HEAT DISPERSION EFFECT ON THERMAL CONVECTION IN ANISOTROPIC POROUS MEDIA

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ABSTRACT

The effect of hydrodynamic dispersion on the onset of thermal convection in flows through anisotropic porous media is studied theoretically. The porous layer is homogeneous and bounded by two infinite, perfectly conducting, impermeable horizontal planes kept at constant temperatures. Horizontal isotropy with respect to permeability and thermal diffusivity is assumed. A pressure-driven basic flow is considered in the limits of small and large Peclet numbers. The analysis shows that the onset of convection in both cases is independent of longitudinal dispersion, while dispersion in lateral directions has stabilizing effects. The preferred mode of disturbance consists of stationary rolls with axes aligned in the direction of the basic flow.

INTRODUCTION

Free convection in porous media is of considerable geophysical interest. This phenomenon may affect groundwater motions in areas with geothermal activity (Wooding, 1957). It has also important technical applications, since the occurrence of convection in porous insulation of buildings increases the loss of heat.

Hydrodynamic dispersion in porous media is important in the theory of miscible fluids. The same phenomenon can give interesting effects in buoyancy-driven convection, see Rubin (1974), Weber (1975) and Neischloss & Dagan (1975). The occurrence of a basic flow has been shown to have a stabilizing effect on the onset of convection in isotropic media.

The relative magnitude of hydrodynamic dispersion to molecular diffusivity in the fluid is an increasing function of the Peclet number. This is true for isotropic as well as anisotropic media with horizontal isotropy. The latter case is considered in this paper.

The stability problem for horizontally isotropic media has been studied by Castinel & Combarnous (1975) and Epherre (1975). In this paper the additional effect of hydrodynamic dispersion caused by a uniform basic flow is taken into account. This has geophysical relevance to groundwater flows in sediments, which usually have this type of anisotropy. Measurements show that the permeability along the plane of sedimentation is usually greater than the crosswise permeability, but the opposite can also be the case, see Bear (1972).

GOVERNING EQUATIONS

Consider a fluid saturated homogeneous porous medium bounded by two infinite, perfectly heat conducting, impermeable horizontal planes kept at constant temperatures. The planes are separated by a distance h and have a constant temperature difference ΔT where the lower plane is the warmer. The horizontal xy-plane is placed in the middle of the porous layer, and the z-axis is directed upwards. The unit vectors are denoted by $\vec{i}$, $\vec{j}$ and $\vec{k}$ in positive x-, y- and z-directions, respectively. The basic flow is in the $\vec{i}$-direction. The saturated porous medium is assumed to have one permeability K_H and one thermal diffusivity κ_{mH} horizontally, and another permeability K_V and thermal diffusivity κ_{mV} vertically.

By choosing the units of dimensionless length, time t, velocity $v = (u,v,w)$, temperature T and pressure p as

$$h,\ (c_p\rho)_m\ h^2/\lambda_{mV}, \kappa_{mV}/h, \Delta T,\ \frac{\rho_0 \nu \kappa_{mV}}{K_V} \tag{1}$$

respectively, the governing equations may be written in dimensionless form as

$$\vec{v} + \mathcal{K}\cdot(\nabla p - Ra\, T\vec{k}) = 0 \tag{2}$$

$$\nabla\cdot\vec{v} = 0 \tag{3}$$

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\nabla T = \nabla\cdot(\mathcal{D}\cdot\nabla T) \tag{4}$$

by utilizing Darcy's law, the Boussinesq approximation and assuming that the density is a linear function of the temperature. As for the derivation of the equations, we refer to Bear (1972) and Katto & Masuoka (1967).

The Rayleigh number is defined with respect to vertical quantities

$$Ra = \frac{K_V \, g \, \alpha \, \Delta T \, h}{\nu \kappa_{mV}} \tag{5}$$

In equations (1) - (5) c_p is the specific heat at constant pressure, ρ the density, ρ_0 a standard density, λ_{mV} the effective vertical thermal conductivity, ν the kinematic viscosity, $\mathcal{K}$ the dimensionless permeability tensor, $\mathcal{D}$ the dimensionless dispersion tensor, g the acceleration of gravity and α the coefficient of thermal volume expansion. The subscripts H and V refer to horizontal and vertical quantities respectively, f to the fluid, and m to the mixture of solid and fluid. The dimensionless permeability tensor introduced in (2) may be written

$$\mathcal{K} = \xi(\vec{i}\vec{i} + \vec{j}\vec{j}) + \vec{k}\vec{k} \tag{6}$$

where

$$\xi = K_H/K_V \tag{7}$$

In equation (2) is applied Darcy's law which requires that the grain Reynolds number

$$Re = \frac{V\ell}{\nu} \tag{8}$$

is of order 1 or less. V is a characteristic dimensional fluid velocity and ℓ a characteristic pore diameter.

The part of the dispersion tensor which is due to molecular diffusion may be written $\eta(\vec{i}\vec{i} + \vec{j}\vec{j}) + \vec{k}\vec{k}$ where we introduce the ratio

$$\eta = \kappa_{mH}/\kappa_{mV} \tag{9}$$

The dispersion tensor $\mathcal{D}$ is generally a function of the fluid velocity, the geometry of the medium and properties of the fluid. Poreh (1965) has derived a general expression for $\mathcal{D}$ by imposing restrictions based on symmetry considerations. This general expression reduces to relatively simple forms in the cases of small and large Peclet numbers, and these two cases will be considered explicitly in the next sections. The Peclet number should be defined as

$$Pe = \frac{V\ell}{\kappa_f} \tag{10}$$

where κ_f is the thermal diffusivity of the fluid.

THE STABILITY OF UNIFORM FLOW

a) Small Peclet numbers.

Poreh (1965) has obtained a dimensional expression for the dispersion tensor which, when $Pe << 1$, reduces to

$$\begin{aligned} \frac{\mathcal{D}}{\kappa_f} &= (\beta_1 + \beta_2 \frac{|\vec{v}|^2\ell^2}{\kappa_f^2})\mathcal{E} + \beta_3\ell^2 \frac{\vec{v}\vec{v}}{\kappa_f^2} \\ &+ (\beta_4+\beta_5 \frac{|\vec{v}|^2\ell^2}{\kappa_f^2})\vec{k}\vec{k} + \beta_6\ell^2|\vec{v}|(\vec{v}\vec{k} + \vec{k}\vec{v})/\kappa_f^2 \end{aligned} \tag{11}$$

Here $\beta_{1,2}$ represent molecular diffusion and β_i $(i = 3,4,5,6)$ are dispersion coefficients. $\mathcal{E}$ is the unit tensor and $\vec{k}$ is introduced as the unit vector along the symmetry axis in a horizontally isotropic medium. We rewrite (11) in a dimensionless form and introduce new dispersion coefficients γ_i $(i = 1,2,3,4)$.

$$\begin{aligned} \mathcal{D} &= (\eta + \gamma_2|\vec{v}|^2)\mathcal{E} + (\gamma_1-\gamma_2)\vec{v}\vec{v} \\ &+ (1 - \eta + (\gamma_3-\gamma_2)|\vec{v}|^2)\vec{k}\vec{k} + \gamma_4 |\vec{v}|(\vec{v}\vec{k}+\vec{k}\vec{v}) \end{aligned} \tag{12}$$

The relations between the old and the new dispersion coefficients are

$$\beta_2 = \gamma_2 \frac{h^2}{\ell^2}\frac{\kappa_f}{\kappa_{mV}} \qquad \beta_3 = (\gamma_1-\gamma_2)\frac{h^2}{\ell^2}\frac{\kappa_f}{\kappa_{mV}}$$

$$\beta_5 = (\gamma_3-\gamma_2)\frac{h^2}{\ell^2}\frac{\kappa_f}{\kappa_{mV}}, \qquad \beta_6 = \gamma_4 \frac{h^2}{\ell^2}\frac{\kappa_f}{\kappa_{mV}} \tag{13}$$

Our choice of dispersion coefficients proves convenient for the case of uniform flow $\vec{v}_s = U\vec{i}$. Then

$$\mathcal{D}_s = \eta(\vec{i}\vec{i}+\vec{j}\vec{j}) + \vec{k}\vec{k} + (\gamma_1\vec{i}\vec{i} + \gamma_2\vec{j}\vec{j} + \gamma_3\vec{k}\vec{k})U^2 \tag{14}$$

The dispersion coefficients γ_i are functions of ξ beside being functions of the angle between $\vec{v}$ and $\vec{k}$. γ_4 is zero for uniform flow.

In an isotropic medium the dispersion coefficients, γ_1 γ_2 and γ_3 are always positive, see Bear (1969). A change from isotropy to anisotropy must be continuous. Then γ_1, γ_2, γ_3 must all be positive when the medium has permeability in all directions ($0 < \xi < \infty$). If not, the dispersion would vanish in certain directions for certain values of ξ, which is physically unacceptable.

Consider the basic state (subscript s)

$$\vec{v}_s = U\vec{i}$$

$$T_s = T_0 - z \tag{15}$$

where U is constant and T_0 is a given temperature. Assume that $Pe = U\frac{\kappa_{mV}}{\kappa_f}\frac{\ell}{h} << 1$.

The basic flow is driven by a uniform pressure gradient. The total velocity is written

$$\vec{v} = U\vec{i} + \vec{v}' \tag{16}$$

where $\vec{v}'$ is a small perturbation. The cosine to the angle between $\vec{v}$ and the vertical may be written

$$\cos(\vec{v},\vec{k}) = \frac{w'}{U} \tag{17}$$

valid to the first order. According to Poreh (1965) γ_1, γ_2 and γ_3 are even functions of $\cos(\vec{v},\vec{k})$. Consequently they enter our linear theory as constants. γ_4, however, is an odd function of $\cos(\vec{v},\vec{k})$

$$\gamma_4 = A\,\frac{w'}{U} \tag{18}$$

valid to the first order. Here $A = A(\xi)$ and $A(1) = 0$.

We further write for the temperature and pressure

$$T = T_s + \theta'$$
$$p = p_s + p' \tag{19}$$

where θ' and p' are small perturbations.

Similar to the isotropic case, it is easily shown that the velocity is a poloidal vector (Weber, 1975). The perturbation velocity can be expressed by a single scalar function ψ:

$$\vec{v}' = \nabla\times(\nabla\times\vec{k}\,\psi) \tag{20}$$

or

$$(u',v',w') = (\psi'_{xz},\psi'_{yz},-\nabla_1^2\psi) \tag{21}$$

where $\nabla_1{}^2$ is the two-dimensional Laplacian. From (2) we then obtain

$$\theta' = -\frac{1}{Ra}\left(\nabla_1{}^2+\frac{1}{\xi}\,\frac{\partial^2}{\partial z^2}\right)\psi \tag{22}$$

By introducing this into the heat equation (4) and utilizing (12) we finally obtain

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} - c_1\frac{\partial^2}{\partial x^2} - c_2\frac{\partial^2}{\partial y^2} - c_3\frac{\partial^2}{\partial z^2}\right)\left(\nabla_1^2 + \frac{1}{\xi}\frac{\partial^2}{\partial z^2}\right)\psi$$
$$= Ra\left[\nabla_1^2 - (\gamma_1 - \gamma_2 + A)U\frac{\partial}{\partial x}\nabla_1^2 + 2\gamma_3 U\frac{\partial^3}{\partial x\partial z^2}\right]\psi \qquad (23)$$

where

$$c_1 = \eta + \gamma_1 U^2, \quad c_2 = \eta + \gamma_2 U^2, \quad c_3 = 1 + \gamma_3 U^2 \qquad (24)$$

The requirements of impermeable, perfectly conducting boundaries lead to the boundary conditions

$$\psi = \left(\nabla_1^2 + \frac{1}{\xi}\frac{\partial^2}{\partial z^2}\right)\psi = 0 \quad \text{for} \quad z = \pm\tfrac{1}{2} \qquad (25)$$

We seek solutions of (23) in the form

$$\psi = F(z)\, e^{i(kx+\ell y)+\sigma t} \qquad (26)$$

where σ may be complex. The boundary conditions for F become

$$F(\pm\tfrac{1}{2}) = F''(\pm\tfrac{1}{2}) = 0 \qquad (27)$$

It is easily seen that the most unstable mode is

$$F(z) = \cos\pi z \qquad (28)$$

The Rayleigh number for the onset of convection is

$$Ra = \left(1 + \frac{\pi^2}{\xi(k^2+\ell^2)}\right)(c_1 k^2 + c_2\ell^2 + c_3\pi^2) \qquad (29)$$

The preferred mode of disturbance is the one which makes Ra an absolute minimum. Minimizing (29) with respect to the wave number k and ℓ, we get two alternatives, depending on the relative magnitude of c_1 to c_2 or equivalently, γ_1 to γ_2. Unfortunately no numerical values of the dispersion coefficients are known in the case of anisotropy. For isotropy the longitudinal dispersion is always considerably greater than the lateral dispersion, see Bear (1969) or Fried & Combarnous (1971). In the present case γ_1 represents the longitudinal and γ_2 the horizontal lateral dispersion. It seems very plausible that the ratio of γ_1 to γ_2 is not significantly affected by the vertical anisotropy. We therefore assume $\gamma_1 > \gamma_2$ and get the critical Rayleigh number

$$Ra_c = \pi^2\left[\sqrt{\frac{\eta+\gamma_2 U^2}{\xi}} + \sqrt{1+\gamma_3 U^2}\,\right]^2 \qquad (30)$$

It is seen from this that the dispersion caused by the basic flow in y- and z-direction gives an increase in the critical Rayleigh number. In the discussion above it is assumed that γ_1, γ_2 and γ_3 in (29) can be interpreted in the way that they enter $\mathscr{D}_s$ (14). This is easily justified by writing $\mathscr{D} = \mathscr{D}_s + \mathscr{D}'$ and checking that $\mathscr{D}'$ gives no contribution to (29).

The values of the wave numbers corresponding to (30) are

$$k_c = 0\ , \qquad \ell_c = \pi\left[\frac{1+\gamma_3 U^2}{\xi(\eta+\gamma_2 U^2)}\right]^{\frac{1}{4}} \qquad (31)$$

The preferred mode consists of rolls with axes aligned in the direction of the basic flow (longitudinal rolls). The dispersion coefficients have opposite effects in (31); γ_2 reduces the critical wave number while γ_3 increases it. This is in contrast to the isotropic case, where the dispersion coefficients do not influence the width of the convection cells, see Weber (1975).

For the case of isotropy, i.e. $\xi = \eta = 1$, formula (30) reduces to the formula given by Weber (1975, eq. 20).

Our results also yield the critical Rayleigh number for fluid at rest $(U = 0)$ in an anisotropic medium

$$Ra_c{}^0 = \pi^2\left(\sqrt{\frac{\eta}{\xi}} + 1\right)^2 \tag{32}$$

for the overall wave number

$$\alpha_c = \pi(\xi\eta)^{-\frac{1}{4}} \tag{33}$$

$(\alpha^2 \equiv k^2+\ell^2)$. These results have been obtained by Epherre (1975).

b) Large Peclet numbers

We now examine the stability of uniform flow when $Pe \gg 1$. The relevant dimensional form of the dispersion tensor is found along the lines of Poreh (1965). The only requirement needed is that molecular diffusion is negligible compared to hydrodynamic dispersion. Then

$$\frac{\mathcal{D}}{\ell|\vec{v}|} = \varepsilon_1 \mathcal{E} + \varepsilon_2 \frac{\vec{v}\vec{v}}{|\vec{v}|^2} + \varepsilon_3 \vec{k}\vec{k} + \varepsilon_4(\vec{v}\vec{k}+\vec{k}\vec{v})/|\vec{v}| \tag{34}$$

Our analysis is restricted to Darcian flows, which implies that ε_i in (34) are independent of the Reynolds number.

We rewrite (34) in a dimensionless form which is relevant to our problem

$$\mathcal{D} = \alpha_2|\vec{v}|\mathcal{E} + (\alpha_1-\alpha_2)\frac{\vec{v}\vec{v}}{|\vec{v}|} + (\alpha_3-\alpha_2)|\vec{v}|\vec{k}\vec{k} + \alpha_4(\vec{v}\vec{k}+\vec{k}\vec{v}) \tag{35}$$

The relations between the old and the new dispersion coefficients are

$$\varepsilon_1 = \alpha_2\frac{h}{\ell}, \quad \varepsilon_2 = (\alpha_1-\alpha_2)\frac{h}{\ell}, \quad \varepsilon_3 = (\alpha_3-\alpha_2)\frac{h}{\ell}, \quad \varepsilon_4 = \alpha_4\frac{h}{\ell} \tag{36}$$

Our dispersion coefficients α_1, α_2 and α_3 are even functions and α_4 an odd function of $\cos(\vec{v},\vec{k})$. The dispersion coefficients α_i are also functions of ξ. The velocity is now written

$$\vec{v} = U\,\vec{i} + \vec{v}' \tag{37}$$

where $\vec{v}'$ is a small perturbation. Now $Pe = U \frac{\kappa_m V}{\kappa_F} \frac{\ell}{h} >> 1$ so that the tensor form (35) is valid. α_1, α_2 and α_3 are constant in our linear theory, while α_4 may be written

$$\alpha_4 = B\,\frac{w'}{U} \tag{38}$$

where $B = B(\xi)$ and $B(1) = 0$. The dispersion tensor for uniform flow $(\vec{v}_s = U\vec{i})$ is

$$\mathcal{D}_s = (\alpha_1 \vec{i}\vec{i} + \alpha_2 \vec{j}\vec{j} + \alpha_3 \vec{k}\vec{k})\,|U| \tag{39}$$

In the same way as for the small Peclet number case,we may now argue that $\alpha_1 > \alpha_2$.

The conditions for neglecting molecular diffusion may be expressed as

$$\alpha_1 |U| >> n \ , \ \alpha_2 |U| >> n \ , \ \alpha_3 |U| >> 1 \tag{40}$$

being more precise than the statement $Pe >> 1$.

A perturbation analysis may be performed by assuming the tensor form (35). The procedure is similar to that of section a) and yields a critical Rayleigh number

$$Ra_c = \pi^2 \left(\sqrt{\frac{\alpha_2 |U|}{\xi}} + \sqrt{\alpha_3 |U|}\right)^2 \tag{41}$$

for the wave numbers

$$k_c = 0 \ , \qquad \ell_c = \left(\frac{\alpha_3}{\xi\alpha_2}\right)^{\frac{1}{4}} \tag{42}$$

The preferred mode consists of longitudinal rolls. Comparing (41) and (40) with (32), we see that the dispersion provides an effective delay of the onset of convection in the case of large Peclet numbers.

SUMMARY AND CONCLUDING REMARKS

We have shown that the dispersion caused by a uniform basic flow U acts stabilizing as it delays the onset of thermal convection in an anisotropic porous medium with horizontal isotropy. The qualitative results are the same for small and large Peclet numbers, but the stabilizing effect is much stronger in the latter case. In both cases the most unstable mode consists of stationary rolls with axes aligned in the direction of the basic flow. It is only the dispersion in directions normal to the basic flow that influences the stability. The dispersion caused by U also affects the size of the convection cells. In the case of large Peclet numbers, which gives largest effect, the width of a roll is increased if $\kappa_{mH}/\kappa_{mV} < \alpha_2/\alpha_3$ and reduced if $\kappa_{mH}/\kappa_{mV} > \alpha_2/\alpha_3$, compared to convection without basic flow.

For heat dispersion in water, which is the most relevant geophysical example, the Peclet number will always be small for Darcian flows. Accordingly dispersion effects will be very small. For convection driven by a solute gradient, however, which may be relevant in connection with waste or fertilizer migrations in soils, dispersion effects will be much more significant even at low Reynolds numbers. This is due to the much smaller molecular diffusivities and thereby much larger Peclet numbers involved in these problems. The formulation of a stability problem is quite analogous to the present one, and leads to the same results.

ACKNOWLEDGEMENT

This work was done as a part of the author's cand.real.thesis at the University of Oslo. He is grateful to Dr. J.E. Weber and Professor E. Palm for valuable discussions during the preparation of this paper.

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