BOUNDS FOR THE AVAILABILITIES AND UNAVAILABILITIES IN A FIXED TIME INTERVAL FOR MULTISTATE MONOTONE SYSTEMS OF MULTISTATE, MAINTAINED, INTERDEPENDENT COMPONENTS

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Abstract

In this paper upper and lower bounds for the availability and unavailability, to any level, in a fixed time interval are arrived at for multistate, monotone systems based on corresponding information on the multistate components. These are assumed to be maintained and interdependent. Such bounds are of great interest when trying to predict the performance process of the system noting that exact expressions are obtainable just for trivial systems. The bounds given generalize the existing bounds known in traditional binary theory and represent improvements of the ones being developed by now in multistate theory.

MULTISTATE MONOTONE SYSTEMS; AVAILABILITIES; UNAVAILABILITIES; BOUNDS; FIXED TIME INTERVAL; MAINTENANCE; INTERDEPENDENT COMPONENTS

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1. Introduction and basic definitions

In reliability theory a key problem is to find out how the reliability of a complex system can be determined from knowledge of the reliabilities of its components. Agrawal and Barlow (1983) demonstrate that computational complexity makes it impossible to arrive at exact reliabilities associated with a large binary coherent system even if the system can be represented by a graph and its components unrealistically are assumed to be independent and not maintained. Therefore bounds on the reliabilities seem to be a necessity. Such bounds are given in Natvig (1980) generalizing and extending work by Esary and Proschan (1970), Bodin (1970) and Barlow and Proschan (1975).

One inherent weakness of traditional reliability theory, as treated in the papers mentioned above, is that the system and the components are always described just as functioning or failed. Fortunately, by now this theory is being replaced by a theory for multistate systems of multistate components. This enables one for instance in a power generation system to let the system state be the amount of power generated, or in a pipeline system the amount of oil running through a crucial point. In both cases the system state is possibly measured on a discrete scale. Three recent papers in this area are Block and Savits (1982), Butler (1982) and Natvig (1982) which independently generalize bounds in Barlow and Proschan (1975) to the multistate case. A summary of the present state of the art of multistate theory, also aiming at a standardization of the terminology, is given in Natvig (1984). The purpose of the present paper is to generalize all bounds in Natvig (1980) to the multistate case.

We start by an introduction to multistate theory as presented in Natvig (1984). Let the set of states of the system be $S = \{0,1,\ldots,M\}$, $M < \infty$. The $M+1$ states represent successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Furthermore, let the set of components be $C = \{1,2,\ldots,n\}$, $n < \infty$, and the set of states of the i th component S_i (i=1,...,n) where $\{0,M\} \subseteq S_i \subseteq S$. Hence the states 0 and M are chosen to represent the endpoints of a performance scale which might be used for both the system and its components. Note that in most cases there is no need for the same detailed description of the components as for the system. If x_i (i=1,...,n) denotes the state or performance level of the i th component and $\underline{x} = (x_1,...,x_n)$, it is furthermore assumed that the state ϕ of the system is given by the <u>structure function</u> $\phi = \phi(\underline{x})$. Here $\underline{x}$ takes values in $S_1 \times S_2 \times \cdots \times S_n$ and ϕ takes values in S. In this paper we will consider the following very large class of systems not involving any assumption on relevance of components:

<u>Definition 1.1</u>. A system is a <u>multistate monotone system</u> (MMS) iff its structure function ϕ satisfies:

(i) $\phi(\underline{x})$ is non-decreasing in each argument,

(ii) $\phi(\underline{0}) = 0$ and $\phi(\underline{M}) = M$ ($\underline{0} = (0,...,0)$, $\underline{M} = (M,...,M)$).

The present definition of an MMS is more general than the one presented in Griffith (1980) and Block and Savits (1982) since they assume $S_i = S$, $i = 1,...,n$. An MMS with structure function ϕ and set of components C is often denoted by (C,ϕ).

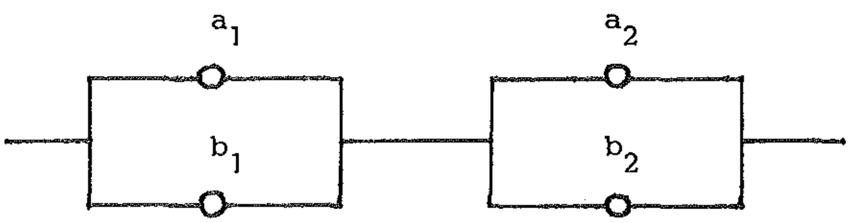

Figure 1

As a simple example of an MMS consider the network of Figure 1. Here component 1 (2) is the parallel module of the branches a_1 and b_1 (a_2 and b_2). Let (i=1,2) $x_i = 3$ if two branches work and 1 (0) if one (no) branch works. The state of the system is given in Table 1.

Component 2	0	1	3
3	0	2	3
1	0	1	2
0	0	0	0
	Component 1		

Table 1

Note for instance that the state 1 is a critical one both for each component and the system as a whole in the sense that the failing of a branch leads to the 0 state. In binary theory the functioning state comprises the states $\{1,2,3\}$ and hence just a rough description of the system's performance is possible.

We next generalize each of the concepts "minimal path set" and "minimal cut set" from binary theory. In the following $\underline{y} < \underline{x}$ means $y_i \leq x_i$ for $i = 1,\ldots,n$ and $y_i < x_i$ for some i.

Definition 1.2. Let ϕ be the structure function of an MMS and let $j \in \{1,\ldots,M\}$. A vector $\underline{x}$ is said to be a minimal path (cut) vector to level j iff $\phi(\underline{x}) \geq j$ and $\phi(\underline{y}) < j$ for all $\underline{y} < \underline{x}$ ($\phi(\underline{x}) < j$ and $\phi(\underline{y}) \geq j$ for all $\underline{y} > \underline{x}$). The corresponding minimal path (cut) sets to level j are given by $C_\phi^j(x) = \{i \mid x_i \geq 1\}$ ($D_\phi^j(\underline{x}) = \{i \mid x_i < M\}$).

We also need the following notation. Let $A \subseteq C$. Then

$\underline{x}^A$ = vector with elements x_i, $i \in A$

A^c = subset of C complementary to A

Most of the following definitions are obvious modifications of the ones listed in Natvig (1980).

Definition 1.3. An MMS (A,χ) is a module of an MMS (C,ϕ) iff

$$\phi(\underline{x}) = \psi[\chi(\underline{x}^A), \underline{x}^{A^c}],$$

where ψ is the structure function of an MMS and $A \subseteq C$. If S_χ denotes the set of states of χ, we assume $S_i \subseteq S_\chi \subseteq S$ for $i \in A$.

Intuitively, a module is a multistate monotone subsystem that acts as if it were just a component.

Definition 1.4. A modular decomposition of an MMS (C,ϕ) is a set of disjoint modules $\{A_\ell, \chi_\ell)\}_{\ell=1}^{r}$ together with an organizing structure function, ψ, of an MMS; i.e.

(i) $C = \bigcup_{\ell=1}^{r} A_\ell$ where $A_i \cap A_j = \emptyset \quad i \neq j$

(ii) $\phi(\underline{x}) = \psi[\chi_1(\underline{x}^{A_1}),\cdots,\chi_r(\underline{x}^{A_r})] = \psi[\underline{\chi}(\underline{x})]$.

Making a modular decomposition of a system is a way of breaking it into a series of subsystems which can be dealt with more easily.

Definition 1.5. Let ϕ be the structure function of an MMS. The dual structure function, ϕ^D, is given by

$$\phi^D(\underline{x}^D) = M-\phi(\underline{x}),$$

where $\underline{x}^D = (x_1^D,\ldots,x_n^D) = (M-x_1,\ldots,M-x_n)$.

It can easily be checked that the dual system of an MMS is an MMS.

Definition 1.6. The performance process of the i th component $(i=1,\ldots,n)$ is a stochastic process $\{X_i(t),\ t\in\tau\}$, where for each fixed $t\in\tau$, $X_i(t)$ is a random variable (r.v.) which takes values in S_i. $X_i(t)$ denotes the state of the i th component at time t.

The index set τ is contained in $[0,\infty)$. We assume that the sample functions of a performance process are continuous from the right on τ.

Definition 1.7. The joint performance process for the components

$$\{\underline{X}(t),\ t\in\tau\} = \{X_1(t),\ldots,X_n(t),\ t\in\tau\}$$

is a vector stochastic process for which the i th marginal process $\{X_i(t),\ t\in\tau\}$ is the performance process of the i th component, $i = 1,\ldots,n$.

Definition 1.8. Let ϕ be the structure function of an MMS. The performance process of the system is a stochastic process $\{\phi(\underline{X}(t)), t \in \tau\}$, where for each fixed $t \in \tau$, $\phi(\underline{X}(t))$ is a r.v. which takes values in S. $\phi(\underline{X}(t))$ denotes the system state at time t.

It follows that the sample functions of $\{\phi(\underline{X}(t)), t \in \tau\}$ are continuous from the right on τ.

Consider now a time interval $I = [t_A, t_B] \subset [0,\infty)$ and let $\tau(I) = \tau \cap I$.

Definition 1.9. The marginal performance processes $\{X_i(t), t \in \tau\}$ $i = 1,\ldots,n$ are independent in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\} \subset \tau(I)$, the random vectors

$$(X_1(t_1),\ldots,X_1(t_m)),\ldots,(X_n(t_1),\ldots,X_n(t_m))$$

are independent.

Definition 1.10. A modular decomposition $\{(A_\ell,\chi_\ell)\}_{\ell=1}^{r}$ consists of totally independent modules in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\} \subset \tau(I)$, the random vectors

$$(\underline{X}^{A_1}(t_1),\ldots,\underline{X}^{A_1}(t_m)),\ldots,(\underline{X}^{A_r}(t_1),\ldots,\underline{X}^{A_r}(t_m))$$

are independent.

Definition 1.11. The r.v.'s $T_1,\ldots,T_n$ are associated iff $\mathrm{Cov}[\Gamma(\underline{T}),\Delta(\underline{T})] \geq 0$ for all pairs of non-decreasing binary functions Γ,Δ. $\underline{T} = (T_1,\ldots,T_n)$.

We list some basic properties of associated r.v.'s.

P_1: Any subset of a set of associated r.v.'s is a set of associated r.v.'s.

P_2: The set consisting of a single r.v. is a set of associated r.v.'s.

P_3: Non-decreasing functions of associated r.v.'s are associated.

P_4: If two sets of associated r.v.'s are independent of each other, then their union is a set of associated r.v.'s.

Definition 1.12. The joint performance process $\{\underline{X}(t), t\in\tau\}$ is associated in the time interval I iff, for any integer m and $\{t_1,\ldots,t_m\} \subset \tau(I)$, the r.v.'s in the array

$$\begin{array}{c} X_1(t_1),\cdots,X_1(t_m) \\ \vdots \\ X_n(t_1),\cdots,X_n(t_m) \end{array}$$

are associated.

This definition obviously applies to a marginal performance process too.

Definition 1.13. Let $j \in \{1,\ldots,M\}$. The availability, $h_\phi^{j(I)}$, and the unavailability, $g_\phi^{j(I)}$, to level j in the time interval I for an MMS with structure function ϕ are given by

$$h_\phi^{j(I)} = P[\phi(\underline{X}(s)) \geq j \quad \forall s\in\tau(I)]$$

$$g_\phi^{j(I)} = P[\phi(\underline{X}(s)) < j \quad \forall s\in\tau(I)]$$

Note that

$$h_\phi^{j(I)} + g_\phi^{j(I)} \leq 1, \qquad (1.1)$$

with equality for the case $I = [t,t]$.

In the present paper we will arrive at bounds for $h_\phi^{j(I)}$ and $g_\phi^{j(I)}$ generalizing the bounds given in Natvig (1980) for the case $M = 1$. The bounds are of great interest when trying to predict the performance process of the system noting that exact expressions are obtainable just for trivial systems. As in the latter paper special bounds are developed for the case $I = [t,t]$; i.e. at a fixed point of time. These latter bounds are improvements of the ones developed in Block and Savits (1982), Butler (1982) and Natvig (1982) and are motivated by the fact that computational complexity makes it impossible to arrive at exact expressions for large systems. It should be admitted that especially some bounds given in Butler (1982) have partly inspired the generalizations of the present paper.

Before proceeding to the bounds we have to establish some basic notation and results. This is done in Section 2. In Section 3 basic bounds on $h_{\phi}^{j(I)}$ and $g_{\phi}^{j(I)}$ are arrived at whereas improved bounds are given in Section 4 by using modular decompositions. Such bounds as the latter ones are in fact the only one of practical interest for large systems as has also been pointed out by Butler (1982). The reason is that the bounds in Section 3 are based on all minimal path and cut vectors of the system. To arrive at these for a system of a large number of components seems impossible for computers of today. However, the number of components of each module, and the number of modules, may be chosen to be moderate making it possible to arrive at the minimal path and cut vectors both of the organizing structure and of each module. The strategy is then to arrive at bounds for the availabilities and unavailabilities for the modules and inserting these into the bounds for the availabilities and unavailabilities for the organizing structure. This finally leads to improved bounds for the availabilities and unavailabilities for the system.

Furthermore, in Section 5 the case $I = [t,t]$ is treated giving improved bounds using modular decompositions. We end up in Section 6 by applying the theory to the simple system given in Figure 1. A more convincing case study is under way. Finally, it should be noted that the present paper represents a thoroughly revised version of a thesis Funnemark (1982) by the first present author. The second author is responsible for the main ideas and guided the work on the thesis. The proofs of theorems that are omitted in this paper can be found in this thesis.

2. Basic results

We start by introducing some basic notation and some basic indicator functions which will be applied all through the paper.

Definition 2.1. Let ϕ be the structure function of an MMS with n components. Furthermore, for $j \in \{1,\ldots,M\}$ let $\underline{y}_{k\phi}^{j} = (y_{1k\phi}^{j},\ldots,y_{nk\phi}^{j})$, $k = 1,\ldots,n_{\phi}^{j}$ $(\underline{z}_{k\phi}^{j}=(z_{1k\phi}^{j},\ldots,z_{nk\phi}^{j})$ $k=1,\ldots,m_{\phi}^{j})$ be its minimal path (cut) vectors to level j and $C_\phi^j(\underline{y}_{k\phi}^j)$ $k = 1,\dots,n_\phi^j$ ($D_\phi^j(\underline{z}_{k\phi}^j)$ $k=1,\dots,m_\phi^j$) the associated minimal path (cut) sets to level j. Then the corresponding minimal path (cut) indicator functions to level j are given by:

$$I_{\underline{y}_{k\phi}^j}(\underline{x}) = \begin{cases} 1 & \text{if } x_i \geq y_{ik\phi}^j \text{ for } i \in C_\phi^j(\underline{y}_{k\phi}^j) \\ 0 & \text{otherwise,} \end{cases} \qquad k = 1,\dots,n_\phi^j \,;$$

$$J_{\underline{z}_{k\phi}^j}(\underline{x}) = \begin{cases} 0 & \text{if } x_i \leq z_{ik\phi}^j \text{ for } i \in D_\phi^j(\underline{z}_{k\phi}^j) \\ 1 & \text{otherwise,} \end{cases} \qquad k = 1,\dots,m_\phi^j \,.$$

The following theorem represents a reformulation of Theorem 2.6 in Natvig (1980).

Theorem 2.2. Consider an MMS with structure function ϕ and let the system state indicator function to level j be given by

$$I_j(\phi(\underline{x})) = \begin{cases} 1 & \text{if } \phi(\underline{x}) \geq j \\ 0 & \text{otherwise}, \end{cases} \qquad j = 1,\dots,M$$

Then

$$I_j(\phi(\underline{x})) = \coprod_{k=1}^{n_\phi^j} I_{\underline{y}_{k\phi}^j}(\underline{x}) \stackrel{\text{def}}{=} 1 - \prod_{k=1}^{n_\phi^j} (1 - I_{\underline{y}_{k\phi}^j}(\underline{x})) \tag{2.1}$$

$$I_j(\phi(\underline{x})) = \prod_{k=1}^{m_\phi^j} J_{\underline{z}_{k\phi}^j}(\underline{x}) \tag{2.2}$$

We now turn to modular decompositions. The following theorem is proved as a part of the proof of Theorem 3.1 in Butler (1982). The proof goes along the same lines as the corresponding part of the proof of Theorem 4.1 in Barlow and Proschan (1975, p.44).

Theorem 2.3. Consider an MMS with structure function ϕ and modular decomposition given by Definition 1.4. Introduce the following binary structure functions of the system's components

$$\phi_k^j(\underline{x}) = J_{\underline{z}_{k\psi}^j}(\underline{\chi}(\underline{x})), \quad j = 1,\dots,M;\ k = 1,\dots,m_\psi^j \,.$$

Then the minimal cut vectors to level j of ϕ are

$$\underline{z}^{1}_{m\phi^j_k} \qquad m = 1,\ldots,m^1_{\phi^j_k} \; ; \; k = 1,\ldots,m^j_\psi \, ,$$

and we hence have:

$$I_j(\phi(\underline{x})) = \prod_{k=1}^{m^j_\psi} \prod_{m=1}^{m^1_{\phi^j_k}} J_{\underline{z}^1_{m\phi^j_k}}$$

We end this section by giving some basic results on dual structure functions. The following theorems are obvious extensions of well-known results in binary theory.

<u>Theorem 2.4</u>. Consider an MMS with structure function ϕ and dual structure function ϕ^D given by Definition 1.5. Let $j \in \{1,\ldots,M\}$ and $\underline{x}$ be a minimal path (cut) vector to level j. Then $\underline{x}^D$ is a minimal cut (path) vector to the <u>dual level</u> $j^D = M-j+1$. We hence have:

$$\underline{y}^{j^D}_{k\phi^D} = \underline{M} - \underline{z}^j_{k\phi}$$

$$C^{j^D}_{\phi^D}(\underline{y}^{j^D}_{k\phi^D}) = D^j_\phi(\underline{z}^j_{k\phi})$$

$$I_{\underline{y}^{j^D}_{k\phi^D}}(\underline{x}^D) = 1 - J_{\underline{z}^j_{k\phi}}(\underline{x}), \qquad k = 1,\ldots,m^j_\phi$$

and

$$\underline{z}^{j^D}_{k\phi^D} = \underline{M} - \underline{y}^j_{k\phi}$$

$$D^{j^D}_{\phi^D}(\underline{z}^{j^D}_{k\phi^D}) = C^j_\phi(\underline{y}^j_{k\phi})$$

$$J_{\underline{z}^{j^D}_{k\phi^D}}(\underline{x}^D) = 1 - I_{\underline{y}^j_{k\phi}}(\underline{x}), \qquad k = 1,\ldots,n^j_\phi \, .$$

Theorem 2.5. Consider an MMS with structure function ϕ and modular decomposition given by Definition 1.4. Then the corresponding modular decomposition of ϕ^D has disjoint modules $\{(A_\ell, \chi_\ell^D)\}_{\ell=1}^r$ and organizing structure function ψ^D; i.e. we have the representation

$$\phi^D(\underline{x}^D) = \psi^D[\chi_1^D((\underline{x}^{A_1})^D),\ldots,\chi_r^D((\underline{x}^{A_r})^D)].$$

Obviously, Theorem 2.4 and the relations listed there can be applied on ψ^D and χ_ℓ^D, $\ell = 1,\ldots,r$.

Finally, we list the following obvious duality relation

$$h_{\phi^D}^{j^D(I)} = g_\phi^{j(I)} \tag{2.3}$$

Note that $(\phi^D)^D = \phi$ and $(j^D)^D = j$. Hence (2.3) implies

$$h_\phi^{j(I)} = g_{\phi^D}^{j^D(I)} \tag{2.4}$$

When we write down duality relations later in this paper, we just present one of the two versions, the other one being straightforward.

3. Basic bounds for the availabilities and unavailabilities in the time interval I

The presentation in this section is very parallel to the one in Section 2 of Natvig (1980). Furthermore, for easy reference we try to keep the notation as close as possible to the notation of this paper.

Theorem 3.1. Let (C,ϕ) be an MMS. Define $(j = 1,\ldots,M)$

$$\ell_\phi^{j''(I)} = \max_{1\le k\le n_\phi^j} P[I_{\underline{y}_{k\phi}^j}(\underline{X}(s))=1 \ \forall s\in\tau(I)]$$

$$u_\phi^{j''(I)} = \min_{1\le k\le m_\phi^j} P[J_{\underline{z}_{k\phi}^j}(\underline{X}(s))=1 \ \forall s\in\tau(I)]$$

$$\bar{\ell}_\phi^{j''(I)} = \max_{1\le k\le m_\phi^j} P[J_{\underline{z}_{k\phi}^j}(\underline{X}(s))=0 \ \forall s\in\tau(I)]$$

$$\bar{u}_\phi^{j''(I)} = \min_{1\le k\le n_\phi^j} P[I_{\underline{y}_{k\phi}^j}(\underline{X}(s))=0 \ \forall s\in\tau(I)]$$

Then

$$\ell_{\phi}^{j''(I)} \leq h_{\phi}^{j(I)} \leq u_{\phi}^{j''(I)}$$

$$\bar{\ell}_{\phi}^{j''(I)} \leq g_{\phi}^{j(I)} \leq \bar{u}_{\phi}^{j''(I)}.$$

Furthermore, $\ell_{\phi}^{j''(I)}$ and $u_{\phi}^{j''(I)}$ ($\bar{\ell}_{\phi}^{j''(I)}$ and $\bar{u}_{\phi}^{j''(I)}$) are non--increasing (non-decreasing) in j as is true for $h_{\phi}^{j(I)}$ ($g_{\phi}^{j(I)}$).

Proof. From Theorem 2.2 we have, for all $s \in \tau(I)$ ($k=1,\ldots,n_{\phi}^{j}$; $\ell=1,\ldots,m_{\phi}^{j}$)

$$I_{\underline{y}_{k\phi}^{j}}(\underline{X}(s)) \leq I_{j}(\phi(\underline{X}(s)) \leq J_{\underline{z}_{\ell\phi}^{j}}(\underline{X}(s))$$

Hence the first two inequalities follow. Applying these inequalities on the dual structure and dual level gives the two last ones remembering Theorem 2.4 and (2.3). Since for all $k \in \{1,\ldots,n_{\phi}^{j+1}\}$ there exists $\ell \in \{1,\ldots,n_{\phi}^{j}\}$, and vice versa, such that $\underline{y}_{k\phi}^{j+1} \geq \underline{y}_{\ell\phi}^{j}$, it follows that $\ell_{\phi}^{j''(I)}$ ($\bar{u}^{j''(I)}$) are non-increasing (non-decreasing) in j. The corresponding properties of the two other bounds follow similarly.

Theorem 3.1 is very general, but seems of little practical value due to the complexity of the bounds. The following corollary is more useful. First denote the availability and unavailability to level j in I for the i th component of an MMS (C,ϕ) by $p_{i\phi}^{j(I)}$ and $q_{i\phi}^{j(I)}$ respectively, $i = 1,\ldots,n$; $j = 1,\ldots,M$. Introduce the $n \times M$ matrices

$$\underline{P}_{\phi}^{(I)} = \left(p_{i\phi}^{j(I)}\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}} \qquad \underline{Q}_{\phi}^{(I)} = \left(q_{i\phi}^{j(I)}\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}} \tag{3.1}$$

We also introduce the dual matrix by the convention

$$\underline{P}_{\phi^D}^{(I)} = \left(p_{i\phi^D}^{j^D(I)}\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,M}} \tag{3.2}$$

and hence have parallel to (2.3)

$$\underline{P}_{\phi^D}^{(I)} = \underline{Q}_{\phi}^{(I)} \tag{3.3}$$

The proofs of the bounds in Corollary 3.2, Theorem 3.3, Corollaries 3.4 and 3.5 to follow are extensions of the proofs of the corresponding results in Section 2 of Natvig (1980). However, the only bound to prove is always the first lower bound since the second lower bound follows by applying the first one on the dual structure and dual level, and the two upper bounds follow from the two lower ones by (1.1). In the following corollary the first lower bound is proved leaving the rest of the proofs to the reader.

Corollary 3.2. Let (C,ϕ) be an MMS and assume that the joint performance process of the system's components is associated in I. Define $(j=1,\ldots,M)$

$$\ell_\phi^{j'}(\underline{P}_\phi^{(I)}) = \max_{1 \leq k \leq n_\phi^j} \prod_{i \in C_\phi^j(\underline{y}_{k\phi}^j)} p_{i\phi}^{y_{ik\phi}^j(I)}$$

$$\bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)}) = \max_{1 \leq k \leq m_\phi^j} \prod_{i \in D_\phi^j(\underline{z}_{k\phi}^j)} q_{i\phi}^{z_{ik\phi}^j+1(I)}$$

Then

$$\ell_\phi^{j'}(\underline{P}_\phi^{(I)}) \leq h_\phi^{j(I)} \leq 1-\bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)})$$

$$\bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)}) \leq g_\phi^{j(I)} \leq 1-\ell_\phi^{j'}(\underline{P}_\phi^{(I)})$$

Furthermore, $\ell_\phi^{j'}(\underline{P}_\phi^{(I)})$ $(\bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)}))$ are non-increasing (non-decreasing) in j.

Proof. Let S be a countable subset of $\tau(I)$ that is dense in $\tau(I)$. Since the sample functions of $\{X_i(t), t\in\tau\}$ $i=1,\ldots,n$ are continuous from the right on τ, then

$$\ell_\phi^{j''(I)} = \max_{1 \leq k \leq n_\phi^j} P[X_i(s) \geq y_{ik\phi}^j \ \forall i \in C_\phi^j(\underline{y}_{k\phi}^j) \text{ and } \forall s \in S]$$

Let $S_m = \{s_1,\ldots,s_m\}$ $m=1,2,\ldots$ be subsets of S such that $S_m \uparrow S$ as $m \to \infty$. By monotone convergence

$$\max_{1 \leq k \leq n_\phi^j} P[X_i(s) \geq y_{ik\phi}^j \ \forall i \in C_\phi^j(\underline{y}_{k\phi}^j) \text{ and } \forall s \in S_m] \downarrow \ell_\phi^{j''(I)}.$$

Now by introducing the component state indicator functions to level j by

$$I_j(x_i) = \begin{cases} 1 & \text{if } x_i \geq j \\ 0 & \text{otherwise}, \quad i=1,\ldots,n;\ j=1,\ldots,M \end{cases}$$

we have

$$\max_{1 \leq k \leq n_\phi^j} P[X_i(s) \geq y_{ik\phi}^j \ \forall i \in C_\phi^j(\underline{y}_{k\phi}^j) \text{ and } \forall s \in S_m]$$

$$= \max_{1 \leq k \leq n_\phi^j} P[\prod_{i \in C_\phi^j(\underline{y}_{k\phi}^j)} \prod_{\ell=1}^{m} I_{y_{ik\phi}^j}(X_i(s_\ell))=1]$$

$$\geq \max_{1 \leq k \leq n_\phi^j} \prod_{i \in C_\phi^j(\underline{y}_{k\phi}^j)} P[X_i(s) \geq y_{ik\phi}^j \ \forall s \in S_m],$$

having applied Theorem 3.1 in Barlow and Proschan (1975). This can be done since the r.v.'s $\prod_{\ell=1}^{m} I_{y_{ik\phi}^j}(X_i(s_\ell)),\ i \in C_\phi^j(\underline{y}_{k\phi}^j)$ are associated by property P_3. Finally by monotone convergence

$$\max_{1 \leq k \leq n_\phi^j} \prod_{i \in C_\phi^j(\underline{y}_{k\phi}^j)} P[X_i(s) \geq y_{ik\phi}^j \ \forall s \in S_m] \uparrow \ell_\phi^{j'}(\underline{P}_\phi^{(I)}),$$

and the first lower bound follows from Theorem 3.1.

Note that the lower bounds of Theorem 3.1 reduce to the corresponding ones of Corollary 3.2 when components are independent. For the case $I = [t,t]$ Butler (1982) proves the bounds above. For this case the next result is also proved in Natvig (1982).

Theorem 3.3. Make the same assumptions as in Corollary 3.2 and define $(j=1,\ldots,M)$

$$\ell_\phi^{j*}(I) = \prod_{k=1}^{m_\phi^j} P[J_{\underline{z}_{k\phi}^j}(\underline{X}(s))=1 \ \forall s \in \tau(I)]$$

$$\bar{\ell}_\phi^{j*}(I) = \prod_{k=1}^{n_\phi^j} P[I_{\underline{y}_{k\phi}^j}(\underline{X}(s))=0 \ \forall s \in \tau(I)]$$

Then

$$\ell_\phi^{j*}(I) \leq h_\phi^j(I) \leq 1-\bar{\ell}_\phi^{j*}(I)$$

$$\bar{\ell}_\phi^{j*}(I) \leq g_\phi^j(I) \leq 1-\ell_\phi^{j*}(I)$$

By combining Corollary 3.2 and Theorem 3.3 we arrive at the following corollary.

Corollary 3.4. Make the same assumptions as in Corollary 3.2 and define $(j=1,\dots,M)$

$$L_\phi^{j(I)} = \max\,[\ell_\phi^{j'}(\underline{P}_\phi^{(I)}), \ell_\phi^{j*(I)}]$$

$$\bar{L}_\phi^{j(I)} = \max\,[\bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)}), \bar{\ell}_\phi^{j*(I)}]$$

Then

$$L_\phi^{j(I)} \leq h_\phi^{j(I)} \leq 1-\bar{L}_\phi^{j(I)}$$

$$\bar{L}_\phi^{j(I)} \leq g_\phi^{j(I)} \leq 1-L_\phi^{j(I)}$$

An objection against these bounds is that $\ell_\phi^{j*(I)}$ and $\bar{\ell}_\phi^{j*(I)}$ seem very complex. This is dealt with in the next corollary, the price paid being stronger assumptions and poorer bounds.

Corollary 3.5. Let (C,ϕ) be an MMS with the marginal performance processes of its components being mutually independent and each of them associated in I. Define $(j=1,\dots,M)$

$$\ell_\phi^{j*}(\underline{P}_\phi^{(I)}) = \prod_{k=1}^{m_\phi^j} \coprod_{i\in D_\phi^j(\underline{z}_{k\phi}^j)} p_{i\phi}^{z_{ik\phi}^j+1\,(I)}$$

$$\bar{\ell}_\phi^{j*}(\underline{Q}_\phi^{(I)}) = \prod_{k=1}^{n_\phi^j} \coprod_{i\in C_\phi^j(\underline{y}_{k\phi}^j)} q_{i\phi}^{y_{ik\phi}^j\,(I)}$$

$$L_\phi^j(\underline{P}_\phi^{(I)}) = \max\,[\ell_\phi^{j'}(\underline{P}_\phi^{(I)}), \ell_\phi^{j*}(\underline{P}_\phi^{(I)})]$$

$$\bar{L}_\phi^j(\underline{Q}_\phi^{(I)}) = \max\,[\bar{\ell}_\phi^{j'}(\underline{Q}_\phi^{(I)}), \bar{\ell}_\phi^{j*}(\underline{Q}_\phi^{(I)})]$$

Then

$$L_\phi^j(\underline{P}_\phi^{(I)}) \leq L_\phi^{j(I)} \leq h_\phi^{j(I)} \leq 1-\bar{L}_\phi^{j(I)} \leq 1-\bar{L}_\phi^j(\underline{Q}_\phi^{(I)})$$

$$\bar{L}_\phi^j(\underline{Q}_\phi^{(I)}) \leq \bar{L}_\phi^{j(I)} \leq g_\phi^{j(I)} \leq 1-L_\phi^{j(I)} \leq 1-L_\phi^j(\underline{P}_\phi^{(I)})$$

Due to a counterexample in Butler (1982) when $I = [t,t]$ the bounds on $h_\phi^{j(I)}$ ($g_\phi^{j(I)}$) given in the two last corollaries are not certain to be non-increasing (non-decreasing) in j. Corollary 3.1 of the latter paper hence inspires the two last corollaries of this section, proofs being straightforward.

Corollary 3.6. Make the same assumptions as in Corollary 3.2 and define $(j=1,\dots,M)$

$$B_\phi^{j(I)} = \max_{j\le k\le M} [L_\phi^{k(I)}]$$
$$\bar{B}_\phi^{j(I)} = \max_{1\le k\le j} [\bar{L}_\phi^{k(I)}]$$

Then

$$L_\phi^{j(I)} \le B_\phi^{j(I)} \le h_\phi^{j(I)} \le 1-\bar{B}_\phi^{j(I)} \le 1-\bar{L}_\phi^{j(I)}$$
$$\bar{L}_\phi^{j(I)} \le \bar{B}_\phi^{j(I)} \le g_\phi^{j(I)} \le 1-B_\phi^{j(I)} \le 1-L_\phi^{j(I)}$$

Corollary 3.7. Make the same assumptions as in Corollary 3.5 and define $(j=1,\dots,M)$

$$B_\phi^{j}(\underline{P}_\phi^{(I)}) = \max_{j\le k\le M} [L_\phi^{k}(\underline{P}_\phi^{(I)})]$$
$$\bar{B}_\phi^{j}(\underline{Q}_\phi^{(I)}) = \max_{1\le k\le j} [\bar{L}_\phi^{k}(\underline{Q}_\phi^{(I)})]$$

Then

$$L_\phi^{j}(\underline{P}_\phi^{(I)}) \le B_\phi^{j}(\underline{P}_\phi^{(I)}) \le B_\phi^{j(I)} \le h_\phi^{j(I)} \le 1-\bar{B}_\phi^{j(I)} \le 1-\bar{B}_\phi^{j}(\underline{Q}_\phi^{(I)}) \le 1-\bar{L}_\phi^{j}(\underline{Q}_\phi^{(I)})$$
$$\bar{L}_\phi^{j}(\underline{Q}_\phi^{(I)}) \le \bar{B}_\phi^{j}(\underline{Q}_\phi^{(I)}) \le \bar{B}_\phi^{j(I)} \le g_\phi^{j(I)} \le 1-B_\phi^{j(I)} \le 1-B_\phi^{j}(\underline{P}_\phi^{(I)}) \le 1-L_\phi^{j}(\underline{P}_\phi^{(I)})$$

We close this section by giving some comments. First of all it is easy to check that all bounds given here lie inside the interval $[0,1]$ as is a minimum claim. On the other hand, the upper bounds given, except in Theorem 3.1 are poor if $h_\phi^{j(I)}+g_\phi^{j(I)}$ is not close to 1. We have, however, not been able to arrive at better upper bounds.

To apply for instance Corollary 3.5 and 3.7 one has to check that the marginal performance process of each component is associated in I. When these processes are Markovian a theorem providing a sufficient condition for each of them to be associated in I is given in Hjort, Natvig and Funnemark (1982), thus generalizing a result from binary theory by Esary and Proschan (1970). Imbedded in the theorem is an equivalent and much more convenient condition in terms of the transition intensities of the Markov process.

4. Improved bounds for the availabilities and unavailabilities in the time interval I using modular decompositions

We start by listing some duality relations which come out from the proofs in the last section

$$ {}_{\ell}j^{D}_{\phi^D}{}''(I) = {}_{\bar{\ell}}j_{\phi}''(I) \tag{4.1} $$

$$ {}_{u}j^{D}_{\phi^D}{}''(I) = {}_{\bar{u}}j_{\phi}''(I) \tag{4.2} $$

$$ {}_{\ell}j^{D}_{\phi^D}{}'(\underline{P}^{(I)}_{\phi^D}) = {}_{\bar{\ell}}j_{\phi}'(\underline{Q}^{(I)}_{\phi}) \tag{4.3} $$

$$ {}_{L}j^{D}_{\phi^D}(I) = {}_{\bar{L}}j_{\phi}(I) \tag{4.4} $$

$$ {}_{L}j^{D}_{\phi^D}(\underline{P}^{(I)}_{\phi^D}) = {}_{\bar{L}}j_{\phi}(\underline{Q}^{(I)}_{\phi}) \tag{4.5} $$

$$ {}_{B}j^{D}_{\phi^D}(I) = {}_{\bar{B}}{}^{(I)}_{\phi} \tag{4.6} $$

$$ {}_{B}j^{D}_{\phi^D}(\underline{P}^{(I)}_{\phi^D}) = {}_{\bar{B}}j_{\phi}(\underline{Q}^{(I)}_{\phi}) \tag{4.7} $$

Now consider an MMS with structure function ϕ and modular decomposition given by Definition 1.4. Obviously, the results of Section 3 and the relations above can be applied on ϕ and χ_ℓ, $\ell = 1,\dots,r$ as well. Introduce the following $r\times M$ matrices

$$ \underline{h}^{(I)}_{\chi} = (h^{j(I)}_{\chi_\ell})_{\substack{\ell=1,\dots,r\\ j=1,\dots,M}} \tag{4.8} $$

and correspondingly define the following $r\times M$ matrices

$$\underline{g}_\chi^{(I)},\ \underline{\ell}_\chi^{\prime\prime(I)},\ \underline{\bar{\ell}}_\chi^{\prime\prime(I)},\ \underline{\ell}_\chi^{\prime}(\underline{P}_\phi^{(I)}),\ \underline{\bar{\ell}}_\chi^{\prime}(\underline{Q}_\phi^{(I)}),\ \underline{\ell}_\chi^{*(I)},\ \underline{\bar{\ell}}_\chi^{*(I)},\ \underline{L}_\chi^{(I)},$$
$$\underline{\bar{L}}_\chi^{(I)},\ \underline{L}_\chi(\underline{P}_\phi^{(I)}),\ \underline{\bar{L}}_\chi(\underline{Q}_\phi^{(I)}),\underline{B}_\chi^{(I)},\ \underline{\bar{B}}_\chi^{(I)},\ \underline{B}_\chi(\underline{P}_\phi^{(I)}),\ \underline{\bar{B}}_\chi(\underline{Q}_\phi^{(I)}).$$

The dual matrices of these are defined by the convention leading to (3.2).

The two theorems to follow are generalizations of Theorem 2.6 and 2.7 in Natvig (1980). In the first one we find lower bounds that are improvements of the lower bounds in Theorem 3.1. This is by no means proved with regard to the upper bounds.

Theorem 4.1. Let (C,ϕ) be an MMS with modular decomposition given by Definition 1.4 consisting of totally independent modules in I. Furthermore, assume that the marginal performance process of each module is associated in I. Then $(j=1,\ldots,M)$

$$\ell_\phi^{j\prime\prime(I)} \overset{5}{\leq} B_\psi^j(\underline{\ell}_\chi^{\prime\prime(I)}) \overset{4}{\leq} B_\psi^j(\underline{h}_\chi^{(I)}) \overset{3}{\leq} h_\phi^{j(I)} \overset{2}{\leq} 1-\bar{B}_\psi^j(\underline{g}_\chi^{(I)}) \overset{1}{\leq} 1-\bar{B}_\psi^j(\underline{\bar{\ell}}_\chi^{\prime\prime(I)})$$

$$u_\phi^{j\prime\prime(I)} \overset{6}{\leq} 1-\bar{\ell}_\psi^{j\prime}(\underline{\bar{\ell}}_\chi^{\prime\prime(I)})$$

$$\bar{\ell}_\phi^{j\prime\prime(I)} \overset{11}{\leq} \bar{B}_\psi^j(\underline{\bar{\ell}}_\chi^{\prime\prime(I)}) \overset{10}{\leq} \bar{B}_\psi^j(\underline{g}_\chi^{(I)}) \overset{9}{\leq} g_\phi^{j(I)} \overset{8}{\leq} 1-B_\psi^j(\underline{h}_\chi^{(I)}) \overset{7}{\leq} 1-B_\psi^j(\underline{\ell}_\chi^{\prime\prime(I)})$$

$$\bar{u}_\phi^{j\prime\prime(I)} \overset{12}{\leq} 1-\ell_\psi^{j\prime}(\underline{\ell}_\chi^{\prime\prime(I)})$$

From the assumptions of the theorem it follows that the marginal performance processes of the modules of (C,ϕ) are mutually independent in I. Hence Corollary 3.7 can be applied by considering the modules as components and the inequalities 2, 3, 8, 9 follow. Furthermore, Theorem 3.1 can be applied on all modules. Hence the inequalities 1 (10) and 4 (7) follow since $\bar{B}_\psi^j(\cdot)$ and $B_\psi^j(\cdot)$ are non-decreasing functions in each matrix element. In addition the inequalities 5 and 12 follow by applying 11 and 6 respectively on the dual structure and dual level remembering (4.1), (4.7), (4.2) and (4.3). The inequalities 6 and 11 are proved in Appendix 1.

<u>Theorem 4.2</u>. Let (C,ϕ) be an MMS with modular decomposition given by Definition 1.4. Assume the marginal performance processes of the modules to be mutually independent in I and furthermore the joint performance process for the components of each module to be associated in I. Then

$$\ell_{\phi}^{j'}(\underline{P}_{\phi}^{(I)}) \overset{6}{\leq} B_{\phi}^{j}(\underline{B}_{\chi}^{(I)}) \overset{5}{\leq} B_{\phi}^{j}(\underline{h}_{\chi}^{(I)}) \overset{4}{\leq} h_{\phi}^{j(I)} \overset{3}{\leq} 1-\bar{B}_{\phi}^{j}(\underline{g}_{\chi}^{(I)})$$

$$\overset{2}{\leq} 1-\bar{B}_{\phi}^{j}(\underline{\bar{B}}_{\chi}^{(I)}) \overset{1}{\leq} 1-\bar{\ell}_{\phi}^{j'}(\underline{Q}_{\phi}^{(I)})$$

$$\bar{\ell}_{\phi}^{j'}(\underline{Q}_{\phi}^{(I)}) \overset{12}{\leq} \bar{B}_{\phi}^{j}(\underline{B}_{\chi}^{(I)}) \overset{11}{\leq} \bar{B}_{\phi}^{j}(\underline{g}_{\chi}^{(I)}) \overset{10}{\leq} g_{\phi}^{j(I)} \overset{9}{\leq} 1-B_{\phi}^{j}(\underline{h}_{\chi}^{(I)})$$

$$\overset{8}{\leq} 1-B_{\phi}^{j}(\underline{B}_{\chi}^{(I)}) \overset{7}{\leq} 1-\ell_{\phi}^{j'}(\underline{P}_{\phi}^{(I)})$$

Note that we have found bounds that are improvements of the bounds in Corollary 3.2. (We have also proved this corollary under somewhat different assumptions.) However, we have not proved that these bounds are improvements of the ones in Corollary 3.4.

From the assumptions of the theorem and property P_3 of associated r.v.'s it follows that the marginal performance process of each module of (C,ϕ) is associated in I. Hence Corollary 3.7 is again applicable and the inequalities 3, 4, 9, 10 follow. Furthermore, Corollary 3.6 can be applied on all modules. Hence the inequalities 2 (11) and 5 (8) follow. Furthermore the inequalities 6 and 7 are equivalent. The same is true for 1 and 12. Since 6 follows by applying 12 on the dual structure and dual level remembering (4.3), (4.6) and (4.7) what remains to prove is inequality 1. This is done in Appendix 2.

5. Improved bounds for the availabilities at a fixed point of time using modular decompositions

In this section we specialize $I = [t,t]$ and obtain improved bounds for system availabilities at time t, denoted by h_ϕ^j $j = 1,\ldots,M$. For simplicity all bounds introduced in the previous section will for the special case treated here appear without I. The bounds obtained will automatically give bounds for the unavailabilities at time t, since as stated in Section 1

$$g_\phi^j = 1-h_\phi^j \qquad j=1,\ldots,M \tag{5.1}$$

Note also that if we assume that all components and hence the system function at $t = 0$, but are not maintained, h_ϕ^j is just the system reliability to level j.

Let $\underline{1}^{a\times b}$ be an $a\times b$ matrix with all elements being 1. We then obviously have

$$\underline{Q}_\phi = \underline{1}^{n\times M} - \underline{P}_\phi$$
$$\underline{g}_\chi = \underline{1}^{r\times M} - \underline{h}_\chi$$

The two first theorems to follow are generalizations of Theorem 3.1 and 3.2 in Natvig (1980).

Theorem 5.1. Let (C,ϕ) be an MMS with modular decomposition given by Definition 1.4 consisting of totally independent modules at time t. Then $(j=1,\ldots,M)$

$$\ell_\phi^{j''} \overset{10}{\leq} B_\phi^j(\underline{\ell}_\chi'') \left\{ \begin{array}{c} \overset{8}{\leq} B_\phi^j(\underline{h}_\chi) \overset{6}{\leq} \\ \overset{9}{\leq} h_\phi^j(\underline{\ell}_\chi'') \overset{7}{\leq} \end{array} \right\} h_\phi^j$$

$$\left\{ \begin{array}{c} \overset{4}{\leq} 1-\bar{B}_\phi^j(\underline{1}^{r\times M}-\underline{h}_\chi) \overset{2}{\leq} \\ \overset{5}{\leq} h_\phi^j(\underline{1}^{r\times M}-\underline{\bar{\ell}}_\chi'') \overset{3}{\leq} \end{array} \right\} 1-\bar{B}_\phi^j(\underline{\bar{\ell}}_\chi'') \overset{1}{\leq} 1-\bar{\ell}_\phi^{j''}$$

Proof. The inequalities 1, 2, 4, 6, 8, 10 follow immediately from the inequalities 11, 10, 9, 3, 4, 5 of Theorem 4.1 by specializing for the case treated in this section. To apply the latter theorem we need the marginal performance process of each module to be associated at time t. This follows from property P_2 of associated r.v.'s. Since the modules are totally independent, extending a part of the proof of Theorem 3.1 in Butler (1982) we have

$$h^j_\phi = h^j_\psi(\underline{h}_\chi) \quad . \tag{5.2}$$

From Theorem 4.2 of El-Neweihi et. al. (1978) it follows that $h^j_\psi(\cdot)$ is non-decreasing in each matrix element. Hence the inequalities 5 and 7 follow from Theorem 3.1 and the inequalities 3 and 9 from Corollary 3.7, noting that due to Theorem 3.1 the matrix arguments $\underline{\ell}''_\chi$ and $\underline{1}^{r\times M}-\underline{\bar{\ell}}''_\chi$ have the same properties as $\underline{h}_\chi$.

Note that due to (5.1) we have no objections against the upper bounds arrived at in the previous sections, when applied here. However, any upper bound arrived at is now equivalent to an established lower bound and nothing is gained. Note also that Theorem 4.1 can be considered as a generalization of the main part of Theorem 5.1. For the remaining theorems given in this section, we have not been able to generalize their main parts to the situation with general I.

Theorem 5.2. Make the same assumptions as in Theorem 5.1. Assume furthermore that for each module the states of the components at time t are associated r.v.'s. Then $(j=1,\ldots,M)$

$$B^j_\phi \overset{10}{\le} B^j_\psi(\underline{B}_\chi) \left\{ \begin{array}{c} \overset{8}{\le} B^j_\psi(\underline{h}_\chi) \overset{6}{\le} \\[2ex] \overset{9}{\le} h^j_\psi(\underline{B}_\chi) \overset{7}{\le} \end{array} \right\} h^j_\phi$$

$$\left\{ \begin{array}{c} \overset{4}{\le} 1-\bar{B}^j_\psi(\underline{1}^{r\times M}-\underline{h}_\chi) \overset{2}{\le} \\[2ex] \overset{5}{\le} \; h^j_\psi(\underline{1}^{r\times M}-\underline{\bar{B}}_\chi) \overset{3}{\le} \end{array} \right\} 1-\bar{B}^j_\psi(\underline{\bar{B}}_\chi) \overset{1}{\le} 1-\bar{B}^j_\phi$$

The inequalities 2, 4, 6, 8 follow immediately from the inequalities 2, 3, 4, 5 of Theorem 4.2 by specializing for the case treated in this section. Remembering (5.2) the inequalities 5 and 7 follow from Corollary 3.6 and the inequalities 3 and 9 from Corollary 3.7. Since 1 follows by applying 10 on the dual structure and dual level remembering (4.6) and (4.7) what remains to prove is inequality 10. This is done in Appendix 3.

<u>Corollary 5.3</u>. Let (C,ϕ) be an MMS with modular decomposition given by Definition 1.4 having independent components at time t. Then $(j=1,\dots,M)$

$$B_\phi^j(\underline{p}_\phi) \overset{10}{\leq} B_\phi^j(\underline{B}_\chi(\underline{p}_\phi)) \left\{ \begin{array}{ll} \overset{8}{\leq} B_\phi^j(\underline{h}_\chi) & \overset{6}{\leq} \\ \overset{9}{\leq} h_\phi^j(\underline{B}_\chi(\underline{p}_\phi)) & \overset{7}{\leq} \end{array} \right\} h_\phi^j$$

$$\left\{ \begin{array}{ll} \overset{4}{\leq} 1-\bar{B}_\phi^j(\underline{1}^{r\times M}-\underline{h}_\chi) & \overset{2}{\leq} \\ \overset{5}{\leq} h_\phi^j(\underline{1}^{r\times M}-\underline{\bar{B}}_\chi(\underline{1}^{n\times M}-\underline{p}_\phi)) & \overset{3}{\leq} \end{array} \right\} 1-\bar{B}_\phi^j(\underline{\bar{B}}_\chi(\underline{1}^{n\times M}-\underline{p}_\phi)) \overset{1}{\leq} 1-\bar{B}_\phi^j(\underline{1}^{n\times M}-\underline{p}_\phi)$$

<u>Proof</u>. Since $I = [t,t]$ and all components are independent at t, it immediately follows that $(j=1,\dots,M)$

$$\ell_\phi^{j*} = \ell_\phi^{j*}(\underline{p}_\phi) \qquad \bar{\ell}_\phi^{j*} = \bar{\ell}_\phi^{j*}(\underline{1}^{n\times M}-\underline{p}_\phi)$$

and hence

$$B_\phi^j = B_\phi^j(\underline{p}_\phi) \qquad \bar{B}_\phi^j = \bar{B}_\phi^j(\underline{1}^{n\times M}-\underline{p}_\phi)$$

These latter relations can obviously also be applied on each module. Hence Corollary 5.3 follows from Theorem 5.2.

Note that even for the special case of the latter corollary our results improve the inequalities of Theorem 3.1 and 3.2 in Butler (1982). The reason is that this author does not utilize the bounds of Corollary 3.2 at all when improving bounds by using modular decompositions.

We end up this section by generalizing Theorem 3.3 and 3.4 in Natvig (1980). The three theorems to follow essentially tell us that it is advantagous to decompose modules with unknown reliabilities and do nothing with the remaining ones.

Theorem 5.4. Make the same assumptions as in Theorem 5.1. Furthermore, for $\ell = 1,\dots,k$ $(1 \le k \le r)$ assume that (A_ℓ, χ_ℓ) has a modular decomposition $\{(B_{\ell m}, \Omega_{\ell m})\}_{m=1}^{s_\ell}$ consisting of totally independent modules at time t and with organizing structure function σ_ℓ. Let

$$\underline{\Omega} = (\Omega_{11},\dots,\Omega_{1s_1},\dots,\Omega_{k1},\dots,\Omega_{ks_k},\chi_{k+1},\dots,\chi_r)$$

and introduce

$$\theta(\underline{\Omega}) = \phi[\sigma_1(\Omega_{11},\dots,\Omega_{1s_1}),\dots,\sigma_k(\Omega_{k1},\dots,\Omega_{ks_k}),\chi_{k+1},\dots,\chi_r].$$

Also block the matrix given by (4.8) into two matrices by

$$\underline{h}_\chi = \begin{bmatrix} \underline{h}_\chi(1) \\ \underline{h}_\chi(2) \end{bmatrix},$$

where $\underline{h}_\chi(1)$ is a $k \times M$ matrix and $\underline{h}_\chi(2)$ is an $(r-k)\times M$ matrix. With obvious notation we have $(j=1,\dots,M)$

$$\left.\begin{array}{l} B^j_\theta(\underline{h}_{\underline{\Omega}_1},\dots,\underline{h}_{\underline{\Omega}_k},\underline{\ell}''_\chi(2)) \overset{7}{\le} B^j_\phi(\underline{h}_\chi(1),\underline{\ell}''_\chi(2)) \overset{5}{\le} \\ h^j_\phi(\underline{\ell}''_\chi(1),\underline{h}_\chi(2)) \overset{8}{\le} h^j_\theta(\underline{\ell}''_{\underline{\Omega}_1},\dots,\underline{\ell}''_{\underline{\Omega}_k},\underline{h}_\chi(2)) \overset{6}{\le} \end{array}\right\} h^j_\phi$$

$$\left\{\begin{array}{l} \overset{3}{\le} 1-\bar{B}^j_\phi(\underline{1}^{k\times M}-\underline{h}_\chi(1),\underline{\bar{\ell}}''_\chi(2)) \overset{1}{\le} 1-\bar{B}^j_\theta(\underline{1}^{s_1\times M}-\underline{h}_{\underline{\Omega}_1},\dots,\underline{1}^{s_k\times M}-\underline{h}_{\underline{\Omega}_k},\underline{\bar{\ell}}''_\chi(2)) \\ \overset{4}{\le} h^j_\theta(\underline{1}^{s_1\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_1},\dots,\underline{1}^{s_k\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_k},\underline{h}_\chi(2)) \overset{2}{\le} h^j_\phi(\underline{1}^{k\times M}-\underline{\bar{\ell}}_\chi(1),\underline{h}_\chi(2)) \end{array}\right.$$

Proof. By applying the inequalities 1 and 2 of Theorem 5.2 and Corollary 3.7 on the structure θ, we get

$$1-\bar{B}^j_\psi(\underline{1}^{k\times M}-\underline{h}_{\underline{\chi}(1)},\underline{\bar{\ell}}_{\underline{\chi}(2)}) \leq 1-\bar{B}^j_\theta \leq 1-\bar{B}^j_\theta(\underline{1}^{s_1\times M}-\underline{h}_{\underline{\Omega}_1},\ldots,\underline{1}^{s_k\times M}-\underline{h}_{\underline{\Omega}_k},\underline{\bar{\ell}}''_{\underline{\chi}(2)})$$

and the inequality 1 follows. From the inequalities 1 and 3 of Theorem 5.1, we have $(\ell=1,\ldots,k)$

$$1-\bar{\ell}^{j''}_{\chi_\ell} \geq h^j_{\sigma_\ell}(\underline{1}^{s_\ell\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_\ell})$$

Since $h^j_\psi(\cdot)$ is non-decreasing in each argument, we then have

$$h^j_\psi(\underline{1}^{k\times M}-\underline{\bar{\ell}}''_{\underline{\chi}(1)},\underline{h}_{\underline{\chi}(2)}) \geq h^j_\psi(\underline{h}_{\underline{\sigma}}(\underline{1}^{s_1\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_1},\ldots,\underline{1}^{s_k\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_k}),\underline{h}_{\underline{\chi}(2)})$$
$$= h_\theta(\underline{1}^{s_1\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_1},\ldots,\underline{1}^{s_k\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_k},\underline{h}_{\underline{\chi}(2)}),$$

and the second inequality follows.

Since $\bar{B}^j_\psi(\cdot)$ is non-decreasing in each argument, we have from Theorem 3.1

$$1-\bar{B}^j_\psi(\underline{1}^{k\times M}-\underline{h}_{\underline{\chi}(1)},\underline{\bar{\ell}}''_{\underline{\chi}(2)}) \geq 1-\bar{B}^j_\psi(\underline{1}^{k\times M}-\underline{h}_{\underline{\chi}(1)},\underline{1}^{(r-k)\times M}-\underline{h}_{\underline{\chi}(2)})$$
$$= 1-\bar{B}^j_\psi(\underline{1}^{r\times M}-\underline{h}_{\underline{\chi}}) \geq h^j_\phi$$

also having applied the inequality 4 of Theorem 5.1. Hence inequality 3 follows. Similarly

$$h^j_\theta(\underline{1}^{s_1\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_1},\ldots,\underline{1}^{s_k\times M}-\underline{\bar{\ell}}''_{\underline{\Omega}_k},\underline{h}_{\underline{\chi}(2)}) \geq h^j_\theta(\underline{h}_{\underline{\Omega}_1},\ldots,\underline{h}_{\underline{\Omega}_k},\underline{h}_{\underline{\chi}(2)})$$
$$= h^j_\psi(\underline{h}_{\underline{\chi}}) = h^j_\phi ,$$

having applied Theorem 3.1 and (5.2) and the inequality 4 follows. Finally by applying the inequalities 1 and 3 on the dual structure and dual level the inequalities 7 and 5 respectively follow. The inequalities 8 and 6 are proved completely parallel to 2 and 4.

Theorem 5.5. Make the same assumptions as in Theorem 5.4. Furthermore, assume that for each of the modules $(B_{\ell m},\Omega_{\ell m})$ $\ell = 1,\ldots,k;$ $m = 1,\ldots,s_\ell$ and (A_ℓ,χ_ℓ) $\ell = k+1,\ldots,r$ the states of the components at time t are associated r.v.'s. Then $(j=1,\ldots,M)$

$$\left.\begin{array}{l} B^j_\theta(\underline{h}_{\underline{\Omega}_1},\ldots,\underline{h}_{\underline{\Omega}_k},\underline{B}_{\underline{\chi}}(2)) \overset{7}{\leq} B^j_\phi(\underline{h}_{\underline{\chi}}(1),\underline{B}_{\underline{\chi}}(2)) \overset{5}{\leq} \\ h^j_\phi(\underline{B}_{\underline{\chi}}(1),\underline{h}_{\underline{\chi}}(2)) \overset{8}{\leq} h^j_\theta(\underline{B}_{\underline{\Omega}_1},\ldots,\underline{B}_{\underline{\Omega}_k},\underline{h}_{\underline{\chi}}(2)) \overset{6}{\leq} \end{array}\right\} h^j_\phi$$

$$\left\{\begin{array}{l} \overset{3}{\leq} 1-\bar{B}^j_\phi(\underline{1}^{k\times M}-\underline{h}_{\underline{\chi}}(1),\bar{\underline{B}}_{\underline{\chi}}(2)) \overset{1}{\leq} 1-\bar{B}^j_\theta(\underline{1}^{s_1\times M}-\underline{h}_{\underline{\Omega}_1},\ldots,\underline{1}^{s_k\times M}-\underline{h}_{\underline{\Omega}_k},\bar{\underline{B}}_{\underline{\chi}}(2)) \\ \overset{4}{\leq} h^j_\theta(\underline{1}^{s_1\times M}-\bar{\underline{B}}_{\underline{\Omega}_1},\ldots,\underline{1}^{s_k\times M}-\bar{\underline{B}}_{\underline{\Omega}_k},\underline{h}_{\underline{\chi}}(2)) \overset{2}{\leq} h^j_\phi(\underline{1}^{k\times M}-\bar{\underline{B}}_{\underline{\chi}}(1),\underline{h}_{\underline{\chi}}(2)) \end{array}\right.$$

Proof. The proof is very similar to the one of Theorem 5.4 applying Theorem 5.2 instead of Theorem 5.1 and Corollary 3.6 instead of Theorem 3.1. Note that due to P_2 also for (A_ℓ,χ_ℓ) $\ell = 1,\ldots,k$ the states of the components at time t are associated r.v.'s. This is needed when proving the inequality 8.

In the same way as Corollary 5.3 follows from Theorem 5.2, the last corollary of this section follows from Theorem 5.5.

Corollary 5.6. Make the same assumptions as in Theorem 5.4. Assume furthermore that all components are independent at time t. Then $(j=1,\ldots,M)$

$$\left.\begin{array}{l} B^j_\theta(\underline{h}_{\underline{\Omega}_1},\ldots,\underline{h}_{\underline{\Omega}_k},\underline{B}_{\underline{\chi}}(2)(\underline{P}_\phi)) \overset{7}{\leq} B^j_\phi(\underline{h}_{\underline{\chi}}(1),\underline{B}_{\underline{\chi}}(2)(\underline{P}_\phi)) \overset{5}{\leq} \\ h^j_\phi(\underline{B}_{\underline{\chi}}(1)(\underline{P}_\phi),\underline{h}_{\underline{\chi}}(2)) \overset{8}{\leq} h^j_\theta(\underline{B}_{\underline{\Omega}_1}(\underline{P}_\phi),\ldots,\underline{B}_{\underline{\Omega}_k}(\underline{P}_\phi),\underline{h}_{\underline{\chi}}(2)) \overset{6}{\leq} \end{array}\right\} h^j_\phi$$

$$\left\{\begin{array}{l} \overset{3}{\leq} 1-\bar{B}^j_\phi(\underline{1}^{k\times M}-\underline{h}_{\underline{\chi}}(1),\bar{\underline{B}}_{\underline{\chi}}(2)(\underline{1}^{n\times M}-\underline{P}_\phi)) \overset{1}{\leq} \\ \qquad 1-\bar{B}^j_\theta(\underline{1}^{s_1\times M}-\underline{h}_{\underline{\Omega}_1},\ldots,\underline{1}^{s_k\times M}-\underline{h}_{\underline{\Omega}_k},\bar{\underline{B}}_{\underline{\chi}}(2)(\underline{1}^{n\times M}-\underline{P}_\phi)) \\ \overset{4}{\leq} h^j_\theta(\underline{1}^{s_1\times M}-\bar{\underline{B}}_{\underline{\Omega}_1}(\underline{1}^{n\times M}-\underline{P}_\phi),\ldots,\underline{1}^{s_k\times M}-\bar{\underline{B}}_{\underline{\Omega}_k}(\underline{1}^{n\times M}-\underline{P}_\phi),\underline{h}_{\underline{\chi}}(2)) \overset{2}{\leq} \\ \qquad h^j_\phi(\underline{1}^{k\times M}-\bar{\underline{B}}_{\underline{\chi}}(1)(\underline{1}^{n\times M}-\underline{P}_\phi),\underline{h}_{\underline{\chi}}(2)) \end{array}\right.$$

Let us try to summarize the implications of Theorem 5.4, 5.5 and Corollary 5.6. Consider an MMS (C,ϕ) with modular decomposition consisting of totally independent modules at time t and with organizing structure function ψ having known availability functions h_ψ^j, $j = 1,\ldots,M$. (These are considered as functions of the availabilities of the modules - again remember (5.2).) Furthermore, assume that each module with unknown availabilities at time t has a modular decomposition consisting of modules having either the specific properties mentioned in Theorem 5.4, 5.5 or Corollary 5.6 and in addition having unknown availabilities at t. Finally, assume that the organizing structure function θ of the refined modular decomposition of (C,ϕ) has known availability functions h_θ^j, $j = 1,\ldots,M$. Then the inequalities 2, 4, 6, 8 tell that the bounds based on the refined modular decomposition are better than the bounds based on the original one.

The remaining inequalities tell us in a way just the opposite. Now the h_ψ^j's and h_θ^j's are supposed to be unknown. However, this time each module with known availabilities at t is decomposed into modules with known availabilities at t. Now the bounds based on the refined modular decomposition are worse. If in this latter case, the h_ψ's and h_θ's are known, for instance in the situation of Theorem 5.4, it readily follows that

$$h_\theta^j(\underline{h}_{\underline{\Omega}_1},\ldots,\underline{h}_{\underline{\Omega}_k},\underline{\ell}''_{\underline{\chi}}(2)) = h_\psi^j(\underline{h}_{\underline{\chi}}(1),\underline{\ell}''_{\underline{\chi}}(2)) \leq h_\phi^j$$

$$\leq h_\psi^j(\underline{h}_{\underline{\chi}}(1),\underline{1}^{(r-k)\times M}-\underline{\bar{\ell}}''_{\underline{\chi}}(2)) = h_\theta^j(\underline{h}_{\underline{\Omega}_1},\ldots,\underline{h}_{\underline{\Omega}_k},\underline{1}^{(r-k)\times M}-\underline{\bar{\ell}}''_{\underline{\chi}}(2)).$$

Parallel relations are proved for the situations in Theorem 5.5 and Corollary 5.6. Hence now the bounds based on the refined modular decomposition and the bounds based on the original one are equally good.

6. A simple example

We now finally return to the simple example given in Figure 1. First of all it is easy to see that the minimal path and cut vectors are given as follows:

$$\underline{y}^1_{1\phi} = (1,1),\ \underline{y}^2_{1\phi} = (3,1),\ \underline{y}^2_{2\phi} = (1,3),\ \underline{y}^3_{1\phi} = (3,3)$$

$$\underline{z}^1_{1\phi} = (3,0),\ \underline{z}^1_{2\phi} = (0,3),\ \underline{z}^2_{1\phi} = (1,1),\ \underline{z}^2_{2\phi} = (3,0),$$

$$\underline{z}^2_{3\phi} = (0,3),\ \underline{z}^3_{1\phi} = (3,1),\ \underline{z}^3_{2\phi} = (1,3).$$

Secondly, we have to work out the availabilities and unavailabilities for the two components in the time interval $I = \tau(I) = [t_A, t_B] \subset [0,\infty)$ letting $\tau = [0,\infty)$. To do this we assume that the marginal performance processes $\{X_i(t);\ t\in[0,\infty)\}$ $i = 1,2$ are independent and identically distributed time-homogeneous Markov processes with state space $\{0,1,3\}$. Denote the transition probabilities by

$$P_{ij}(t) = P[X_1(t)=j \mid X_1(0)=i]\ , \qquad i,j \in\{0,1,3\}$$

and the corresponding transition intensities by μ_{ij}. Assuming more specifically that the two branches of each component fail and are repaired/replaced independently of each other, both having the same instantaneous failure rate λ and repair/replacement rate μ, we have

$$\mu_{01} = 2\mu,\ \mu_{13} = \mu,\ \mu_{31} = 2\lambda,\ \mu_{10} = \lambda,\ \mu_{03} = \mu_{30} = 0. \qquad (6.1)$$

By standard techniques, using for instance Laplace Transforms, one arrives at the expressions

$$P_{30}(t) = (\lambda^2/\xi^2)(1+\exp(-2\xi t)-2\exp(-\xi t))$$

$$P_{31}(t) = (2\lambda/\xi^2)(\mu-\lambda\exp(-2\xi t)+(\lambda-\mu)\exp(-\xi t)),$$

where $\xi = \lambda+\mu$.

We now assume that both components function perfectly at time 0. The availability to level 3 and unavailability to level 1 in I for the components are straightforward $(i=1,2)$:

$$p_{i\phi}^{3(I)} = [1-P_{30}(t_A)-P_{31}(t_A)]\exp(-2\lambda(t_B-t_A))$$

$$q_{i\phi}^{1(I)} = P_{30}(t_A)\exp(-2\mu(t_B-t_A))$$

By considering a Markov process with state space $\{0,1,3\}$, letting first $\{0\}$ be an absorbing state, and then $\{3\}$, we get by using Sverdrup (1965) (i=1,2):

$$p_{i\phi}^{1(I)} = [1-P_{30}(t_A)-P_{31}(t_A)][r_2\exp(r_1(t_B-t_A))-r_1\exp(r_2(t_B-t_A))]/(r_2-r_1)$$
$$+ P_{31}(t_A)[(r_2+\lambda)\exp(r_1(t_B-t_A))-(r_1+\lambda)\exp(r_2(t_B-t_A))]/(r_2-r_1),$$

where $\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = (-3\lambda-\mu \pm \sqrt{(\lambda-\mu)^2+8\lambda\mu})/2$

$$q_{i\phi}^{3(I)} = P_{30}(t_A)[r_2'\exp(r_1'(t_B-t_A))-r_1'\exp(r_2'(t_B-t_A))]/(r_2'-r_1')$$
$$+ P_{31}(t_A)[(r_2'+\mu)\exp(r_1'(t_B-t_A))-(r_1'+\mu)\exp(r_2'(t_B-t_A))]/(r_2'-r_1'),$$

where $\left.\begin{matrix} r_1' \\ r_2' \end{matrix}\right\} = (-3\mu-\lambda \pm \sqrt{(\lambda-\mu)^2+8\lambda\mu})/2$

Note that the second factor and denominator in each addend of $q_{i\phi}^{1(I)}$ ($q_{i\phi}^{3(I)}$) are obtained from the corresponding expressions of $p_{i\phi}^{3(I)}$ ($p_{i\phi}^{1(I)}$) by interchanging λ and μ. This is obvious from symmetry considerations.

To be allowed to apply Corollary 3.5 and 3.7 the marginal performance process of each component must be associated in I. From Hjort, Natvig and Funnemark (1982) Theorem 2.1 a sufficient condition for this to hold is given by

$$\mu_{03} \leq \mu_{13}, \quad \mu_{30} \leq \mu_{10},$$

which is trivially satisfied due to (6.1). Since the marginal performance processes of the components are assumed mutually independent, it furthermore follows from property P_4 of associated r.v.'s that their joint performance process is associated in I.

Hence, Corollary 3.2, Theorem 3.3, Corollary 3.4 and 3.6 can be applied as well in addition to Theorem 3.1.

In the following we concentrate on bounds on the availabilities. It turns out that several bounds coincide. The lower bounds are given by:

$$\ell_\phi^{1''(I)} = \ell_\phi^{1'}(\underline{P}_\phi^{(I)}) = \ell_\phi^{1*(I)} = L_\phi^{1(I)} = L_\phi^{1}(\underline{P}_\phi^{(I)})$$
$$= B_\phi^{1}(\underline{P}_\phi^{(I)}) = (p_{1\,\phi}^{1(I)})^2$$

$$\ell_\phi^{2''(I)} = \ell_\phi^{2'}\ (\underline{P}_\phi^{(I)}) = p_{1\,\phi}^{1(I)} \cdot p_{1\,\phi}^{3(I)}$$
$$\leq p_{1\,\phi}^{1(I)} \cdot p_{1\,\phi}^{3(I)} \max[1, p_{1\,\phi}^{1(I)}(2-p_{1\,\phi}^{3(I)})] = L_\phi^{2}(\underline{P}_\phi^{(I)}) = B_\phi^{2}(\underline{P}_\phi^{(I)})$$

$$\ell_\phi^{3''(I)} = \ell_\phi^{3'}(\underline{P}_\phi^{(I)}) = \ell_\phi^{3*(I)} = L_\phi^{3(I)} = L_\phi^{3}(\underline{P}_\phi^{(I)})$$
$$= B_\phi^{3(I)} = B_\phi^{3}(\underline{P}_\phi^{(I)}) = (p_{1\,\phi}^{3(I)})^2$$

The bound $\ell_\phi^{2*(I)}$ can not be established as a function of component availabilities. Hence the same is true for $L_\phi^{2(I)}$, $B_\phi^{1(I)}$ and $B_\phi^{2(I)}$.

The upper bounds are given by:

$$u_\phi^{1''(I)} = p_{1\,\phi}^{1(I)} \leq 1-q_{1\,\phi}^{1(I)} = 1-\bar{\ell}_\phi^{1'}\ (\underline{Q}_\phi^{(I)}) \geq (1-q_{1\,\phi}^{1(I)})^2$$
$$= 1-\bar{L}_\phi^{1}(\underline{Q}_\phi^{(I)}) = 1-\bar{B}_\phi^{1}(\underline{Q}_\phi^{(I)})$$

$$1-\bar{\ell}_\phi^{2'}\ (\underline{Q}_\phi^{(I)}) = 1-\max[q_{1\,\phi}^{1(I)}, (q_{1\,\phi}^{3(I)})^2]$$
$$\geq 1-\max[q_{1\,\phi}^{1(I)}, (q_{1\,\phi}^{1(I)}+q_{1\,\phi}^{3(I)}-q_{1\,\phi}^{1(I)} \cdot q_{1\,\phi}^{3(I)})^2] = 1-\bar{L}_\phi^{2}(\underline{Q}_\phi^{(I)})$$
$$\geq 1-\bar{B}_\phi^{2}(\underline{Q}_\phi^{(I)}) = \min[(1-q_{1\,\phi}^{1(I)})^2, 1-(q_{1\,\phi}^{1(I)}+q_{1\,\phi}^{3(I)}-q_{1\,\phi}^{1(I)} \cdot q_{1\,\phi}^{3(I)})^2]$$

$$u_\phi^{3''(I)} = p_{1\,\phi}^{3(I)} \leq 1-q_{1\,\phi}^{3(I)} = 1-\bar{\ell}_\phi^{3'}\ (\underline{Q}_\phi^{(I)}) \geq (1-q_{1\,\phi}^{3(I)})^2$$
$$= 1-\bar{L}_\phi^{3}(\underline{Q}_\phi^{(I)}) = 1-\bar{B}_\phi^{3}(\underline{Q}_\phi^{(I)})$$

The bounds $1-\bar{\ell}_{\phi}^{j*(I)}$, j = 1,2,3 can not be established as a function of component availabilities. Hence the same is true for $1-\bar{L}_{\phi}^{j(I)}$ and $1-\bar{B}_{\phi}^{j(I)}$, j = 1,2,3. In addition $u_{\phi}^{2''(I)}$ can not be obtained in a simple way.

In Table 3 we give numerical values for the bounds for all combinations of λ = 0.001, 0.01; μ = 0.001, 0.01 and $I = [t_A, t_B]$ = [100,110],[100,200],[1000,1100]. The bounds are presented as in Table 2.

	LEVEL 1		LEVEL 2		LEVEL 3	
	LOWER	UPPER	LOWER	UPPER	LOWER	UPPER
λ = μ = I = [,]	$\ell_{\phi}^{1''(I)}$	$u_{\phi}^{1''(I)}$ $1-\bar{\ell}_{\phi}^{1'}(\underline{Q}_{\phi}^{(I)})$ $1-\bar{L}_{\phi}^{1}(\underline{Q}_{\phi}^{(I)})$	$\ell_{\phi}^{2''(I)}$ $L_{\phi}^{2}(\underline{P}_{\phi}^{(I)})$	$1-\bar{\ell}_{\phi}^{2'}(\underline{Q}_{\phi}^{(I)})$ $1-\bar{L}_{\phi}^{2}(\underline{Q}_{\phi}^{(I)})$ $1-\bar{B}_{\phi}^{2}(\underline{Q}_{\phi}^{(I)})$	$\ell_{\phi}^{3''(I)}$	$u_{\phi}^{3''(I)}$ $1-\bar{\ell}_{\phi}^{3'}(\underline{Q}_{\phi}^{(I)})$ $1-\bar{L}_{\phi}^{3}(\underline{Q}_{\phi}^{(I)})$

Table 2

	LEVEL 1		LEVEL 2		LEVEL 3	
	LOWER	UPPER	LOWER	UPPER	LOWER	UPPER
λ = 0.001 μ = 0.001 I = [100,110]	0.9802	0.9901 0.9919 0.9840	0.8025 0.9451	0.9706 0.9683 0.9683	0.6570	0.8106 0.8286 0.6865
λ = 0.001 μ = 0.001 I = [100,200]	0.9400	0.9695 0.9933 0.9866	0.6564 0.8420	0.9750 0.9732 0.9732	0.4584	0.6770 0.8420 0.7090
λ = 0.001 μ = 0.001 I=[1000,1100]	0.5862	0.7656 0.8470 0.7174	0.2020 0.2685	0.6012 0.5268 0.5268	0.0696	0.2638 0.3685 0.1358
λ = 0.001 μ = 0.01 I = [100,110]	0.9903	0.9952 0.9970 0.9940	0.8607 0.9723	0.9886 0.9880 0.9880	0.7481	0.8649 0.8932 0.7978
λ = 0.001 μ = 0.01 I = [100,200]	0.9667	0.9832 0.9995 0.9990	0.7103 0.8923	0.9979 0.9979 0.9979	0.5219	0.7224 0.9543 0.9108
λ = 0.001 μ = 0.01 I=[1000,1100]	0.9522	0.9758 0.9989 0.9978	0.6603 0.8526	0.9954 0.9952 0.9952	0.4578	0.6766 0.9320 0.8686

Table 3 (continued)

	LEVEL 1		LEVEL 2		LEVEL 3	
	LOWER	UPPER	LOWER	UPPER	LOWER	UPPER
$\lambda = 0.01$	0.3429	0.5856	0.0742	0.2934	0.0161	0.1268
$\mu =$		0.6395	0.0814	0.1935		0.1594
$I = [100,110]$		0.4089		0.1935		0.0254
$\lambda = 0.01$	0.0762	0.2761	0.0058	0.3394	0.0004	0.0210
$\mu = 0.001$		0.6989	0.0058	0.2445		0.1872
$I = [100,200]$		0.4884		0.2445		0.0350
$\lambda = 0.01$	0.0046	0.0680	0.0001	0.0478	0.0000	0.0011
$\mu = 0.001$		0.3234	0.0001	0.0156		0.0242
$I = [1000,1100]$		0.1046		0.0156		0.0006
$\lambda = 0.01$	0.5862	0.7656	0.2020	0.6012	0.0696	0.2638
$\mu = 0.01$		0.8470	0.2685	0.5268		0.3685
$I = [100,110]$		0.7174		0.5268		0.1358
$\lambda = 0.01$	0.2024	0.4499	0.0196	0.8705	0.0019	0.0436
$\mu = 0.01$		0.9747	0.0196	0.8585		0.6401
$I = [100,200]$		0.9500		0.8585		0.4097
$\lambda = 0.01$	0.1651	0.4063	0.0137	0.8349	0.0011	0.0338
$\mu = 0.01$		0.9662	0.0137	0.8182		0.5937
$I = [1000,1100]$		0.9335		0.8182		0.3525

Table 3

As we see from Table 3 the bounds for this simple example are amazingly good. Only some bounds for the availability to level 2 are getting really bad; i.e. for $\lambda = \mu = 0.01$ and $I = [100,200]$, $[1000,1100]$. In these cases we expect more fluctuations between the states of the two components and hence of the system during I. Since our bounds are based on component availabilities/unavailabilities, a lot of information is lost in these cases, and the bounds are hence poor.

Note also that we always have

$$1-\bar{L}_{\phi}^{2}(\underline{\Omega}_{\phi}^{(I)}) = 1-\bar{B}_{\phi}(\underline{\Omega}_{\phi}^{(I)})$$

in Table 3. In fact we have not been able at all to find values of λ, μ, I such that this equality does not hold. For the special case $t_A = t_B = \infty$ it is not too hard to show this analytically. Hence the bounds of Corollary 3.6, 3.7 have not been to any help in this example.

APPENDIX 1. PROOF OF THE INEQUALITIES 6 AND 11 OF THEOREM 4.1

We start by proving inequality 6 and an equality which immediately leads to inequality 11, for the special case that ϕ has only one single minimal cut vector to each level.

Lemma A1.1. Make the first part of assumptions of Theorem 4.1. In addition assume that ϕ has only one single minimal cut vector to level j, $\underline{z}^j_\phi = (z^j_{1\phi},\dots,z^j_{r\phi})$, with corresponding minimal cut set $D^j_\phi(\underline{z}^j_\phi)$. Then $(j=1,\dots,M)$

$$u^{j''(I)}_\phi \leq 1 - \prod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} \bar{\ell}^{z^j_{\ell\phi}+1''(I)}_{\chi_\ell}$$

$$\bar{\ell}^{j''(I)}_\phi = \prod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} \bar{\ell}^{z^j_{\ell\phi}+1''(I)}_{\chi_\ell}$$

Proof. We have by Definition 2.1 and Theorem 2.2

$$I_j(\phi(\underline{x})) = J_{\underline{z}^j_\phi}(\underline{\chi}(\underline{x})) = \coprod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} I_{z^j_{\ell\phi}+1}(\chi_\ell(\underline{x}^{A_\ell}))$$

$$= \coprod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} \prod_{k=1}^{m_{\chi_\ell}^{z^j_{\ell\phi}+1}} J_{\underline{z}^{z^j_{\ell\phi}+1}_{k\chi_\ell}}(\underline{x}^{A_\ell})$$

Now the minimal cut indicator functions for ϕ to level j are given by

$$\coprod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} J_{\underline{z}^{z^j_{\ell\phi}+1}_{k_\ell\chi_\ell}}(\underline{x}^{A_\ell}), \tag{A1.1}$$

where for each $\ell \in D^j_\phi(\underline{z}^j_\phi)$ we can choose $1 \leq k_\ell \leq m_{\chi_\ell}^{z^j_{\ell\phi}+1}$. Hence

$$u_{\phi}^{j''}(I) = \min_{\substack{1 \leq k_i \leq m_{\chi_i}^{z^j_{i\psi}+1} \\ i \in D^j_{\psi}(\underline{z}^j_{\psi})}} P\Big[\coprod_{\ell \in D^j_{\psi}(\underline{z}^j_{\psi})} J_{\underline{z}^{z^j_{\ell\psi}+1}_{k_\ell \chi_\ell}} (\underline{X}^{A_\ell}(s)) = 1 \ \forall s \in \tau(I) \Big]$$

$$= 1 - \max_{\substack{1 \leq k_i \leq m_{\chi_i}^{z^j_{i\psi}+1} \\ i \in D^j_{\psi}(\underline{z}^j_{\psi})}} P\Big[\exists s \in \tau(I) \ni J_{\underline{z}^{z^j_{\ell\psi}+1}_{k_\ell \chi_\ell}} (\underline{X}^{A_\ell}(s)) = 0 \ \forall \ell \in D^j_{\psi}(\underline{z}^j_{\psi}) \Big]$$

$$\leq 1 - \max_{\substack{1 \leq k_i \leq m_{\chi_i}^{z^j_{i\psi}+1} \\ i \in D^j_{\psi}(\underline{z}^j_{\psi})}} P\Big[J_{\underline{z}^{z^j_{\ell\psi}+1}_{k_\ell \chi_\ell}} (\underline{X}^{A_\ell}(s)) = 0 \ \forall \ell \in D^j_{\psi}(\underline{z}^j_{\psi}) \forall s \in \tau(I) \Big]$$

$$= 1 - \max_{\substack{1 \leq k_i \leq m_{\chi_i}^{z^j_{i\psi}+1} \\ i \in D^j_{\psi}(\underline{z}^j_{\psi})}} \prod_{\ell \in D^j_{\psi}(\underline{z}^j_{\psi})} P\Big[J_{\underline{z}^{z^j_{\ell\psi}+1}_{k_\ell \chi_\ell}} (\underline{X}^{A_\ell}(s)) = 0 \ \forall s \in \tau(I) \Big]$$

$$= 1 - \prod_{\ell \in D^j_{\psi}(\underline{z}^j_{\psi})} \max_{1 \leq k \leq m_{\chi_\ell}^{z^j_{\ell\psi}+1}} P\Big[J_{\underline{z}^{z^j_{\ell\psi}+1}_{k \chi_\ell}} (\underline{X}^{A_\ell}(s)) = 0 \ \forall s \in \tau(I) \Big]$$

$$= 1 - \prod_{\ell \in D^j_{\psi}(\underline{z}^j_{\psi})} \bar{\ell}_{\chi_\ell}^{z^j_{\ell\psi}+1\,''}(I),$$

having used the fact that the modules are totally independent in I. Finally

$$\bar{\ell}_{\phi}^{j''}(I) = \max_{\substack{1 \leq k_i \leq m_{\chi_i}^{z^j_{i\psi}+1} \\ i \in D^j_{\psi}(\underline{z}^j_{\psi})}} P\Big[J_{\underline{z}^{z^j_{\ell\psi}+1}_{k_\ell \chi_\ell}} (\underline{X}^{A_\ell}(s)) = 0 \ \forall \ell \in D^j_{\psi}(\underline{z}^j_{\psi}) \ \forall s \in \tau(I) \Big]$$

$$= \prod_{\ell \in D^j_{\psi}(\underline{z}^j_{\psi})} \bar{\ell}_{\chi_\ell}^{z^j_{\ell\psi}+1\,''}(I),$$

and the proof of the lemma is completed.

Now from Theorem 2.3 we have

$$u_{\phi}^{j''}(I) = \min_{1\leq k\leq m_{\phi}^{j}} \min_{1\leq m\leq m^{1}_{\phi_k^j}} P[J_{\underline{z}^{1}_{m\phi_k^j}}(\underline{X}(s))=1 \ \forall s\in\tau(I)]$$

$$= \min_{1\leq k\leq m_{\phi}^{j}} u^{1''}_{\phi_k^j}(I) \leq \min_{1\leq k\leq m_{\phi}^{j}}[1-\prod_{\ell\in D_{\phi}^{j}(\underline{z}^{j}_{k\phi})} \bar{\ell}_{\chi_\ell}^{z^j_{\ell k\phi}+1''}(I)]$$

$$= 1-\max_{1\leq k\leq m_{\phi}^{j}} \prod_{\ell\in D_{\phi}^{j}(\underline{z}^{j}_{k\phi})} \bar{\ell}_{\chi_\ell}^{z^j_{\ell k\phi}+1''}(I) = 1-\bar{\ell}_{\phi}^{j'}(\underline{\bar{\ell}}_{\chi}''(I)),$$

having applied Lemma A1.1, and the inequality 6 is proved. Furthermore,

$$\bar{\ell}_{\phi}^{j''}(I) = \max_{1\leq k\leq m_{\phi}^{j}} \max_{1\leq m\leq m^{1}_{\phi_k^j}} P[J_{\underline{z}^{1}_{m\phi_k^j}}(\underline{X}(s))=0 \ \forall s\in\tau(I)]$$

$$= \max_{1\leq k\leq m_{\phi}^{j}} \bar{\ell}^{1''}_{\phi_k^j}(I) = \max_{1\leq k\leq m_{\phi}^{j}} \prod_{\ell\in D_{\phi}^{j}(\underline{z}^{j}_{k\phi})} \bar{\ell}_{\chi_\ell}^{z^j_{\ell k\phi}+1''}(I) = \bar{\ell}_{\phi}^{j'}(\underline{\bar{\ell}}_{\chi}''(I)),$$

again having applied Lemma A1.1, and the inequality 11 is readily proved.

APPENDIX 2. PROOF OF THE INEQUALITY 1 OF THEOREM 4.2

We start by proving an equality which immediately leads to inequality 1, for the special case that ϕ has only one single minimal cut vector to each level.

Lemma A2.1. Let (C,ϕ) be an MMS with modular decomposition given by Definition 1.4. In addition assume that ϕ has only one single minimal cut vector to level j, $\underline{z}_{\phi}^{j} = (z^{j}_{1\phi},\ldots,z^{j}_{r\phi})$, with corresponding minimal cut set $D_{\phi}^{j}(\underline{z}_{\phi}^{j})$. Then $(j=1,\ldots,M)$

$$\bar{\ell}_{\phi}^{j'}(\underline{Q}_{\phi}^{(I)}) = \prod_{\ell\in D_{\phi}^{j}(\underline{z}_{\phi}^{j})} \bar{\ell}_{\chi_\ell}^{z^j_{\ell\phi}+1'}(\underline{Q}_{\phi}^{(I)})$$

Proof. The proof is straightforward when remembering the notation of the proof of Lemma A1.1.

$$\bar{\ell}_{\phi}^{j'}(\underline{Q}_{\phi}^{(I)}) = \max_{\substack{1 \leq k_s \leq m_{\chi_s}^{z_{s\phi}^{j}+1} \\ s \in D_{\phi}^{j}(\underline{z}_{\phi}^{j})}} \prod_{\ell \in D_{\phi}^{j}(\underline{z}_{\phi}^{j})} \prod_{i \in D_{\chi_\ell}^{z_{\ell\phi}^{j}+1}(\underline{z}_{k_\ell \chi_\ell}^{z_{\ell\phi}^{j}+1})} q_{i\phi}^{z_{ik_\ell\chi_\ell}^{z_{\ell\phi}^{j}+1}+1\,(I)}$$

$$= \prod_{\ell \in D_{\phi}^{j}(\underline{z}_{\phi}^{j})} \max_{1 \leq k \leq m_{\chi_\ell}^{z_{\ell\phi}^{j}+1}} \prod_{i \in D_{\chi_\ell}^{z_{\ell\phi}^{j}+1}(\underline{z}_{k\chi_\ell}^{z_{\ell\phi}^{j}+1})} q_{i\phi}^{z_{ik\chi_\ell}^{z_{\ell\phi}^{j}+1}+1\,(I)}$$

$$= \prod_{\ell \in D_{\phi}^{j}(\underline{z}_{\phi}^{j})} \bar{\ell}_{\chi_\ell}^{z_{\ell\phi}^{j}+1\,'}(\underline{Q}_{\phi}^{(I)})$$

and the proof of the lemma is completed.

Now remembering the main part of the proof from Appendix 1, we get by applying Lemma A2.1.

$$\bar{\ell}_{\phi}^{j'}(\underline{Q}_{\phi}^{(I)}) = \max_{1 \leq k \leq m_{\phi}^{j}} \max_{1 \leq m \leq m^{1}_{\phi_k^j}} \prod_{i \in D^{1}_{\phi_k^j}(\underline{z}^{1}_{m\phi_k^j})} q_{i\phi}^{z^{1}_{im\phi_k^j}+1\,(I)}$$

$$= \max_{1 \leq k \leq m_{\phi}^{j}} \bar{\ell}^{1\,'}_{\phi_k^j}(\underline{Q}_{\phi}^{(I)}) = \max_{1 \leq k \leq m_{\phi}^{j}} \prod_{\ell \in D_{\phi}^{j}(\underline{z}_{k\phi}^{j})} \bar{\ell}_{\chi_\ell}^{z_{\ell k\phi}^{j}+1\,'}(\underline{Q}_{\phi}^{(I)})$$

$$= \bar{\ell}_{\phi}^{j'}(\underline{\bar{\ell}}_{\chi}^{'}(\underline{Q}_{\phi}^{(I)})). \tag{A2.1}$$

Hence we have:

$$\bar{\ell}_{\phi}^{j'}(\underline{Q}_{\phi}^{(I)}) \leq \bar{L}_{\phi}^{j}(\underline{\bar{\ell}}_{\chi}^{'}(\underline{Q}_{\phi}^{(I)})) \leq \bar{L}_{\phi}^{j}(\underline{\bar{L}}_{\chi}^{(I)}) \leq \bar{L}_{\phi}^{j}(\underline{\bar{B}}_{\chi}^{(I)}) \leq \bar{B}_{\phi}^{j}(\underline{\bar{B}}_{\chi}^{(I)}),$$

and the inequality 1 is proved.

APPENDIX 3. PROOF OF THE INEQUALITY 10 OF THEOREM 5.2

We start by proving the following lemma which is a corrected and slightly generalized version of Lemma 3.2 of Butler (1982).

Lemma A3.1

Make the same assumptions as in Theorem 5.1. In addition assume that ϕ has only one single minimal cut vector to level j, $\underline{z}^j_\phi = (z^j_{1\phi},\ldots,z^j_{r\phi})$, with corresponding minimal cut set $D^j_\phi(\underline{z}^j_\phi)$. Then $(j=1,\ldots,M)$

$$\ell^{j*}_\phi \leq \coprod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} \ell^{z^j_{\ell\phi}+1*}_{\chi_\ell}$$

Proof. Consider the structure function ϕ^* of an MMS defined by

$$I_j(\phi^*(\underline{Y}(t))) = \coprod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} \prod_{k=1}^{m_{\chi_\ell}^{z^j_{\ell\phi}+1}} Y_{k\ell}^{z^j_{\ell\phi}+1}(t),$$

where $\underline{Y}(t)$ is a vector of binary, mutually independent (and hence associated) components at time t with availabilites (to level 1) at this point of time $(k=1,\ldots,m^j_{\chi_\ell};\ell=1,\ldots,r;j=1,\ldots,M)$

$$P(Y^j_{k\ell}(t)=1) = P[J_{\underline{z}^j_{k\chi_\ell}}(\underline{X}^{A_\ell}(t))=1].$$

We have

$$h^j_{\phi^*} = E[\coprod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} \prod_{k=1}^{m_{\chi_\ell}^{z^j_{\ell\phi}+1}} Y_{k\ell}^{z^j_{\ell\phi}+1}(t)]$$

$$= \coprod_{\ell\in D^j_\phi(\underline{z}^j_\phi)} \prod_{k=1}^{m_{\chi_\ell}^{z^j_{\ell\phi}+1}} P[J_{\underline{z}^{z^j_{\ell\phi}+1}_{k\chi_\ell}}(\underline{X}^{A_\ell}(t))=1] = \coprod_{\ell\in D^j_\psi(\underline{z}^j_\psi)} \ell^{z^j_{\ell\phi}+1*}_{\chi_\ell} \tag{A3.1}$$

Now all the minimal cut indicator functions for ϕ^* to level j are of the same form as the ones for ϕ, given by (A1.1), just replacing $J_{\underline{z}^{z^j_{\ell\psi}+1}_{k_\ell \chi_\ell}}(\underline{X}^{A_\ell}(t))$ by $Y^{z^j_{\ell\psi}+1}_{k_\ell \ell}(t)$. Since the modules of ϕ are totally independent at time t, $\{J_{\underline{z}^{z^j_{\ell\psi}+1}_{k_\ell \chi_\ell}}(\underline{X}^{A_\ell}(t))\}_{\ell \in D^j_\psi(\underline{z}^j_\psi)}$ are independent, exactly as is true for for $\{Y^{z^j_{\ell\psi}+1}_{k_\ell \ell}(t)\}_{\ell \in D^j_\psi(\underline{z}^j_\psi)}$.

We then have

$$\ell^{j*}_{\phi^*} = \ell^{j*}_{\phi}.$$

Hence from Theorem 3.3 and (A3.1) we get

$$\ell^{j*}_{\phi} = \ell^{j*}_{\phi^*} \leq h^j_{\phi^*} = \coprod_{\ell \in D^j_\psi(\underline{z}^j_\psi)} \ell^{z^j_{\ell\psi}+1*}_{\chi_\ell},$$

and the proof of the lemma is completed.

Now from Theorem 2.3 we have

$$\ell^{j*}_{\phi} = \prod_{k=1}^{m^j_\psi} \prod_{m=1}^{m^1_{\phi^j_k}} P[J_{\underline{z}^1_{m\phi^j_k}}(\underline{X}(t))=1]$$

$$= \prod_{k=1}^{m^j_\psi} \ell^{1*}_{\phi^j_k} \leq \prod_{k=1}^{m^j_\psi} \coprod_{\ell \in D^j_\psi(\underline{z}^j_{k\psi})} \ell^{z^j_{\ell k\psi}+1*}_{\chi_\ell} = \ell^{j*}_\psi(\underline{\ell}^*_\chi), \tag{A3.2}$$

having applied Lemma A3.1.

On the other hand by applying (A2.1) on the dual structure and dual level, specializing $I = [t,t]$ and remembering (4.3) we get

$$\ell^{j'}_\phi(\underline{P}_\phi) = \ell^{j'}_\psi(\underline{\ell}'_\chi(\underline{P}_\phi)) \tag{A3.3}$$

From (A3.2) and (A3.3) we now get

$$L^j_\phi = \max[\ell^{j'}_\phi(\underline{P}_\phi), \ell^{j*}_\phi] \leq \max[\ell^{j'}_\psi(\underline{L}_\chi), \ell^{j*}_\psi(\underline{L}_\chi)]$$

$$\leq \max[\ell^{j'}_\psi(\underline{B}_\chi), \ell^{j*}_\psi(\underline{B}_\chi)] = L^j_\psi(\underline{B}_\chi),$$

and the inequality 10 immediately follows.

ACKNOWLEDGEMENT

We are thankful to Odd Aalen for providing the reference to Sverdrup (1965).

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