The effect of viscosity on long capillary-gravity waves

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The effect of viscosity on capillary-gravity waves is examined by a long wave expansion. The phase velocity differs little from that for an inviscid fluid unless the depth is close to the boundary layer thickness. Numerical results for water are in agreement with measurements.

In a recent paper by Walbridge and Woodward [1)] measurements on surface capillary-gravity waves in shallow water are reported. The waves are generated by a harmonically vibrating wave maker and for a certain frequency, (20 Hz), the wavelength for waves on water layers of different depths is measured. Their experimental values agree, within the error of their measurements, with those derived from the inviscid dispersion relation down to depths of about 0.012 cm. For smaller depths, however, they observed a lower wavelength than predicted by the inviscid theory, and they suggested that this decrease may be due to the effect of viscosity.

The damping effect of viscosity on capillary-gravity waves on deep water was studied theoretically many years ago [2)]. To our knowledge the case when the depth becomes of the order of the bottom boundary layer thickness as in the experiments quoted above, has not been treated. For long waves, as in the case reported in Ref. 1) a solution to the equations of motion which include the effect of viscosity can easily be found, and we will proceed to show that in agreement with the observations viscosity affects the phase velocity of the waves.

Consider two-dimensional wave motion with frequency ω on a layer of an incompressible viscous fluid with kinematic viscosity ν, and density ρ, which is laying on a horizontal plane. The mean layer thickness is h. A coordinate system $(\tilde{x},\tilde{z})$ with axes along the plane and normal to the plane, respectively, is introduced and the velocity components along these axes are $\tilde{u}$ and $\tilde{w}$. The surface deflection is described by $\tilde{\zeta} = h + \tilde{\eta}(\tilde{x},\tilde{t})$, where $\tilde{t}$ is time. With a given frequency ω the dispersion relation for capillary-gravity waves on an inviscid fluid layer determines the dimensionless wavenumber α

$$\alpha = 2\pi \frac{h}{\lambda} ,$$

where λ denotes the wavelength. For the smallest depth treated in Ref. 1), $\alpha = 0.12$, so that in this case a long wave expansion is appropriate. Consequently we assume $\alpha \ll 1$ and we define dimensionless variables (without a tilda) with the following transformation; $\tilde{u} = c_s u$, $\tilde{w} = \alpha c_s w$, $\tilde{x} = \frac{h}{\alpha} x$, $\tilde{z} = hz$, $\tilde{t} = \frac{h}{\alpha c_s} t$, $\tilde{p} = \rho c_s{}^2 p$, $\tilde{\zeta} = h\zeta$, where $\tilde{p}$ denotes the pressure and $c_s = \sqrt{gh}$. We also assume the amplitude of the motion which is determined by the amplitude of the wave maker to be of order ε, where $\varepsilon \ll 1$. Hence the equations of motion can be written :

$$u_t = -p_x + \frac{1}{\alpha R}(u_{zz} + \alpha^2 u_{xx}) + O(\varepsilon^2), \tag{1a}$$

$$(p - \frac{\alpha}{R} w_z + z)_z = O(\alpha^2 \varepsilon), \tag{1b}$$

where a variable as index denotes differentiation with respect to that variable and

$$R = \frac{c_s h}{\nu}$$

is a Reynolds number. If terms of order $\alpha^2\varepsilon$ are neglected, Eq.(1b) shows that $p - \frac{\alpha}{R} w_z + z$ is determined by the dynamical boundary conditions at $z = \zeta$. In the case of a free surface these must express the continuity in normal stress component and vanishing of the tangential stress component. By neglecting higher order terms in α and ε in these conditions we obtain

$$p - \frac{\alpha}{R} w_z + z = \frac{\alpha}{R} w_z\Big|_{z=1} + \eta - \alpha^2 W\eta_{xx}, \tag{2}$$

and

$$u_z + \alpha^2 w_x = 0 \quad \text{at} \quad z = 1. \tag{3}$$

In (2) a Weber number,

$$W = \frac{T}{\rho g h^2} ,$$

where T denotes the surface tension, is introduced. We assume $\alpha^2 W$ to be at least of order unity which implies that capillarity is at least of the same importance as gravity. This is for example the case for the range of α and h treated in Ref. 1). The boundary condition at the bottom plane is

$$u = 0 \quad \text{at} \quad z = 0, \tag{4}$$

and conservation of mass requires

$$\eta_t = - \int_0^1 u_x dz \; . \tag{5}$$

Eq. (1a) with the conditions (2), (3), (4), and (5) can easily be solved. We assume a solution of the form

$$u = U(z) \exp i(\beta x - \sigma t) \tag{6}$$

where β and σ are constants and i is the imaginary unit. Hence for given values of σ, R and W we have an eigenvalue problem with β as the complex eigenvalue. According to the definition of α, $\beta = 1$ for an inviscid fluid. With $\text{Im}\beta > 0$ the solution (6) represents a spatially damped wave solution. On the other hand, for a given wavenumber an eigenvalue problem leading to a complex eigenvalue frequency could equally have been posed. However, for weakly spatially damped waves, properties for temporally damped waves can be obtained from the solution (6) by similar arguments as given in Ref. 3).

The complex dispersion relation and the eigenfunction $U(z)$ are given by lengthy algebraic expressions. Therefore only the result when terms of order α^2 are neglected will be given here :

$$\sigma^2 = \beta^2(1 + \beta^2\alpha^2 W)(1 - \frac{\mathrm{Tan}\,\Omega}{\Omega}), \qquad (7)$$

$$U(z) = A\left[1 - \frac{\exp(-\Omega z) + \exp(\Omega(z-2))}{1 + \exp(-2\Omega)}\right], \qquad (8)$$

where $\Omega = (1-i)\frac{h}{h_b}$; $h_b = (\frac{2\nu}{\omega})^{\frac{1}{2}}$, the thickness of the boundary layer at the bottom; and A is an integration constant. For $h_b \to 0$ (7) reduces to the wellknown dispersion relation from inviscid surface wave theory.

For $|\Omega| >> 1$ $(h >> h_b)$ Eq (7) can be solved approximately. With an ordinary perturbation technique we find

$$\beta = 1 + \frac{1+i}{4}\,\frac{1+\alpha^2 W}{1+2\alpha^2 W}\,\frac{h_b}{h} + O(\frac{h_b}{h})^2. \qquad (9)$$

Eq. (9) shows that for a given frequency the effect of viscosity leads to a reduction of the wavelength (or equivalently a reduction in the phase velosity) compared to the inviscid values. This effect of viscosity is most pronounced for wave motion dominated by gravity i.e. $\alpha^2 W << 1$.

In order to find the relation between frequency and wavelength i in cases where h is of order h_b we have solved Eq. (9) numerically. This is done for water with ω in the range 10 - 40 Hz.

Since β is found to vary little with frequency, only the results for two cases, $\omega = 20$ Hz, and $\omega = 10$ Hz is given in Figure 1. It is a striking feature of these graphs that the viscous correction to the wavelength is small unless h becomes close to h_b, and that the magnitude of the correction increases rapidly for $h \lesssim h_b$.

If we assume that the approximations used here are valid at least at some distance away from the wave maker our result for $\omega = 20$ Hz could be compared with the observation in Ref. 1). It is found that the viscous correction to the wavelength for values of h down to h_b is within the error of their measurement. For the smallest value of h ($h \cong 0.5\ h_b$) reported in Ref. 1) the computed wavelength seems to be somewhat less than the measured value. The reason for this discreapancy for the smallest depth might be due to non-linear effects, for in an experiment where the amplitude of the wave generator is kept constant for the various depths treated (which seems to be the case in the quoted experiments) non-linear effects will obviously be more pronounced at the smallest depth.

Finally it should be noted that the condition (5) requires that the mean volume flux with respect to time is constant in the x-direction. Therefore the mean volume flux produced by the waves must compensate that due to viscous effects. This implies that an adjustment of the mean layer thickness occurs and also that a circulation (of order ε^2) is generated. This circulation can be computed from the solution given above. Observation of the circulation pattern is obviously a difficult task and has, to our knowledge, not been reported.

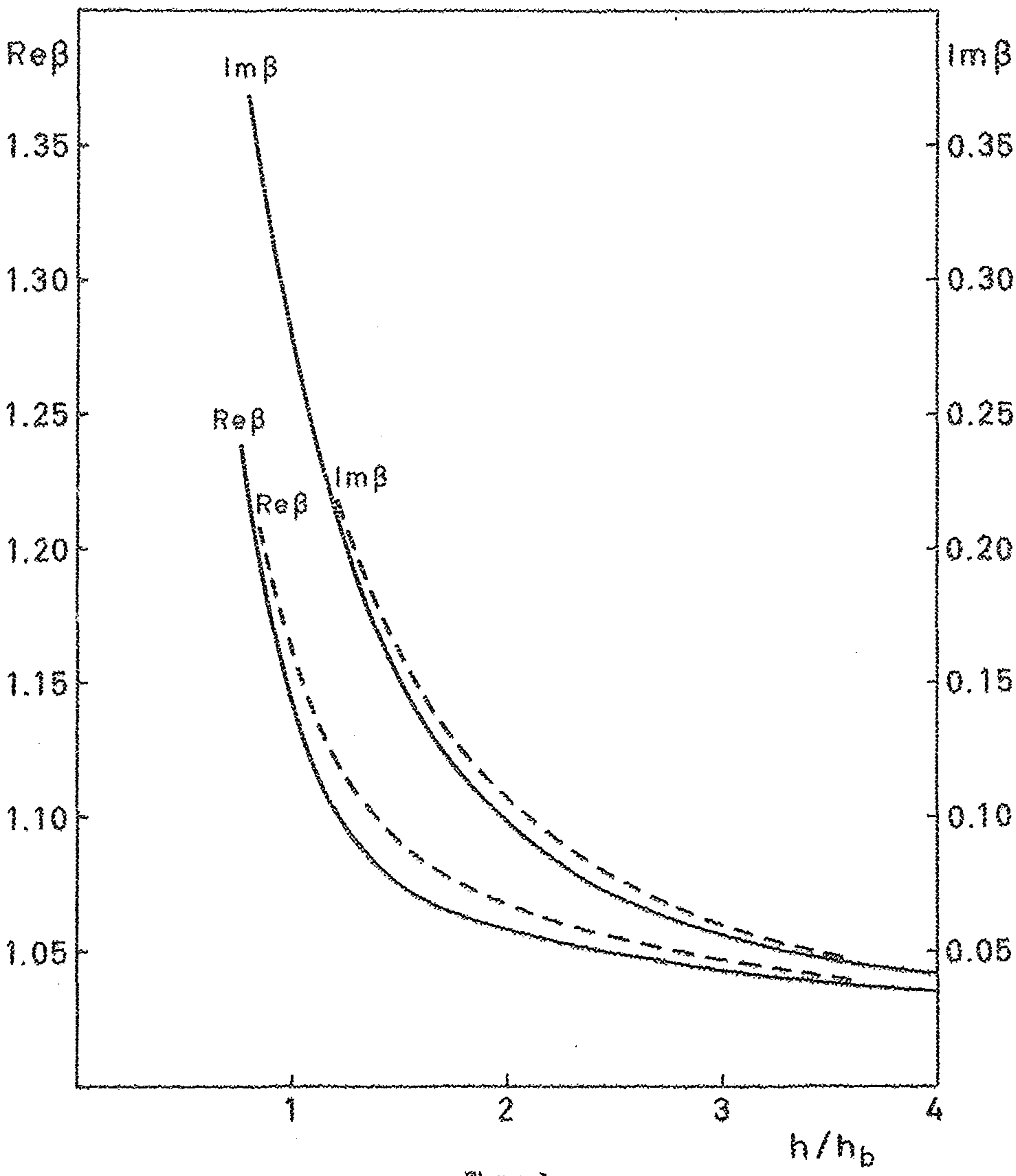
References and Table Caption

1) N.L.Walbridge and L.A.Woodward, Phys.Fluids 13,10,2461,(1970)

2) H.Lamb, Hydrodynamics, pp. 626, 6 ed. Dover, New York

3) M.Gaster, J.Fluid Mech. 14, 222, (1962)

Figure 1 The complex eigenvalue β for water, with $\nu = 0.01$ cm^2/s, and $T = 75$ g/cm^2, as function of the ratio h/h_b. Solid curves; $\omega = 10$ Hz, $h_b = 0.013$ cm. Dotted curves; $\omega = 20$ Hz, $h_b = 0.018$ cm.



Reβ
Imβ
Imβ
Reβ
Imβ
Reβ
1.35
1.30
1.25
1.20
1.15
1.10
1.05
0.35
0.30
0.25
0.20
0.15
0.10
0.05
1
2
3
4
h/h_b

Figure 1