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Derivatives Pricing in Energy Markets: An Infinite-Dimensional Approach*

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Abstract. Based on forward curves modelled as Hilbert-space valued processes, we analyze the pricing of various options relevant in energy markets. In particular, we connect empirical evidence about energy forward prices known from the literature to propose stochastic models. Forward prices can be represented as linear functions on a Hilbert space, and options can thus be viewed as derivatives on the whole curve. The value of these options are computed under various specifications, in addition to their deltas. In a second part, cross-commodity models are investigated, leading to a study of square integrable random variables with values in a two-dimensional Hilbert space. We analyze the covariance operator and representations of such variables, as well as presenting applications to the pricing of spread and energy quanto options.

Key words. energy markets, forward curves, Hilbert-valued stochastic processes, option pricing, hedging, spread options

AMS subject classifications. 60G60, 60G51, 60H15, 91G20, 91G80

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1. Introduction.

In energy markets like NYMEX, CME, EEX, and NordPool, there is a large trade in forwards and futures contracts. Forwards and futures on power and gas are delivering the underlying commodity over a period of time rather than at a fixed delivery time, as is the case for oil, say. Related markets, like shipping and weather, also trade in futures and forwards settled on an index measured over a time period. We refer to Burger, Graeber, and Schindlmayr [19], Eydeland and Wolyniec [26], and Geman [30] for a presentation and discussion of different energy markets and the traded derivatives contracts. For a more technical analysis on modelling aspects of energy prices, we refer to Benth, Benth, and Koekebakker [12].

Typically, many of the energy markets trade in European call and put options written on the forward and futures contracts, including, for example, the power exchanges EEX in Germany and NordPool in the Nordic area. At NYMEX, one finds options on the spread between futures on different refined oil blends. Other cross-commodity derivatives include options on the spread between power and fuels (dark and spark spreads, say; see Eydeland and Wolyniec [26]) or quanto options which are settled on the product between a power price and a weather index (see Benth, Lange, and Myklebust [10]).

In this paper, we analyze the pricing of options in the framework of forward curves mod-

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elled as Hilbert-space valued stochastic processes. Empirical studies reveal that energy forwards show a high degree of idiosyncratic risk across maturities. For example, a principal component analysis of the NordPool power forward and futures market by Benth, Benth, and Koekebakker [12] reveals that more than ten factors are needed to explain 95% of the volatility. (This confirms earlier studies of the same market by Frestad [28] and Koekebakker and Ollmar [32].) Using methods from spatial statistics (see Frestad [28], Frestad, Benth, and Koekebakker [29], and Andresen, Koekebakker, and Westgaard [3]), studies of NordPool forward and futures prices show a clear correlation structure across times to maturity. These empirical studies point toward the need for modelling the time dynamics of the forward curve by means of a Hilbert-space valued process. Moreover, the abovementioned studies also highlight the leptokurtic behavior of price returns, motivating the introduction of infinite-dimensional Lévy processes as the noise in the forward dynamics.

This paper significantly extends the analysis of forward curves by Benth and Krühner [15] toward a theory for pricing options in energy markets. In particular, the present paper contributes in two different, but related, directions. First, we provide a detailed analysis of the pricing of typical European options traded in various energy markets. Our results include expressions for the deltas. Second, we lay the theoretical foundation for a modelling of cross-commodity forwards and futures markets in an infinite-dimensional framework. In addition, we present a broad analysis of delivery period forwards much more far-reaching than what is found in Benth and Krühner [15].

A European option of a forward contract can, in our context, be viewed as an option on the forward curve. The payoff of the option will be represented as a linear functional acting on the curve, followed by a nonlinear payoff function. We provide a detailed analysis on how to view forwards and futures contracts as linear functionals on the forward curve, set in a Hilbert space of absolutely continuous function on $\mathbb{R}_+$. We present the explicit functionals based on various typical contracts traded in power and weather (temperature) markets. Using a representation theorem from Benth and Krühner [15], one can derive a real-valued stochastic process for the forward contract underlying the option, which in some special cases can be further computed to provided simple expressions for the option price. For example, for arithmetic (linear) forward curve models, we can find expression of the option price, either analytical in the Gaussian case or computable via fast Fourier transform in the more general Lévy case. The prices will depend on the realized volatility of the infinite-dimensional forward curve dynamics, which involves some linear functionals and their duals. In particular, we need to have available the dual of the shift operator and some integral operators, which we derive explicitly in our chosen Hilbert space.

Also, we derive the delta of these options. The delta of the option will be defined as the derivative of the price with respect to the initial forward curve. Interestingly, the delta will provide information on how sensitive the price is toward inaccuracies on the initial forward curve. As we need to construct this curve from discretely observed data, the delta provides valuable information on the robustness of the option price toward mis-specification in the forward curve. We also show that the option price is Lipschitz continuous as a function of the initial forward curve as long as the payoff function is Lipschitz. In this part of our paper, we also discuss options written on the spread between two forward contracts on the same commodity but with different delivery periods. This spread can effectively be represented as the difference of two linear functionals on the forward curve extracting two different pieces of this curve. With such options, the covariance structure along the forward curve becomes an important ingredient in the pricing.

In the second part of the paper, we turn the focus to modelling and pricing in cross-commodity energy markets. Typically, one is interested in modelling the joint forward dynamics in two energy markets, for example, in two connected power markets or the markets for gas and power. Alternatively, one may be interested in modelling the joint forward dynamics between temperature contracts and power. We express a bivariate forward price dynamics through a stochastic process with values in a two-dimensional Hilbert space. More specifically, we assume that the process is the mild solution of two Musiela stochastic partial differential equations, each taking values in a Hilbert space of absolutely continuous functions on $\mathbb{R}_+$, where the dynamics is driven by two dependent Hilbert-space valued Wiener processes. Furthermore, we allow for functional dependency in the volatility specifications of the two stochastic partial differential equations. The crucial point in our analysis is the covariance operator for the bivariate Hilbert-space valued Wiener process. We show that the covariance operator can be expressed as a 2×2 matrix of operators, where we find the respective marginal covariance operators on the diagonal and an operator describing the covariance between the two Wiener processes on the off-diagonal, analogous to the situation of a bivariate Gaussian random variable on $\mathbb{R}^2$. We derive a decomposition of two square-integrable Hilbert space valued random variables in terms of a common factor and an independent random variable. This "linear regression" decomposition is expressed in terms of an operator which resembles the correlation.

Our theoretical considerations are applied to the pricing of spread options. (See Carmona and Durrleman [21] for an extensive account on the zoology of spread options in energy and commodity markets.) Another interesting class of derivatives is the so-called energy quanto options, which offer the holder a payoff depending on price and volume. The volume component is measured in terms of some appropriate temperature index, which means that the energy quanto option can be viewed as an option written on the forward prices of energy and temperatures. We remark that there is a weather market at the Chicago Mercantile Exchange trading in temperature futures.

Our infinite-dimensional approach to forward price modelling in energy markets builds on the extensive theory in fixed-income markets. We refer to Filipovic [27] and Carmona and Tehranchi [22] for an analysis of forward rates modelled as infinite-dimensional stochastic processes. In Benth and Krühner [15], a particular Hilbert space proposed by Filipovic [27] to realize forward curves plays a central role. Audet et al. [4] are, to the best of our knowledge, the first to model power forward prices using infinite-dimensional processes. Exponential and arithmetic energy forward curve models are analyzed in Barth and Benth [9] with an emphasis on introducing numerical schemes to simulate the dynamics. Another path is taken in Benth and Lempa [11], where optimal portfolio selection in commodity forward markets is studied. Barndorff-Nielsen, Benth, and Veraart [7] propose using ambit fields, a class of spatio-temporal random fields, as an alternative modelling approach to the dynamic specification of forward curves used in the present paper. In a recent paper, Barndorff-Nielsen, Benth, and

Veraart [8] have extended the ambit field idea to cross-commodity market modelling and the pricing of spread options. We remark that there is a close relationship between ambit fields and stochastic partial differential equations (see Barndorff-Nielsen, Benth, and Veraart [6]).

We present our results as follows: in section 2, we express energy forward and futures delivering over a settlement period as linear operators on a Hilbert space of functions. European options on energy futures are analyzed in section 3, while we consider cross-commodity futures price modelling and option pricing in section 4.

1.1. Some notation. As a final note in this introduction, throughout this paper we let $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ be a filtered probability space, where Q denotes the risk-neutral probability. We are working directly under risk-neutrality, as we have pricing of financial derivatives in mind. Furthermore, we use the notation $L(U, V)$ for the space of bounded linear operators from the Hilbert space U into the Hilbert space V. In the case $U = V$, we use the shorthand notation $L(U)$ for $L(U, U)$. Throughout this paper, the Hilbert-spaces that we shall use will all be assumed separable. Finally, $L_{\mathrm{HS}}(U, V)$ denotes the space of Hilbert–Schmidt operators from U to V, and $L_{\mathrm{HS}}(U) = L_{\mathrm{HS}}(U, U)$.

2. Hilbert-space realization of energy forwards and futures.

In this section, we aim at representing the forward and futures prices in energy markets as an element of a Hilbert space of functions. Motivated from results in Benth and Krühner [15], we will see that various relevant futures contracts traded in energy markets, which deliver the underlying over a period rather than at a fixed time in the future, can be understood as a bounded operator on a suitable Hilbert space.

Let us first introduce the Filipovic space (see Filipovic [27]), which will be the Hilbert space appropriate for our considerations. Let H_α be defined as the space of all absolutely continuous functions $g : \mathbb{R}_+ \to \mathbb{R}$ for which

$$\int_0^\infty \alpha(x) g'(x)^2 \, dx < \infty$$

for a given continuous and increasing weight function $\alpha : \mathbb{R}_+ \to [1, \infty)$ with $\alpha(0) = 1$. The norm of H_α is $\|g\|_\alpha^2 = \langle g, g \rangle$ for the inner product

$$\langle f, g \rangle = f(0)g(0) + \int_0^\infty \alpha(x) g'(x) f'(x) \, dx \, .$$

Here, $f, g \in H_\alpha$. We assume that $\int_0^\infty \alpha^{-1}(x) \, dx < \infty$. Remark that the typical choice of weight function is that of an exponential function, $\alpha(x) = \exp(\widetilde{\alpha} x)$ for a constant $\widetilde{\alpha} > 0$, in which case the integrability condition on the inverse of α is trivially satisfied. From Filipovic [27], we know that H_α is a separable Hilbert space. As we shall see, one can realize energy forward and futures prices as linear operators on H_α and in fact interpret energy forward and futures prices as stochastic processes with values in this space.

Let us consider a simple example motivating the appropriateness of the choice of H_α. The classical model for the dynamics of energy spot prices is the so-called Schwartz dynamics. (See Schwartz [36] and Benth, Benth, and Koekebakker [12, Chap. 3] for an extension to the Lévy case.) Here, the spot price $S(t)$ at time $t \geq 0$ is given by

$$S(t) = \exp(X(t))$$

for $X(t)$ being an Ornstein–Uhlenbeck (OU) process

$$dX(t) = \rho(\theta - X(t))\,dt + dL(t)\,,$$

driven by a Lévy process L. We assume that $L(1)$ has exponential moments, $\rho > 0, \theta$ are constants, and $\ln S(0) = X(0) = x \in \mathbb{R}$. From Benth, Benth, and Koekebakker [12, Prop. 4.6], we find that the forward price $f(t,T)$ at time $t \geq 0$, for a contract delivering at time $T \geq t$, is

$$f(t,T) = \exp\left(\mathrm{e}^{-\rho(T-t)}X(t) + \theta(1 - \mathrm{e}^{-\rho(T-t)}) + \int_0^{T-t} \phi(\mathrm{e}^{-\rho s})\,ds\right)$$

with ϕ being the logarithm of the moment generating function of $L(1)$. Recall that we model the spot price directly under the pricing measure Q. Letting $x = T - t \geq 0$, we find (by slightly abusing the notation)

$$f(t,x) = \exp\left(\mathrm{e}^{-\rho x}X(t) + \theta(1 - \mathrm{e}^{-\rho x}) + \int_0^{x} \phi(\mathrm{e}^{-\rho s})\,ds\right)\,.$$

It is simple to see that $x \mapsto f(t,x)$ is continuously differentiable for every t, and

$$\frac{\partial f}{\partial x}(t,x) = f(t,x)\left(\rho\mathrm{e}^{-\rho x}(\theta - X(t)) + \phi(\mathrm{e}^{-\rho x})\right)\,.$$

Assume that the weight function α is such that

$$\alpha(x)\mathrm{e}^{-2\rho x} \in L^1(\mathbb{R}_+)\,, \qquad \alpha(x)\phi^2(\mathrm{e}^{-2\rho x}) \in L^1(\mathbb{R}_+)\,.$$

Then it follows that $\int_0^\infty |\phi(\exp(-\rho s))|\,ds < \infty$ from the Cauchy–Schwartz inequality and the assumption $\int_0^\infty \alpha^{-1}(x)\,dx < \infty$. Hence, f is uniformly bounded in x since

$$|f(t,x)| \leq \exp\left(X(t) + \theta + \int_0^\infty |\phi(\mathrm{e}^{-\alpha s})|\,ds\right)\,.$$

But then

$$\begin{aligned}\|f(t,\cdot)\|_\alpha^2 &= |\exp(X(t))|^2 + \int_0^\infty \alpha(x)f^2(t,x)(\rho\mathrm{e}^{-\rho x}(\theta - X(t)) + \phi(\mathrm{e}^{-\rho x}))^2\,dx \\ &\leq c\mathrm{e}^{2X(t)}\left(1 + \int_0^\infty \alpha(x)\mathrm{e}^{-2\rho x}\,dx + \int_0^\infty \alpha(x)\phi^2(\mathrm{e}^{-\rho x})\,dx\right)\,,\end{aligned}$$

which shows that $f(t,\cdot) \in H_\alpha$. If L is a driftless Lévy process, the exponential moment condition on $L(1)$ yields that $\phi(x)$ has the representation

$$\phi(x) = \frac{1}{2}\sigma^2x^2 + \int_{\mathbb{R}} \{\mathrm{e}^{xz} - 1 - xz\}\,\ell(dz)$$

for a constant $\sigma \geq 0$ and Lévy measure $\ell(dz)$. But by the monotone convergence theorem and L'Hopital's rule, we find that

$$\lim_{x\searrow 0}\frac{1}{x^2}\int_{\mathbb{R}}\{\mathrm{e}^{xz} - 1 - xz\}\,\ell(dz) = \frac{1}{2}\int_{\mathbb{R}} z^2\,\ell(dz)\,,$$

and therefore $\phi(x) \sim x^2$ when x is small. Thus, a sufficient condition for $f(t,\cdot) \in H_\alpha$ is $\alpha(x)\exp(-2\rho x) \in L^1(\mathbb{R}_+,\mathbb{R})$.

We now move our attention to the main theme of this section, namely, the realization in H_α of general energy forward and futures contracts with a delivery period. Suppose that $F(t,T_1,T_2)$ is the swap price at time t of a contract on energy delivering over the time interval $[T_1,T_2]$, where $0 \le t \le T_1 < T_2$. Then one can express (see Benth, Benth, and Koekebakker [12, Prop. 4.1]) this price as

$$F(t,T_1,T_2) = \int_{T_1}^{T_2} \widetilde{w}(T;T_1,T_2) f(t,T)\,dT, \tag{2.1}$$

where $f(t,T)$, $t \le T$ is the forward price for a contract "delivering energy" at the fixed time time T, and $\widetilde{w}(T;T_1,T_2)$ is a deterministic weight function. We will later make precise assumptions on $\widetilde{w}$, but for now we implicitly assume that the integral in (2.1) makes sense. For example, at the NordPool and EEX power exchanges, swap contracts deliver electricity over specific weeks, months, quarters, and even years and are of either forward or futures style. The delivery is financial, meaning that the seller of a contract receives the accumulated spot price of power over the specified period of delivery (forward style) or the interest-rate discounted accumulated spot price (futures style), i.e., for these power swap contracts, we have the weight function

$$\widetilde{w}(T;T_1,T_2) = \frac{1}{T_2 - T_1} \tag{2.2}$$

for the forward-style contracts and

$$\widetilde{w}(T;T_1,T_2) = \frac{\mathrm{e}^{-rT}}{\int_{T_1}^{T_2} \mathrm{e}^{-rs}\,ds} \tag{2.3}$$

for the futures style. Here, $r > 0$ is the risk-free interest rate which we suppose to be constant. The reason for the averaging is the market convention of denominating forward and futures (swap) prices in terms of mega watt hours (MWh). In the gas market on NYMEX, say, gas is delivered physically at a location (Henry Hub in the case of NYMEX) over a given delivery period like month or quarter. We will therefore have the same expression (2.1) for the gas swap prices as in the case of power swaps.

Futures on temperature indices like HDD, CDD, and CAT[1] deliver the money-equivalent from the aggregated index value over a specified period. Hence, the futures price can be expressed as

$$F(t,T_1,T_2) = \int_{T_1}^{T_2} f(t,T)\,dT\,,$$

where $f(t,T)$ is the futures price of a contract that delivers the corresponding temperature index at the fixed delivery time $T \ge t$, i.e., temperature futures can be expressed by (2.1) with

$$\widetilde{w}(T;T_1,T_2) = 1 \tag{2.4}$$

[1]HDD is shorthand for heating-degree days, CDD for cooling-degree days, and CAT for cumulative average temperature.

as the weight function. We refer to Benth and Benth [13] for a discussion on weather futures as well as the definition of various temperature indices. Here, one may also find a discussion of the more recent wind futures, which can be expressed as the temperature futures except for a different index interpretation of f.

We aim at a so-called Musiela representation of $F(t,T_1,T_2)$ in (2.1). Define $x := T_1 - t$ as the time until start of delivery of the swap and $\ell = T_2 - T_1 > 0$ as the length of delivery of the swap. With the notation $g(t,y) := f(t,t+y)$, one easily derives

$$G_\ell^w(t,x) := F(t,t+x,t+x+\ell) = \int_x^{x+\ell} w_\ell(t,x,y)g(t,y)\,dy \tag{2.5}$$

for the weight function $w_\ell(t,x,y)$ defined by

$$w_\ell(t,x,y) := \widetilde{w}(t+y;t+x,t+x+\ell)\,, \tag{2.6}$$

where $y \in [x, x+\ell]$, $x \geq 0$ and $t \geq 0$. Referring to the different cases of the weight function $\widetilde{w}$, we find that $w_\ell(t,x,y) = 1$ for a temperature (wind) contract (with $\widetilde{w}$ as in (2.4)) and $w_\ell(t,x,y) = 1/\ell$ for the forward-style power (gas) swap (using $\widetilde{w}$ as in (2.2)). Slightly more interesting are the future-style power swaps, yielding

$$w_\ell(t,x,y) = \frac{r}{1-\mathrm{e}^{-r\ell}}\mathrm{e}^{-r(y-x)}\,. \tag{2.7}$$

Here, we used (2.3). Note that all these cases result in a weight function w_ℓ which is independent of time. Furthermore, the only case that depend on x and y is given in (2.7), which becomes in fact stationary in the sense that w_ℓ depends on $y - x$. We shall for simplicity restrict ourselves to the situation for which w_ℓ is time-independent and stationary. By slightly abusing notation, we consider weight functions $w_\ell : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$G_\ell^w(t,x) = \int_x^{x+\ell} w_\ell(y-x)g(t,y)\,dy\,. \tag{2.8}$$

Based on the different cases above, we assume that the weight function $u \mapsto w_\ell(u)$ is positive, bounded, and measurable.

Following Benth and Krühner [15, sect. 4], we can represent G_ℓ^w as a linear operator on g after performing a simple integration-by-parts, that is,

$$G_\ell^w(t) = \mathcal{D}_\ell^w(g(t)),$$

where, for a generic function $g \in H_\alpha$,

$$\mathcal{D}_\ell^w(g) = W_\ell(\ell)\mathrm{Id}(g) + \mathcal{I}_\ell^w(g)\,. \tag{2.9}$$

Here, Id is the identity operator, and the function $u \mapsto W_\ell(u)$, $u \geq 0$, is defined as

$$W_\ell(u) = \int_0^u w_\ell(v)\,dv\,. \tag{2.10}$$

As w_ℓ is a measurable and bounded function, W_ℓ is well-defined for every $u \geq 0$. Note that the limit of $W_\ell(u)$ does not necessarily exist when $u \to \infty$. For example, W_ℓ tends to infinity with u for $w_\ell = 1/\ell$ or $w_\ell(u) = 1$. However, when w_ℓ is as in (2.7), the limit of W_ℓ exists. Since w_ℓ is positive, the function $u \mapsto W_\ell(u)$ is increasing. Hence, $W_\ell(\ell) > 0$, and the first term of $\mathcal{D}_\ell^w$ in (2.9) is simply the indicator operator on H_α scaled by the positive number $W_\ell(\ell)$. Furthermore, $\mathcal{I}_\ell^w$ in (2.9) is an integral operator

$$\mathcal{I}_\ell^w(g) = \int_0^\infty q_\ell^w(\cdot, y) g'(y)\, dy \tag{2.11}$$

with kernel

$$q_\ell^w(x,y) = (W_\ell(\ell) - W_\ell(y-x)) 1_{[0,\ell]}(y-x)\,. \tag{2.12}$$

Before we show that $\mathcal{I}_\ell^w$ is a bounded operator on H_α, we look at a special case.

Consider a simple forward-style power swap, i.e., $w_\ell(u) = 1/\ell$. We get $W_\ell(u) = u/\ell$, and therefore $W_\ell(\ell) = 1$ yielding that first term in (2.9) is simply the identity operator on H_α. The integral operator $\mathcal{I}_\ell^w$ has the kernel

$$q_\ell^w(x,y) = \frac{1}{\ell}(x + \ell - y) 1_{[x, x+\ell]}(y)\,.$$

This example is analyzed in Benth and Krühner [15, sect. 4]. They show that the integral operator I_ℓ^w in this case is a bounded linear operator on H_α, implying that $t \mapsto G_\ell^w(t)$ is a stochastic process with values in H_α as long as $t \mapsto g(t)$ is an H_α-valued process. It turns out that the boundedness property of the integral operator $\mathcal{I}_\ell^w$ holds also for our class of more general weight functions. This is shown in the next proposition.

Proposition 2.1. *Under the assumption that $u \mapsto w_\ell(u)$ for $u \in \mathbb{R}_+$ is positive, bounded, and measurable, it holds that $\mathcal{I}_\ell^w$ is a bounded linear operator on H_α.*

Proof. Obviously, $q_\ell^w(x,y)$ is measurable on $\mathbb{R}_+^2$. Moreover, it is bounded since for $y \in [x, x+\ell]$

$$0 \leq W_\ell(\ell) - W_\ell(y-x) = \int_{y-x}^{\ell} w_\ell(u)\, du \leq c\ell\,,$$

where c is the constant majorizing w_ℓ. Hence, $0 \leq q_\ell^w(x,y) \leq c\ell$. It follows that

$$\int_0^\infty \alpha^{-1}(y)(q_\ell^w(x,y))^2\, dy \leq c^2\ell^2 \int_0^\infty \alpha^{-1}(y)\, dy < \infty$$

and part 1 of Corollary 4.5 in Benth and Krühner [15] holds. This implies that the integral operator $\mathcal{I}_\ell^w$ is defined for all $g \in H_\alpha$. We continue to demonstrate that part 2 of the same corollary also holds.

As shorthand notation, let, for a given $g \in H_\alpha$,

$$\xi(x) := \int_0^\infty q_\ell^w(x,y) g'(y)\, dy = \int_x^{x+\ell} (W_\ell(\ell) - W_\ell(y-x)) g'(y)\, dy\,.$$

In particular,

$$\xi(0) = \int_0^\ell (W_\ell(\ell) - W_\ell(y))g'(y)\,dy = \int_0^\ell \int_y^\ell w_\ell(u)\,du\,g'(y)\,dy\,.$$

Hence, we find

$$\begin{aligned}
\xi^2(0) &= \left(\int_0^\ell \int_y^\ell w_\ell(u)\,du\,g'(y)\,dy\right)^2 \\
&\le \left(\int_0^\ell \int_y^\ell w_\ell(u)\,du|g'(y)|\,dy\right)^2 \\
&\le \left(\int_0^\ell w_\ell(u)\,du\right)^2 \left(\int_0^\ell |g'(y)|\,dy\right)^2 \\
&= W_\ell^2(\ell)\left(\int_0^\ell \sqrt{\alpha(y)}|g'(y)|\sqrt{\alpha(y)}^{-1}\,dy\right)^2 \\
&\le W_\ell^2(\ell)\int_0^\ell \alpha^{-1}(y)\,dy \int_0^\ell \alpha(y)g'(y)^2\,dy \\
&\le W_\ell^2(\ell)\int_0^\ell \alpha^{-1}(y)\,dy\|g\|_\alpha^2\,,
\end{aligned}$$

where in the second inequality we used that w_ℓ is positive and in the third the Cauchy–Schwartz inequality. Recall that by assumption, $\int_0^\infty \alpha^{-1}(y)\,dy < \infty$. Furthermore, it holds that

$$\begin{aligned}
\xi'(x) &= \frac{d}{dx}\int_x^{x+\ell} (W_\ell(\ell) - W_\ell(y-x))g'(y)\,dy \\
&= (W_\ell(\ell) - W_\ell(\ell))g'(x+\ell) - (W_\ell(\ell) - W_\ell(0))g'(x) \\
&\quad + \int_x^{x+\ell} (-W_\ell'(y-x))(-1)g'(y)\,dy \\
&= \int_x^{x+\ell} w_\ell(y-x)g'(y)\,dy - W_\ell(\ell)g'(x)\,,
\end{aligned}$$

and therefore ξ has a (weak) derivative. By the triangle inequality,

$$\xi'(x)^2 \le 2W_\ell(\ell)g'(x)^2 + 2\left(\int_x^{x+\ell} w_\ell(y-x)g'(y)\,dy\right)^2\,.$$

We consider the second term on the right-hand side: By the Cauchy–Schwartz inequality and boundedness of w_ℓ,

$$\begin{aligned}
\int_0^\infty \alpha(x)\left(\int_x^{x+\ell} w_\ell(y-x)g'(y)\,dy\right)^2 dx &\le \int_0^\infty \alpha(x)\left(\int_x^{x+\ell} w_\ell(y-x)|g'(y)|\,dy\right)^2 dx \\
&\le \int_0^\infty \alpha(x)\int_x^{x+\ell} w_\ell^2(y-x)\,dy \int_x^{x+\ell} g'(y)^2\,dy\,dx
\end{aligned}$$

$$\leq c^2\ell \int_0^\infty \int_x^{x+\ell} \alpha(y)g'(y)^2\,dy\,dx$$
$$\leq c^2\ell^2\|g\|_\alpha^2\,,$$

after using Fubini's theorem and that α is nondecreasing. Wrapping up these estimates, we majorize the H_α-norm of ξ

$$\begin{aligned}\|\xi\|_\alpha^2 &= |\xi^2(0)| + \int_0^\infty \alpha(x)\xi'(x)^2\,dx \\ &\leq W_\ell^2(\ell)\int_0^\ell \alpha^{-1}(y)\,dy\|g\|_\alpha^2 + 2W_\ell^2(\ell)\|g\|_\alpha^2 + 2c^2\ell^2\|g\|_\alpha^2 \\ &\leq C\|g\|_\alpha^2\end{aligned}$$

for a positive constant C. But then $\xi \in H_\alpha$, and we can conclude from Corollary 4.5 of Benth and Krühner [15] that $\mathcal{I}_\ell^w$ is a continuous linear operator on H_α. The proposition follows. ∎

From Proposition 2.1, it follows immediately that $\mathcal{D}_\ell^w$ in (2.9) is a continuous linear operator on H_α, as it is the sum of the scaled identity operator and the integral operator $\mathcal{I}_\ell^w$. Moreover, for $g \in H_\alpha$, it holds (by inspection of the proof of Proposition 2.1) that

$$\||\mathcal{D}_\ell^w(g)\|_\alpha \leq \left\{ W_\ell(\ell) + \sqrt{W_\ell^2(\ell)(2 + \int_0^\ell \alpha^{-1}(y)\,dy) + 2c^2\ell^2} \right\} \|g\|_\alpha\,,$$

which provides us with an upper bound on the operator norm of $\mathcal{D}_\ell^w$. Furthermore, it follows immediately from Proposition 2.1 that we can realize the dynamics of swap price curves in H_α, e.g., if $g(t)$ is an H_α-valued stochastic process, then $t \mapsto G_\ell^w(t)$ will be a stochastic process with values in H_α as well.

3. European options on energy forwards and futures. At the energy exchanges, plain vanilla call and put options are offered for trade on futures and forward contracts. For example, at NordPool, one can buy and sell options on the quarterly settled power futures contracts, while at CME, one can trade in options on weather futures, including HDD/CDD and CAT temperature futures. NYMEX offers trade in options on gas futures, among a number of other derivatives on energy and commodity futures (including different blends of oil).

Consider a European option on an energy forward contract delivering over the period $[T_1, T_2]$ and price $F(t, T_1, T_2)$ at time t, where the option has exercise time $0 \leq \tau \leq T_1$ and payoff $p(F(\tau, T_1, T_2))$ for some function $p : \mathbb{R} \to \mathbb{R}$. For plain vanilla call and put options, we have $p(x) = \max(x - K, 0)$ or $p(x) = \max(K - x, 0)$, respectively, with the strike price denoted K. We assume in general p to be a measurable function of at most linear growth. We recall the representation $F(t, T_1, T_2) = \mathcal{D}_\ell^w(g(t))(T_1 - t)$. The following proposition provides the link to the infinite-dimensional swap prices.

Proposition 3.1. *Suppose that p is of at most linear growth. It holds that*

$$p(F(\tau, T_1, T_2)) = \mathcal{P}_\ell(T_1 - \tau, g(\tau))$$

for a nonlinear functional $\mathcal{P}_\ell^w : \mathbb{R}_+ \times H_\alpha \to \mathbb{R}$ *defined by*

$$\mathcal{P}_\ell^w(x,g) = p \circ \delta_x \circ \mathcal{D}_\ell^w(g)\,.$$

Here, $\ell = T_2 - T_1$. *Moreover, there exists a constant* $c_\ell > 0$ *depending on* ℓ *such that*

$$|\mathcal{P}_\ell^w(\cdot,g)|_\infty \le c_\ell(1+\|g\|_\alpha)\,.$$

Proof. Since we have $F(\tau,T_1,T_2) = G^w_{T_2-T_1}(\tau,T_1-\tau)$, the first claim follows. From the linear growth of p , we find

$$|\mathcal{P}_\ell^w(x,g)| = |p(\mathcal{D}_\ell^w(g)(x))| \le c_1\,(1+|\mathcal{D}_\ell^w(g)(x)|)$$

for a positive constant c_1. Since $\int_0^\infty \alpha^{-1}(y)\,dy < \infty$, we find by Lemma 3.2 in Benth and Krühner [15]

$$|\mathcal{P}_\ell^w(\cdot,g)|_\infty = \sup_{x\in\mathbb{R}_+} |\mathcal{P}_\ell(x,g)| \le c_2\,(1+\|\mathcal{D}_\ell^w(g)\|_\alpha)$$

for a positive constant $c_2 > 0$. But $\mathcal{D}_\ell^w$ is a continuous linear operator on H_α by Proposition 2.1, and hence so is $\mathcal{D}_\ell^w$. The last claim follows, and the proof is complete. ■

Consider the special case of power forwards, for which we recall that $w_\ell(u) = 1/\ell$. In this case, we observe

$$\lim_{\ell\downarrow 0} G_\ell^w(t,x) = \frac{\partial}{\partial \ell}\int_x^{x+\ell} g(t,y)\,dy|_{\ell=0} = g(t,x)\,.$$

Hence, we can make sense out of $\mathcal{P}_0^w$ for $w_\ell(u) = 1/\ell$ as

$$\mathcal{P}_0(x,g) = p\circ\delta_x(g)\,. \tag{3.1}$$

Here, $x \in \mathbb{R}_+$ and $g \in H_\alpha$, and we use the simplified notation $\mathcal{P}_0$ instead of $\mathcal{P}_0^w$ in this particular case. We note that the nonlinear operator $\mathcal{P}_0$ will be the payoff from an option on a forward with fixed time to delivery x instead of a delivery period which lasts $\ell > 0$, since it holds that

$$p(f(\tau,T)) = \mathcal{P}_0(T-\tau, g(\tau)) \tag{3.2}$$

for $\tau \le T$. The markets for oil at NYMEX, for example, trade in forwards and futures with fixed delivery times and options on these contracts. It is straightforward from Lemma 3.2 in Benth and Krühner [15] that

$$|\mathcal{P}_0(\cdot,g)|_\infty = \sup_{x\in\mathbb{R}_+}|p(g(x))| \le c_1(1+\sup_{x\in\mathbb{R}_+}|g(x)|) \le c_2\,(1+\|g\|_\alpha)$$

for $g \in H_\alpha$ and a payoff function p with at most linear growth.

Suppose now that $g(t)$ is a stochastic process in H_α satisfying

$$\mathbb{E}\,[\|g(t)\|_\alpha] < \infty \tag{3.3}$$

for all $t \geq 0$. The price $V(t)$ at time $0 \leq t \leq \tau$ of the option with payoff $p(F(\tau; T_1, T_2)$ at time $0 < \tau \leq T_1$ is given as

$$V(t) = \mathrm{e}^{-r(\tau - t)} \mathbb{E}\left[\mathcal{P}_\ell^w(T_1 - \tau, g(\tau)) \,|\, \mathcal{F}_t\right] . \tag{3.4}$$

The expectation is well-defined by Proposition 3.1 for any given $\ell > 0$. If we select $w_\ell(u) = 1/\ell$, then the option value in (3.4) also incorporates contracts written on fixed-delivery forwards, that is, options with payoff $p(f(\tau, T))$,

$$V(t) = \mathrm{e}^{-r(\tau - t)} \mathbb{E}\left[\mathcal{P}_0(T - \tau, g(\tau)) \,|\, \mathcal{F}_t\right] . \tag{3.5}$$

This is also well-defined under the assumption (3.3).

3.1. Markovian forward curves. We want to analyze option prices for a class of Markovian forward curve dynamics, where the process $g(t)$ is specified as the solution of a (first-order) stochastic partial differential equation. We shall be concerned with dynamics driven by an infinite-dimensional Lévy process.

Before proceeding, let us first introduce some general notions (see, e.g., Peszat and Zabczyk [34] for what follows): A random variable X with values in a separable Hilbert space H is *square integrable* if $\mathbb{E}(\|X\|^2) < \infty$. If X is square integrable, $\mathcal{Q} \in L(H)$ is called the *covariance operator* of X if

$$\mathbb{E}\left(\langle X, u\rangle \langle X, v\rangle\right) = \langle \mathcal{Q}u, v\rangle$$

for any $u, v \in H$. Here, $\langle \cdot, \cdot \rangle$ is the inner product in H and $\| \cdot \|$ the associated norm. The following result can be found in Peszat and Zabczyk [34, Thm. 4.44] and is stated here for convenience.

Lemma 3.2. *Let X be a square integrable H-valued random variable where H is a separable Hilbert space. Then there is a unique operator $\mathcal{Q} \in L(H)$ such that $\mathcal{Q}$ is the covariance operator of X. Moreover, $\mathcal{Q}$ is a positive semidefinite trace class operator. Consequently, there is an orthonormal basis $(e_n)_{n \in I}$ of H and a sequence $(\lambda_n)_{n \in I} \in l^1(I, \mathbb{R}_+)$ such that*

$$\mathcal{Q}u = \sum_{n \in \mathbb{N}} \lambda_n \langle e_n, u\rangle e_n$$

for any $u \in H$.

For a separable Hilbert space H, $\mathbb{L} := \{\mathbb{L}(t)\}_{t \geq 0}$ is an H-valued Lévy process if $\mathbb{L}$ has independent and stationary increments and stochastically continuous paths and $\mathbb{L}(0) = 0$. This definition is found in Peszat and Zabczyk [34, Chap. 4] and can in fact be formulated on a general Banach space. We remark in passing that Theorem. 4.44 in Peszat and Zabczyk [34] is formulated for Lévy processes.

Let us now move our attention back to modelling the forward rate dynamics and suppose that $\mathbb{L}$ is a square-integrable H-valued Lévy process with zero mean and denote its covariance operator by $\mathcal{Q}$. Furthermore, let $\sigma : \mathbb{R}_+ \times H_\alpha \to L(H, H_\alpha)$ be a measurable map, and assume there exists an increasing function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following Lipschitz continuity and linear growth holds: for any $f, h \in H_\alpha$ and $t \in \mathbb{R}_+$,

$$\|\sigma(t,f) - \sigma(t,h)\|_{\text{op}} \leq K(t)\|f - h\|_\alpha \,, \tag{3.6}$$

$$\|\sigma(t,f)\|_{\text{op}} \leq K(t)(1 + \|f\|_\alpha) \,. \tag{3.7}$$

Consider the dynamics of the H_α-valued stochastic process $\{g(t)\}_{t\geq 0}$ defined by the stochastic partial differential equation

$$dg(t) = \partial_x g(t)\, dt + \sigma(t, g(t))\, d\mathbb{L}(t) \,. \tag{3.8}$$

Let $\mathcal{S}_x, x \geq 0$ denote the right-shift operator on H_α, i.e., $\mathcal{S}_x f = f(x + \cdot)$. Then $\mathcal{S}_x$ is the C_0-semigroup generated by the operator ∂_x (see Filipovic [27, Theorem. 5.1.1]). From Lemma 3.5 in Benth and Krühner [15], $\mathcal{S}_x$ is quasi-contractive, i.e., there exists a positive constant c such that $\|\mathcal{S}_x\|_{\text{op}} \leq \exp(ct)$ for $t > 0$. Hence, referring to Theorem 4.5 in Tappe [37] (or Albeverio, Mandrekar, and Rüdiger [2] for the path-dependent case), there exists a unique mild solution of (3.8) for $s \geq t$, that is, a càdlàg process $g \in H_\alpha$ satisfying

$$g(s) = \mathcal{S}_{s-t} g(t) + \int_t^s \mathcal{S}_{s-u} \sigma(u, g(u))\, d\mathbb{L}(u) \,. \tag{3.9}$$

The shift and the pricing operator for $F(t, T_1, T_2)$ commute, which allows one to find the dynamics of $F(\cdot, T_1, T_2)$. Moreover, this dynamics reveals that $t \mapsto F(t, T_1, T_2)$ is a martingale in our setup, as desired.

Lemma 3.3. *We have $\mathcal{S}_x \mathcal{D}_\ell^w = \mathcal{D}_\ell^w \mathcal{S}_x$ for any $x \geq 0$. Consequently, we have*

$$F(s, T_1, T_2) = \delta_{T_1 - t} \mathcal{D}_\ell^w g(t) + \int_t^s \delta_{T_1 - u} \mathcal{D}_\ell^w \sigma(u, g(u))\, d\mathbb{L}(u) \tag{3.10}$$

for any $0 \leq t \leq s$.

Proof. The first equality follows from a straightforward computation. Applying the mild solution in equation (3.9) to $F(s, T_1, T_2) = \delta_{T_1 - s} \mathcal{D}_\ell^w g(s)$, the claim follows after using the commutation property. ■

Remark 3.4. Lemma 3.3 reveals that the dynamics of $s \mapsto F(s, T_1, T_2)$ are martingales. Conversely, if one assumes that $dg(s) = \beta(s)ds + \sigma(s, g(s))d\mathbb{L}(s)$ and that the dynamics of $s \mapsto F(s, T_1, T_2)$ are martingales for any $0 \leq T_1 < T_2$, then sending $T_2 \searrow T_1$ shows that

$$g(s) = \mathcal{S}_s g(0) + \int_0^s \mathcal{S}_{s-u} \sigma(u, g(u))\, d\mathbb{L}(u)$$

for any $s \geq 0$ which reveals that g is a solution to the stochastic partial differential equation (3.8). Hence, it is somewhat equivalent to assume that the dynamics of $s \mapsto F(s, T_1, T_2)$ are martingales and that the drift of g at time t is given by $\partial_x g(t)$.

Below, it will be convenient to know that $\mathcal{S}_x$ is uniformly bounded in the operator norm.

Lemma 3.5. *It holds that $\|\mathcal{S}_x\|_{op}^2 \leq 2\max(1, \int_0^\infty \alpha^{-1}(y)\, dy)$ for $x \geq 0$.*

Proof. This follows by a direct calculation: By the fundamental theorem of calculus, the elementary inequality $2ab \leq a^2 + b^2$ and α being nondecreasing, we find for $f \in H_\alpha$

$$\begin{aligned}\|\mathcal{S}_x f\|_\alpha^2 &= f^2(x) + \int_0^\infty \alpha(y)|f'(x+y)|^2\,dy \\ &= \left(f(0) + \int_0^x f'(y)\,dy\right)^2 + \int_x^\infty \alpha(y-x)|f'(y)|^2\,dy \\ &\leq 2f^2(0) + 2\left(\int_0^x \alpha^{-1/2}(y)\alpha^{1/2}(y)f'(y)\,dy\right)^2 + \int_x^\infty \alpha(y)|f'(y)|^2\,dy\,.\end{aligned}$$

Appealing to the Cauchy–Schwartz inequality, we find

$$\|\mathcal{S}_x f\|_\alpha^2 \leq 2f^2(0) + 2\int_0^x \alpha^{-1}(y)\,dy \int_0^x \alpha(y)|f'(y)|^2\,dy + \int_x^\infty \alpha(y)|f'(y)|^2\,dy\,.$$

Hence, $\|\mathcal{S}_x f\|_\alpha^2 \leq \max(2, 2\int_0^\infty \alpha^{-1}(y)\,dy)\|f\|_\alpha^2$, and the lemma follows. ■

From (3.9), the dynamics of g becomes Markovian. This means in particular that $V(t)$ defined in (3.4) can be expressed as $V(t) = V(t, g(t))$ (with a slight abuse of notation) for

$$V(t, \overline{g}) = \mathrm{e}^{-r(\tau - t)}\mathbb{E}\left[\mathcal{P}_\ell(g^{t,\overline{g}}(\tau))\right]\,. \tag{3.11}$$

Here, we have used the notation $g^{t,\overline{g}}(s)\, s \geq t$ for the process $g(s), s \geq t$, starting in $\overline{g}$ at time t, e.g., $g^{t,\overline{g}}(t) = \overline{g}$, $\overline{g} \in H_\alpha$.

We shall use the continuity of the translation operator as a linear operator on H_α to prove Lipschitz continuity of the functional $\overline{g} \mapsto V(t, \overline{g})$, uniformly in $t \leq \tau$. Recall that τ is the exercise time of the option in question.

Proposition 3.6. *Assume that the payoff function p is Lipschitz continuous and volatility functional $g \mapsto \sigma(s, g)$ satisfies the Lipschitz and linear growth conditions in* (3.6), (3.7)*. Then there exists a positive constant C (depending on τ) such that*

$$\sup_{t \leq \tau} |V(t, \overline{g}) - V(t, \widetilde{g})| \leq C\|\overline{g} - \widetilde{g}\|_\alpha$$

for $\overline{g}, \widetilde{g} \in H_\alpha$.

Proof. As p is Lipschitz continuous, it follows that $\overline{g} \mapsto \mathcal{P}_\ell(x, \overline{g})$ is Lipschitz continuous since $\mathcal{P}_\ell(x, \cdot) = p \circ \delta_x \circ \mathcal{D}_\ell^w$, and $\delta_x, \mathcal{D}_\ell^w$ are bounded linear operators. Moreover, the Lipschitz continuity is uniform in x, as it follows from Lemma 3.1 in Benth and Krühner [15] that the operator norm of δ_x satisfies

$$\|\delta_x\|_{\mathrm{op}}^2 = 1 + \int_0^x \alpha^{-1}(y)\,dy \leq 1 + \int_0^\infty \alpha^{-1}(y)\,dy < \infty\,.$$

Hence, there exists a constant $C_{\mathcal{P}} > 0$ such that

$$|\mathcal{P}_\ell(x, \overline{g}) - \mathcal{P}_\ell(x, \widetilde{g})| \leq C_{\mathcal{P}}\|\overline{g} - \widetilde{g}\|_\alpha\,.$$

Therefore,

$$|V(t, \overline{g}) - V(t, \widetilde{g})| \leq C_{\mathcal{P}}\mathbb{E}\left[\|g^{t,\overline{g}}(\tau) - g^{t,\widetilde{g}}(\tau)\|_\alpha\right]\,.$$

Since

$$g^{t,\overline{g}}(\tau) = \mathcal{S}_{\tau-t}\overline{g} + \int_t^\tau \mathcal{S}_{\tau-s}\sigma(s, g^{t,\overline{g}}(s))\, d\mathbb{L}(s),$$

we have by the triangle inequality and Lemma 3.5

$$\begin{aligned}\|g^{t,\overline{g}}(\tau) - g^{t,\widetilde{g}}(\tau)\|_\alpha &\leq \|\mathcal{S}_{\tau-t}(\overline{g} - \widetilde{g})\|_\alpha + \|\int_t^\tau \mathcal{S}_{\tau-s}\left(\sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))\right) d\mathbb{L}(s)\|_\alpha \\ &\leq c\|\overline{g} - \widetilde{g}\|_\alpha + \|\int_t^\tau \mathcal{S}_{\tau-s}\left(\sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))\right) d\mathbb{L}(s)\|_\alpha\,,\end{aligned}$$

where the constant c is positive and in fact given explicitly in Lemma 3.5. By the Itô isometry, it follows that

$$\begin{aligned}&\mathbb{E}\left[\|\int_t^\tau \mathcal{S}_{\tau-s}\left(\sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))\right) d\mathbb{L}(s)\|_\alpha^2\right] \\ &\qquad = \int_t^\tau \mathbb{E}\left[\|\mathcal{S}_{\tau-s}\left(\sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))\right)\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H,H_\alpha)}\right] ds\,.\end{aligned}$$

Now let $\mathcal{T} \in L(H, H_\alpha)$. Then, we have

$$\begin{aligned}\|\mathcal{S}_x\mathcal{T}\mathcal{Q}^{1/2}\|_{L_{HS}(H,H_\alpha)} &\leq \|\mathcal{S}_x\|_{\mathrm{op}}\|\mathcal{T}\|_{\mathrm{op}}\|\mathcal{Q}^{1/2}\|_{L_{HS}(H)} \\ &\leq c\|\mathcal{T}\|_{\mathrm{op}}\|\mathcal{Q}^{1/2}\|_{L_{HS}(H)}.\end{aligned}$$

Letting $\mathcal{T} = \sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))$ and $x = \tau - s$, we find from the Lipschitz continuity of σ in (3.6)

$$\begin{aligned}&\|\mathcal{S}_{\tau-s}\left(\sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))\right)\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H,H_\alpha)} \\ &\qquad \leq c^2\|\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H)}\|\sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))\|^2_{\mathrm{op}} \\ &\qquad \leq c^2K^2(s)\|\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H)}\|g^{t,\overline{g}}(s) - g^{t,\widetilde{g}}(s)\|^2_\alpha\,.\end{aligned}$$

But K is an increasing function in the Lipschitz continuity of σ, so $K(s) \leq K(\tau)$. Hence,

$$\begin{aligned}&\mathbb{E}\left[\|\int_t^\tau \mathcal{S}_{\tau-s}\left(\sigma(s, g^{t,\overline{g}}(s)) - \sigma(s, g^{t,\widetilde{g}}(s))\right) d\mathbb{L}(s)\|_\alpha^2\right] \\ &\qquad \leq c^2K^2(\tau)\|\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H)}\int_t^\tau \mathbb{E}\left[\|g^{t,\overline{g}}(s) - g^{t,\widetilde{g}}(s)\|^2_\alpha\right] ds\,.\end{aligned}$$

If we now apply the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we derive

$$\begin{aligned}\mathbb{E}\left[\|g^{t,\overline{g}}(\tau) - g^{t,\widetilde{g}}(\tau)\|^2_\alpha\right] &\leq 2c^2\|\overline{g} - \widetilde{g}\|^2_\alpha \\ &\quad + 2c^2\|\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H)}K^2(\tau)\int_t^\tau \mathbb{E}\left[\|g^{t,\overline{g}}(s) - g^{t,\widetilde{g}}(s)\|^2_\alpha\right] ds\,.\end{aligned}$$

Grönwall's inequality then yields

$$\mathbb{E}\left[\|g^{t,\overline{g}}(\tau) - g^{t,\widetilde{g}}(\tau)\|^2_\alpha\right] \leq 2c\mathrm{e}^{2c\|\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H)}K^2(\tau)(\tau-t)}\|\overline{g} - \widetilde{g}\|^2_\alpha\,.$$

From Jensen's inequality, we thus derive

$$|V(t,\overline{g})-V(t,\widetilde{g})|\leq C_{\mathcal{P}}\sqrt{2c}\mathrm{e}^{(2cK^2(\tau)\|\mathcal{Q}^{1/2}\|^2_{L_{\mathrm{HS}}(H)}\tau}\|\overline{g}-\widetilde{g}\|_\alpha\,,$$

and the result follows. ■

The proposition shows that the option price is uniformly Lipschitz continuous in the initial forward curve as long as we consider Lipschitz continuous payoff functions and volatility operators σ. We remark that put and call options have Lipschitz continuous payoff functions. One immediate interpretation of the uniform Lipschitz property of the functional $g\mapsto V(t,g)$ is that the option price is stable with respect to small perturbations in the initial curve g. This means, in practical terms, that the option price is robust toward small errors in the specification of the initial curve. It is important to notice that we only have available a discrete set of forward prices in practice, and thus the specification of the initial curve g may be prone to error as it is not perfectly observable.

Another interesting application of Proposition 3.6 is the majorization of the option pricing error in case we wish to compute the price for a finite-dimensional projection of the infinite-dimensional curve g. Recall that from a practical market perspective, we only have knowledge of a finite subset of values from the whole curve g. This is the situation we discuss now.

Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of H_α, and define the projection operator $\Gamma_n:H_\alpha\to H_\alpha^n$ by

$$\Gamma_n g=\sum_{k=1}^n\langle g,e_k\rangle_\alpha e_k\,, \tag{3.12}$$

where H_α^n is the n-dimensional subspace of H_α spanned by the basis $\{e_1,\dots,e_n\}$. The option price with $\Gamma_n g$ as initial curve becomes $V_n(t,g):=V(t,\Gamma_n g)$, and we find from Proposition 3.6 that

$$\sup_{t\leq\tau}|V(t,g)-V_n(t,g)|\leq C\|g-\Gamma_n g\|_\alpha\,.$$

But, when $n\to\infty$, it follows from Parseval's identity

$$\|g-\Gamma_n g\|_\alpha^2=\sum_{k=n+1}^\infty|\langle g,e_k\rangle_\alpha|^2\to 0\,,$$

and we can approximate $V(t,g)$ within a desirable error by choosing n sufficiently big. Note that with

$$\widehat{V}(t,x_1,\dots,x_n):=V(t,\sum_{k=1}^n x_k e_k),$$

we have that $V_n(t,g)=\widehat{V}(t,\langle g,e_1\rangle_\alpha,\dots,\langle g,e_n\rangle_\alpha)$. We can view $\widehat{V}(t,x_1,\dots,x_n)$ as the option price on the H_α-valued stochastic process g, which is started in the finite-dimensional subspace H_α^n at time t with the values $\langle\Gamma_n g,e_k\rangle_\alpha=x_k, k=1,\dots,n$. By the dynamics of g, we have no guarantee that the process g will remain in H_α^n, so that at time τ we have in general that $g^{t,\Gamma_n g}(\tau)\notin H_\alpha^n$. Indeed, it may truly be an infinite-dimensional object and thus not in any H_α^m, $m\in\mathbb{N}$. Furthermore, it is important to note that such an approximation $\Gamma_n g$ typically fails to be a martingale under the pricing measure Q, and hence the option price $V_n(t,g)$ will not be arbitrage-free. In a forthcoming paper [17], we study arbitrage-free finite-dimensional approximations.

3.2. The arithmetic Gaussian case. Suppose that g solves the simple linear Musiela equation

$$dg(t) = \partial_x g(t)\,dt + \sigma(t)\,d\mathbb{B}(t), \tag{3.13}$$

where $\mathbb{B}$ is an H-valued Wiener process with covariance operator $\mathcal{Q}$ and H being a separable Hilbert space. The volatility σ is assumed to be a stochastic process $\sigma : \mathbb{R}_+ \mapsto L(H, H_\alpha)$, where $\sigma \in \mathcal{L}^2_{\mathbb{B}}(H_\alpha)$, the space of integrands for the stochastic integral with respect to $\mathbb{B}$ (see section 8.2 in Peszat and Zabczyk [34]). This is indeed a special case of the general Markovian dynamics presented above, and the mild solution becomes

$$g(\tau) = \mathcal{S}_{\tau-t}g(t) + \int_t^\tau \mathcal{S}_{\tau-s}\sigma(s)\,d\mathbb{B}(s) \tag{3.14}$$

for $\tau \geq t$. We now analyze $V(t)$ defined in (3.4) and (3.5) for this particular dynamics. First, recall from Lemma 3.3 that

$$F(\tau, T_1, T_2) = \delta_{T_1-t}\mathcal{D}^w_\ell g(t) + \int_t^\tau \delta_{T_1-s}\mathcal{D}^w_\ell \sigma(s)\,d\mathbb{B}(s)$$

for any $t \in [0, \tau]$.

It follows from Theorem 2.1 in Benth and Krühner [15] that

$$F(\tau, T_1, T_2) = \delta_{T_1-t}\mathcal{D}^w_\ell g(t) + \int_t^\tau \widetilde{\sigma}(s)\,dB(s) \tag{3.15}$$

for any $t \in [0, \tau]$ where

$$\widetilde{\sigma}^2(s) = (\delta_{T_1-s}\mathcal{D}^w_\ell \sigma(s)\mathcal{Q}\sigma^*(s)(\delta_{T_1-s}\mathcal{D}^w_\ell)^*)(1)$$

and B is a standard Brownian motion.

This implies

$$V(t, g(t)) = e^{-r(\tau-t)}\mathbb{E}[p(F(\tau, T_1, T_2))].$$

We find the following particular result for V in the case of a nonrandom volatility.

Proposition 3.7. *Let σ be nonrandom. Then we have*

$$V(t, g) = \mathrm{e}^{-r(\tau-t)}\mathbb{E}\left[p(m(g) + \xi X)\right]$$

for any for $t \leq \tau \leq T_1$. Here, X is a standard normal distributed random variable,

$$\xi^2 := \int_t^\tau \widetilde{\sigma}^2(s)\,ds = \int_t^\tau (\delta_{T_1-s}\mathcal{D}^w_\ell \sigma(s)\mathcal{Q}\sigma^*(s)(\delta_{T_1-s}\mathcal{D}^w_\ell)^*)(1)\,ds$$

for any $t \in [0, \tau]$ *and*

$$m(g) := (\delta_{T_1 - t} \circ \mathcal{D}_\ell^w)(g)\,, \quad g \in H_\alpha.$$

Proof. In the case of σ being nonrandom, we find that the stochastic integral $\int_t^\tau \widetilde{\sigma}(s)\,dB(s)$ in (3.15) is a centered normal distributed random variable. The variance is ξ^2, which follows straightforwardly by the Itô isometry. By the independent increment property of Brownian motion, the result follows. ■

In order to compute the realized variance ξ^2 in the proposition above, we must find the dual operator of $\delta_{T_1-s} \circ \mathcal{D}_\ell^w$. Obviously, it holds that

$$(\delta_{T_1-s} \circ \mathcal{D}_\ell^w)^* = \mathcal{D}_\ell^{w*} \circ \delta_{T_1-s}^*\,.$$

The dual operator of δ_y is found in Filipovic [27] (see also Lemma 3.1 in [15]) and is the mapping $\delta_y^* : \mathbb{R} \mapsto H_\alpha$ defined as

$$\delta_y^*(c) : x \mapsto c + c\int_0^{y\wedge x} \alpha^{-1}(u)\,du := ch_y(x) \tag{3.16}$$

for $c \in \mathbb{R}$ and $x \geq 0$ and

$$h_y(x) = 1 + \int_0^{y \wedge x} \alpha^{-1}(u)\,du\,. \tag{3.17}$$

Thus, $\delta_{T_1-s}^*(1)$ is the function

$$\delta_{T_1-s}^*(1)(x) = h_{T_1-s}(x) = 1 + \int_0^{(T_1-s)\wedge x} \alpha^{-1}(u)\,du \tag{3.18}$$

for $x \geq 0$. Now we are left to derive the function $\mathcal{D}_\ell^{w*}(h_{T_1-s})$.

Proposition 3.8. *With the preceding notation we have*

$$\mathcal{D}_\ell^{w*}(h_{T_1-s})(x) = W_\ell(\ell)h_{T_1-s}(x) + \int_0^x \frac{q_\ell^w(T_1-s,z)}{\alpha(z)}dz$$

for any $s \in [0, T_1]$, $x \geq 0$.

Proof. Let $x \geq 0$ and $s \in [0, T_1]$. Then we have

$$\begin{aligned}
\mathcal{D}_\ell^{w*}(h_{T_1-s})(x) &= \langle \mathcal{D}_\ell^{w*}(h_{T_1-s}), h_x\rangle \\
&= \langle h_{T_1-s}, \mathcal{D}_\ell^w h_x\rangle \\
&= \mathcal{D}_\ell^w h_x(T_1 - s) \\
&= W_\ell(\ell)h_{T_1-s}(x) + \int_0^x \frac{q_\ell^w(T_1-s,z)}{\alpha(z)}dz.
\end{aligned}$$ ■

If we define $\Sigma(s) := \widetilde{\sigma}(s)\mathcal{Q}\widetilde{\sigma}^*(s)$ and if we want to apply Proposition 3.7, then we need to calculate

$$\zeta^2 = \int_t^\tau (\delta_{T_1-s}\mathcal{D}_\ell^w)\Sigma(s)(\delta_{T_1-s}\mathcal{D}_\ell^w)^*(1)ds.$$

With a representation for $(\delta_{T_1-s}\mathcal{D}_\ell^w)^*(1) = \mathcal{D}_\ell^{w*}(h_{T_1-s})$ at hand, we still need to calculate the operator $\delta_{T_1-s}\mathcal{D}_\ell^w$. However, we simply have

$$\delta_{T_1-s}\mathcal{D}_\ell^w g = W_\ell(\ell)g(T_1-s) + \int_0^\infty q_\ell^w(T_1-s,y)g'(y)dy$$

for any $g \in H_\alpha$. Q and σ are—of course—up to the modeller's choice. However, after σ and Q have been picked, one does need to calculate $\sigma^*(s)$. The following proposition gives a simple formula for calculating the dual operator of a given operator. As a side remark, the next proposition also shows that any linear operator $\mathcal{T} = (\mathcal{T}^*)^*$ on H_α is the sum of an integral operator and an operator which "only" acts on the initial value of the inserted function.

Proposition 3.9. *Let $\mathcal{T} \in L(H_\alpha)$. Then*

$$\mathcal{T}^* g(x) = g(0)\eta(x) + \int_0^\infty q(x,y)g'(y)dy, \quad g \in H_\alpha,$$

where

$$\begin{aligned}\eta(x) &:= (\mathcal{T}h_x)(0) = (\mathcal{T}^* h_0)(x),\\ q(x,y) &:= (\mathcal{T}h_x)'(y)\alpha(y)\end{aligned}$$

for any $x, y \geq 0$ and h_x is defined in (3.17).

Proof. Filipovic [27, Lem. 5.3.1] shows that $g(x) = \langle g, h_x\rangle$ for any $g \in H_\alpha$, $x \geq 0$. Hence,

$$\begin{aligned}\mathcal{T}^* g(x) &= \langle \mathcal{T}^* g, h_x\rangle\\ &= \langle g, \mathcal{T}h_x\rangle\\ &= g(0)\mathcal{T}h_x(0) + \int_0^\infty g'(y)(\mathcal{T}h_x)'(y)\alpha(y)\,dy\\ &= g(0)\eta(x) + \int_0^\infty q(x,y)g'(y)\,dy\end{aligned}$$

for any $g \in H_\alpha$, $x \geq 0$. This proves the result. ■

Let us next move our attention to the so-called delta of the option price in Proposition 3.7. We define the delta to be the Gâteaux derivative of the price $V(t, g(t))$ along some direction $h \in H_\alpha$. This will measure how sensitive the price functional is to perturbations along h of the forward curve $g(t)$. We have the following result.

Proposition 3.10. *Assume that σ is nonrandom. Then the Gâteaux derivative of $V(t, g(t))$ in direction $h \in H_\alpha$ is*

$$D_h V(t,g) = \frac{1}{\xi} m(h)\mathbb{E}\left[p(m(g)+\xi X)X\right]$$

with $m(g)$ and ξ defined in Proposition 3.7.

Proof. We apply the so-called density method (see Glasserman [31]) along with properties of the Gâteaux derivative. For $g \in H_\alpha$, it holds after a change of variables,

$$V(t,g) = \int_{\mathbb{R}} p(m(g)+\xi x)\phi(x)\,dx = \frac{1}{\xi}\int_{\mathbb{R}} p(y)\phi\left(\frac{y-m(g)}{\xi}\right)dy,$$

where ϕ is the standard normal probability density function. By the linear growth of p and integrability properties of the normal density function ϕ, it follows that

$$\begin{aligned} D_h V(t,g) &= \frac{1}{\xi}\int_{\mathbb{R}} p(y) D_h \phi\left(\frac{y-m(g)}{\xi}\right)\,dy \\ &= \frac{1}{\xi}\int_{\mathbb{R}} p(y)\phi'\left(\frac{y-m(g)}{\xi}\right)\left(-\frac{1}{\xi}\right) D_h m(g)\,dy \\ &= \frac{1}{\xi^2} D_h m(g)\int_{\mathbb{R}} p(y)\left(\frac{y-m(g)}{\xi}\right)\phi\left(\frac{y-m(g)}{\xi}\right)\,dy \\ &= \frac{1}{\xi} D_h m(g)\int_{\mathbb{R}} p(m(g)+\xi x)x\phi(x)\,dx \\ &= \frac{1}{\xi} D_h m(g)\mathbb{E}\left[p(m(g)+\xi X)X\right]\,. \end{aligned}$$

But obviously

$$D_h m(g) = \frac{d}{d\epsilon} m(g+\epsilon h)_{\epsilon=0} = \frac{d}{d\epsilon}(m(g)+\epsilon m(h))_{\epsilon=0} = m(h)\,,$$

and the proposition follows. ■

It is interesting to note here that the delta computed in the proposition above gives the sensitivity of the option price to perturbations in the direction of a forward curve h. As mentioned earlier, the market only quotes forward prices for a finite set of delivery periods, and not for all delivery times. Hence, we do not have the forward curve accessible. Indeed, we do not know $g(t)$ at time t, but only a finite set of values of swap prices, which is equivalent to a finite set of linear functionals on integral operators applied to g. It is market practice to extract such a curve by appealing to some smoothing techniques. (See, for example, Benth, Koekebakker, and Ollmar [14] for a spline approach.) From given observations of delivery-period swap prices, one constructs a forward curve of continuous delivery times. This will then give the observed curve $g(t)$ at time t. Note that we need to have this curve accessible to price the option at time t, as we can see from Proposition 3.7. The extraction of such a curve from observations is by far a uniquely defined object (one can choose several different ways to produce such a curve), and as such it is crucial to use the expression for the delta to see how sensitive the price is toward perturbations of it.

We find the following explicit result for the price and sensitivity (delta) of call options.

Proposition 3.11. *The price of a call option with strike K and exercise time $\tau \le T_1$ is*

$$V(t,g(t)) = \xi\phi\left(\frac{m(g(t))-K}{\xi}\right) + (m(g(t))-K)\Phi\left(\frac{m(g(t))-K}{\xi}\right)\,,$$

where ξ and $m(g)$ are defined in Proposition 3.7, $\Phi(x)$ is the cumulative standard normal distribution function, and ϕ is its density, i.e., $\Phi'(x)=\phi(x)$. Moreover,

$$D_h V(t,g(t)) = m(h)\Phi\left(\frac{m(g(t))-K}{\xi}\right)$$

for any $h \in H_\alpha$.

Proof. For a call option with strike K, we have $p(F) = \max(F - K, 0)$. Hence, from Proposition 3.7

$$V(t, g(t)) = \int_{\mathbb{R}} \max\left(m(g(t)) + \xi x - K, 0\right) \phi(x)\, dx\,.$$

The formula for $V(t, g(t))$ follows from standard calculations using the properties of the normal distribution. As for the Gâteaux derivative of V, we calculate this directly from $V(t, g(t))$. ■

Note that the expression for the sensitivity of V with respect to g is the classical delta of a call option, scaled by $m(h)$.

As a slight extension of the option pricing theory above, we discuss a class of spread options written on forwards with different delivery periods. To this end, consider an option written on two forwards with delivery periods being $[T_1^1, T_2^1]$ and $[T_1^2, T_2^2]$, respectively, where the option pays $p(F(\tau, T_1^1, T_2^1), F(\tau, T_1^2, T_2^2))$ at exercise time $\tau \leq \min(T_1^1, T_1^2)$. We assume that $p : \mathbb{R}^2 \to \mathbb{R}$ is a measurable function of at most linear growth. For example, $p(x, y) = \max(x - y, 0)$ will be the payoff from the spread between two forwards of different delivery periods, a kind of calendar spread option. By following the arguments of Proposition 3.1, we find that

$$p((F(\tau, T_1^1, T_2^1), F(\tau, T_1^2, T_2^2)) = \mathcal{P}_{\ell_1,\ell_2}(T_1^1 - \tau, T_1^2 - \tau, g(\tau)) \tag{3.19}$$

for $\mathcal{P}_{\ell_1,\ell_2} : \mathbb{R}_+^2 \times H_\alpha \to \mathbb{R}$ defined as

$$\mathcal{P}_{\ell_1,\ell_2}(x, y, g) = p \circ (\delta_x \circ \mathcal{D}_{\ell_1}^w(g), \delta_y \circ \mathcal{D}_{\ell_2}^w(g))\,. \tag{3.20}$$

Here, $\ell_i = T_2^i - T_1^i$, $i = 1, 2$. By the linear growth of p, we can show that $\mathcal{P}_{\ell_1,\ell_2}$ is at most linearly growing in $\|g\|_\alpha$, uniformly in x, y. By following the arguments for the univariate case above, the price of the option at time $t \leq \tau$ can be computed as follows:

$$\begin{aligned} V(t, g(t)) &= \mathrm{e}^{-r(\tau-t)}\mathbb{E}\left[p\left(\delta_{T_1^1-\tau} \circ \mathcal{D}_{\ell_1}^w(g(\tau)), \delta_{T_1^2-\tau} \circ \mathcal{D}_{\ell_2}^w(g(\tau))\right) \mid \mathcal{F}_t\right] \\ &= \mathrm{e}^{-r(\tau-t)}\mathbb{E}\left[p(F(\tau, T_1^1, T_2^1), F(\tau, T_1^2, T_2^2)) \,|\, g(t)\right]\,. \end{aligned}$$

Yet again, we find

$$\begin{aligned} (F(\tau, T_1^1, T_2^1), F(\tau, T_1^2, T_2^2)) &= (\delta_{T_1^1-\tau} \circ \mathcal{D}_{\ell_1}^w(g(\tau)), \delta_{T_1^2-\tau} \circ \mathcal{D}_{\ell_2}^w(g(\tau))) \\ &= (\delta_{T_1^1-t} \circ \mathcal{D}_{\ell_1}^w(g(t)), \delta_{T_1^2-t} \circ \mathcal{D}_{\ell_2}^w(g(t))) \\ &\quad + \int_t^\tau (\delta_{T_1^1-s}\mathcal{D}_{\ell_1}^w \sigma(s)\, d\mathbb{B}(s), \delta_{T_1^2-t}\mathcal{D}_{\ell_2}^w \sigma(s)\, d\mathbb{B}(s)) \\ &= (\delta_{T_1^1-t} \circ \mathcal{D}_{\ell_1}^w(g(t)), \delta_{T_1^2-t} \circ \mathcal{D}_{\ell_2}^w(g(t))) + \int_t^\tau \Sigma(s) dB(s), \end{aligned}$$

where B is some two-dimensional standard Brownian motion and $\Sigma(s)$ is the positive semidefinite root of

$$\left((\delta_{T_1^i-s}\mathcal{D}_{\ell_i}^w \sigma(s))Q(\delta_{T_1^j-s}\mathcal{D}_{\ell_j}^w \sigma(s))^*\right)_{i,j=1,2}$$

for any $s \geq 0$. The matrix $\Sigma^2(s)$ can be computed as before and appears in the formula for the realized variance. Hence,

$$V(t,g) = \mathbb{E}\left[p\left((\delta_{T_1^1-t} \circ \mathcal{D}_{\ell_1}^w(g), \delta_{T_1^2-t} \circ \mathcal{D}_{\ell_2}^w(g)) + \int_t^\tau \Sigma(s)dB(s)\right)\right]$$

for any $t \in [0,\tau]$, $g \in H_\alpha$.

In conclusion, we see that we get a two-dimensional stochastic Itô integral of a deterministic integrand in the expectation defining the price $V(t,g)$, yielding a bivariate Gaussian random variable. Therefore, we can—after computing the correlation—represent the option price as an expectation of a function of a bivariate Gaussian random variable. The correlation will depend on $\mathcal{Q}$, the spatial covariance structure of the noise $\mathbb{B}$, the volatility $\sigma(s)$ of the forward curve σ, and the delivery periods of the two forwards. Roughly explained, we are extracting two pieces of the forward curve (defined by the delivery periods), and constructing a bivariate Gaussian random variable of it. Although the expression involved becomes rather technical, we can obtain rather explicit option prices which honor the spatial dependency structure of the forward curve.

3.3. The geometric Gaussian case. First, we show that the Hilbert space H_α is closed under exponentiating.

Lemma 3.12. *If $g \in H_\alpha$, then $\exp(g) \in H_\alpha$, where $\exp(g) = \sum_{n=0}^\infty g^n/n!$.*

Proof. First, if $g \in H_\alpha$, then $x \mapsto \exp(g(x))$ is an absolutely continuous function from $\mathbb{R}_+$ into $\mathbb{R}_+$. Due to Proposition 4.18 in Benth and Krühner [15], H_α is a Banach algebra with respect to the norm $\|\cdot\| := k_1\|\cdot\|_\alpha$, where $k_1 = \sqrt{5+4k^2}$ and $k^2 = \int_0^\infty \alpha^{-1}(x)\,dx$, i.e., if $f,g \in H_\alpha$, then $\|fg\| \leq \|f\|\|g\|$. By the triangle inequality, we therefore have $\|\exp(g)\| \leq \exp(\|g\|) < \infty$ for any $g \in H_\alpha$, or, in other words,

$$\|\exp(g)\|_\alpha \leq \frac{1}{k_1}\exp(k_1\|g\|_\alpha) < \infty\,.$$

Hence, $\exp(g) \in H_\alpha$, and the lemma follows. ■

Suppose that the forward prices are given as the exponential of a stochastic process in H_α, i.e., of the form

$$g(t) = \exp(\widetilde{g}(t))\,, \tag{3.21}$$

where

$$d\widetilde{g}(t) = (\partial_x\widetilde{g}(t) + \mu(t))\,dt + \sigma(t)\,d\mathbb{B}(t)\,, \tag{3.22}$$

where σ and $\mathbb{B}$ are as for the stochastic partial differential equation in (3.13), and μ a predictable H_α-valued stochastic process which is Bochner integrable on any finite time interval. To have a no-arbitrage dynamics, we must impose the drift condition (see Barth and Benth [9])

$$x \mapsto \mu(t,x) := -\frac{1}{2}\|\delta_x\sigma(t)\mathcal{Q}^{1/2}\|_{L_{HS}(H,\mathbb{R})}\,. \tag{3.23}$$

We assume that this drift condition holds from now on. The following simplification of the drift condition holds true.

Lemma 3.13. *The drift condition for μ in (3.23) can be expressed as*

$$\mu(t,x) = -\frac{1}{2}\delta_x\sigma(t)\mathcal{Q}\sigma^*(t)\delta_x^*(1)\,.$$

Proof. It follows from the definition of the Hilbert–Schmidt norm that

$$\mu(t,x) = -\frac{1}{2}\sum_{k=1}^{\infty}(\delta_x\sigma(t)\mathcal{Q}^{1/2}e_k)^2\,,$$

where $\{e_k\}_k$ is a basis of H. But,

$$(\delta_x\sigma(t)\mathcal{Q}^{1/2})(e_k)\cdot 1 = \langle e_k, (\delta_x\sigma(t)\mathcal{Q}^{1/2})^*(1)\rangle_H = \langle e_k, \mathcal{Q}^{1/2}\sigma^*(t)\delta_x^*(1)\rangle_H\,.$$

Hence, by linearity of operators,

$$\begin{aligned}\mu(t,x) &= -\frac{1}{2}\sum_{k=1}^{\infty}(\delta_x\sigma(t)\mathcal{Q}^{1/2}e_k)\langle e_k, \mathcal{Q}^{1/2}\sigma^*(t)\delta_x^*(1)\rangle_H \\ &= \delta_x\sigma(t)\mathcal{Q}^{1/2}(\sum_{k=1}^{\infty}\langle e_k, \mathcal{Q}^{1/2}\sigma^*(t)\delta_x^*(1)\rangle_H e_k) \\ &= \delta_x\sigma(t)\mathcal{Q}^{1/2}(\mathcal{Q}^{1/2}\sigma^*(t)\delta_x^*(1))\,.\end{aligned}$$

The result follows. ■

We recall that $\delta_x^*(1) = h_x$, with the function $y \mapsto h_x(y)$ defined in (3.17). Thus, we can write $\mu(t,x) = -\delta_x\sigma(t)\mathcal{Q}\sigma^*(t)h_x(\cdot)/2$.

As for (3.13) in the subsection above, we have a mild solution of the stochastic partial differential equation (3.22) satisfying for $\tau \geq t$

$$\widetilde{g}(\tau) = \mathcal{S}_{\tau-t}\widetilde{g}(t) + \int_t^{\tau}\mathcal{S}_{\tau-s}\mu(s)\,ds + \int_t^{\tau}\mathcal{S}_{\tau-s}\sigma(s)\,d\mathbb{B}(s)\,.$$

The following lemma states the dynamics of the curve-valued process $g(t) := \exp(\widetilde{g}(t))$, $t \geq 0$, revealing that g is Markovian as in section 3.1.

Lemma 3.14. *Under the drift condition* (3.23)*, we have*

$$g(\tau) = \mathcal{S}_{\tau-t}g(t) + \int_t^{\tau}\mathcal{S}_{\tau-s}\widehat{\sigma}(s,g(s))\,d\mathbb{B}(s)$$

for any $0 \leq t \leq \tau$, where $\widehat{\sigma}(s,g)h(x) := g(x)\sigma(s)h(x)$ for any $x \geq 0$, $g,h \in H_\alpha$. Consequently, the forward dynamics are given by

$$F(\tau,T_1,T_2) = \delta_{T_1-t}\mathcal{D}_\ell^w g(t) + \int_t^{\tau}\delta_{T_1-s}\mathcal{D}_\ell^w\widehat{\sigma}(s,g(s))\,d\mathbb{B}(s),$$

Proof. Recall that $G(\tau,T) = g(\tau)(T-\tau)$ and define $\widetilde{G}(\tau,T) := \widetilde{g}(\tau)(T-\tau)$. Then we have

$$\begin{aligned} G(\tau,T) &= g(\tau)(T-\tau) \\ &= \exp(\widetilde{g}(\tau)(T-\tau)) \\ &= \exp(\widetilde{G}(\tau,T)) \end{aligned}$$

for any $0 \le \tau \le T$. Moreover, we have

$$\widetilde{G}(\tau,T) = \delta_{T-t}\widetilde{g}(t) + \int_t^\tau \delta_{T-s}(\mu(s)ds + \sigma(s)\,d\mathbb{B}(s)),$$

and hence Itô's formula together with the drift condition (3.23) yields

$$\begin{aligned} G(\tau,T) &= \exp(\widetilde{G}(\tau,T)) \\ &= \delta_{T-t}g(t) + \int_t^\tau G(s,T)\delta_{T-s}\sigma(s)\,d\mathbb{B}(s) \\ &= \delta_{T-t}g(t) + \int_t^\tau \delta_{T-s}\widehat{\sigma}(s,g(s))\,d\mathbb{B}(s) \end{aligned}$$

for any $0 \le \tau \le T$. Since $g(\tau)(x) = G(\tau,\tau+x)$, we conclude that

$$g(\tau) = \mathcal{S}_{\tau-t}g(t) + \int_t^\tau \mathcal{S}_{\tau-s}\widehat{\sigma}(s,g(s))\,d\mathbb{B}(s)$$

for any $0 \le t \le \tau$. ■

The price of a European option with exercise time $\tau \ge t$ on a forward delivering at time T when σ is nonrandom can be easily derived as in the arithmetic case. Indeed, it holds that

$$V(t,\widetilde{g}) = \mathrm{e}^{-r(\tau-t)}\mathbb{E}\left[p(\exp(\widehat{m}(\widetilde{g}) + \xi X))\right], \tag{3.24}$$

where X is a standard normal distributed random variable, ξ is as in Proposition 3.7 (using the T instead of T_1), and

$$\widehat{m}(g) = \widetilde{g}(T-t) - \frac{1}{2}\int_t^\tau \mu(s)(T-s)\,ds. \tag{3.25}$$

If we let p be the payoff function of a call option, then a simple calculation shows that we recover the Black-76 formula. (See Black [18], or Benth, Benth, and Koekebakker [12] for a more general version.)

Finally, we remark that if we are interested in pricing options written on a forward delivering over a period, the payoff function will become

$$p((\delta_{T-\tau} \circ \mathcal{D}_\ell^w)(g(\tau))) = p(F(\tau,T_1,T_2)).$$

The integral operator $\mathcal{D}_\ell^w$ maps $\exp(\widetilde{g}(\tau)) \in H_\alpha$ into H_α, but we do not have any nice representation of it. The problem is, of course, that the integral of the exponent of a general function is not analytically known. Thus, it seems difficult to obtain any tractable expression yielding simple pricing formulas.

3.4. Lévy models. We include a brief discussion of the pricing of options when the forward curve is driven by a Lévy process $\mathbb{L}$. We confine our analysis to the arithmetic model

$$dg(t) = \partial_x g(t)\,dt + \sigma(t)\,d\mathbb{L}(t)\,, \tag{3.26}$$

where $\mathbb{L}$ is a Lévy process with values in a separable Hilbert space H, having zero mean and being square integrable. The stochastic process $\sigma : \mathbb{R}_+ \to L(H, H_\alpha)$ is integrable with respect to $\mathcal{L}$, i.e., $\sigma \in \mathcal{L}^2_{\mathbb{L}}(H_\alpha)$. (See section 8.2 in Peszat and Zabczyk [34] for this notation.)

The price of an option given in (3.4) requires the computation of $(\delta_{T_1-\tau} \circ \mathcal{D}^w_\ell)(g(\tau))$. As for the Gaussian models, there exists a mild solution of (3.26) which for $\tau \geq t \geq 0$ is given by

$$g(\tau) = \mathcal{S}_{\tau-t}g(t) + \int_t^\tau \mathcal{S}_{\tau-s}\sigma(s)\,d\mathbb{L}(s)\,. \tag{3.27}$$

From the linearity of the operators, it holds that

$$(\delta_{T_1-\tau} \circ \mathcal{D}^w_\ell)g(\tau) = (\delta_{T_1-t} \circ \mathcal{D}^w_\ell)g(t) + \int_t^\tau (\delta_{T_1-s} \circ \mathcal{D}^w_\ell)\sigma(s)\,d\mathbb{L}(s))\,.$$

The first term on the right-hand side is, not surprisingly, $m(g(t))$ with m defined in Proposition 3.7. For the Gaussian model, we used a result in Benth and Krühner [15] that provided us with an explicit representation of a linear functional applied on a H_α-valued stochastic integral with respect to a H-valued Wiener process. One can write this functional as a stochastic integral of a real-valued stochastic integrand with respect to a real-valued Brownian motion. The integrand is, moreover, explicitly known. Something similar is known for the special class of Lévy processes being subordinated Wiener processes.

Following Benth and Krühner [16], we introduce H-valued subordinated Brownian motion: Denote by $U(t)\}_{t\geq 0}$ a Lévy process with values on the positive real line, that is, a nondecreasing Lévy process. These processes are frequently called *subordinators* (see Sato [35]). Let $\mathbb{L}(t) := \mathbb{B}(U(t))$, which then becomes a Lévy process with values in H. In Benth and Krühner [16] one finds conditions on U implying that $\mathbb{L}$ is a zero-mean square integrable Lévy process.

From Theorem 2.5 in Benth and Krühner [15], we find that

$$\int_t^\tau (\delta_{T_1-s} \circ \mathcal{D}^w_\ell)\sigma(s)\,d\mathbb{L}(s)) = \int_t^\tau \widetilde{\sigma}(s)\,dL(s)\,,$$

where L is a real-valued subordinated Brownian motion $L(t) := B(U(t))$, B being a standard Brownian motion. Moreover, the process $\widetilde{\sigma}(s)$ is given by

$$\widetilde{\sigma}^2(s) = (\delta_{T_1-s} \circ \mathcal{D}^w_\ell)\sigma(s)\mathcal{Q}\sigma^*(s)(\delta_{T_1-s} \circ \mathcal{D}^w_\ell)^*(1)\,,$$

which is identical to the Gaussian case studied above.

For the problem of pricing options, we see that we are back to computing the expectation of a functional of a univariate stochastic integral. If σ is nonrandom, we can use, for example, Fourier techniques to compute this expectation, as we know the cumulant function of L from the cumulant of U and Brownian motion. (See Carr and Madan [23] for an account on Fourier methods in derivatives pricing, and Benth, Benth, and Koekebakker [12] for the application to energy markets.) This will provide us with an expression for the option price that can be efficiently computed using fast Fourier transform techniques.

We next consider an example on a subordinated Lévy process of particular interest in energy markets. Assume that U is an inverse Gaussian subordinator, that is, a Lévy process with nondecreasing paths, and $U(1)$ is inverse Gaussian distributed. Then $\mathbb{L}(t) = \mathbb{B}(U(t))$ becomes an H-valued normal inverse Gaussian (NIG) Lévy process in the sense defined by Benth and Krühner [16, Def. 4.1]. In fact, for any functional $\mathcal{L} \in L(H, \mathbb{R}^n)$, $t \mapsto \mathcal{L}(\mathbb{L}(t))$ will be an n-variate NIG Lévy process with the particular case $L(t)$ introduced above defining an NIG Lévy process on the real line. We refer to Barndorff-Nielsen [5] for details on the inverse Gaussian subordinator and NIG Lévy processes. Several empirical studies have demonstrated that returns of energy forward and futures prices can be conveniently modelled by the NIG distribution. (See Benth, Benth, and Koekebakker [12] and the references therein for the case of NordPool power prices.) Frestad, Benth, and Koekebakker [29] and Andresen, Koekebakker, and Westgaard [3] find that the NIG distribution fits power forward returns with fixed time to maturity and given delivery period. Their analysis covers time series of prices with different times to maturity and different delivery periods (weekly, monthly, quarterly, say), where these time series are constructed from a nonparametric smoothing of the original price data observed in the market. In fact, in our modelling context, they are looking at time series observations of the stochastic process $t \mapsto (\delta_x \circ \mathcal{D}_\ell^w)(g(t))$. From the analysis above, we see that choosing $\mathbb{L}$ to be an H-valued NIG (HNIG) Lévy process and g to be an arithmetic dynamics will give price increments being NIG distributed. Of course, this is not the same as the returns being NIG. As we have mentioned earlier, it is not straightforward to model the price of forward with delivery period using an exponential dynamics. Frestad, Benth, and Koekebakker [29] and Andresen, Koekebakker, and Westgaard [3] also estimate empirically the volatility term structure and the spatial (in time to maturity) correlation structure, which provides information on the volatility $\sigma(t)$ and the covariance operator $\mathcal{Q}$. Indeed, Andresen, Koekebakker, and Westgaard [3] propose a multivariate NIG distribution to model the returns.

We end this section with a note on market completeness. The energy market seems to be incomplete when analysing real data, e.g., a futures with a very short maturity cannot be replicated by futures with delivery far in the future. If a model for all futures is driven by a low-dimensional Brownian motion, then the resulting theoretical model is in most cases complete and one needs as many futures as there are independent driving Brownian motions to explain the theoretical behavior of any other security, like a future on the short end. Clearly, if one either uses a high- or infinite-dimensional Brownian motion, this is no longer the case. Also, jump-type noise excludes market completeness (except for some very simplistic cases), even if the driving noise is only one-dimensional. Consequently, the only thing that remains from the risk-neutral approach is the existence of at least one pricing measure in order to ensure arbitrage-free price dynamics of the futures. However, this does not at all imply replication.

4. Cross-commodity modeling. In this section, we want to analyze a joint model for the forward curve evolution in two commodity markets. For example, European power markets are interconnected, and thus forward prices will be dependent. Also, the markets for gas and coal will influence the power market, since gas and coal are important fuels for power generation in many countries like for example UK and Germany. This links forward contracts on gas and coal to those traded in the power markets. Finally, weather clearly affects the demand (through temperature) and supply (through precipitation and wind) of energy, and one can therefore also claim a dependency between weather futures (traded at Chicago Mercantile Exchange (CME), say) and power futures. These examples motivate the introduction of multivariate dynamic models for the time evolution of forward curves across different markets. We will restrict our attention merely to the bivariate case here and make some detailed analysis of a two-dimensional forward curve dynamics.

Consider two commodity forward markets. We model the bivariate forward curve dynamics $t \mapsto (g_1(t), g_2(t))$ as the $H_\alpha \times H_\alpha$-valued stochastic process being the solution of the stochastic partial differential equation

$$\begin{aligned} dg_1(t) &= \partial_x g_1(t)\,dt + \sigma_1(t, g_1(t), g_2(t))\,d\mathbb{L}_1(t), \\ dg_2(t) &= \partial_x g_2(t)\,dt + \sigma_2(t, g_1(t), g_2(t))\,d\mathbb{L}_2(t) \end{aligned} \tag{4.1}$$

with $(g_1(0), g_2(0) = (g_1^0, g_2^0) \in H_\alpha \times H_\alpha$ given. We suppose that $(\mathbb{L}_1, \mathbb{L}_2)$ is an $H_1 \times H_2$-valued square-integrable zero-mean Lévy process, where $H_i, i = 1,2$ are two separable Hilbert spaces and $\mathcal{Q}_i, i = 1,2$ are the respective (marginal) covariance operators, i.e., $\mathbb{E}[\langle L_i(t), g\rangle\langle L_i(s), h\rangle] = (t \wedge s)\langle Q_i g, h\rangle$ for any $t, s \geq 0$, $g, h \in H_\alpha$, and $i = 1,2$. Furthermore, we assume that $\sigma_i : \mathbb{R}_+ \times H_\alpha \times H_\alpha \to L(H_i, H_\alpha)$ for $i = 1,2$ are measurable functions and that there exists an increasing function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that σ_i, $i = 1,2$ are Lipschitz and of linear growth, that is, for any $(f_1, f_2), (h_1, h_2) \in H_\alpha \times H_\alpha$ and $t \in \mathbb{R}_+$,

$$\|\sigma_i(t, f_1, f_2) - \sigma_i(t, h_1, h_2)\|_{\text{op}} \leq K(t)\|(f_1, f_2) - (h_1, h_2)\|_{H_\alpha \times H_\alpha}\,, \tag{4.2}$$

$$\|\sigma_i(t, f_1, f_2)\|_{\text{op}} \leq K(t)(1 + \|(f_1, f_2)\|_{H_\alpha \times H_\alpha})\,. \tag{4.3}$$

Note that since the product of two (separable) Hilbert space again is a (separable) Hilbert space (using the canonical 2-norm, i.e., $\|(f,g)\|^2_{H_\alpha \times H_\alpha} := \|f\|^2_{H_\alpha} + \|g\|^2_{H_\alpha}$), we can relate to the theory of existence and uniqueness of mild solutions of stochastic partial differential equations given by Tappe [37]: there exists a unique mild solution satisfying the integral equations

$$\begin{aligned} g_1(t) &= \mathcal{S}_t g_1^0 + \int_0^t \mathcal{S}_{t-s}\sigma_1(s, g_1(s), g_2(s))\,d\mathbb{L}_1(s), \\ g_2(t) &= \mathcal{S}_t g_2^0 + \int_0^t \mathcal{S}_{t-s}\sigma_2(s, g_1(s), g_2(s))\,d\mathbb{L}_2(s)\,. \end{aligned} \tag{4.4}$$

Observe that $t \mapsto (F_1(t,T), F_2(t,T)) := (\delta_{T-t} g_1(t), \delta_{T-t} g_2(t))$, $t \leq T$, will be an $H_\alpha \times H_\alpha$-valued (local) martingale. Moreover, the marginal H_α-valued processes $t \mapsto F_i(t,T) := \delta_{T-t} g_i(t), i = 1,2, t \leq T$ will also be (local) martingales, ensuring that we have an arbitrage-free model for the forward price dynamics in the two commodity markets.

Our main concern in the rest of this section is to analyze in detail the bivariate Lévy process $(\mathbb{L}_1, \mathbb{L}_2)$. We are interested in its probabilistic properties in terms of representation of the covariance operator and linear decomposition. Since $(\mathbb{L}_1(1), \mathbb{L}_2(1))$ is an $H_1 \times H_2$-valued square-integrable variable, we analyze general square-integrable random variables (X_1, X_2) in $H_1 \times H_2$.

Before we set off, we recall the spectral theorem for normal compact operators on Hilbert spaces (see, e.g., [24, Stat. 7.6]).

Proposition 4.1. *Let H be a separable Hilbert space and $\mathcal{T}$ be a symmetric compact operator. Then there is an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H and a family $\{\lambda_i\}_{i\in\mathbb{N}}$ of real numbers such that*

$$\mathcal{T}f = \sum_{i\in\mathbb{N}} \lambda_i \langle e_i, f\rangle e_i$$

for any $f \in H$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be measurable. Then

$$\phi(\mathcal{T}) : \left\{ f \in H : \sum_{i\in\mathbb{N}} |\phi(\lambda_i)|^2 \langle e_i, f\rangle^2 < \infty \right\} \to H, f \mapsto \sum_{i\in\mathbb{N}} \phi(\lambda_i)\langle e_i, f\rangle e_i\,,$$

defines a closed linear symmetric operator which is bounded and everywhere defined if ϕ is bounded on $\{\lambda_i : i \in \mathbb{N}\}$. For measurable $\phi, \psi : \mathbb{R} \to \mathbb{R}$ with ψ bounded, we have $(\phi+\psi)(\mathcal{T}) = \phi(\mathcal{T}) + \psi(\mathcal{T})$ and $(\phi\psi)(\mathcal{T}) = \phi(\mathcal{T})\psi(\mathcal{T})$.

We will apply this result in particular to define the square-root and the pseudoinverse of a compact operator. We shall use the definition of a pseudoinverse given in Albert [1].

Definition 4.2. *Let $\mathcal{P}$ be a positive semidefinite compact operator on a separable Hilbert space H. Then $\mathcal{R} := \sqrt{\mathcal{P}}$ is the* square-root *of $\mathcal{P}$. The* pseudoinverse *$\mathcal{J}$ of $\mathcal{P}$ is defined by $\mathcal{J} := \phi(\mathcal{P})$, where $\phi : \mathbb{R} \to \mathbb{R}, x \mapsto 1_{\{x\neq 0\}}/x$.*

Next, we want to represent covariance operators of square-integrable random variables in $H_1 \times H_2$ in terms of operators on H_1, H_2, and between thoose spaces. To this end, we will need the natural projectors $\Pi_1 : H_1 \times H_2 \to H_1, (x, y) \mapsto x$ and $\Pi_2 : H_1 \times H_2 \to H_2, (x, y) \mapsto y$. We have the following general statement on the representation of the covariance operator of square-integrable random variables in $H_1 \times H_2$.

Theorem 4.3. *For $i = 1, 2$, let X_i be a square integrable H_i-valued random variable and $\mathcal{Q}_i$ be its covariance operator. Denote the positive semidefinite square-root of $\mathcal{Q}_i$ by $\mathcal{R}_i$ for $i = 1, 2$. Then there is a linear operator $\mathcal{Q}_{12} \in L(H_1, H_2)$ such that*

(i) $\mathcal{Q} := \begin{pmatrix} \mathcal{Q}_1 & \mathcal{Q}_{12}^* \\ \mathcal{Q}_{12} & \mathcal{Q}_2 \end{pmatrix}$ *is the covariance operator of the $H_1 \times H_2$-valued square integrable random variable (X_1, X_2),*

(ii) $|\langle \mathcal{Q}_{12}u, v\rangle| \leq \|\mathcal{R}_1 u\|_1 \|\mathcal{R}_2 v\|_2$ *for any $u \in H_1$, $v \in H_2$, and*

(iii) $\mathrm{ran}(\mathcal{Q}_{12}) \subseteq \overline{\mathrm{ran}(\mathcal{Q}_2)}$ *and* $\mathrm{ran}(\mathcal{Q}_{12}^*) \subseteq \overline{\mathrm{ran}(\mathcal{Q}_1)}$.

Proof.

(i) Let $\mathcal{Q}$ be the covariance operator of (X_1, X_2) and $\Phi_i : H_i \to H_1 \times H_2$ be the natural embedding, i.e., $\Phi_i = \Pi_i^*$ for $i = 1, 2$. Define

$$\mathcal{Q}_{12} := \Pi_2 \mathcal{Q} \Phi_1\,.$$

Then the first assertion is evident.

(ii) Let $u \in H_1$, $v \in H_2$, and $\beta \in \mathbb{R}$. We have

$$\begin{aligned} 0 &\leq \langle \mathcal{Q}(\beta u, v), (\beta u, v)\rangle \\ &= \beta^2 \langle \mathcal{Q}_1 u, u\rangle + \langle \mathcal{Q}_2 v, v\rangle + 2\beta\langle \mathcal{Q}_{12}u, v\rangle \end{aligned}$$

$$= \beta^2 \|\mathcal{R}_1 u\|_1^2 + \|\mathcal{R}_2 v\|_2^2 + 2\beta \langle \mathcal{Q}_{12} u, v \rangle \,,$$

and hence

$$-\beta \langle \mathcal{Q}_{12} u, v \rangle \leq \frac{1}{2} \left(\beta^2 \|\mathcal{R}_1 u\|_1^2 + \|\mathcal{R}_2 v\|_2^2 \right) .$$

Letting β have the same sign as $-\langle \mathcal{Q}_{12} u, v \rangle$ yields

$$|\beta| |\langle \mathcal{Q}_{12} u, v \rangle| \leq \frac{1}{2} \left(\beta^2 \|\mathcal{R}_1 u\|_1^2 + \|\mathcal{R}_2 v\|_2^2 \right) .$$

If $\|\mathcal{R}_1 u\|_1 = 0$, then with $|\beta| \to \infty$ we see that $|\langle \mathcal{Q}_{12} u, v \rangle| = 0$ and hence the claimed inequality holds. Thus, we may assume that $\|\mathcal{R}_1 u\|_1 \neq 0$. Choosing $\beta = \frac{\|\mathcal{R}_2 v\|_2}{\|\mathcal{R}_1 u\|_1}$ yields

$$|\langle \mathcal{Q}_{12} u, v \rangle| \leq \|\mathcal{R}_1 u\|_1 \|\mathcal{R}_2 v\|_2,$$

as claimed.

(iii) We show that $\mathcal{Q}_{12} u$ is orthogonal to any $v \in \mathrm{Kern}(\mathcal{Q}_2)$ for any $u \in H_1$. If that is done, then the claim follows because $\mathcal{Q}_2$ is positive semidefinite, and hence its kernel and the closure of its range are closed orthogonal spaces. Let $u \in H_1$, $v \in H_2$ such that $\mathcal{Q}_2 v = 0$. Then, $\mathcal{R}_2 v = 0$, and hence (ii) yields

$$|\langle \mathcal{Q}_{12} u, v \rangle| \leq \|\mathcal{R}_1 u|\|_1 \|\mathcal{R}_2 v\|_2 = 0.$$

The corresponding arguments show that $\mathcal{Q}_{12}^*$ maps into the closure of the range of $\mathcal{Q}_1$. ■

Consider now $H_i = H_{\alpha_i}$, H_{α_i} being the Filipovic space with weight function α_i, $i = 1, 2$. We suppose that both weight functions α_1, α_2 satisfy the hypotheses stated at the beginning of section 2. We first demonstrate that the operator $\mathcal{Q}_{12}$ yields the covariance between $\mathbb{L}_1(t)$ and $\mathbb{L}_2(t)$ evaluated at two different maturities x and y with $x, y \in \mathbb{R}_+$. To this end, recall the function h_x in (3.17). Then we have for any $x \in \mathbb{R}_+$ and $X \in H_\alpha$,

$$\delta_x X = \langle X, \delta_x^*(1) \rangle = \langle X, h_x \rangle \,,$$

by (3.16). Hence, with h_x^i being the function h_x defined in (3.17) using the weight function α_i,

$$\delta_z^i(\mathbb{L}_i(t)) = \langle \mathbb{L}_i(t), h_z^i \rangle \,.$$

Thus, with $(\mathbb{L}_1, \mathbb{L}_2)$ being a zero mean Lévy process, we find for $x, y \in \mathbb{R}_+$,

$$\begin{aligned}
\mathrm{Cov}(\mathbb{L}_1(t,x), \mathbb{L}_2(t,y)) &= \mathbb{E}\left[\delta_x^1(\mathbb{L}_1(t)) \delta_y^2(\mathbb{L}_2(t)) \right] \\
&= \mathbb{E}\left[\langle \mathbb{L}_1(t), h_x^1 \rangle \langle \mathbb{L}_2(t), h_y^2 \rangle \right] \\
&= \mathbb{E}\left[\langle (\mathbb{L}_1(t), \mathbb{L}_2(t)), \Pi_1^* h_x^1 \rangle \langle (\mathbb{L}_1(t), \mathbb{L}_2(t)), \Pi_2^* h_y^2 \rangle \right] \\
&= t \langle \mathcal{Q} \Pi_1^* h_x^1, \Pi_2^* h_y^2 \rangle \\
&= t \langle \Pi_2 \mathcal{Q} \Pi_1^* h_x^1, h_y^2 \rangle \,.
\end{aligned}$$

We have $\Pi_2 \mathcal{Q} \Pi_1^* = \mathcal{Q}_{12}$, and it follows that

$$\mathrm{Cov}(\mathbb{L}_1(t,x), \mathbb{L}_2(t,y)) = t \langle \mathcal{Q}_{12} h_x^1, h_y^2 \rangle \,, \tag{4.5}$$

as claimed.

Let us analyze a very simple case of the bivariate forward dynamics in (4.1), where $\alpha_1 = \alpha_2 = \alpha$ and $\sigma_i = \mathrm{Id}$, the identity operator on H_α, $i = 1, 2$, and $(\mathbb{L}_1, \mathbb{L}_2) = (\mathbb{B}_1, \mathbb{B}_2)$ is a Wiener process. The mild solution in (4.4) takes the form

$$g_i(t) = \mathcal{S}_t g_i^0 + \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_i(s)$$

for $i = 1, 2$. We find similar to above that, for $x, y \in \mathbb{R}_+$,

$$\begin{aligned}\mathrm{Cov}(g_1(t,x), g_2(t,y)) &= \mathbb{E}\left[\delta_x \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_1(s) \cdot \delta_y \int_0^t \mathcal{S}_{t-s}\mathbb{B}_2(s)\right] \\ &= \mathbb{E}\left[\left\langle \left(\int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_2(s)\right), \Pi_1^* h_x \right\rangle\right. \\ &\qquad \left. \times \left\langle \left(\int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_2(s)\right), \Pi_2^* h_y \right\rangle\right].\end{aligned}$$

We show that $(\int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}(s))$ is a Gaussian $H_\alpha \times H_\alpha$-valued stochastic process.

Lemma 4.4. *Suppose that $H_i = H_\alpha$ for $i = 1, 2$. The process $t \mapsto (\int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_2(s))$ is a mean-zero Gaussian $H_\alpha \times H_\alpha$-valued process with covariance operator $\mathcal{Q}_t$ for each $t \geq 0$ given by*

$$\mathcal{Q}_t = \begin{bmatrix} \int_0^t \mathcal{S}_s \mathcal{Q}_1 \mathcal{S}_s^*\, ds & \int_0^t \mathcal{S}_s \mathcal{Q}_{12}^* \mathcal{S}_s\, ds \\ \int_0^t \mathcal{S}_s \mathcal{Q}_{12} \mathcal{S}_s^*\, ds & \int_0^t \mathcal{S}_s \mathcal{Q}_2 \mathcal{S}_s^*\, ds \end{bmatrix}.$$

The integrals in $\mathcal{Q}_t$ are interpreted as Bochner integrals in the space of Hilbert–Schmidt operators.

Proof. First, note that all the integrals in $\mathcal{Q}_t$ are well-defined as Bochner integrals because the operator norm of the involved operators are bounded uniformly in time by Lemma 3.5.

Consider the characteristic function of the process at time $t \geq 0$. A straightforward computation gives

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\mathrm{i}\left\langle\left(\int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}(s)\right), (u,v)\right\rangle\right)\right] \\ &\qquad = \exp\left(-\frac{1}{2}\int_0^t \langle \mathcal{Q}(\mathcal{S}_{t-s}^* u, \mathcal{S}_{t-s}^* v), (\mathcal{S}_{t-s}^* u, \mathcal{S}_{t-s}^* v)\rangle\, ds\right).\end{aligned}$$

Using the definition of $\mathcal{Q}$ shows that

$$\mathbb{E}\left[\exp\left(\mathrm{i}\left\langle\left(\int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s}\, d\mathbb{B}(s)\right), (u,v)\right\rangle\right)\right] = \exp\left(-\frac{1}{2}\langle \mathcal{Q}_t(u,v), (u,v)\rangle\right),$$

and the result follows. ■

It follows from this lemma that

$$\begin{aligned}\mathrm{Cov}(g_1(t,x), g_2(t,y)) &= \langle \mathcal{Q}_t \Pi_1^* h_x, \Pi_2^* h_y\rangle \\ &= \langle \Pi_2 \mathcal{Q}_t \Pi_1^* h_x, h_y\rangle \\ &= \left\langle \int_0^t \mathcal{S}_s \mathcal{Q}_{12} \mathcal{S}_s h_x\, ds, h_y\right\rangle\end{aligned}$$

$$= \int_0^t \langle \mathcal{S}_s \mathcal{Q}_{12} \mathcal{S}_s^* h_x, h_y \rangle \, ds$$
$$= \int_0^t \delta_y \mathcal{S}_s \mathcal{Q}_{12} \mathcal{S}_s^* \delta_x^*(1) \, ds$$
$$= \int_0^t \delta_{y+s} \mathcal{Q}_{12} \delta_{x+s}^*(1) \, ds \, .$$

This provides us with an explicit expression for the covariance between the forward prices $g_1(t)$ and $g_2(t)$ at two different maturities x and y.

An application of the above considerations is the pricing of so-called energy quanto options. Such options have gained some attention in recent years since they offer a hedge against both price and volume risk in energy production. A typical payoff function at exercise time τ from a quanto option takes the form

$$p(F_{\text{energy}}(\tau, T_1, T_2)) \times q(F_{\text{temp}}(\tau, T_1, T_2)) \, ,$$

where F_{energy} is the forward price on some energy like power or gas, and F_{temp} is the forward price on some temperature index. Both forwards have a delivery[2] period $[T_1, T_2]$, and it is assumed that $\tau \leq T_1$. The functions p and q are real-valued and of linear growth and typically given by call and put option payoff functions. Temperature is closely linked to the demand for power, and the quanto options are structured to yield a payoff which depends on the product of price and volume. We refer to Caporin, Pres, and Torro [20] and Benth, Lange, and Myklebust [10] for a detailed discussion of energy quanto options. From the considerations in section 2, we can express the price at $t \leq \tau$ of the quanto options as

$$V(t, g_1(t), g_2(t)) = \mathrm{e}^{-r(\tau - t)} \mathbb{E} \left[p(\mathcal{L}_{\text{energy}}(g_1(\tau)) q(\mathcal{L}_{\text{temp}}(g_2(\tau)) \, | \, g_1(t), g_2(t) \right] . \tag{4.6}$$

Here, we have assumed that

$$F_{\text{energy}}(t, T_1, T_2) := \mathcal{L}_{\text{energy}}(g_1(t)) := \delta_{T_1 - t} \circ \mathcal{D}_\ell^{w,1}(g_1(t)), \tag{4.7}$$
$$F_{\text{temp}}(t, T_1, T_2) := \mathcal{L}_{\text{temp}}(g_1(t)) := \delta_{T_1 - t} \circ \mathcal{D}_\ell^{w,2}(g_2(t)) \tag{4.8}$$

with $\mathcal{D}_\ell^{w,i}$ defined as in (2.9) using the obvious meaning of the indexing by $i = 1, 2$. Since $\mathcal{L}_{\text{energy}}$ and $\mathcal{L}_{\text{temp}}$ are linear functionals on H_α, it follows from Theorem 2.1 in Benth and Krühner [15] that

$$(F_{\text{energy}}(t, T_1, T_2), F_{\text{temp}}(t, T_1, T_2))$$

is a bivariate Gaussian random variable on $\mathbb{R}^2$. From Lemma 4.4, we can compute the covariance as

$$\begin{aligned} \mathrm{Cov}(F_{\text{energy}}(t, T_1, T_2), F_{\text{temp}}(t, T_1, T_2)) &= \mathbb{E} \left[\mathcal{L}_{\text{energy}} \int_0^t \mathcal{S}_{t-s} \, d\mathbb{B}_1(s) \cdot \mathcal{L}_{\text{temp}} \int_0^t \mathcal{S}_{t-s} \, d\mathbb{B}_2(s) \right] \\ &= \mathbb{E} \left[\langle (\int_0^t \mathcal{S}_{t-s} \, d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s} \, d\mathbb{B}_2(s)), \Pi_1^* \mathcal{L}_{\text{energy}}^*(1) \rangle \right. \end{aligned}$$

[2] Obviously, temperature is not *delivered*, but the temperature futures are settled against the measured temperature index over this period.

$$
\begin{aligned}
&\times\langle(\int_0^t \mathcal{S}_{t-s}\,d\mathbb{B}_1(s), \int_0^t \mathcal{S}_{t-s}\,d\mathbb{B}_2(s)), \Pi_2^*\mathcal{L}_{\text{temp}}^*(1)\rangle\Big] \\
&= \langle \mathcal{Q}_t(\Pi_1^*\mathcal{L}_{\text{energy}}^*(1), \Pi_2^*\mathcal{L}_{\text{temp}}^*(1)), (\Pi_1^*\mathcal{L}_{\text{energy}}^*(1), \Pi_2^*\mathcal{L}_{\text{temp}}^*(1))\rangle \\
&= \int_0^t C_{12}(s)\,ds\,,
\end{aligned}
$$

where

$$
\begin{aligned}
C_{12}(s) = \mathcal{L}_{\text{energy}}\Pi_1\mathcal{S}_s\mathcal{Q}_1\mathcal{S}_s^*\Pi_1^*\mathcal{L}_{\text{energy}}^*(1) + \mathcal{L}_{\text{energy}}\Pi_1\mathcal{S}_s\mathcal{Q}_{12}^*\mathcal{S}_s^*\Pi_2^*\mathcal{L}_{\text{temp}}^*(1) \\
+ \mathcal{L}_{\text{temp}}\Pi_2\mathcal{S}_s\mathcal{Q}_{12}\mathcal{S}_s^*\Pi_1^*\mathcal{L}_{\text{energy}}^*(1) + \mathcal{L}_{\text{temp}}\Pi_2\mathcal{S}_s\mathcal{Q}_2\mathcal{S}_s^*\Pi_2^*\mathcal{L}_{\text{temp}}^*(1)\,.
\end{aligned}
$$

Thus, we can obtain a price $V(t, g_1(t), g_2(t))$ in terms of an integral with respect to a Gaussian bivariate probability distribution, involving similar operators (and their duals) as for the European options studied in section 3. We remark in passing that Benth, Lange, and Myklebust [10] derive a Black and Scholes-like pricing formula for a call-call quanto options, which is applied to price such derivatives written on Henry Hub gas futures traded at NYMEX and HDD/CDD temperature futures traded at CME.

We next return back to the general considerations on the factorization of the covariance operator $\mathcal{Q}$ of a bivariate square-integrable random variable in $H_1\times H_2$. If we want to construct an operator $\mathcal{Q}$ as in Theorem 4.3, then the operator $\mathcal{Q}_{12}$ appearing there necessarily has to satisfy condition (ii). As we will show in the next theorem, condition (ii) of Theorem 4.3 is sufficient as well.

Theorem 4.5. *Let H_i be a separable Hilbert space and $\mathcal{Q}_i$ be a positive semidefinite trace class operator on H_i and define $\mathcal{R}_i := \sqrt{\mathcal{Q}_i}$ for $i = 1, 2$. Let $\mathcal{Q}_{12} \in L(H_1, H_2)$ such that*

$$
|\langle \mathcal{Q}_{12}u, v\rangle| \le \|R_1u\|_1\|R_2v\|_2
$$

for any $u \in H_1$, $v \in H_2$. Then

$$
\mathcal{Q} := \begin{pmatrix} \mathcal{Q}_1 & \mathcal{Q}_{12}^* \\ \mathcal{Q}_{12} & \mathcal{Q}_2 \end{pmatrix}
$$

defines a positive semidefinite operator on $H_1 \times H_2$. Moreover, $\mathcal{Q}$ is positive definite if and only if $\mathcal{Q}_1$, $\mathcal{Q}_2$ are positive definite and

$$
|\langle \mathcal{Q}_{12}u, v\rangle| < \|\mathcal{R}_1u\|_1\|\mathcal{R}_2v\|_2
$$

for any $u \in H_1\backslash\{0\}$, $v \in H_2\backslash\{0\}$.

Proof. Let $u \in H_1$ and $v \in H_2$. Then

$$
\begin{aligned}
\langle \mathcal{Q}(u, v), (u, v)\rangle &= \langle \mathcal{Q}_1u, u\rangle + \langle \mathcal{Q}_2v, v\rangle + 2\langle \mathcal{Q}_{12}u, v\rangle \\
&\ge \|\mathcal{R}_1u\|_1^2 + \|\mathcal{R}_2v\|_2^2 - 2\|\mathcal{R}_1u\|_1\|\mathcal{R}_2v\|_2 \\
&= (\|\mathcal{R}_1u\|_1 - \|\mathcal{R}_2v\|_2)^2 \\
&\ge 0.
\end{aligned}
$$

Under the additional assumptions, the first inequality is strict. ■

We now analyze the pricing of spread options in a simple setting: Let us consider a bivariate exponential model $g_i(t) = \exp(\widetilde{g}_i(t))$, $i = 1, 2$, defined on the space $H_\alpha \times H_\alpha$ by a dynamics similar to (4.1) (but with a drift) driven by $(\mathbb{L}_1(t), \mathbb{L}_2(t)) = (\mathbb{B}_1(t), \mathbb{B}_2(t))$:

$$d\widetilde{g}_1(t) = \partial_x \widetilde{g}_1(t)\,dt + \mu_1(t)\,dt + \sigma_1(t)\,d\mathbb{B}_1(t),$$
$$d\widetilde{g}_2(t) = \partial_x \widetilde{g}_2(t)\,dt + \mu_2(t)\,dt + \sigma_2(t)\,d\mathbb{B}_2(t)$$

Here, we suppose that $\sigma_i : \mathbb{R}_+ \to L(H_\alpha)$ is nonrandom and $\sigma_i \in \mathcal{L}^2_{\mathbb{B}_i}(H_\alpha)$, $i = 1, 2$. Thus, we have the forward price dynamics $f_i(\tau, T)$ given $f_i(t, T)$ for $t \le \tau \le T$,

$$f_i(\tau, T) = f_i(t, T)\exp\left(\int_t^\tau \delta_{T-s}\mu_i(s)\,ds + \delta_{T-\tau}\int_t^\tau \mathcal{S}_{\tau-s}\sigma_i(s)\,d\mathbb{B}_i(s)\right) \tag{4.9}$$

for $i = 1, 2$. Introduce the notation

$$\widetilde{\sigma}_i^2(s, T) = \delta_{T-s}\sigma_i(s)\mathcal{Q}_i\sigma_i^*(s)\delta_{T-s}^*(1) \tag{4.10}$$

for $i = 1, 2$. From Theorem 2.1 in Benth and Krühner [15], it follows for $i = 1, 2$,

$$\delta_{T-\tau}\int_t^\tau \mathcal{S}_{\tau-s}\sigma_i(s)\,d\mathbb{B}_i(s) = \int_t^\tau \widetilde{\sigma}_i(s, T)\,dB_i(s)\,,$$

where B_i is a real-valued Brownian motion. By Lemma 3.13, we have the no-arbitrage drift condition

$$\widetilde{\mu}_i(s, T) := \delta_{T-s}\mu_i(s) = -\frac{1}{2}\widetilde{\sigma}_i^2(s, T)\,. \tag{4.11}$$

Remark that, as a consequence of the nonrandom assumption on $\sigma_i(s)$, $\int_t^\tau \widetilde{\sigma}_i(s, T)\,dB_i(s), i = 1, 2$, are two Gaussian random variables on $\mathbb{R}$ with mean zero and variance $\int_t^\tau \widetilde{\sigma}_i^2(s, T)\,ds, i = 1, 2$, respectively Moreover, a direct computation using the above theory reveals the covariance between these two random variables:

$$\begin{aligned}
&\mathbb{E}\left[\int_t^\tau \widetilde{\sigma}_1(s, T)\,dB_1(s)\int_t^\tau \widetilde{\sigma}_2(s, T)\,dB_2(s)\right] \\
&\quad = \mathbb{E}\left[\delta_{T-\tau}\int_t^\tau \mathcal{S}_{\tau-s}\sigma_1(s)\,d\mathbb{B}_1(s) \times \delta_{T-\tau}\int_t^\tau \mathcal{S}_{\tau-s}\sigma_2(s)\,d\mathbb{B}_2(s)\right] \\
&\quad = \mathbb{E}\left[\langle\int_t^\tau \mathcal{S}_{\tau-s}\sigma_1(s)\,d\mathbb{B}_1(s), h_{T-\tau}\rangle\langle\int_t^\tau \mathcal{S}_{\tau-s}\sigma_2(s)\,d\mathbb{B}_2(s), h_{T-\tau}\rangle\right] \\
&\quad = \int_t^\tau \langle \mathcal{Q}\Pi_1^*\sigma_1^*(s)\mathcal{S}_{\tau-s}^* h_{T-\tau}, \Pi_2^*\sigma_2^*(s)\mathcal{S}_{\tau-s}^* h_{T-\tau}\rangle\,ds \\
&\quad = \int_t^\tau \delta_{T-s}\sigma_2(s)\mathcal{Q}_{12}\sigma_1^*(s)\delta_{T-s}^*(1)\,ds \\
&\quad := \int_t^\tau \widetilde{\sigma}_{12}(s, T)\,ds\,.
\end{aligned}$$

Hence, for $i = 1, 2$,

$$f_i(\tau, T) = f_i(t, T) \exp\left(-\frac{1}{2}\int_t^\tau \widetilde{\sigma}_i^2(s, T)\, ds + \int_t^\tau \widetilde{\sigma}_i(s)\, dB_i(s)\right), \tag{4.12}$$

where we know that the two stochastic integrals form a bivariate Gaussian random variable with known variance-covariance matrix. The price at time t of a call option written on the spread between the two forwards with exercise at time $t \leq \tau \leq T$ will be

$$V(t) = \mathrm{e}^{-r(\tau-t)}\mathbb{E}\left[\max\left(f_1(\tau, T) - f_2(\tau, T), 0\right) \mid \mathcal{F}_t\right].$$

Using the representation of the forward prices in (4.12), we find the spread option pricing formula

$$V(t) = \mathrm{e}^{-r(\tau-t)}\left\{f_1(t, T)\Phi(d_+) - f_2(t, T)\Phi(d_-)\right\}, \tag{4.13}$$

where Φ is the cumulative standard normal distribution function,

$$d_\pm = \frac{\ln(f_1(t, T)/f_2(t, T)) \pm \Sigma^2(t, \tau, T)/2}{\Sigma(t, \tau, T)},$$

and

$$\Sigma^2(t, \tau, T) = \int_t^\tau \widetilde{\sigma}_1^2 - 2\widetilde{\sigma}_{12}(s, T) + \widetilde{\sigma}_2^2(s, T)\, ds.$$

We have recovered the Margrabe formula (see Margrabe [33]) with time-dependent volatility and correlation. Observe that the spread option price becomes a function of the initial forward prices at time t with delivery at time T.

We proceed with some more general considerations on bivariate random variables in Hilbert spaces and their representation. If (X, Y) is a two-dimensional Gaussian random variable, we know from classical probability theory that there exist a Gaussian random variable Z being independent of X and $a \in \mathbb{R}$ such that $Y = aX + Z$. The next Proposition is a generalization of this statement to square-integrable Hilbert-space valued random variables.

Proposition 4.6. *Let X_i be an H_i-valued square-integrable random variable with covariance $\mathcal{Q}_i$ and let $\mathcal{Q}_{12} \in L(H_1, H_2)$ be the operator given in Theorem 4.3 such that*

$$\mathcal{Q} := \begin{pmatrix} \mathcal{Q}_1 & \mathcal{Q}_{12}^* \\ \mathcal{Q}_{12} & \mathcal{Q}_2 \end{pmatrix}$$

is the covariance operator of (X_1, X_2). Assume that $\mathrm{ran}(\mathcal{Q}_{12}^*) \subseteq \mathrm{ran}(\mathcal{Q}_1)$. Then the closure $\mathcal{B}$ of the densely defined operator $\mathcal{Q}_{12}\mathcal{Q}_1^{-1}$ is in $L(H_1, H_2)$, where $\mathcal{Q}_1^{-1}$ denotes the pseudoinverse of $\mathcal{Q}_1$. Define $Z := X_2 - \mathcal{B}X_1$. Then Z is a centered, square integrable, and H_2-valued random variable with $\mathbb{E}(\langle X_1, u\rangle\langle Z, v\rangle) = 0$ for any $u \in H_1$, $v \in H_2$, i.e., X_1 and Z are uncorrelated.

In particular, the covariance operator of (X_1, Z) is given by

$$\mathcal{Q}_{X_1,Z} := \begin{pmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_Z \end{pmatrix},$$

where $\mathcal{Q}_Z$ denotes the covariance operator of Z.

Proof. $\mathcal{Q}_{12}\mathcal{Q}_1^{-1}$ is densely defined because its domain is the domain of $\mathcal{Q}_1^{-1}$. Define $\mathcal{C} := \mathcal{Q}_1^{-1}\mathcal{Q}_{12}^*$, which is a closed operator whose domain is H_2 by assumption. The closed graph theorem yields that $\mathcal{C}$ is continuous and linear. Consequently, its dual is a continuous linear continuation of $\mathcal{Q}_{12}\mathcal{Q}_1^{-1}$. However, the latter operator is densely defined, and hence $\mathcal{B} := \mathcal{C}^*$ is its closure. Now, let $u \in H_1$, $v \in H_2$. Then

$$\begin{aligned}\mathbb{E}[\langle X_1, u\rangle\langle \mathcal{B}X_1, v\rangle] &= \langle \mathcal{Q}_1 u, \mathcal{B}^* v\rangle \\ &= \langle \mathcal{Q}_1 u, \mathcal{Q}_1^{-1}\mathcal{Q}_{12}^* v\rangle \\ &= \langle \mathcal{Q}_{12} u, v\rangle \\ &= \mathbb{E}[\langle X_1, u\rangle\langle X_2, v\rangle].\end{aligned}$$

Thus, X_1 and Z are uncorrelated, and the claim follows. ■

This result can be applied to state a representation of the $H_1 \times H_2$-valued Wiener process $(\mathbb{B}_1, \mathbb{B}_2)$.

Proposition 4.7. *Let $\mathbb{B}_1$, $\mathbb{B}_2$ be H_1, respectively H_2-valued Brownian motions where H_1, H_2 are separable Hilbert spaces. Suppose that the random variables $\mathbb{B}_i(1)$, $i = 1, 2$, satisfy the conditions in Proposition 4.6. Then, there exists an operator $\mathcal{B} \in L(H_1, H_2)$ such that $\mathbb{W} := \mathbb{B}_2 - \mathcal{B}\mathbb{B}_1$ is an H_2-valued Brownian motion which is independent of H_1.*

Proof. Let $\mathcal{B}$ be the operator given in Proposition 4.6 for the random variables $\mathbb{B}_1$ and $\mathbb{B}_2$. Then, $(\mathbb{B}_1, \mathbb{W})$ is an other Brownian motion. Moreover,

$$\mathbb{E}[\langle \mathbb{B}_1(t), u\rangle\langle \mathbb{W}(t), v\rangle] = t\mathbb{E}[\langle \mathbb{B}_1(1), u\rangle\langle \mathbb{W}(1), v\rangle] = 0$$

for any $t \geq 0$. The claim follows. ■

The proposition allows us to model a bivariate forward dynamics driven by two dependent Brownian motions

$$\begin{aligned}dg_1(t) &= \partial_x g_1(t)\,dt + \sigma_1(t, g_1(t), g_2(t))\,d\mathbb{B}_1(t), \\ dg_2(t) &= \partial_x g_2(t)\,dt + \sigma_2(t, g_1(t), g_2(t))\,d\mathbb{B}_2(t)\end{aligned}$$

by a dynamics driven by two independent Brownian motions,

$$\begin{aligned}dg_1(t) &= \partial_x g_1(t)\,dt + \sigma_1(t, g_1(t), g_2(t))\,d\mathbb{B}_1(t), \\ dg_2(t) &= \partial_x g_2(t)\,dt + \sigma_2(t, g_1(t), g_2(t))\,d\mathbb{W}(t) - \sigma_2(t, g_1(t), g_2(t))\mathcal{B}\,d\mathbb{B}_1(t)\,.\end{aligned}$$

Here, the operator $\mathcal{B}$ plays the role of a correlation coefficient, describing how the two noises $\mathbb{B}_1$ and $\mathbb{B}_2$ depend. Indeed, choosing $H_i = H_\alpha$, $i = 1, 2$ to be the Filipovic space, we see that

$$\begin{aligned}\mathbb{E}[\delta_x\mathbb{B}_1(t)\delta_y\mathbb{B}_2(t)] &= \mathbb{E}[\langle \mathbb{B}_1(t), h_x\rangle\langle \mathbb{B}_2(t), h_y\rangle] \\ &= \mathbb{E}[\langle \mathbb{B}_1(t), h_x\rangle\langle \mathcal{B}\mathbb{B}_1(t), h_y\rangle] \\ &= t\langle \mathcal{B}\mathcal{Q}_1 h_x, h_y\rangle \\ &= t\delta_y\mathcal{B}\mathcal{Q}_1\delta_x^*(1)\end{aligned}$$

for $x, y \in \mathbb{R}_+$. Hence, the correlation between $\mathbb{B}_1(t, x)$ and $\mathbb{B}_2(t, y)$ is modelled by the operator $\mathcal{B}$. We can derive a similar representation for two Lévy processes, but they will not be independent but only uncorrelated in most cases.

As a final remark, we like to note that the "odd" range condition in Proposition 4.7 is needed to ensure the existence of a linear operator from H_1 to H_2. However, in the Gaussian case it is possible to find a linear operator $\mathcal{T}$ from $L^2(\Omega, \mathcal{A}, P, H_1)$ to $L^2(\Omega, \mathcal{A}, P, H_2)$ yielding an independent decomposition of the second factor. We now give the precise statement.

Proposition 4.8. *Let H_1, H_2 be separable Hilbert spaces and (X_1, X_2) be an $H_1 \times H_2$-valued Gaussian random variable. Let $\mathcal{B}$ be the closure of $Q_{12}^* Q_1^{-1}$. Then, $P(X_1 \in \mathrm{dom}(\mathcal{B})) = 1$ and $Z := X_2 - \mathcal{B}X_1$ is Gaussian and X_1, Z are independent.*

Proof. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H_1 such that $X_1 = \sum_{n=1}^\infty \sqrt{\lambda_n}\Phi_n e_n$, where $(\Phi_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. standard normal random variables $\lambda_n \geq 0$ and $\sum_{n\in\mathbb{N}} \lambda_n < \infty$; cf. Peszat and Zabczyk [34, Thm. 4.20]. Define $Y_k := \sum_{n=1}^k \sqrt{\lambda_n}\Phi_n e_n$ for any $k \in \mathbb{N}$. Clearly, we have $Y_k \to X_1$ for $k \to \infty$. We now want to show that $\mathcal{B}Y_k$ converges to $\mathbb{E}[X_2|X_1]$, which will complete the proof.

Let $(p_j)_{j\in\mathbb{N}}$ be the hermite polynomials on $\mathbb{R}$. Then $\mathbb{E}[p_j(\Phi_1)p_i(\Phi_1)] = 1_{\{i=j\}}$ for any $i, j \in \mathbb{N}$. For an H_2-valued square integrable random variable A, we have

$$\mathbb{E}[A|X_1] = \sum_{n,m,j=1}^{\infty} \mathbb{E}[\langle A, f_m\rangle p_j(\Phi_n)]p_j(\Phi_n)f_m,$$

where $(f_m)_{m\in\mathbb{N}}$ is an orthonormal basis of H_2. Since (X_1, X_2) is Gaussian, $(\Phi_n, \langle X_2, f_m\rangle)$ is Gaussian for any $n, m \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\mathbb{E}[X_2|X_1] &= \sum_{n,m,j=1}^{\infty} \mathbb{E}[\langle X_2, f_m\rangle p_j(\Phi_n)]p_j(\Phi_n)f_m \\ &= \sum_{n,m=1}^{\infty} \mathbb{E}[\langle X_2, f_m\rangle \Phi_n]\Phi_n f_m\end{aligned}$$

because $\mathbb{E}[Ap_j(B)] = 0$ whenever (A, B) is a normal random variable in $\mathbb{R}^2$, B is standard normal, and $j \neq 1$. Moreover, $\Phi_n = \frac{\langle X_1, e_n\rangle}{\sqrt{\lambda_n}}$, and hence

$$\begin{aligned}\mathbb{E}[\langle X_2, f_m\rangle \Phi_n] &= \frac{\langle \mathcal{Q}_{12}e_n, f_m\rangle}{\sqrt{\lambda_n}}, \\ \Phi_n &= \sqrt{\lambda_n}\langle X_1, \mathcal{Q}_1^{-1}e_n\rangle, \\ \mathbb{E}[X_2|X_1] &= \sum_{n,m=1}^{\infty} \langle \mathcal{Q}_{12}e_n, f_m\rangle\langle X_1, \mathcal{Q}_1^{-1}e_n\rangle f_m \\ &= \sum_{n=1}^{\infty}\langle X_1, \mathcal{Q}_1^{-1}e_n\rangle \mathcal{Q}_{12}e_n.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathcal{B}Y_k &= \sum_{n=1}^{k} \langle Y_k, \mathcal{Q}_1^{-1} e_n \rangle \mathcal{Q}_{12} e_n \\ &= \sum_{n=1}^{k} \langle X_1, \mathcal{Q}_1^{-1} e_n \rangle \mathcal{Q}_{12} e_n \\ &\to \mathbb{E}[X_2 | X_1]\end{aligned}$$

for $k \to \infty$, where we used Parseval's identity for the first equality. Since $\mathcal{B}$ is closed, we have $X_1 \in \mathrm{dom}(\mathcal{B})$ P-a.s. and $\mathcal{B}X_1 = \mathbb{E}[X_2|X_1]$. ■

5. Numerical illustration. In this section, we look at a simplified setup to illustrate the power of the tools developed in this paper. We do compare three different models, namely, a classical "sum of two OU-processes spot model," a truly infinite-dimensional Gaussian model with clear interpretation of its parameters and an infinite-dimensional HNIG with the same correlation structure as the infinite-dimensional Gaussian model (cf. [16, sect. 5.1] for the definition of HNIG processes).

In each of the three cases, the model for the underlying forward curve can be written as

$$df(t) = \partial_x f(t) dt + d\mathbb{L}(t), f(0) = f_0 \in H_\alpha \,, \tag{5.1}$$

under the pricing measure, where $\mathbb{L}$ is some H_α-valued Lévy process. The futures prices of contracts with delivery period $[T_1, T_2]$ can be recovered by

$$F(t, T_1, T_2) = F(0, T_1, T_2) + \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_0^t \delta_{\tau - s} \, d\mathbb{L}(s) \, d\tau$$

for any $t \le T_1$, where $L_{[T_1,T_2]}(t) := \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_0^t \delta_{\tau-s} \, d\mathbb{L}(s) \, d\tau$ is a centered one-dimensional Lévy process. In particular, if $\mathbb{L} = \mathbb{W}$ is Gaussian, then $W_{[T_1,T_2]}(t) := L_{[T_1,T_2]}(t)$ is a one-dimensional Brownian motion and

$$\mathbb{E}\left[W_{[T_1,T_2]}(t) W_{[S_1,S_2]}(t)\right] = \frac{1}{T_2 - T_1} \frac{1}{S_2 - S_1} \int_{T_1}^{T_2} \int_{S_1}^{S_2} \int_0^t q(\tau_1 - s, \tau_2 - s) \, ds \, d\tau_1 \, d\tau_2 \tag{5.2}$$

$$= \int_0^t \left(\frac{1}{T_2 - T_1} \frac{1}{S_2 - S_1} \int_{T_1}^{T_2} \int_{S_1}^{S_2} q(\tau_1 - s, \tau_2 - s) \, d\tau_1 \, d\tau_2 \right) ds \tag{5.3}$$

for any $t \le T_1 \wedge S_1$, $0 \le S_1 \le S_2$, $0 \le T_1 \le T_2$, where

$$q(x, y) := \langle \mathcal{Q} h_x, h_y \rangle, \quad x, y \ge 0,$$

and $\mathcal{Q}$ is the covariance operator of $\mathbb{W}$.

In order to get simple structural formulae, we define the two *covariance functions*

$$q_1(x,y) := a_1 e^{-\lambda_1(x+y)} + a_2 e^{-\lambda_2(x+y)}\,,$$
$$q_2(x,y) := a e^{-\lambda(x+y)-\mu|x-y|}$$

for any $x, y \geq 0$, where $a_1, a_2, a \geq 0$, $\lambda_1, \lambda_2, \lambda, \mu > 0$, and their associated covariance operators

$$\mathcal{Q}_1 h(x) := \langle h, q_1(x,\cdot)\rangle\,,$$
$$\mathcal{Q}_2 h(x) := \langle h, q_2(x,\cdot)\rangle$$

for $x \geq 0$. The covariance function $q_j(x,y)$ roughly determines the local quadratic covariation between two futures with instantaneous delivery and time to delivery x and y; cf. (5.3).

We record that the first covariance function does indeed belong to a sum of OU spot-model. The following proposition has an obvious generalisation to multi-factor OU spot-models.

Proposition 5.1. *Assume that* (5.1) *holds and that* $\mathbb{W} := \mathbb{L}$ *is an* H_α*-valued Brownian motion with covariance operator* $\mathcal{Q}_1$*. Then, the spot price process* $S(t) := \delta_0(f(t))$ *is given by*

$$S(t) = f_0(t) + \sum_{i=1}^{2} \sqrt{a_i} \int_0^t e^{-\lambda_i(t-s)} dW_i(s)$$

for any $t \geq 0$*, where* W_1*,* W_2 *are two independent standard Brownian motions.*

Proof. Let (W_1, W_2) be independent standard Brownian motions and define

$$\tilde{\mathbb{W}}(t,x) = \sum_{j=1}^{2} \sqrt{a_j} e^{-\lambda_j x} W_j(t).$$

Then $\tilde{\mathbb{W}}$ is an H_α-valued Brownian motion, and we have

$$E\left[\langle h_1, \tilde{\mathbb{W}}\rangle\langle h_2, \tilde{\mathbb{W}}\rangle\right] = \sum_{j=1}^{2}\langle h_1, b_j\rangle\langle h_2, b_j\rangle \tag{5.4}$$
$$= \langle \mathcal{Q}_1 h_1, h_2\rangle \tag{5.5}$$

for any $h_1, h_2 \in H_\alpha$, where $b_j(x) := \sqrt{a_j} e^{-\lambda_j x}$ for $x \geq 0$ and obviously $b_j \in H_\alpha$. Thus, $\mathcal{Q}_1$ is the covariance operator $\tilde{\mathbb{W}}$. Since $\mathbb{W}$ is an other Brownian motion with covariance operator $\mathcal{Q}_1$, we have that $\tilde{\mathbb{W}}$ and $\mathbb{W}$ have the same law. Hence, we may assume $\tilde{\mathbb{W}} = \mathbb{W}$. Now, we get

$$S(t) = \delta_0(\mathcal{S}_t f_0) + \int_0^t \delta_0 \mathcal{S}_{t-s} d(b_1 W_1 + b_2 W_2)(s)$$
$$= f_0(t) + \int_0^t b_1(t-s)\, dW_1(s) + \int_0^t b_2(t-s)\, dW_2(s)$$

for any $t \geq 0$, as claimed. ■

Next, we show that a Wiener process with covariance $\mathcal{Q}_2$ cannot be finite-dimensional. We would like to stress that in practice the aim is to explain the presented correlation structure given by real world data. Applying techniques from spatial statistics (see Cressie and Wikle [25]), one can estimate the parameters of low-parametric correlation functions like q_2. Such correlation functions typically do not belong to a finite-dimensional noise term representation, and it is possible that reasonable finite-dimensional approximations might need a lot of factors. Since the numerical handling of a lot of factors (like ≥ 10 as argued for by Benth, Benth, and Koekebakker [12] in the Nordic power market NordPool) compared to theoretically infinitely many is not very different, we see no advantage in restricting oneself to a finite amount of driving factors. Indeed, the infinite-dimensional case is based on low-parametric correlation structures. Moreover, one has to bear in mind that a finite-dimensional approximation induces an additional approximation error.

Lemma 5.2. *Let $\mathbb{W} := \mathbb{L}$ be an H_α-valued Brownian motion with covariance operator $\mathcal{Q}_2$. Then, there is no finite-dimensional subspace $V \subseteq H_\alpha$ such that $P(\mathbb{W}(t) \in V) = 1$ for any $t > 0$.*

Proof. Assume by contradiction that there is a finite-dimensional subspace $V \subseteq H_\alpha$ such that $P(\mathbb{W}(t) \in V) = 1$ for any $t > 0$. Let $b_1, \dots, b_d$ be the eigenvectors of the positive operator $\mathcal{Q}_2$, where $d := \dim(V)$. Then, we have

$$E(\langle b_j, \mathbb{W}(t)\rangle\langle b_k, \mathbb{W}(t)\rangle = \langle \mathcal{Q} b_j, b_k\rangle = 1_{\{j=k\}}\lambda_j\langle b_j, b_j\rangle\,,$$

where λ_j denotes the eigenvalue corresponding to b_j. Hence, $W_j(t) := \langle b_j, \mathbb{W}(t)\rangle$ is an $\mathbb{R}$-valued Brownian motion and Parseval's identity yields

$$\mathbb{W}(t) = \sum_{j=1}^{d} b_j W_j(t), \quad t \geq 0\,.$$

As in the proof of Propositon 5.1, we can now deduce that

$$q_2(x, y) = \sum_{j=1}^{d} b_j(x) b_j(y),$$

which obviously is not the case. ■

It is easily seen that more summands for a covariance function, i.e.,

$$q_3(x, y) = \sum_{j=1}^{N} a_j e^{-\lambda_j(x+y)}\,,$$

leads to more OU-factors for the spot price process. However, instead of adding an appropriate number of factors — either for the spot or the futures dynamics directly — we aim at a specific covariance structure. The intuitive meaning of the parameters of the covariance function q_2 are as follows. The parameter λ determines how quickly the local covariance of an asset decays the further its delivery date/period is in the future. It is quite clear that a security with a delivery far in the future has less volatility than a derivative with delivery in short

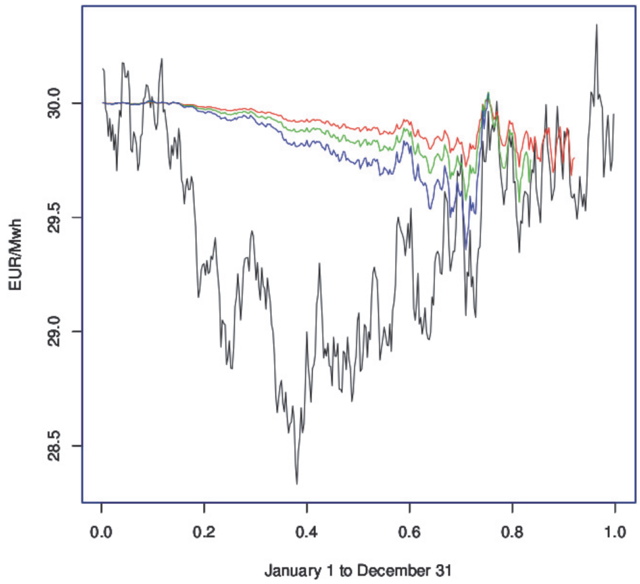

Figure 1. *Spot (black) and futures prices for contracts with delivery in October (blue), November (green), and December (red); generated with the model* (5.1) *based on a covariance operator $\mathcal{Q}_1$ and a Brownian driving noise. Here, the spot is driven by two independent OU processes; cf. Proposition* 5.1. *The parameters have been chosen to be $a_1 = 0.5 = a_2$, $\lambda_1 = 5$, $\lambda_2 = 50$, and $f_0(t) = 30$.*

time. The parameter μ determines the covariance between derivatives with different delivery dates/periods, namely, the higher μ is, the quicker the covariance between the assets vanishes. Finally, the parameter a determines the overall volatility level, i.e., the higher it is, the higher is the total volatility of any security.

The covariance function q_2, however, belongs to an infinite-dimensional Brownian motion, while the proof of Proposition 5.1 reveals that the covariance function q_1 essentially belongs to a two-dimensional Brownian motion in the sense that it takes values in a two-dimensional subspace of H_α, namely, in the span of b_1, b_2, where b_1, b_2 are the basis functions appearing in the proof of Proposition 5.1.

Since the joint law of the futures is jointly Gaussian and its covariance is determined by the covariance operator if we assume (5.1) and that the driving law is an H_α-valued Brownian motion $\mathbb{W} := \mathbb{L}$, it is simple and straightforward to run simulations or to find simulated values for spread options on futures with different delivery periods.

At the end of the paper, we provide our simulations, all of them with the same total theoretical volatility of 100% annually for the spot. In our simulations, we have not taken into account any seasonal effects—which of course should be captured in the initial forward curve f_0. The model run by two OU-type processes (see Figure 1) clearly shows that the futures with different delivery periods are strongly correlated, and in fact any two of them can be used to perfectly hedge any other instrument on that market until the first delivery period of the two futures starts. This market completeness effect can only be removed by either having

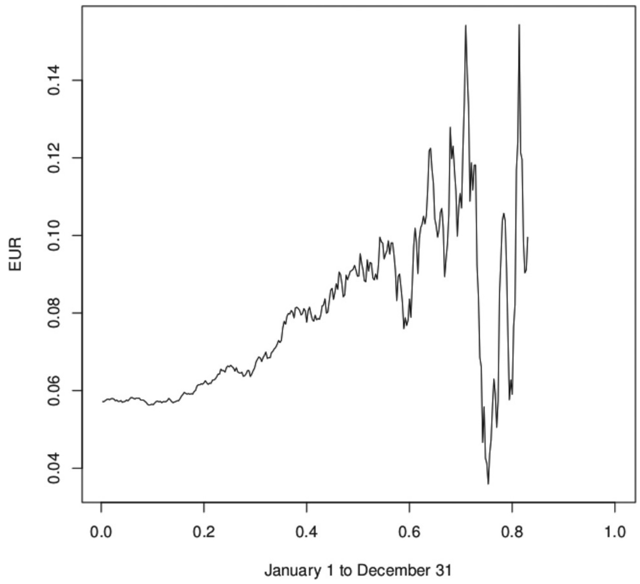

Figure 2. *Spread option price dynamics between the December and November futures with delivery on October* 31 *and strike equal to zero. Dynamics is computed based on the OU model.*

more Brownian motions than there are securities or by introducing jumps. The Gaussian driven model with covariance operator $\mathcal{Q}_2$ shows (see Figure 3) a very clear decoupling of the futures prices which is even more clear for futures with far distant delivery periods—or with a higher μ parameter. Finally, we added simulation with HNIG processes which can be easily generated by subordination, i.e., in a first step, one generates an inverse Gaussian random variable with mean 1, uses its value as a factor on the covariance operator for the next simulation step, and generates one Gaussian increment. Then, one repeats this for the next time step. Visually, the paths look as random as for the Gaussian case (see Figure 5), but a finer analysis would reveal NIG distributed price differentials. These are likely to be more in line with actual market behavior (see Andresen, Koekebakker, and Westgaard [3], Frestad [28], Frestad, Benth, and Koekebakker [29] for evidence of NIG distributed returns in power markets). The leptokurtic behavior is more pronounced in the spot price than in the smoothed (by delivery period averaging) monthly contracts. The Samuelson effect is evident in all three models.

The values for the spread options can be calculated directly because the conditional laws of the futures are Gaussian if the driving noise was a Wiener process. Indeed, if X, Y are $N\left((\mu_1,\mu_2), \begin{pmatrix} c_1 & c_{12} \\ c_{12} & c_2 \end{pmatrix}\right)$ distributed, then

$$E((X-Y-K)^+) = \mu\Phi(\mu/\sqrt{c}) + \sqrt{c}\phi(\mu/\sqrt{c}),$$

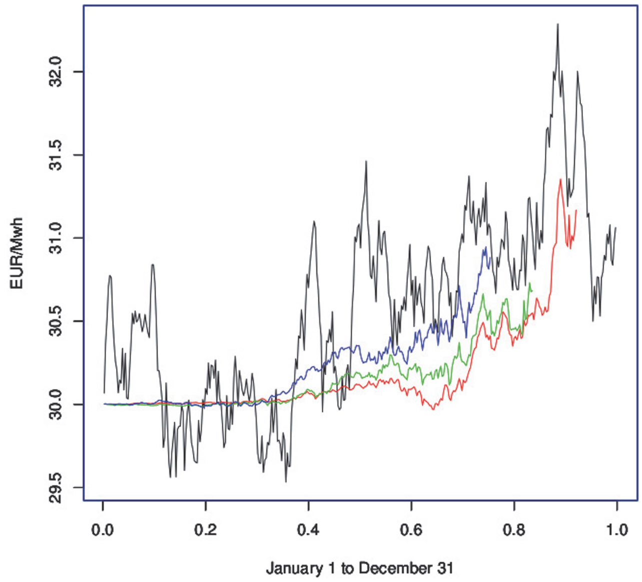

Figure 3. *Spot (black) and futures prices for contracts with delivery in October (blue), November (green), and December (red); generated with the model (5.1) based on a covariance operator $\mathcal{Q}_2$ and a Brownian driving noise. The parameters have been chosen to be $a = 1$, $\lambda = 5$, $\mu = 5$, and $f_0(t) = 30$.*

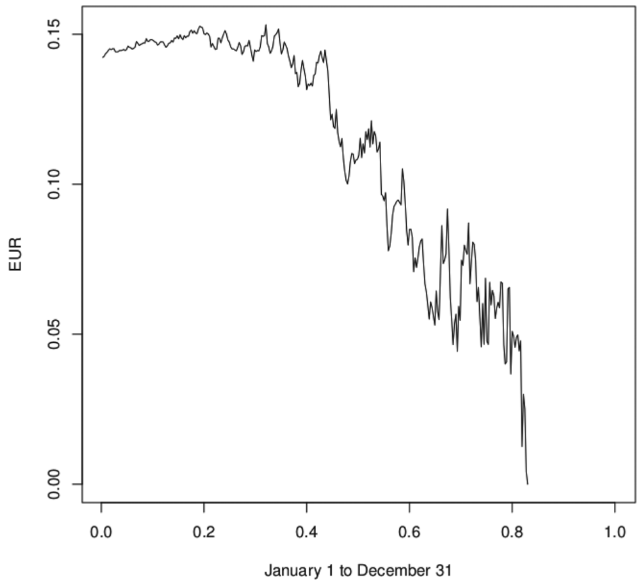

Figure 4. *Spread option price dynamics between the December and November futures with delivery on October 31 and strike equal to zero. Dynamics is computed based on the Gaussian model.*

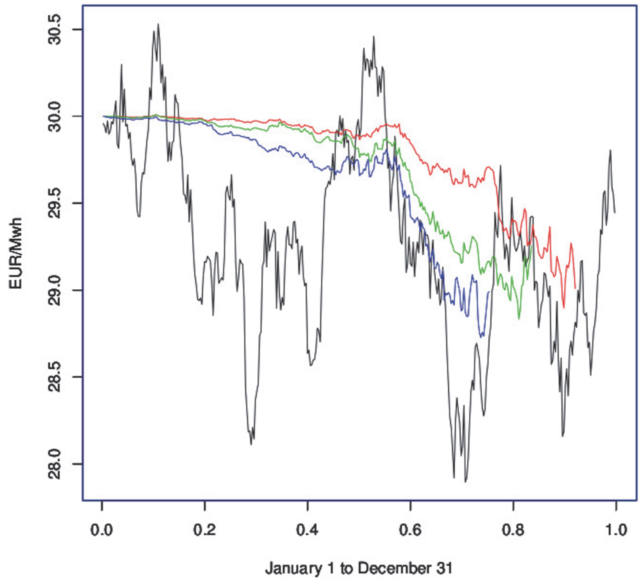

Figure 5. *Spot (black) and futures prices for contracts with delivery in October (blue), November (green), and December (red); generated with the model* (5.1) *based on a covariance operator* $\mathcal{Q}_2$ *and an HNIG driving noise. The parameters have been chosen to be* $a = 1$, $\lambda = 5$, $\mu = 5$, *and* $f_0(t) = 30$.

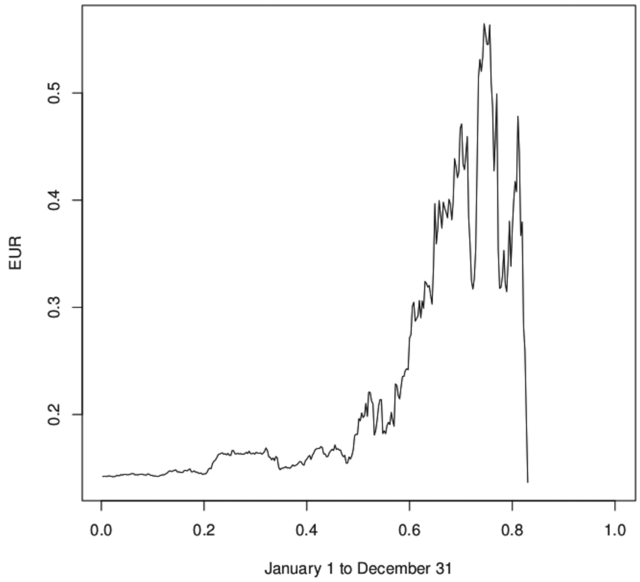

Figure 6. *Spread option price dynamics (cyan) between the December and November futures with delivery on October* 31 *and strike equal to zero. Dynamics is computed based on the HNIG model.*

where $c := c_1 + c_2 - c_{12}$ is the variance of $X - Y$ and $\mu := \mu_1 - \mu_2 - K$ is the mean of $X - Y - K$. Since the law of the NIG distribution is known, similar calculations lead to a semiclosed form expression, i.e., up to one integral expression which has to be solved numerically; compare Proposition 3.11 for the Gaussian case and Proposition 3.7 for the more general price formula. The dynamics of the spread option price is shown for the three different spot-futures models above in Figures 2, 4, and 6. The price dynamics look qualitatively slightly different, with the OU model more volatile than the Gaussian and HNIG. This is an implication of the stronger correlation implied by the infinite-dimensional models compared with the two-factor OU model. We have used the simulated paths from the examples above as the underlying futures prices.

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