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ON THE WAITING TIME DISTRIBUTION OF BULK QUEUES

by

John Dagsvik

Abstract

The waiting time process of the n-th arriving group is considered for the general bulk queueing model $GI^X/G^Y/1$.

A generalisation of Lindley's waiting time equation is established.

By a generalisation of Kingman's method [3], this equation is solved for the models $GI^X/E_k{}^Y/1$ and $E_k{}^X/G^Y/1$.

When the service time is Erlang distributed E_k , the results are applied to the case where the service- and the arrival groups are of constant size.

Key words: Wendel projection, Group Waiting time, Restbatch, Waiting time equation, Erlang distributions, Hyperexponential distribution, Stationary distributions.

Contents

1. Introduction

In the present paper we shall assume that customers arrive in groups C_n , $n = 0,1,2,\ldots$. The group size is a stochastic variable X , with probability distribution $f(\circ)$. The inter-arrival intervals A_n , $n = 0,1,\ldots$ are independent and have the same distribution $a(\circ)$. The service mechanism is described as follows: At the end of a service period the server accepts Y customers from the waiting line, or a smaller number if the line is shorter. Y is called the service group capasity. The length of the service time B , has the distribution $b(\circ)$. We shall assume the existence of two integers m, l such that $X \leq l$, $Y \leq m$.

The most general works on bulk queues seems to be those of Keilson [2], Cohen [1], Le Gall [5], Lambotte and Teghem [4]. They obtain the distribution of the queue length from which the waiting time distribution is derived. However, there exists no such results for general distributions $a(\circ)$, $b(\circ)$, $f(\circ)$ and $g(\circ)$. Earlier works are restricted to the case where $a(\circ)$ or $b(\circ)$ are the exponential distribution. Even if $b(\circ)$ is exponential the analysis are only limited to bulk service models (Cohen [1], Le Gall [5]).

2. The algebraic formalism.

Let Ω_n denote the set of $n \times n$ matrices whose components are finite complex measures on the Borel subsets of the real line. According to Kingman [3] the product of two measures is defined as their convolution. An operator $T : \Omega_1 \to \Omega_1$ is defined by

(2.1) $\quad (T\nu)(E) = \nu(E \cap R^+) + \nu(-R^+)\epsilon(E) \; , \quad \nu \in \Omega_1 \; ,$

where ϵ is the measure

$$\epsilon(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases}$$

and $R^+ = (0, \infty)$.

This operator has the property that if X is a random variable with distribution

$$\nu(E) = \Pr\{X \in E\},$$

then

$$(T\nu)(E) = \Pr\{X^+ \in E\} \; .$$

Kingman shows that Ω_1 is an commutative algebra over the complex field $\mathbb{C}$ with identity ϵ . Furthermore, he shows that the image Ω_1^+ and the kernel Ω_1^- of T are both disjoint subalgebras of Ω_1 . T is extended to Ω_n by $T\{\nu_{ij}\} = \{T\nu_{ij}\}$, $\{\nu_{ij}\} \in \Omega_n$ and multiplication in Ω_n is defined in the obvious way. With $I_n\epsilon$ as identity it is easy to verify that Ω_n has the same properties as Ω_1 except that Ω_n is no longer commutative. The norm on Ω_n is defined by

$$\|\nu\| = \max_j \sum_i \int |d\nu_{ij}| \; , \quad \nu = \{\nu_{ij}\} \; .$$

The set whose elements are Fourier-Stieltjes transform of elements from Ω_1 we denote $\hat{\Omega}_1$, and extend the definition to $\hat{\Omega}_n$ in the obvious manner.

3. Group waiting time.

We shall study the group waiting time process defined as follows:

Definition 3.1

By the group waiting time W_n , $n = 0,1,2,\ldots$ we mean the waiting time (excluding service time) of the first customer from C_n who is taken into service.

It is convenient to work with the random variable Z_n which is defined as the time C_n spends in the queue until there is no one ahead, except the ones being served. If L_n is the time interval from the instant when Z_n becomes zero and until the last service group with customers from C_n starts, then the sequence $\{Z_n\}$, $n = 0,1,2,\ldots$ satisfies

$$(3.1) \quad Z_{n+1} = (Z_n+L_n-A_n)^+ ,$$

with the initial condition $Z_o = z$. We recognize the expression above as an equation of the similar type as the waiting time equation found by Lindley [6]. However, there is an important difference: The random variables Z_n and L_n are no longer independent as in the model GI/G/1 .

Consider now the servicing of the n-th arrival group C_n . Service may be performed in one or several groups. The last service group which has customers from C_n will be called the n-th rest batch. The rest batch may be filled with customers from C_n or there may be places available for customers from C_{n+1} .

We define two random variables T_n and J_n, as follows: If the n-th restbatch can accept s customers and contains only $t \leq s$ customers from C_n, then $|J_n| = s-t$. If the n-th restbatch contains customers from C_{n-1}; $J_n = -|J_n|$. Otherwise $J_n = |J_n|$. T_n is the number of customers which can be accepted from C_{n+1} in the first service group with customers from C_{n+1}. We define $J_{-1} = T_{-1}$; the capasity of the initial service group.

If $S(T_{n-1})$ denotes service time of C_n, excluding the service time of the n-th rest batch, we realize that $Z_n > 0$ implies that

$$(3.2)\quad L_n = \begin{cases} S(T_{n-1})+B\delta(0,T_{n-1}) = S'(T_{n-1}) & \text{when } J_n \geq 0 \\ 0 & \text{when } J_n < 0\,, \end{cases}$$

and $\quad T_{n-1} = J_{n-1}$,

since C_n will have to wait an extra service period when $J_{n-1} = 0$, $J_n \geq 0$. When $Z_n = 0$ the first service group from C_n has capacity Y and C_n must wait for the time W_n before service starts, hence

$$(3.3)\quad L_n = \begin{cases} S(T_{n-1})+W_n & \text{when } J_n \geq 0 \\ 0 & \text{when } J_n < 0\,, \end{cases}$$

and $\quad T_{n-1} = Y$.

We define matrices of distribution functions

$U_n(t) = \{U_n^i(t)\delta(j,0)\}$, $V_n(t) = \{V_n^i(t)\delta(j,0)\}$, $K(t) = \{K_{ij}\}$

and $H(t) = \{H_{ij}\}$

for $i,j = -(m-2),-(m-1),\ldots,m-1$,

by

$$U_n^i(t) = \Pr\{Z_n \leq t,\ J_{n-1} = i\},$$

$$V_n^i(t) = \Pr\{W_n \leq t,\ J_{n-1} = i\},$$

$$K_{ij} = \begin{cases} \Pr\{S'(T_{n-1}) \leq t,\ J_n=i \mid J_{n-1}=j, Z_n > 0\} & \text{when } i \geq 0 \\ \Pr\{0 \leq t,\ J_n=i \mid J_{n-1}=j\} & \text{when } i < 0 , \end{cases}$$

$$H_{ij} = \begin{cases} \Pr\{S'(T_{n-1}) \leq t,\ J_n=i \mid J_{n-1}=j, Z_n=0\} & \text{when } i \geq 0 \\ 0 & \text{when } i < 0 . \end{cases}$$

If μ is the probability distribution of a random variable X, μ^* denote the distribution of $(-X)$.

Let

$$M_n = Z_n + S'(J_{n-1}) - A_n,\quad N_n = W_n + S(Y) - A_n,\quad D_{ij} = \{J_n=i,\ J_{n-1}=j\} .$$

Considering the possible events between the n-th and the n+1-th arrival we find

(3.8 a) $$U_{n+1}^i(t) = \sum_j \Pr\{(Z_n - A_n)^+ \leq t,\ Z_n > 0,\ D_{ij}\} \quad \text{when } i < 0 ,$$

(3.8 b) $$U_{n+1}^i(t) = \sum_j [\Pr\{M_n^+ \leq t, Z_n > 0, D_{ij}\} + \Pr\{N_n^+ \leq t, Z_n = 0, D_{ij}\}]$$

when $i \geq 0$,

(3.9 a) $$V_{n+1}^i(t) = \sum_j [\Pr\{(Z_n + \delta(0,J_n)B - A_n)^+ \leq t, Z_n - A_n > 0, D_{ij}\}$$

$$+ \Pr\{(Z_n + B - A_n)^+ \leq t, Z_n - A_n \leq 0, Z_n > 0, D_{ij}\}]$$

when $i < 0$. (3.9 a) can be written

$$V^i_{n+1}(t) = \sum_j [\Pr\{(Z_n - A_n)^+ + \delta(0,J_n)B \leq t,\ D_{ij}\}$$

$$-\Pr\{\delta(0,J_n)B \leq t, (Z_n - A_n)^+ = 0,\ D_{ij}\}$$

$$+\Pr\{(Z_n - A)^+ + B \leq t,\ Z_n > 0,\ D_{ij}\}$$

$$-\Pr\{B \leq t, (Z_n - A_n)^+ = 0,\ Z_n > 0,\ D_{ij}\}\ ]$$

$$= ((b^{\delta(0,i)} - b)(U^i_{n+1} - \epsilon U^i_{n+1}(0)))(t)$$

$$+ \sum_j \Pr\{(Z_n - A_n + B)^+ \leq t,\ Z_n > 0,\ D_{ij}\}.$$

When $i \geq 0$ the expression is more complicated;

$$(3.9\ b) \quad V^i_{n+1}(t) = \sum_j [\Pr\{M_n + B\delta(0,J_n) \leq t,\ M_n > 0,\ Z_n > 0,\ D_{ij}\}$$

$$+\Pr\{(M_n + B)^+ \leq t,\ M_n \leq 0,\ Z_n > 0,\ D_{ij}\}$$

$$+\Pr\{N_n + B\delta(0,J_n) \leq t,\ N_n > 0,\ Z_n = 0,\ D_{ij}\}$$

$$+\Pr\{(N_n + B)^+ \leq t,\ N_n \leq 0,\ Z_n = 0,\ D_{ij}\}\ ].$$

If (3.9 b) is rewritten in the same way as (3.9 a) we obtain

$$V^i_{n+1}(t) = ((b^{\delta(0,i)} - b)(U^i_{n+1} - \epsilon U^i_{n+1}(0)))(t)$$

$$+ \sum_j [\Pr\{(M_n + B)^+ \leq t,\ Z_n > 0,\ D_{ij}\}$$

$$+ \Pr\{(N_n + B)^+ \leq t,\ Z_n = 0,\ D_{ij}\}],\ i \geq 0\ .$$

Since $Z_n > 0$ implies $W_n = Z_n + B\delta(0,J_{n-1})$, the probability of $\{(N_n+B)^+ \leq t,\ Z_n = 0\}$ can be written

$$\Pr\{N_n+B)^+ \leq t,\ Z_n = 0\} = \Pr\{(N_n+B)^+ \leq t\}$$

$$-\Pr\{(Z_n+S(Y)+(1+\delta(0,J_{n-1}))B-A_n)^+ \leq t,\ Z_n > 0\},$$

whence

$$(3.10\ a)\quad U^i_{n+1} = \sum_j T(a{*}K_{ij}(U^j_n - \epsilon U^j_n(0)) \quad \text{when} \quad i < 0 ,$$

$$(3.10\ b)\quad U^i_{n+1} = \sum_j [T(a{*}(K_{ij}-b^{\delta(0,j)}H_{ij})(U^j_n - \epsilon U^j_n(0)))$$

$$+ T(a{*}H_{ij}V^j_n)] \quad \text{when} \quad i \geq 0$$

and

$$(3.10\ c)\quad V^i_{n+1} = (b^{\delta(0,i)}-b)(U^i_{n+1}-\epsilon U^i_{n+1}(0))$$

$$+ \sum_j T(a{*}bK_{ij}(U^j_n - \epsilon U^j_n(0))) \quad \text{when} \quad i < 0 ,$$

$$(3.10\ d)\quad V^i_{n+1} = (b^{\delta(0,i)}-b)(U^i_{n+1}-\epsilon U^i_{n+1}(0))$$

$$+ \sum_j [T(a{*}b(K_{ij}-b^{\delta(0,j)}H_{ij})(U^j_n - \epsilon U^j_n(0))]$$

$$+ T(a{*}bH_{ij}V^j_n) \quad \text{when} \quad i \geq 0 .$$

Hence we have established a set of equation for U_n and V_n .

Lemma 3.1

$$\text{(i)} \quad K_{ij} = \begin{cases} f(|j|+i)\ , & \text{when } i < 0 \\ \sum_{r\geq 1} b^r \sum_{p>0} (f(g^*)^{r-1})(p+|j|)g(i+p)+\delta(0,i)f(|j|)\epsilon & \\ \quad \text{when } i \geq 0\ , & \end{cases}$$

$$\text{(ii)} \quad H_{ij} = \begin{cases} \sum_{r\geq 0} b^r \sum_{p>0} (f(g^*)^r)(p)g(i+p)\ , & \text{when } i \geq 0 \\ 0 & \text{when } i < 0\ , \end{cases}$$

$$\text{(iii)} \quad bH_{ij} = K_{io} \quad \text{when } i \geq 0\ .$$

Proof:

Let $R(J_{n-1})+1$ denote the number of service groups with customers from C_n. When $J_n \geq 0$, $J_{n-1} = j \neq 0$, $Z_n > 0$, $R(J_{n-1})$ must satisfy

$$\text{(3.11 a)} \quad J_n+X = \sum_{s=1}^{R(j)} Y_s+|j|\ , \ X > \sum_{s=1}^{R(j)-1} Y_s+|j|$$

because

$$\sum_{s=1}^{R(j)} Y_a+|j|$$

customers are served in $R(j)+1$ groups and the rest of C_n is served in the n-th restbatch. When $J_{n-1} = 0$, the restbatch is complete and $T_{n-1} = Y$, so that

$$\text{(3.11 b)} \quad J_n+X = \sum_{s=1}^{R(0)} Y_s+Y\ , \ X > \sum_{s=1}^{R(0)-1} Y_s+Y\ .$$

By an elementary argument

$$\Pr\{R(J_{n-1})=r,\ J_n=i\,|\,J_{n-1}=j\} = \sum_{p>0}(f(g*)^{r-1})(p+|j|)g(i+p),$$

$$i \geq 0\ ,\ j \neq 0\ ,\ r \geq 1\ ,$$

$$\Pr\{R(J_{n-1})=r,\ J_n=i\,|\,J_{n-1}=0\} = \sum_{p>0}(f(g*)^{r})(p)g(i+p),\ i \geq 0,\ r \geq 0\ ,$$

$$\Pr\{S'(J_{n-1}) \leq t\,|\,R(J_{n-1})=r,\ J_n=i\,|\,J_{n-1}=j,\ Z_n > 0\} = b^{r+\delta(0,j)}(t)\ .$$

When $J_n < 0$, $Z_n > 0$, the relation

$$X + |J_n| = |J_{n-1}|$$

must be valid. Furthermore $S(Y)$ is seen to have the same distribution as $S(0)$. The theorem now follows easily.

It is convenient to introduce some further matrix notations. Let $F = \{F_{ij}\}$, $G = \{G_{ij}\}$, $E = \{E_{ij}\}$, $K' = \{K'_{ij}\}$ and $\nu_r = \{\nu_r^{ij}\}$ be the matrices with entries

$$F_{ij} = f(j-i),\ K'_{ij} = \sum_{r\geq 0} b^r \sum_{p>0}(f(g*)^r)(p+j)g(i+p),$$

for $i,j = 0,1,2,\ldots m-1$,

and $F_{ij} = K'_{ij} = 0$ for $i,j = m,\ldots\ m+(1-m)^{+}-1$.

$$G_{ij} = g(i-j),\ E_{ij} = \sum_{p>0} g(i+p)f(p+j),\ \text{for}\ i,j = 0,1,\ldots m+(1-m)^{+}-1\ ,$$

$$\nu_r^{ij} = \delta(i,0)\ \text{ for }\ i,j = 0,1,\ldots r-1\ .$$

Lemma 3.2

(i) $K' = E(\epsilon I_m - bG)^{-1} I_m$

(ii) $$K_{ij} = \begin{cases} \epsilon F_{-ij}, & \text{when } i < 0,\ j \geq 0 \\ bK_{ij} + \delta(0,i) F_{ij}\epsilon, & \text{when } i \geq 0,\ j \geq 0 . \end{cases}$$

Proof:

(ii) is seen immediately.

(i): Since $EG^r = \{(EG^r)_{ij}\} = \{\sum_{p>0} (f(g^*)^r)(p+j)g(i+p)\}$,

$i,j = 0,1,\ldots m-1$, the theorem follows. Observe that $G^r = 0$ when $r > m+(l-m)^+$, hence there is no convergence problem.

Remark:

Even if the matrices involved are defined m dimensional they are understood to be $m+(n-m)^+$ dimensional with the undefined entries equal to zero.

By the substitutions $Q_n = \{Q_n^{ik}\}$, $P_n = \{P_n^{ik}\}$, where

(3.12 a) $$Q_n^{ik} = (V_n^i + (b - b^{\delta(0,i)})(U_n^i - \epsilon U_n^i(0)))\delta(k,0)$$

and

(3.12 b) $$P_n^{ik} = (U_n^{-i} + U_n^i - \delta(0,i)\sum_r (U_n^{-r} + U_n^r))\delta(k,0) ,$$

$$i,k = 0,1,\ldots,m-1 ,$$

it is possible to write (3.10) on the form

$$(3.13\ \mathrm{a})\quad U_{n+1} = T(a*\begin{bmatrix}0\\K'\end{bmatrix}\nu_{2m-1}Q_n)+T(a*\begin{bmatrix}F\\bK'\end{bmatrix}(P_n-\epsilon P_n(0))),$$

$$(3.13\ \mathrm{b})\quad Q_{n+1} = T(a*b\begin{bmatrix}0\\K'\end{bmatrix}\nu_{2m-1}Q_n)+T(a*b\begin{bmatrix}F\\bK'\end{bmatrix}(P_n-\epsilon P_n(0))),$$

whence theorem 3.3 follows.

Theorem 3.3

Let

$$\tilde{P}_n = \begin{bmatrix}P_n\\ \nu_{2m-1}Q_n\end{bmatrix}$$

and assume that $U_o(t) = I_{2m-1}\epsilon(z+t)$, $V_o(t) = I_{2m-1}\epsilon(w+t)$. Then U_n and V_n are uniquely determined by (3.13) and

$$(3.14)\quad \tilde{P}_{n+1} = T(a*K_1(\tilde{P}_n-\epsilon\tilde{P}_n(0)))+T(a*K_2)\tilde{P}_n(0),\ n = 0,1,\ldots\ ,$$

where

$$K_1 = \begin{bmatrix}(I_m-\nu_m)(F+bK'),(I_m-\nu_m)K'\\ b\nu_m(F+bK'),\ b\nu_mK'\end{bmatrix},\ K_2 = \begin{bmatrix}0,(I_m-\nu_m)K'\\ 0,\ b\nu_mK'\end{bmatrix}.$$

Equation (3.14) is called the waiting time equation.

Corollary 3.4

When the trafic intensity $\rho = E(X)E(B)/E(Y)E(A)$ is less than unity the stationary distribution $\tilde{P} = \lim_{n\to\infty}\tilde{P}_n$ exists and is determined by

$$(3.15)\quad \tilde{P} = T(a*K_1(\tilde{P}-\epsilon\tilde{P}(0)))+T(a*K_2)\tilde{P}(0)\ .$$

A proof of the existence of the stationary distribution is given in [7] and [2].

We will assume that the customers in an arrival group are ordered and the queuedisiplin is first come first served. Let W_n^p be the waiting time of customer nr. p in C_n given that C_n contains at least p customers. Let

$\Gamma_n^p = Pr\{W_n^p \leq t\}$.

The problem of finding Γ_n^p can be solved by the following argument: If C_n contains exactly p customer then obviously

$$\Gamma_n{}^p = \sum_i U_{n+1}^i \ .$$

Hence we must have

$$\Gamma_n{}^p = T(a{*}\nu_m K'_p \nu_{2m-1} Q_n) + T(a{*}\nu_{2m-1} K_p (P_n - \epsilon P_n(0))) \ ,$$

where K'_p and K_p are K' and K respectively, when $f(\cdot)$ is replaced by $\delta(p,\circ)$.

4. The model $GI^X/E_k{}^Y/1$.

Within this model it is possible to obtain solutions of the waiting time equation (by using the approach suggested in [3] p.p. 312-313). Let $e_\theta;\ \theta \in \mathbb{C}$, denote the complex exponential measure defined by

$$e_\theta(E) = \int_{E\cap R^+} \theta \exp(-\theta x)dx \ , \quad \text{if} \quad Re\ \theta > 0 \ ,$$

$$e_\theta(E) = \int_{E\cap -R^+} \theta \exp(-\theta x)dx \ , \quad \text{if} \quad Re\ \theta < 0 \ ,$$

The following lemma is proved in the same way as the analogous results in [3] p.p. 312-313.

Lemma 4.1

(i) $$e_\theta e_\mu^n = \beta^n e_\theta + (1-\beta) \sum_{k=1}^{n} \beta^{n-k} e_\mu^k ,$$

where

$$\beta = \frac{\mu}{\mu-\theta} .$$

(ii) $$T(a*e_\mu^n) = \sum_{k=1}^{n} (e_\mu^k - \epsilon)\gamma_{n-k} + \epsilon ,$$

where

$$\gamma_m = \int_0^\infty \frac{e^{-\mu y}(\mu y)^m}{m!} a(dy) .$$

By the introduction of the generating functions

$$\psi_x(t) = \sum_{n\geq 0} P_n(t)x^n , \quad \varphi_x(t) = \sum_{n\geq 0} \nu_{2m-1} Q_n(t)x^n .$$

$$\tilde{\psi}_x = \begin{pmatrix} \psi_x \\ \varphi_x \end{pmatrix} = \sum_{n\geq 0} \tilde{P}_n x^n .$$

(3.14) is transformed to the equivalent equation

(4.1) $$\tilde{\psi}_x = xT(a*K_1(\tilde{\psi}_x - \epsilon\tilde{\psi}_x(0))) + xT(a*K_2)\tilde{\psi}_x(0) + \tilde{P}_o .$$

By the transformation

$$\omega_x = (\epsilon I_m - bG)^{-1}(b(\psi_x - \psi_x(0)\epsilon) + \varphi_x)$$

we get

$$(4.2\ \mathrm{i})\quad \psi_x = x(I_m - \nu_m)T(a*(F(\psi_x - \epsilon\psi_x(0)) + E\omega_x)) + P_o \ ,$$

$$(4.2\mathrm{ii})\quad \omega_x - b(G\omega_x + \psi_x - \epsilon\psi_x(0)) = x\nu_m T(a*b(F(\psi_x - \epsilon\psi_x(0)) + E\omega_x) + \nu_m Q_o \ .$$

By assumption $b = e^k_\mu$. Assume that (4.2) has a solution of hyperexponential type; i.e,

$$(4.3)\quad \begin{pmatrix}\psi_x\\ \omega_x\end{pmatrix} = \begin{pmatrix}q'_o\\ q''_o\end{pmatrix}\epsilon + \sum_{j\geq 1}^{h}\begin{pmatrix}q'_j\\ q''_j\end{pmatrix}e_{\theta_j} \ .$$

If $U_o = \epsilon U_o(0)$ and $V_o = \epsilon V_o(0)$, it follows from the definition of $\tilde{P}_n$ that $\tilde{P}_o = \begin{pmatrix}(I_m-\nu_m)U_o\\ \nu_m V_o\end{pmatrix}$.

Inserting (4.3) in (4.2) gives

$$\epsilon q'_o + \sum_{j>0} e_{\theta_j} q'_j = x(I_m-\nu_m)\sum_{j>0} T(a*e_{\theta_j}(Fq'_j+Eq''_j)) + x(I_m-\nu_m)Eq''_o\epsilon + \\ +(I_m-\nu_m)U_o \ ,$$

$$(\epsilon I_m - bG)q''_o + \sum_{j>0}(e_{\theta_j}q''_j - e_{\theta_j}b(Gq''_j + q'_j))$$

$$= x\nu_m\sum_{j>0} T(a*be_{\theta_j}(Fq'_j+Eq''_j)) + x\nu_m T(a*bEq''_o) + \nu_m V_o \ .$$

- From Lemma 4.1 follows

$$(4.4\text{ i})\quad \epsilon q_o' + \sum_{j>0} e_{\theta_j} q_j' - (I_m - \nu_m) x \sum_{j>0} (Fq_j' + Eq_j'')(\alpha_j e_{\theta_j} + (1-\alpha_j)\epsilon)$$

$$-x(I_m - \nu_m) Eq_o''\epsilon + (I_m - \nu_m) U_o = 0 \;,$$

$$(4.4\text{ii})\quad \sum_{j>0} ((\epsilon I_m - \beta_j^k G) q_j'' - \beta_j^k q_j') e_{\theta_j} - (e_\mu^k - \epsilon) G q_o'' - G q_o''\epsilon + q_o''\epsilon$$

$$- \sum_{j>0} (Gq_j'' + q_j')((1-\beta_j) \sum_{r=1}^{k} \beta_j^{k-r} (e_\mu^r - \epsilon) + (1-\beta_j^k)\epsilon)$$

$$-x\nu_m \sum_{j>0} (Fq_j' + Eq_j'') \beta_j^k (\alpha_j e_{\theta_j} + (1-\alpha_j)\epsilon)$$

$$-x\nu_m \sum_{j>0} (Fq_j' + Eq_j'')((1-\beta_j) \sum_{r=1}^{k} \beta_j^k \eta_{rj} (e_\mu^r - \epsilon) + (1-\beta_j^k)\epsilon)$$

$$-x\nu_m E q_o'' (\sum_{r=1}^{k} (e_\mu^r - \epsilon)\gamma_{k-r} + \epsilon) - \nu_m V_o = 0 \;,$$

where

$$\eta_{rj} = \sum_{s\geq r}^{k} \beta_j^{-r} \gamma_{s-r} \;, \quad \alpha_j = \alpha(\theta_j) \;.$$

The equations above imply that the coefficients of ϵ, $(e_\mu^r - \epsilon)$, $r = 1,2,..,k$, and e_{θ_j}, $j = 1,2,...$ are zero. Hence, we are led to the equations

$$(4.5\text{ i})\quad q_o' = \sum_{j>0} \frac{1-\alpha_j}{\alpha_j} q_j' + x(I_m - \nu_m) E q_o'' + (I_m - \nu_m) U_o \;,$$

$$(4.5\text{ii})\quad q_j' = \beta_j^{-k} (I_m - \nu_m)(I_m - \beta_j^k G) q_j'' \;,$$

(4.5iii) $(I_m - G - x\nu_m E)q_0'' = \sum_{j>0} \left(\frac{1-\alpha_j}{\alpha_j \beta_j^k}\nu_m (I - \beta_j^k G) - 1 + \beta_j^{-k}\right) q_j'' + \nu_m V_0$,

(4.5iv) $(\delta(r,k)G + x\gamma_{k-r}\nu_m E)q_0'' = \sum_{j>0} [(\beta_j - 1)\beta_j^{-r}$,

$+(\beta_j - 1)(\eta_{rj}\alpha_j^{-1} - \beta_j^{-r})\nu_m (I_m - \beta_j^k G)] q_j''$, $r = 1,2,\ldots,k$,

(4.5 v) $((I_h - x\alpha_j F)(I_m - \beta_j^k G) - x\alpha_j \beta_j^k E)q_j'' = 0$, $j = 1,2,\ldots$,

where $\beta_j, j = 1,2,\ldots hk$, are the roots of

(4.5vi) $\det\{(I_h - x\alpha F)(I_m - \beta^k G) - x\alpha\beta^k E\}$,

where $h = m + (1-m)^+$.

Observe that (4.5 iv-v) constitutes a set of $2kh$ equations. Hence it is sufficient that (4.5 vi) has kh roots.

5. Special cases.

Unfortunately, the assumptions (4.3) are not always fullfilled. For instance when $Y = m > 1$. However, in this case we are able to slighten the assumptions (4.3). Observe that $Y = m > 1$ implies $G = 0$. Let

(5.1) $\xi_x = (F + bE)(\psi_x - \epsilon\psi_x(0)) + E\varphi_x$

whence

(5.2a) $\psi_x = x(I_m - \nu_m)T(a * \xi_x) + (I_m - \nu_m)U_0$, (5.2b) $\varphi_x = x\nu_m T(a*b\xi_x)+\nu_m V_o$

and

(5.3) $\xi_x = x(F+bE)(I_m-\nu_m)(Ta*\xi_x-\epsilon T(a*\xi_x)(0))+xE\nu_m T(a*b\xi_x)+E\nu_m V_o$.

We now suppose that (5.3) has a solution of the form

$$\xi_x = p_o\epsilon+\sum_{j>0} e_{\theta_j} p_j \ ,$$

that is; $p_o, p_j, j > 0$, must satisfy

$$p_o\epsilon+\sum_{j>0} e_{\theta_j} p_j = x(F+bE)(I_m-\nu_m)\sum_{j>0} T(a*e_{\theta_j})p_j+xE\nu_m\sum_{j>0} T(a*be_{\theta_j})p_j$$

$$+xE\nu_m T(a*b)p_o+E\nu_m V_o \ .$$

Exactly the same calculations as in the preceeding section give

(5.4 i) $p_j = x\alpha_j(F+\beta_j^k E)p_j$, $j = 1,2,\dots$,

(5.4ii) $\sum_{j>0}\{(\beta_j-1)\beta_j^{k-r}\alpha_j-\delta(r,k)(1-\alpha_j))E(I_m-\nu_m)$

$$+(\beta_j-1)\beta_j^k\eta_{rj}E\nu_m\}p_j = \gamma_{k-r}E\nu_m p_o \ , \quad r = 1,2,\dots,k \ ,$$

(5.4iii) $(I_m-xE\nu_m)p_o = \sum_j\{(\alpha_j^{-1}-1)(I_m-\nu_m)+xE\nu_m+x\alpha_j E(I_m-\nu_m)$

$$-x\alpha_j\beta_j^k E\}p_j+E\nu_m V_o \ .$$

We shall now consider the case when both the arrival and the service group capacity are of constant size; i.e. $X = l$, $Y = m$. Assume first that $m < l$. Then $F = 0$ and

$$I_l-\beta^k G-x\alpha\beta^k E = I_l-\beta^k \begin{bmatrix} 0 & ,x\alpha I_m \\ I_{l-m}, & 0 \end{bmatrix} .$$

It follows that

$$(5.5)\quad \det(I_l-\beta^k G-x\alpha\beta^k E) = 1-(x\alpha)^m\beta^{kl} .$$

From Lemma 1 in Takács [8] page 82 we conclude that (5.5) has kl distinct roots $\beta_j = \beta(\theta_j)$, $j = 1,2,\ldots,kl$ for $\mathrm{Re}\,\theta \geq 0$. Hence equations (4.5) have a solution. Assume that $l \leq m$. Then $G = 0$;

$$I_m-x\alpha(F+\beta^k E) = I_m-x\alpha \begin{bmatrix} 0 & ,I_{m-l} \\ \beta^k I_l, & 0 \end{bmatrix} ,$$

and

$$(5.6)\quad \det(I_m-x\alpha F-x\alpha\beta^k E) = 1-(x\alpha)^m\beta^{kl} .$$

When p_0 is eliminated, (5.4ii) is a set of kl equations because

$$E = \begin{bmatrix} 0 & , & 0 \\ I_l, & & 0 \end{bmatrix}.$$

Accordingly, (5.4) has a solution since (5.6) has kl distinct roots.

The stationary solution is obtained by multiplying (4.5) and (5.4) by $1-x$ and let x tend to $1-$.

6. The model $E_k^X/G^Y/1$.

The "key" to the solution is the fact that the assumption $a = e_\lambda^k$ enables us to express the operator T in a special form

Lemma 6.1

Suppose $\mu \in \Omega_m^+$. Then

$$T((e*)^k\mu) = (e*)^k\mu+(\epsilon-e*)\sum_{r=1}^{k}\alpha_r(e*)^{k-r} ,$$

where

$$\alpha_r = ((e*)^r\mu)(\sim \mathbb{R}^+) .$$

Proof:

For $k = 1$ the Lemma reduces to equation (72) in Kingman [3]. Assume (6.1) valid for $k = 1,2,\ldots p$. Since $T(e*)^r = \epsilon$ we get

$$T((e*)^{p+1}\mu) = T(e*T((e*)^p\mu)) .$$

Now $\nu = T((e*)^p\mu) \in \Omega_m^+$ implies

$$T((e*)^{p+1}\mu) = T(e_\lambda^* \nu) = e*\nu+(\epsilon-e*)\alpha' ,$$

where $\alpha' = (e*\nu)(\sim \mathbb{R}^+)$.

Obviously $T(e*\nu)(\sim R^+) = (e*\nu)(\sim R^+)$ which yields $\alpha' = \alpha_{p+1}$. Hence, by the assumption

$$T(e*\nu) = (e*)^{p+1}\mu+(\epsilon-e*)\sum_{r=1}^{p}\alpha_r(e*)^{p+1-r}+(\epsilon-e*)\alpha_{p+1}$$

and the Lemma follows by induction.

With $\mu = K_1\tilde{\psi}_x$ and $e = e_\lambda$ application of (6.1) on (4.1) gives (with $I = I_{2m-1}$)

$$(6.2)\quad (\epsilon I - xa*K_1)\tilde{\psi}_x = x(\epsilon - e*)\sum_{r=1}^{k}\alpha_r(e*)^{k-r} + xT(a*(K_2-K_1))\tilde{\psi}_x(0) + \tilde{P}_o \ .$$

Let $\Delta_x = \epsilon I - xa*K_1$.

Inserting $t = 0$ gives an expression for α_k , viz.,

$$(6.3)\quad x\alpha_k = (\epsilon I - x(a*(K_2-K_1))(0))\tilde{\psi}_x - \tilde{P}_o(0) \ .$$

Since $\|a*K_1\| \leq 1$, $\Delta_x^{-1} = \sum_{r\geq 0}(xa*K_1)^r$ exists when $|x| < 1$. Thus (6.3) and (6.2) are equivalent to

$$(6.4)\quad \tilde{\psi}_x - \epsilon\tilde{\psi}_x(0) = \Delta_x^{-1}(\tilde{P}_o - (\epsilon - e_\lambda*)\tilde{P}_o(0)) + (\epsilon - e_\lambda*)\Delta_x^{-1}\sum_{r=1}^{k-1}\alpha_r(e_\lambda*)^{k-r}$$

$$-e_\lambda*\Delta_x^{-1}(I\epsilon - x(e_\lambda*)^{k-1}K_2)\tilde{\psi}_x(0) \ .$$

If $k = 1$, $t = 0$ determines $\tilde{\psi}_x(0)$ by

$$(6.5)\quad (e_\lambda*\Delta_x^{-1}(\epsilon I - x(e_\lambda*)^{k-1}K_2)(0)\tilde{\psi}_x(0) = (\Delta_x^{-1}(\tilde{P}_o - (\epsilon - e_\lambda*)\tilde{P}_o(0)))(0) \ .$$

When $k > 1$ let

$$c_r = (\epsilon - e_\lambda*)(e_\lambda*)^{k-r} \ , \quad r = 1,2,\ldots,k-1 \ ,$$

$$C_x = e_\lambda*(\epsilon I - x(e_\lambda*)^{k-1}K_2) \ ,$$

$$D = (\tilde{P}_o - (\epsilon - e_\lambda*)\tilde{P}_o(0)) \ ,$$

$$E_j = (e_\lambda*)^j K_1\Delta_x^{-1} \ .$$

Thus, (6.4) can be written

$$(6.6)\quad \tilde{\psi}_x = (\epsilon I-\Delta_x^{-1}C_x)\tilde{\psi}_x(0)+\sum_{r=1}^{k-1} c_r\Delta_x^{-1}\alpha_r+\Delta_x^{-1}D \;.$$

Furthermore $(E_j\tilde{\psi}_x)(0) = \alpha_j$ leads to

$$(6.7\ i)\quad \alpha_j = (E_j(\Delta_x^{-1}-C_x))(0)\tilde{\psi}_x(0)+\sum_{r=1}^{k-1}(E_jc_r)(0)\alpha_r+E_jD \;,$$

$$j = 1,2,\ldots,k-1 \;,$$

$$(6.7ii)\quad (\Delta_x^{-1}C_x)(0)\tilde{\psi}_x(0) = \sum_{r=1}^{k-1} c_r(0)\alpha_r+(E_jD)(0)$$

which determines α_j, $j = 1,2,\ldots k-1$, and $\tilde{\psi}_x(0)$.

7. The stationary solution

In this section we shall demonstrate how the stationary solution of (6.2) can be obtained by use of the Fourier-Stieltjes transform. In the stationary case we have

$$(7.1)\quad (\epsilon I-a{*}K_1)\tilde{P} = (\epsilon-e_\lambda{*})\sum_{r=1}^{k}\alpha_r(e_\lambda{*})^{k-r}+T(a{*}(K_2-K_1))\tilde{P}(0) \;,$$

where now

$$\alpha_r = (e_\lambda{*}^r K_1\tilde{P})(0) \;.$$

After the introduction of the Fourier transform an equation analogous to (6.6) is obtained

$$(7.2)\quad \hat{\Delta}_1(z)(\hat{\tilde{P}}(z)-\tilde{P}(0)) = -\hat{C}_1(z)\tilde{P}(0)+\sum_{r=1}^{k-1}\hat{c}_r(z)\alpha_r \;.$$

Let $\varkappa$ be defined by

$$\hat{\varkappa}\hat{\Delta}_1 = \det \hat{\Delta}_1 ,$$

whence

$$(7.3) \quad \det(\Delta_1(z))(\hat{\tilde{P}}(z)-\tilde{P}(0)) = -\hat{\varkappa}(z)\hat{C}_1(z)\tilde{P}(0)+\sum_{r=1}^{k-1}\hat{c}_r(z)\hat{\varkappa}(z)\alpha_r .$$

Suppose that $\det \Delta_1(z)$ has k roots $z_1, z_2, \ldots, z_k$. Then $\tilde{P}(0), \alpha_1, \ldots, \alpha_{k-1}$, are determined except for a constant by

$$(7.4) \quad \hat{\varkappa}(z_i)\hat{C}_1(z_i)\tilde{P}(0) = \sum_{r=1}^{k-1}\hat{\varkappa}(z_i)\hat{c}_r(z_i)\alpha_r , \quad i = 1,2,\ldots k .$$

Premultiplicating (7.2) by

$$\upsilon = \begin{bmatrix} \nu_m, 0 \\ 0, \nu_m \end{bmatrix}$$

gives an equation where both sides become zero when $z = 0$. By l'Hospitals rule,

$$(7.5) \quad (\hat{a}*\upsilon\hat{K}_1)'(0)(\hat{\tilde{P}}-\hat{\tilde{P}}(0)) = \upsilon\hat{C}_1'(0)\tilde{P}(0)-i\sum_{r=1}^{k-1}\upsilon\alpha_r .$$

Consider $(a*\upsilon\hat{K}_1)'(0)\hat{\tilde{P}}(0)$.

The process $\{J_n\}$ is recurrent and therefore J_n converges in distribution to J, say. Since

$$\upsilon K_1 = (b\nu_m K, \nu_m(K-F))$$

and

$$\nu_m K(t) = (k_o(t), k_1(t), \ldots, k_{m-1}(t)) ,$$

where

$$k_j(t) = \sum_i k_{ij}(t) ,$$

it follows that

$$(\hat{a}*\hat{b}\nu_m\hat{K})'(0) = i(E(S'(j))|J=j)-iE(A)+iE(B) .$$

Thus

$$(\hat{a}*\upsilon\hat{K}_1)'(0)\hat{\tilde{P}}(0) = \nu_m(\hat{a}*\hat{b}\hat{K})'(0)\hat{P}(0)+\nu_m(\hat{a}*\hat{K})'(0)\nu_{2m-1}\hat{Q}(0)$$

$$= i \sum_j E(S'(j))|J=j)Pr(J=j)-iE(A)+iE(B) .$$

By (3.11) we find

$$E(S'(J)-A+B) = (\rho-1)E(A)+E(B) = k\lambda^{-1}(\rho-1)+E(B) .$$

Equation (6.12) therefore reduces to

$$(7.6) \quad E(B)-k(1-\rho) = -i\lambda(\hat{a}*\hat{b}\hat{K})'(0)P(0)+\nu_{2m-1}Q(0)-\lambda\sum_{r=1}^{k-1}\upsilon\alpha_r .$$

Together with (7.4) we have a set of $k+1$ matrix equations to determine the $k+1$ unknowns $\tilde{P}(0), \alpha_1, \ldots, \alpha_{k-1}$.

It is known that a probability distribution can be approximated by a linear combination of Erlang distributions. From Lemma 6.1 it is clear that the results in the last section can be generalized to the case when $a(\circ)$ is a linear combination of Erlang distributions. Accordingly, it is possible to obtain approximate solutions of the waiting time equation for general $a(\circ)$.

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