	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	$-e_3$	$-u_2$	$-u_1$	0	0	0

4. $\lambda = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - e_1$	0	0	u_1	pe_3
e_2	$e_1 - e_3$	0	e_3	u_1	0	u_2
e_3	0	$-e_3$	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	u_1
u_2	$-u_1$	0	0	0	0	u_2
u_3	$-pe_3$	$-u_2$	$-u_1$	$-u_1$	$-u_2$	0

5. $\lambda = \frac{1}{2}$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3-\frac{1}{2}e_1$	0	0	u_1	0
e_2	$\frac{1}{2}e_1-e_3$	0	$\frac{1}{2}e_3$	u_1	$e_1+\frac{1}{2}u_2$	$u_2+\frac{1}{2}u_3$
e_3	0	$-\frac{1}{2}e_3$	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	0
u_2	$-u_1$	$-e_1-\frac{1}{2}u_2$	0	0	0	0
u_3	0	$-u_2-\frac{1}{2}u_3$	$-u_1$	0	0	0

Proof. Let $\lambda = 1$. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is nilpotent endomorphism. Then by statement 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ for $\lambda = 1$.

In the sequel we assume that $\lambda \neq 1$.

Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & \lambda-1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; A(e_2) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1-\lambda \end{pmatrix}, \; A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda-1 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on the generalized module 3.27 is equivalent to one of the following:*

a) $\lambda = 0$

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_1(e_2) = C_1(e_3) = 0;$$

b) $\lambda = \frac{1}{2}$

$$C_2(e_1) = C_2(e_3)0, \quad C_2(e_2) = \begin{pmatrix} 0 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

c) $\lambda \notin \{0, \frac{1}{2}, 1\}$

$$C_3(e_1) = C_3(e_2) = 0 = C_3(e_2) = 0.$$

Proof. Put

$$C(e_i) = (c^i_{jk})_{1 \leqslant j,k \leqslant 3}, \quad i = 1, 2, 3.$$

Consider the following cases:

1°. $\lambda = 0$.

Since for any virtual structure q condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ 0 & c^1_{11} & 0 \\ -c^1_{11} & c^1_{32} & c^1_{33} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c^2_{12} & c^2_{13} \\ c^1_{11} & 0 & c^2_{23} \\ c^1_{12} - c^2_{23} & c^2_{32} & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c^1_{12} - c^2_{23} \\ 0 & 0 & c^1_{11} \\ c^1_{11} & c^2_{23} & c^2_{23} + c^1_{13} + c^1_{32} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^1_{12} - c^2_{23} & -c^2_{12} & -c^2_{13} - c^2_{12} \\ -c^1_{11} & c^2_{23} & c^1_{13} \\ c^1_{32} + c^2_{23} & -c^2_{32} - c^2_{12} & -c^2_{32} - 2c^2_{12} - c^2_{13} - c^2_{33} \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^1_{23} + c^1_{13} \end{pmatrix}, \quad C_1(e_2) = C_1(e_3) = 0.$$

By corollary 3, Chapter II, the virtual structures C and C_1 are equivalent.

2°. $\lambda = \frac{1}{2}$.

Since for any virtual structure q condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1) = \begin{pmatrix} \frac{1}{2}c^1_{22} & c^1_{12} & c^1_{13} \\ 0 & c^1_{22} & 0 \\ -c^1_{22} & c^1_{32} & -2c^1_{13} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} \frac{1}{2}(c^1_{12} - c^2_{22}) & c^2_{12} & c^2_{13} \\ c^1_{22} & c^2_{22} & 2c^2_{22} + c^1_{13} \\ c^1_{12} + \frac{1}{2}c^1_{32} & c^2_{32} & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c^1_{12} - c^2_{22} \\ 0 & 0 & c^1_{22} \\ \frac{1}{2}c^1_{22} & c^2_{22} & 2c^2_{32} + c^1_{32} + c^1_{13} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} -c^2_{22} - c^1_{12} & c^2_{32} & c^2_{33} - a \\ c^1_{22} & 2c^2_{22} & 2c^1_{13} \\ c^1_{32} + 2c^2_{22} & a & b \end{pmatrix}.$$

Now put $C_2(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_2(e_1) = C_2(e_3), \quad C_2(e_2) = \begin{pmatrix} 0 & c^2_{12} & c^2_{23} - c^2_{32} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By corollary 3, Chapter II, the virtual structures C and C_2 are equivalent.

3°. $\lambda \notin \{0, \frac{1}{2}, 1\}$.
Direct calculation shows that

$$C(e_1) = \begin{pmatrix} (1-\lambda)c^1_{22} & c^1_{12} & c^1_{13} \\ 0 & c^1_{22} & 0 \\ -c^1_{22} & c^1_{32} & \frac{1}{\lambda-1}c^1_{13} \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} \lambda c^1_{12}-(1-\lambda)c^1_{22} & c^2_{12} & c^2_{13} \\ c^1_{22} & c^2_{22} & \frac{1}{\lambda}c^2_{22}+\frac{\lambda}{1-\lambda}c^1_{13} \\ c^1_{12}+\lambda c^1_{32}+\frac{2\lambda-1}{\lambda}c^2_{22} & c^2_{32} & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c^1_{12}+\frac{\lambda-1}{\lambda}c^2_{22} \\ 0 & 0 & c^1_{22} \\ (1-\lambda)c^1_{22} & \frac{1-\lambda}{\lambda}c^2_{22} & \frac{1}{\lambda}c^2_{22}+c^1_{32}+c^1_{13} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^1_{12}-\frac{1-\lambda}{\lambda}c^2_{22} & \frac{1}{2\lambda-1}c^2_{12} & \frac{1}{2\lambda-1}(c^2_{13}+\frac{1}{1-2\lambda}c^2_{12}) \\ c^1_{22} & \frac{1}{\lambda}c^2_{22} & \frac{1}{1-\lambda}c^1_{13} \\ c^1_{32}+\frac{1}{\lambda}c^2_{22} & \frac{1}{2\lambda-1}(c^2_{32}+\frac{1}{1-2\lambda}c^2_{12}) & \frac{1}{2\lambda-1}(c^2_{33}+\frac{1}{1-2\lambda}(c^2_{13}+c^2_{32})) \end{pmatrix}$$

and $C_3(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_3(e_1) = C_3(e_2) = C_3(e_3) = 0.$$

By corollary 3, Chapter II, the virtual structures C and C_3 are equivalent.
This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.27. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Put

$$\begin{aligned} [u_1, u_2] &= a_1e_1 + a_2e_2 + a_3e_3 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3, \\ [u_1, u_3] &= b_1e_1 + b_2e_2 + b_3e_3 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3, \\ [u_2, u_3] &= c_1e_1 + c_2e_2 + c_3e_3 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3. \end{aligned}$$

Consider the following cases:

1°. $\lambda = 0$. Then

$$\begin{aligned} &[e_1, e_2] = e_3 - e_1, \\ &[e_1, e_3] = 0, \quad [e_2, e_3] = e_3, \\ &[e_1, u_1] = 0, \quad [e_2, u_1] = u_1, \quad [e_3, u_1] = 0, \\ &[e_1, u_2] = u_1, \quad [e_2, u_2] = 0, \quad [e_3, u_2] = 0, \\ &[e_1, u_3] = pe_3, \quad [e_2, u_3] = u_2, \quad [e_3, u_3] = u_1. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - e_1$	0	0	u_1	pe_3
e_2	$e_1 - e_3$	0	e_3	u_1	0	u_2
e_3	0	e_3	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	$\beta_1 u_1$
u_2	$-u_1$	0	0	0	0	$\beta_1 u_2$
u_3	$-pe_3$	$-u_2$	$-u_1$	$-\beta_1 u_1$	$-\beta_1 u_2$	0

1.1°. $\beta_1 = 0$.

1.1.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.1.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_3$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \frac{1}{p} u_j, \ j = 1, 2, 3.$$

Since the virtual structure $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ is trivial, and the virtual structure $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ is not trivial, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

1.2°. $\beta_1 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_4$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \frac{1}{\beta_1} u_j, \ j = 1, 2, 3.$$

It is possible to show that the pairs $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ with different values of parameter p are not equivalent.

2°. $\lambda = \frac{1}{2}$. Then

$$[e_1, e_2] = e_3 - \frac{1}{2} e_1,$$
$$[e_1, e_3] = 0, \quad [e_2, e_3] = \frac{1}{2} e_3,$$
$$[e_1, u_1] = 0, \quad [e_2, u_1] = u_1, \quad [e_3, u_1] = 0,$$
$$[e_1, u_2] = u_1, \quad [e_2, u_2] = re_1 + \frac{1}{2} u_2, \quad [e_3, u_2] = 0,$$
$$[e_1, u_3] = 0, \quad [e_2, u_3] = se_1 + u_2 + \frac{1}{2} u_3, \quad [e_3, u_3] = u_1.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - \frac{1}{2} e_1$	0	0	u_1	0
e_2	$\frac{1}{2} e_1 - e_3$	0	$\frac{1}{2} e_3$	u_1	$re_1 + \frac{1}{2} u_2$	$u_2 + \frac{1}{2} u_3$
e_3	0	$-\frac{1}{2} e_3$	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	0
u_2	$-u_1$	$-re_1 - \frac{1}{2} u_2$	0	0	0	$\gamma_1 u_1$
u_3	0	$-u_2 - \frac{1}{2} u_3$	$-u_1$	0	$-\gamma_1 u_1$	0

2.1°. $r = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ with $\lambda = \frac{1}{2}$.

2.2°. $r \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_5$, where

$$\pi(e_i) = e_i,\ i = 1, 2, 3,$$
$$\pi(u_1) = \frac{1}{r}u_1,$$
$$\pi(u_2) = -\frac{\gamma_1}{2}e_3 + \frac{1}{r}u_2,$$
$$\pi(u_3) = \frac{\gamma_1}{2}e_1 + \frac{1}{r}u_3.$$

Since the virtual structure $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ is trivial, and the virtual structure $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ is not trivial, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ are not equivalent.

3°. $\lambda \notin \{0, \frac{1}{2}, 1\}$.

$$\begin{array}{lll}
[e_1, e_2] = e_3 - (1-\lambda)e_1, & & \\
[e_1, e_3] = 0, & [e_2, e_3] = (1-\lambda)e_3, & \\
[e_1, u_1] = 0, & [e_2, u_1] = u_1, & [e_3, u_1] = 0, \\
[e_1, u_2] = u_1, & [e_2, u_2] = \lambda u_2, & [e_3, u_2] = 0, \\
[e_1, u_3] = 0, & [e_2, u_3] = u_2 + \lambda u_3, & [e_3, u_3] = u_1.
\end{array}$$

3.1°. $\lambda \notin \{0, \frac{1}{2}, \frac{1}{3}, 1\}$.

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

3.2°. $\lambda = \frac{1}{3}$.

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - \frac{2}{3}e_1$	0	0	u_1	0
e_2	$\frac{2}{3}e_1 - e_3$	0	$\frac{2}{3}e_3$	u_1	$\frac{1}{3}u_2$	$u_2 + \frac{1}{3}u_3$
e_3	0	$-\frac{2}{3}e_3$	0	0	0	u_1
u_1	0	$-u_1$	0	0	0	0
u_2	$-u_1$	$-\frac{1}{3}u_2$	0	0	0	c_3e_3
u_3	0	$-u_2 - \frac{1}{3}u_3$	$-u_1$	0	$-c_3e_3$	0

3.2.1°. $c_3 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

3.2.2°. $c_3 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_2$, where

$$\pi(e_1) = \frac{1}{c_3} e_1,$$
$$\pi(e_2) = e_2,$$
$$\pi(e_3) = \frac{1}{c_3} e_3,$$
$$\pi(u_1) = \frac{1}{c_3^2} u_1,$$
$$\pi(u_2) = \frac{1}{c_3} u_2,$$
$$\pi(u_3) = \frac{1}{c_3} u_3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_2$ we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Let $\lambda = 0$. Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(3, \mathbb{R})$, $i = 1, 3, 4$; where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\bar{\mathfrak{g}}_i} x$, $x \in \bar{\mathfrak{g}}_i$, in the basis $\{e_1, e_3, u_1, u_3\}$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, 3, 4$, are not conjugate, we see that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ for $i = 1, 3, 4$, are not equivalent to each other.

Similarly, we prove that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ (for $\lambda = 1/2$) and $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ are not equivalent.

This completes the proof of the Proposition.

Proposition 3.28. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.28 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - e_2$	$-e_3$	0	u_1	u_3
e_2	$e_2 - e_3$	0	0	0	0	u_2
e_3	e_3	0	0	0	0	u_1
u_1	0	0	0	0	0	0
u_2	$-u_1$	0	0	0	0	0
u_3	$-u_3$	$-u_2$	$-u_1$	0	0	0

2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - e_2$	$-e_3$	0	u_1	u_3
e_2	$e_2 - e_3$	0	0	e_3	$2e_3$	$2e_1 + u_2$
e_3	e_3	0	0	0	$-e_3$	u_1
u_1	0	$-e_3$	0	0	$-u_1$	0
u_2	$-u_1$	$-2e_3$	e_3	u_1	0	0
u_3	$-u_3$	$-2e_1 - u_2$	$-u_1$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \ A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on generalized module 3.28 is equivalent to one of the following:*

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 2p \\ 0 & 0 & 0 \\ p & r & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -p & 0 \end{pmatrix}.$$

Proof.

Put $C(e_i) = (c^i_{jk})_{1 \leqslant j,k \leqslant 3}$, $i = 1,2,3$. Since for any virtual structure q condition (2), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1) = \begin{pmatrix} 0 & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} 0 & 0 & 2c^2_{31} - 2c^1_{12} - c^2_{22} \\ -c^1_{12} & c^2_{22} & c^1_{22} - c^1_{13} - c^1_{21} \\ c^2_{31} & c^2_{32} & c^1_{22} + c^1_{13} + c^1_{32} - 2c^1_{21} - c^1_{31} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c^1_{12} \\ 0 & 0 & c^1_{21} \\ -c^1_{12} & c^2_{22} + c^1_{12} - c^2_{31} & c^1_{31} + c^1_{21} - c^1_{13} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^1_{12} & -c^2_{22} & c^1_{13} \\ c21^1 & c^1_{22} - c^1_{21} & \frac{1}{2}c^1_{23} \\ c^1_{31} + c^1_{21} & c^1_{32} + c^1_{22} + c^1_{31} - 2c^1_{21} & \frac{1}{4}c^1_{23} + \frac{1}{2}c^1_{33} \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = 0, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & 2(c^2_{31} - c^1_{12}) \\ 0 & 0 & 0 \\ c^2_{31} - c^1_{12} & c^2_{32} + c^2_{22} & 0 \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -(c^2_{31} - c^1_{12}) & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent. This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.28. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{lll} [e_1, e_2] = e_3 - e_2, & & \\ [e_1, e_3] = -e_3, & [e_2, e_3] = 0, & \\ [e_1, u_1] = 0, & [e_2, u_1] = pe_3, & [e_3, u_1] = 0, \\ [e_1, u_2] = u_1, & [e_2, u_2] = re_3, & [e_3, u_2] = -pe_3, \\ [e_1, u_3] = u_3, & [e_2, u_3] = 2pe_1 + u_2, & [e_3, u_3] = u_1. \end{array}$$

Put

$$\begin{aligned} [u_1, u_2] &= a_1 e_1 + a_2 e_2 + a_3 e_3 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\ [u_1, u_3] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$e_3 - e_2$	$-e_3$	0	u_1	u_3
e_2	$e_2 - e_3$	0	0	pe_3	$2pe_3$	$2pe_1 + u_2$
e_3	e_3	0	0	0	$-pe_3$	u_1
u_1	0	$-pe_3$	0	0	$-pu_1$	0
u_2	$-u_1$	$-2pe_3$	pe_3	pu_1	0	0
u_3	$-u_3$	$-2pe_1 - u_2$	$-u_1$	0	0	0

Consider the following cases:

1°. $p = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}_2$, where

$$\pi(e_i) = e_i, \; i = 1, 2, 3,$$
$$\pi(u_i) = \frac{1}{p} u_i, \; i = 1, 2, 3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

This proves the Proposition.

Proposition 3.29. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.29 is equivalent to one and only one of the following pairs:*

1.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$(1-\mu)e_2$	$e_2 + (1-\mu)e_3$	u_1	$u_1 + u_2$	μu_3
e_2	$(\mu - 1)e_2$	0	0	0	0	u_1
e_3	$(\mu - 1)e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_1 - u_2$	0	0	0	0	0
u_3	$-\mu u_3$	$-u_1$	$-u_2$	0	0	0

2. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$e_2 + e_3$	u_1	$u_1 + u_2$	0
e_2	$-e_2$	0	0	0	0	u_1
e_3	$-e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_1 - u_2$	0	0	0	0	e_2
u_3	0	$-u_1$	$-u_2$	0	$-e_2$	0

3. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$e_2 + e_3$	u_1	$u_1 + u_2$	0
e_2	$-e_2$	0	0	0	0	u_1
e_3	$-e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_1 - u_2$	0	0	0	0	$-e_2$
u_3	0	$-u_1$	$-u_2$	0	e_2	0

4. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$e_2 + e_3$	u_1	$u_1 + u_2$	0
e_2	$-e_2$	0	0	0	0	u_1
e_3	$-e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_1 - u_2$	0	0	0	0	$u_1 + \alpha e_2$
u_3	0	$-u_1$	$-u_2$	0	$-u_1 - \alpha e_2$	0

5. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$e_2 + e_3$	u_1	$u_1 + u_2$	0
e_2	$-e_2$	0	0	0	0	u_1
e_3	$-e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	$-e_2$
u_2	$-u_1 - u_2$	0	0	0	0	$\alpha e_2 - e_3 + \beta u_1$
u_3	0	$-u_1$	$-u_2$	e_2	$-\alpha e_2 + e_3 - \beta u_1$	0

, $\beta \geqslant 0$

6. $\mu = 0$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$e_2 + e_3$	u_1	$u_1 + u_2$	0
e_2	$-e_2$	0	0	0	0	u_1
e_3	$-e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	e_2
u_2	$-u_1 - u_2$	0	0	0	0	$\alpha e_2 + e_3 + \beta u_1$
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-\alpha e_2 - e_3 - \beta u_1$	0

, $\beta \geqslant 0$

7. $\mu = \frac{1}{2}$

[,]	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{1}{2}e_2$	$e_2 + \frac{1}{2}e_3$	u_1	$u_1 + u_2$	$\frac{1}{2}u_3 + e_3$
e_2	$-\frac{1}{2}e_2$	0	0	0	0	u_1
e_3	$-\frac{1}{2}e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_1 - u_2$	0	0	0	0	0
u_3	$-\frac{1}{2}u_3 - e_3$	$-u_1$	$-u_2$	0	0	0

Proof. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\mu & 1 \\ 0 & 0 & 1-\mu \end{pmatrix}, \ A(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ \mu-1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ \mu-1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with x.

Lemma. *Any virtual structure q on the generalized module 3.29 is equivalent to one of the following:*

a) $\mu = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & p \\ -p & 0 & 0 \\ 0 & -p & 0 \end{pmatrix}, \quad C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -p \end{pmatrix};$$

b) $\mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = C(e_3) = 0.$$

c) $\mu \notin \{0, \frac{1}{2}\}$.

$$C(e_i) = 0, \quad 1 \leqslant i \leqslant 3.$$

Proof. Put

$$C(e_i) = (c^i_{jk})_{1 \leqslant j,k \leqslant 3}, \quad 1 \leqslant i \leqslant 3.$$

Checking condition (6), Chapter II, we obtain:

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} \\ \mu c^2_{23} - c^3_{33} - (1-\mu)c^1_{13} & c^1_{22} & c^1_{23} \\ \mu c^2_{33} & c^2_{33} + \mu c^2_{23} + \mu c^3_{23} - c^1_{13} - \mu c^1_{22} & c^1_{33} \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} 0 & 0 & c^1_{11} \\ (1-\mu)c^1_{11} & (1-\mu)(c^1_{12} - c^1_{11}) & c^2_{23} \\ 0 & 0 & c^2_{33} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c^1_{12} - c^1_{11} \\ c^1_{11} & c^1_{12} - c^1_{11} & c^3_{23} \\ (1-\mu)c^1_{11} & (1-\mu)(c^1_{12} - c^1_{11}) & c^2_{23} + \mu c^3_{23} - c^1_{13} - c^1_{22} \end{pmatrix}.$$

Let $\mu = 0$. Then put

$$H = \begin{pmatrix} c^1_{11} & c^1_{12} - c^1_{11} & 0 \\ c^2_{23} & c^3_{23} & c^1_{33} - c^1_{23} \\ c^2_{33} & c^2_{23} - c^1_{22} & -c^1_{33} \end{pmatrix}$$

and $C_1 = C(x) + A(x)H - HB(x)$. We obtain

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ -c^1_{13} & 0 & 0 \\ 0 & -c^1_{13} & 0 \end{pmatrix}, \quad C_1(e_2) = 0, \quad C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c^1_{13} \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures C and C_1 are equivalent.
Similarly we obtain the other results of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 3.29. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\mu = 0$. Then

$$\begin{aligned}
&[e_1, e_2] = e_2, \\
&[e_1, e_3] = e_2 + e_3, && [e_2, e_3] = 0, \\
&[e_1, u_1] = u_1 - pe_2, && [e_2, u_1] = 0, && [e_3, u_1] = 0, \\
&[e_1, u_2] = u_1 + u_2 - pe_3, && [e_2, u_2] = 0, && [e_3, u_2] = 0, \\
&[e_1, u_3] = pe_1, && [e_2, u_3] = u_1, && [e_3, u_3] = u_2 - pe_3.
\end{aligned}$$

Now put

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1 + a_2 e_2 + a_3 e_3 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\
[u_1, u_3] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\
[u_2, u_3] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$e_2 + e_3$	u_1	$u_1 + u_2$	0
e_2	$-e_2$	0	0	0	0	u_1
e_3	$-e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	B
u_2	$-u_1 - u_2$	0	0	0	0	C
u_3	0	$-u_1$	$-u_2$	$-B$	$-C$	0 ,

where $B = b_2 e_2 + \beta_1 u_1$, $C = c_2 e_2 + b_3 e_3 + \gamma_1 u_1 + \beta_1 u_2$.

The mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, 2, 3, \\
\pi(u_1) &= u_1 + \frac{\beta_1}{2} e_2, \\
\pi(u_2) &= u_2 + \frac{\beta_1}{2}(e_2 + e_3), \\
\pi(u_3) &= u_3 - \frac{\beta_1}{2} e_1,
\end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}', \mathfrak{g}')$ and $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}'$ has the form:

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$e_2 + e_3$	u_1	$u_1 + u_2$	0
e_2	$-e_2$	0	0	0	0	u_1
e_3	$-e_3 - e_2$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	$b_2 e_2$
u_2	$-u_1 - u_2$	0	0	0	0	$\gamma_1 u_1 + c_2 e_2 + b_2 e_3$
u_3	0	$-u_1$	$-u_2$	$-e_2$	$-e_3 - \beta u_1 - \alpha e_2$	0

Consider the following cases:

1.1°. $b_2 = 0$.

1.1.1°. $\gamma_1 = 0$.

1.1.1.1°. $c_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

1.1.1.2°. $c_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ or $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, if $c_2 > 0$, or $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, if $c_2 < 0$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \sqrt{|c_2|}u_j, \ j = 1, 2, 3.$$

1.1.2°. $\gamma_1 \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \gamma_1 u_j, \ j = 1, 2, 3.$$

1.2°. $b_2 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ or $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, if $b_2 > 0$, or $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, if $b_2 < 0$, where

$$\pi(e_i) = e_i, \ i = 1, 2, 3,$$
$$\pi(u_j) = \sqrt{|b_2|}u_j, \ j = 1, 2, 3.$$

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R})$, $i = 1, \dots, 6$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{e_2, e_3, u_1, u_2\}$ of $\mathcal{D}\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, \dots, 6$, are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \dots, 6$, are not equivalent.

2°. $\mu = \frac{1}{2}$. Then

$$[e_1, e_2] = \frac{1}{2}e_2,$$
$$[e_1, e_3] = e_2 + \frac{1}{2}e_3, \quad [e_2, e_3] = 0,$$
$$[e_1, u_1] = u_1, \quad [e_2, u_1] = 0, \quad [e_3, u_1] = 0,$$
$$[e_1, u_2] = u_1 + u_2, \quad [e_2, u_2] = 0, \quad [e_3, u_2] = 0,$$
$$[e_1, u_3] = \frac{1}{2}u_3 + pe_3, \quad [e_2, u_3] = u_1, \quad [e_3, u_3] = u_2,$$

Put

$$[u_1, u_2] = a_1e_1 + a_2e_2 + a_3e_3 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,$$
$$[u_1, u_3] = b_1e_1 + b_2e_2 + b_3e_3 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,$$
$$[u_2, u_3] = c_1e_1 + c_2e_2 + c_3e_3 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

Consider the following cases:

2.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2°. $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ by means of the mapping $\pi\colon \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1, 2, 3$$
$$\pi(u_j) = pu_j,\ j = 1, 2, 3.$$

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R})$, $i = \overline{1, 7}$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{e_2, e_3, u_1, u_2\}$ of $\mathcal{D}\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i), i = \overline{1, 7}$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i), i = \overline{1, 7}$ are not equivalent.

3°. $\mu \notin \{0, \frac{1}{2}\}$. Then

$$[e_1, e_2] = \frac{1}{2}e_2,$$
$$[e_1, e_3] = e_2 + \frac{1}{2}e_3, \quad [e_2, e_3] = 0,$$
$$[e_1, u_1] = u_1, \quad [e_2, u_1] = 0, \quad [e_3, u_1] = 0,$$
$$[e_1, u_2] = u_1 + u_2, \quad [e_2, u_2] = 0, \quad [e_3, u_2] = 0,$$
$$[e_1, u_3] = \frac{1}{2}u_3, \quad [e_2, u_3] = u_1, \quad [e_3, u_3] = u_2.$$

Put

$$[u_1, u_2] = a_1e_1 + a_2e_2 + a_3e_3 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,$$
$$[u_1, u_3] = b_1e_1 + b_2e_2 + b_3e_3 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,$$
$$[u_2, u_3] = c_1e_1 + c_2e_2 + c_3e_3 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

This completes the proof of the Proposition.

Proposition 3.30. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.30 is trivial.*

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$-e_3$	0	u_1	u_2	$u_2 + u_3$
e_2	e_3	0	0	0	u_1	u_2
e_3	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0
u_3	$-u_2 - u_3$	$-u_2$	$-u_1$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where φ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Proposition 3.30. *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 3.30 is trivial.*

$[,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2	u_3
e_2	0	0	e_3	u_1	0	$-u_3$
e_3	0	$-e_3$	0	0	u_1	u_2
u_1	$-u_1$	$-u_1$	0	0	0	0
u_2	$-u_2$	0	$-u_1$	0	0	0
u_3	$-u_3$	$-u_3$	$-u_2$	0	0	0

Proof. Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.