# RELIABILITY ANALYSIS IN SHOCK MODELS

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## 1 Introduction

We begin by reviewing some of the basic concepts of reliability theory. A binary monotone system is an ordered pair $(E, \phi)$, where $E = \{1, \ldots, n\}$ is a set of components and $\phi$ is the structure function. $\phi$ is a binary, non-decreasing function defined on the random vector $\mathbf{X} = (X_1, \ldots, X_n)$ of binary component states at a certain fixed point of time. $X_i = 1$ if the $i$-th component is functioning, otherwise $X_i = 0$. Similarly $\phi$ takes the value 1 if the system is functioning and 0 otherwise.

The reliability $p_i$ of the $i$-th component is the probability that the component is functioning. The reliability $h$ of the system is the probability that the system is functioning. The reliability function $h(\cdot)$ of the system is a multilinear function in the variables $p_1, \ldots, p_n$ whose value is the reliability of the system if the component states are independent. $h$ can be expressed as

$$h(\mathbf{p}) = \sum_{A \subseteq E} \delta(A) \prod_{i \in A} p_i \tag{1}$$

The function $\delta$ is called the signed domination function and is defined through the equation

$$\phi(\mathbf{X}) = \sum_{A \subseteq E} \delta(A) \prod_{i \in A} X_i \tag{2}$$

A subset $P$ of $E$ is called a path set for the system if $X_i = 1$ for all $i \in P$ implies $\phi(\mathbf{X}) = 1$. $P$ is called a minimal path set if no proper subset of $P$ is a path set. A subset $K$ of $E$ is called a cut set for the system if $X_i = 0$ for all $i \in K$ implies $\phi(\mathbf{X}) = 0$. $K$ is called a minimal cut set if no proper subset of $K$ is a cut set.

Many theoretical results are known, and many methods exist for analysing systems with independent components. The analysis of systems with dependent components is much more difficult in general. In practice the asumption of independence is often unrealistic. It is therefore important to study different types of dependence models that can be used to model systems of dependent components in different situations. One such model is the shock model introduced by Boyles and Samaniego (1984). This model is particularly well suited to analyse systems where the dependence is due to common cause failures.

In section 2 we define the shock model, and discuss how the reliability of shock systems can be calculated. Section 3 of this paper is devoted to a discussion of bounds for the reliability of a shock system. In section 4 we discuss Bayesian analysis of shock systems in relation to a framework for such analysis for general reliability systems introduced by

Natvig and Eide (1987). In section 5 we discuss Bayesian estimation of the life distribution of a shock system in analogy with the procedure of Natvig and Eide (1987).

# 2 The shock model

In the shock model we assume that there exists a set of common shocks corresponding to a family $\mathcal{A}$ of subsets of $E$. The shock corresponding to the set $A \in \mathcal{A}$ destroys all components in $A$ and no others. The common shocks are also referred to as external shocks. The individual failure of a component is also thought of as a shock (corresponding to a set consisting of one element) and is called an individual shock. We denote by $E_i$ the set of shocks destroying the $i$-th component, and more generally, by $E_A$ the set of shocks $B$ such that $B$ intersects $A$. In order to keep track of the shock status we introduce the random variables $Y_A$. $Y_A = 1$ if the shock corresponding to $A$ has not yet occurred, and 0 if it has occurred. The probability that the shock $A$ has not occurred is denoted by $q_A$, and the vector with components $q_A$ is denoted by $\mathbf{q}$. The variables $Y_A$ are assumed to be independent. Now we have the following obvious relations:

$$X_i = \prod_{A \in E_i} Y_A \tag{3}$$

$$p_i = \prod_{A \in E_i} q_A \tag{4}$$

Since the variables $Y_A$ are binary, and since the state of the system clearly is a non-decreasing function of these variables, it is obvious that the $Y_A$'s can be used to define a binary, monotone system that gives an alternative description of the state of the original system. More precisely, we summarize the equations (3) and (4) in the equations

$$\mathbf{X} = \eta(\mathbf{Y}) \qquad \mathbf{p} = \eta(\mathbf{q}) \tag{5}$$

and define the system $(F, \psi)$ by letting $F$ consist of all the sets in $\mathcal{A}$ together with the one point sets $\{i\}, i \in E$. Thus we identify the components of the system with subsets of $E$ rather than with natural numbers. The structure function $\psi$ is defined by

$$\psi(\mathbf{Y}) = \phi(\eta(\mathbf{Y})) \tag{6}$$

The advantage of this is that the components of the system $(F, \psi)$ are independent. Note that $\mathbf{q}$ is the vector of component reliabilities for the new system. We will denote by $g(\cdot)$ the reliability function of $(F, \psi)$. The independence of the components in $(F, \psi)$ implies that $g(\mathbf{q})$ is the reliability of the system.

Of course, knowledge of $\mathbf{q}$ is required in order to perform a reliability analysis by means of the system $(F, \psi)$. However, data on the reliabilities of the components will often be in a form directly relevant to the quantities $q_A$, with $A$ a one point set. In laboratory experiments the components will usually not be subject to the common shocks that can occur when the system is operating.

By (2) and (3) the structure function of $(F, \psi)$ can be expressed as

$$\psi(\mathbf{Y}) = \phi(\eta(\mathbf{Y})) = \sum_{A \subseteq E} \delta(A) \prod_{i \in A} (\prod_{B \in E_i} Y_B) = \sum_{A \subseteq E} \delta(A) \prod_{B \in E_A} Y_B \tag{7}$$

It follows that the reliability function $g$ is given by

$$g(\mathbf{q}) = \sum_{A \subseteq E} \delta(A) \prod_{B \in E_A} q_B \tag{8}$$

Thus if the reliability function $h$ of $(E, \phi)$ is known analytically through knowledge of the signed domination function $\delta$, the same is true for g, and the reliabibity of the system can be calculated by (8).

For some systems it is possible to calculate the functional state of the system, given the functional state of the components, even if the signed domination function has not been derived. It is in principle always possible to compute $\phi$ if the minimal path sets or cut sets are known, since every binary, monotone system can be regarded as a series system of the minimal cut parallell structures or a parallell system of the minimal path series structures (see Barlow and Proschan (1975)). For some types of systems there exist algorithms for calculating the structure function. It is then in principle possible to compute the reliability function $h$ by going through all possible state vectors $\mathbf{X}$. This gives the formula

$$h(\mathbf{p}) = \sum_{\mathbf{x} \in \mathcal{X}} \phi(x) (\prod_{i=1}^{n} p_i^{x_i} (1 - p_i)^{1 - x_i}) \tag{9}$$

where $\mathcal{X}$ denotes the set $(\{0, 1\})^n$. By instead enumerating all possible values for $\mathbf{Y}$ and evaluating $\phi(\eta(\mathbf{Y}))$ for each $\mathbf{Y}$, the reliability can be calculated by the formula

$$g(\mathbf{q}) = \sum_{\mathbf{y} \in \mathcal{Y}} \phi(\eta(\mathbf{y})) (\prod_{A \in \mathcal{B}} q_A^{y_A} (1 - q_A)^{1 - y_A}) \tag{10}$$

where $\mathcal{Y}$ denotes the set of possible values for $\mathbf{Y}$, and $\mathcal{B}$ denotes the family $\mathcal{A} \cup \{1\} \cup \cdots \cup \{n\}$ of subsets of $E$ corresponding to shocks. The number of summands in (9) is $2^n$ and $2^{n+p}$ in (10), if the number of common shocks is $p$. Thus it is impossible to use the method for very complicated systems. In some cases, however, it may be easy to find the subsets $\eta^{-1}(\mathbf{x})$ of $\mathcal{Y}$, and if the system is sufficiently small such that $h$ can be calculated by (9), it may also be possible to calculate $g$ despite the added complexity due to the shocks. We then have the formula

$$g(\mathbf{q}) = \sum_{\mathbf{x} \in \mathcal{X}} \phi(\mathbf{x}) \Big( \sum_{\mathbf{y} \in \eta^{-1}(\mathbf{x})} \prod_{A \in \mathcal{B}} q_A^{y_A} (1 - q_A)^{1 - y_A} \Big) \tag{11}$$

If the system is too complicated to use (10) or (11), one may estimate the reliability by drawing a large number of vectors $\mathbf{Y}$ from the distribution defined by $\mathbf{q}$ and calculate the average of $\psi(\mathbf{Y})$ over the sample.

# 3 Bounds for the reliability

If the minimal path sets and minimal cut sets of a binary, monotone system are known, it is possible to calculate bounds for the reliability of the system (see Barlow and Proschan (1975)). The components must be associated in order for these bounds to be valid, and for some of the bounds independence of the components is needed. In a shock model the component states will always be associated since they are nondecreasing functions of the independent shock variables. The bounds based on independence, however, can not be used. In the case of a shock model an obvious alternative to the standard approach is to base the bounds on the minimal path and cut sets for the system $(F, \psi)$. One advantage of this is that the bounds based on independence of the components can be used. Furthermore, we shall see that some of the bounds based on association are improved by this approach. On the other hand, it may be more difficult to find the minimal path and cut sets for $(F, \psi)$ than for $(E, \phi)$. For the minimal path sets, however, the connection is very simple, as seen by the following proposition:

**Proposition 1** *If $P$ is a minimal path set for $(E, \phi)$ then the set $E_P$ is a minimal path set for $(F, \psi)$. If $Q$ is a minimal path set for $(F, \psi)$ then the set $P(Q) = \{i | \{i\} \in Q\}$ is a minimal path set for $(E, \phi)$. This establishes a one to one correspondence between the families $\mathcal{P}$ and $\mathcal{Q}$ of minimal path sets for $(E, \phi)$ and $(F, \psi)$ respectively.*

**Proof:** The proof is straightforward and is omitted.

The connection between the families $\mathcal{K}$ and $\mathcal{L}$ of minimal cut sets for $(E, \phi)$ and $(F, \psi)$ respectively, is more complicated. Each minimal cut set for $(E, \phi)$ gives rise to several minimal cut sets for $(F, \psi)$, but cut sets for $(F, \psi)$ generated from different minimal cut sets for $(E, \phi)$ may coincide. In order to simplify the formulation of the following proposition we define a family $M$ of subsets of $E$ to be irreducible if for each $A \in M$ we have $A \cap (\cup_{B \in M, B \neq A} B) \neq A$. In other words, a family is irreducible if the union of the sets in the family becomes smaller if any of the sets in the family is removed. The union of the sets in $M$ will be denoted by $A_M$.

**Proposition 2** *Let $\mathcal{L}'$ be the set of irreducible families $L$ of subsets of $E$ such that $A_L = K$ for some minimal cut set $K$ for $(E, \phi)$. Let $\mathcal{L}''$ be the set of families $L$ of subsets of $E$ such that $A_L$ contains a minimal cut set $K$ for $(E, \phi)$, and such that $A_M$ does not contain $K$ for any proper subfamily $M$ of $L$ ($L$ is then necessarily irreducible). Then $\mathcal{L}' \subset \mathcal{L} \subset \mathcal{L}''$. $\mathcal{L}'$ is a disjoint set of minimal cut sets for $(F, \psi)$, $\mathcal{L}''$ is a possibly nondisjoint set of cut sets for $(F, \psi)$ containing all minimal cut sets.*

**Proof:** The proof is fairly straightforward and is omitted.

Two different minimal cut sets $K_1, K_2$ for $(E, \phi)$ may give rise to the same cut set $L \in \mathcal{L}''$. Moreover, $K_1$ may give rise to a cut set $L_1$ properly contained in a cut set $L_2$ generated by $K_2$. However, $\mathcal{L}$ can be obtained from $\mathcal{L}''$ by removing all sets properly containing some other set in $\mathcal{L}''$ and all unnecessary copies of other members of $\mathcal{L}''$.

Note that there exist algorithms for constructing the minimal path and cut sets for a binary, monotone system based on a fault tree representation of the system. Constructing the fault tree representation of $(F,\psi)$ from that of $(E,\psi)$ is a trivial matter. Thus in practise one may use such an algorithm on $(F,\psi)$ to find $\mathcal{Q}$ and $\mathcal{L}$ directly, rather than deriving these sets from $\mathcal{P}$ and $\mathcal{K}$.

The following theorem shows that for the min-max bounds in Barlow and Proschan (1975) the lower bounds are improved by passing to $(F,\psi)$.

**Theorem 3** $l_1(\mathbf{p}) = \max_{P\in\mathcal{P}} \prod_{i\in P} p_i \leq l_1(\mathbf{q}) = \max_{Q\in\mathcal{Q}} \prod_{A\in Q} q_A \leq h \leq u_1(\mathbf{q}) = \min_{L\in\mathcal{L}} \coprod_{A\in L} q_A = \min_{L\in\mathcal{L}''} \coprod_{A\in L} q_A \leq \min_{L\in\mathcal{L}'} \coprod_{A\in L} q_A$

*In the last two expressions $\mathcal{L}''$ and $\mathcal{L}'$ may be replaced by any family of cut sets containing $\mathcal{L}$ and contained in $\mathcal{L}$ respectively.*

**Proof:** By proposition 1, if $P \in \mathcal{P}$, then $Q = E_P = \bigcup_{i\in P} E_i \in \mathcal{Q}$. By (4) $\prod_{i\in P} p_i = \prod_{i\in P} \left(\prod_{A\in E_i} q_A\right) \leq \prod_{A\in\bigcup_{i\in P} E_i} q_A = \prod_{A\in E_P} q_A$. By maximizing over $\mathcal{P}$ the first inequality follows. The second and third inequalities are applications of standard results in Barlow and Proschan (1975). The next equality is true because the reliability of any cut parallell structure of a nonminimal cut set $L$ majorises the reliability of the cut parallell structure of a minimal cut set $M \subset L$. The last inequality is obvious.

Note that we have a corresponding upper bound $u_1(\mathbf{p})$ based on $\mathcal{K}$, since the result of Barlow and Proschan (1975) only requires associated components.

The "product – ip" bounds of Barlow and Proschan (1975) applied directly to $(E,\phi)$ are not valid, since the components of $(E,\phi)$ are not independent. By passing to $(F,\psi)$ we obtain the following bounds:

$$\prod_{L\in\mathcal{L}''} \coprod_{A\in L} q_A \leq l_2(\mathbf{q}) = \prod_{L\in\mathcal{L}} \coprod_{A\in L} q_A \leq h \leq u_2(\mathbf{q}) = \coprod_{Q\in\mathcal{Q}} \prod_{A\in Q} q_A \tag{12}$$

In the first of these inequalities $\mathcal{L}''$ may obviously be replaced by any family of cut sets containing $\mathcal{L}$.

The last two inequalities in (12) follow by applying Barlow and Proschan (1975).

By expanding $P(\bigcup_{Q\in\mathcal{Q}} Q)$ and $P(\bigcup_{L\in\mathcal{L}} L)$ according to the formula for the probability of a union of sets up to a certain order, alternate upper and lower bounds for $h$ are obtained (see Barlow and Proschan (1975)). These are again based on independence of components, and therefore similar bounds based on $\mathcal{P}$ and $\mathcal{K}$ are not valid. For instance, $\sum_{Q\in\mathcal{Q}} \prod_{A\in Q} q_A$ is an upper bound, $\sum_{Q\in\mathcal{Q}} \prod_{A\in Q} q_A - \sum_{Q,R\in\mathcal{Q},Q\neq R} \prod_{A\in Q\cup R} q_A$ is a lower bound, and so on. When expanding according to cut sets up to order one, one gets alternative bounds by using $\mathcal{L}''$ instead of $\mathcal{L}$, since we obviously have

$$1 - \sum_{L\in\mathcal{L}''} \prod_{A\in L} (1-q_A) \leq 1 - \sum_{L\in\mathcal{L}} \prod_{A\in L} (1-q_A) \leq h \tag{13}$$

Of course, all bounds discussed in this section are based on knowledge of $\mathbf{q}$. If $\mathbf{p}$ is known, but $\mathbf{q}$ is unknown, only the standard bounds of Barlow and Proschan (1975) based on association are available. However, if the reliability function is known, it is possible to get other bounds in some situations by applying results of Egeland and Huseby

(1991). They showed that knowledge of the shock structure in some cases can be used to determine whether $h$ is overestimated or underestimated by $h(\mathbf{p})$. Loosely, one may say that $h(\mathbf{p})$ tends to underestimate $h$ (and thus act as a lower bound) if the sets $A \in \mathcal{A}$ define subsystems (socalled minors) $(A, \chi)$ of $(E, \phi)$ containing cut sets of order 1, and overestimate $h$ otherwise (see section 4 of Egeland and Huseby (1991)). This generalises the well-known fact that $\prod_{i=1}^{n} p_i$ underestimates the reliability of a series system, whereas $\coprod_{i=1}^{n} p_i$ overestimates the reliability of a parallell system of associated components.

# 4 Bayesian estimation of system reliability

Suppose for a while that $(E, \phi)$ is a binary, monotone system, but not necessarily a shock system, with vector of component reliabilities $\mathbf{p}$. Suppose that $\mathbf{p}$ is not known, but that the uncertainty about $\mathbf{p}$ is expressed in terms of a prior distribution $\pi$. Then the system reliability $h$ must be estimated within a Bayesian framework. More ambitiously, Natvig and Eide (1987) describe a procedure for calculating or estimating the distribution $\pi_0$ of $h$ from $\pi$. One important reason for doing this, besides getting a description of the uncertainty of $h$, is that the procedure provides a framework for updating the knowledge about $h$ when acquiring data on the system level. On the other hand, data on the component level may also be used to adjust the distribution of $\mathbf{p}$ before the distribution of $h$ is calculated. Information obtained from experts' opinions may also be incorporated, both on component and system level. This may be done by applying the retrospective method of Huseby (1986, 1988) reconstructing as far as possible the experts' background information as imaginary data (see Gåsemyr and Natvig (1991, 1992) for an example relevant to binary systems).

The procedure of Natvig and Eide (1987) is in principle straightforward if the components of $(E, \phi)$ are independent. It involves repeated use of Bayes theorem together with the transformation formula for probability distributions (see section 1 of Natvig and Eide (1987) and also section 1 of Gåsemyr and Natvig (1992)). If the $p_i$'s have independent prior distributions, updating of $\pi$ by data on the component level is also straightforward (see Natvig and Eide (1987)).

Since the distribution calculated by the above procedure is the distribution of $h(\mathbf{p})$, it can not be used if $h \neq h(\mathbf{p})$, as is the case when the components of $(E, \phi)$ are not independent. Theoretically, there could exist a function $h$ such that $h = h(\mathbf{p})$, in which case a similar procedure might be used. Knowing such a function $h$ amounts to having very specific knowledge of the dependence structure of $(E, \phi)$. In many cases such detailed information on the dependence structure is not available, but one would nevertheless be interesed in getting as much information on the distribution $\pi_0$ of $h$ as possible. For instance, one could hope to find a class of possible distributions for $h$ containing all distributions consistent with the available information.

Besides these problems, the integration needed to arive at $\pi_0$ may be very complicated, even if the components are independent. In order to address these difficulties Natvig and Eide (1987) suggest "estimating" $\pi_0$ indirectly from bounds on the moments of the distribution derived from the moments of the distributions $\pi_i$ of $p_i$. "Estimating" here means choosing within the class of possible distributions consisting of all distributions whose moments lie within the bounds calculated. Note also that even with independent components

the direct procedure can not be used if the function $\mathbf{p} \to h(\mathbf{p})$ is not known analytically, since use of the transformation formula requires use of a Jacobian determinant involving this function. The indirect method of Natvig and Eide (1987) may then be used instead, since the bounds used by this procedure are based on knowledge only of the minimal path and cut sets of $(E, \phi)$. These bounds are analogues of the standard bounds for the reliability of a binary system in Barlow and Proschan (1975) (see also the previous section), and are derived in section 2 of Natvig and Eide (1987) (see theorems 2.7 and 2.8). For convenience we quote these theorems here (other bounds are found in Lindqvist (1991)):

**Theorem 2.7** *Let $h$ be the reliability of a monotone system of $n$ associated components with independent reliabilities $p_1, \ldots, p_n$. Then for $m = 1, 2, \ldots$*

$$\max_{1 \le j \le p} \prod_{i \in P_j} Ep_i^m \le Eh^m \le \min_{1 \le j \le k} \sum_{r=0}^{m} \binom{m}{r} (-1)^r \prod_{i \in K_j} \sum_{s=0}^{r} \binom{r}{s} (-1)^s Ep_i^s$$

**Theorem 2.8** *Let $h(p)$ be the reliability of a monotone system of $n$ components with independent reliabilities $p_1, \ldots, p_n$. Let the components be independent given $\mathbf{p}$. Then for $m = 1, 2, \ldots$*

$$\prod_{j=1}^{k} \sum_{r=0}^{m} \binom{m}{r} (-1)^r \prod_{i \in K_j} \sum_{s=0}^{r} \binom{r}{s} (-1)^s Ep_i^s \le E\{h(\mathbf{p})\}^m$$

$$Eh(\mathbf{p}) \le \coprod_{j=1}^{p} \prod_{i \in P_j} Ep_i$$

We observe that only the first of these theorems can be used if the components of $(E, \phi)$ are not independent. Moreover, only the lower bounds can be used if the component reliabilities $p_i$ are not independent. When using the retrospective method of Huseby (1986) to incorporate experts' opinions into the distribution of $\mathbf{p}$, dependence between the $p_i$'s can easily arise (an example of this is found in section 2 of Gåsemyr and Natvig (1992) in the case of the Marshall-Olkin model (Marshall and Olkin (1967))). Thus in the case of dependent components the quoted theorems may give a poor foundation for estimating the distribution of $h$. Another difficulty in the case of dependent components, concerns the updating of the distributions of the $p_i$'s by real data $D_i$ on the $i$-th component. These data will not be independent if they are obtained from operation of the system, which complicates the updating. In order to be independent they must typically result from laboratory experiments. However, under laboratory conditions the $i$-th component is not influenced by the behaviour of the other components, and consequently its reliability will be different from $p_i$.

We now turn to the shock model. Thus we consider a binary system $(E, \phi)$ whose state is determined by the independent shock indicator variables $Y_A$, where $A$ runs through the set $\mathcal{B} = \{1\} \cup \cdots \cup \{n\} \cup \mathcal{A}$, $n$ is the number of components and $\mathcal{A}$ is the set of common shocks (just as in section 2). However, the vector $\mathbf{q}$ of "shock reliabilities" is considered as a random quantity, with prior distribution $\rho$. We now consider the $Y_A$'s as the components of a binary system $(F, \psi)$, as in section 2. The distribution $\rho_0$ of the reliability $h = g(\mathbf{q})$

of $(F, \psi)$ can then in principle be calculated by the direct procedure of Natvig and Eide (1987), provided that the function $g(\cdot)$ is known analytically. This procedure could not be applied directly to $(E, \phi)$ since the components are dependent. In fact, knowledge of a distribution $\pi$ for $\mathbf{p}$ together with knowledge of the shock structure is not sufficient information for determining a distribution $\pi_0$ for h, since obviously different distributions $\rho$ producing different distributions $\rho_0$ for $h$ are consistent with $\pi$ (any conditional joint distribution for the $q_A$'s with $A \in \mathcal{A}$, given $\mathbf{p}$, will produce such a distribution).

Since $\mathbf{p} = \eta(\mathbf{q})$, where $\eta$ is a non-bijective function (see (5)), the decision maker is required to provide more information by this procedure than just specifying a distribution for $\mathbf{p}$. However, we think that specifying a distribution for $\mathbf{q}$ is at least equally natural in most cases. In many situations it is even reasonable to think of the $q_A$'s as independent, so that $\rho$ is determined by the marginal distributions $\rho_A$ for $q_A$, $A \in \mathcal{B}$. In that case, data from laboratory experiments providing information on the reliabilities $q_{\{i\}}, i \in E$ can easily be used to update $\rho$. Also, the independence between the $q_A$'s makes the upper bounds of theorem 2.7 of Natvig and Eide (1987) available. Gåsemyr and Natvig (1992) gives an example where this independence is preserved when $\rho$ is updated by imaginary data from expert opinions, even though the experts are challenged to provide information reflecting the dependence structure of the model. In addition, the bounds in theorem 2.8 of Natvig and Eide (1987) can also be used. Thus even when one has to use the method of bounds for moments, a significant improvement is obtained, by basing the analysis on a prior distribution for $\mathbf{q}$ rather than for $\mathbf{p}$. Moreover, in analogy with theorem 3 the following theorem shows that the lower bound of theorem 2.7 of Natvig and Eide (1987) is improved:

**Theorem 4** *Let $(E, \phi)$ be a shock system, and $(F, \psi)$ the corresponding system of independent shock variables. Let $\mathcal{P}, \mathcal{Q}$ denote the set of minimal path sets for $(E, \phi)$ and $(F, \psi)$ respectively. We then have $\lambda_1^m(\mathbf{p}) = \max_{P \in \mathcal{P}} \prod_{i \in P} E(p_i^m) \leq \lambda_1^m(\mathbf{q}) = \max_{Q \in \mathcal{Q}} \prod_{A \in Q} E(q_A^m) \leq E(g^m)$*

**Proof:** Note that by (4) we have

$$E(p_i^m) = E\Big(\prod_{A \in E_i} q_A^m\Big) = \prod_{A \in E_i} E(q_A^m)\,. \tag{14}$$

Applying this, the first inequality follows by an argument similar to the first part of the proof of theorem 3. The second inequality follows from theorem 2.7 in Natvig and Eide (1987).

# 5 Bayesian estimation of life distribution

In this section we asume that $(E, \phi)$ is a system of n independent components with reliability function $h$ and vector of component reliabilities $\mathbf{p}$. In the light of the previous sections it is clear that the ideas can be applied to a shock system as well. However, for notational simplicity we chose to formulate the material in this section in terms of systems with independent components.

If the decision maker is content with a simple numerical estimate for the reliability of the system, and if the uncertainty about $\mathbf{p}$ is expressed by the prior distribution $\pi$, it is natural to choose the Bayes estimator subject to quadratic loss, i.e.

$$\hat{h} = E(h(\mathbf{p})) = h(E(\mathbf{p})) \tag{15}$$

This number may also be considered as the true probability that the system functions, given all the uncertainty in the model (i. e. the uncertainty in $\mathbf{p}$, and the uncertainty in $\mathbf{X}$ given $\mathbf{p}$). Indeed, we have

$$P(\phi(\mathbf{X}) = 1) = E(\phi(\mathbf{X})) = E(E(\phi(\mathbf{X})|\mathbf{p})) = E(h(\mathbf{p})) = \hat{h} \tag{16}$$

Thus, if the goal is a numerical estimate, it is not necessary to provide a full distribution for $\mathbf{p}$. It is sufficient to specify an expected value for $\mathbf{p}$. However, a distribution $\pi$ facilitates the updating procedure if data is acquired.

If the uncertainty about the model instead is expressed in terms of a distribution for parameters describing the life distribution for the components, the life distribution of the system can be estimated in analogy with (14). To be specific, let $T$ be the life time of the system, and define $S(t) = P(T > t)$. Denote by $S_i(t|\boldsymbol{\theta})$ the probability that the $i$-th component survives time t, where $\boldsymbol{\theta}$ is an unknown parameter vector belonging to some set $\Theta$. We suppose that $\boldsymbol{\theta}$ has the prior distribution $\pi$. Conditional on $\boldsymbol{\theta}$, the survival probability for the system beyond $t$ is denoted by $S(t|\boldsymbol{\theta})$. We then get

$$S(t) = \int_\Theta S(t|\boldsymbol{\theta})\pi(d\boldsymbol{\theta}) = \int_\Theta h(S_1(t|\boldsymbol{\theta}), \ldots, S_n(t|\boldsymbol{\theta}))\pi(d\boldsymbol{\theta}), t \in (0, \infty) \tag{17}$$

If the distribution of $T$ given $\theta$ is absolutely continuous, this can be updated by data on the system level by means of Bayes' theorem. We denote by $f(t|\boldsymbol{\theta}) = -S'(t|\boldsymbol{\theta})$ the conditional density for $T$ given $\boldsymbol{\theta}$. Let $Z$ be the life time of the system in an experiment on the system level. We asume that $T$ is independent of $Z$ given $\boldsymbol{\theta}$. We then have

$$\begin{aligned} &P(T > t|Z = z) = \\ &= \int_\Theta h(S_1(t|\boldsymbol{\theta}), \ldots, S_n(t|\boldsymbol{\theta}))\pi(d\boldsymbol{\theta}|Z = z) \propto \int_\Theta h(S_1(t|\boldsymbol{\theta}), \ldots, S_n(t|\boldsymbol{\theta}))f(z|\boldsymbol{\theta})\pi(d\boldsymbol{\theta}) \end{aligned} \tag{18}$$

Here the last expression must be adjusted by introducing a suitable normalizing constant so that it takes the value 1 if $t = 0$.

**Example 1.** Suppose that $(E, \phi)$ is a series system of n independent components, each of which is exponentially distributed with failure rate $\theta_i$. We asume that the $\theta_i$'s are independent and gamma distributed with parameters $a_i, b_i$ respectively. Suppose furthermore that in an experiment the system obtains the life time $z$. We then have $S(t|\boldsymbol{\theta} = \exp(-(\sum_{i=1}^n \theta_i)t), f(z|\boldsymbol{\theta}) = (\sum_{i=1}^n \theta_i)\exp(-(\sum_{i=1}^n \theta_i)z)$. Applying (18) we get

$$\begin{aligned} &P(T > t|Z = z) \propto \int_0^\infty \ldots \int_0^\infty \Big(\sum_{i=1}^n \theta_i\Big)\Big(\prod_{i=1}^n \theta_i^{a_i - 1}\Big) \\ &\exp\Big(-\sum_{i=1}^n (b_i + z + t)\theta_i)\Big)d\theta_1 \ldots d\theta_n \propto \Big(\sum_{i=1}^n (a_i/(b_i + z + t))\Big)\Big(\prod_{j=1}^n (1/(b_j + z + t)^{a_j})\Big) \end{aligned} \tag{19}$$

By a slight modification of the argument, this example can be generalized to the case when $k$ independent experiments are performed on the system level, producing the data $Z_1 = z_1, \ldots, Z_k = z_k$. The $Z_i$'s are supposed to be independent given $\boldsymbol{\theta}$. Define

$$\mathcal{E} = \left\{ \mathbf{r} = (r_1, \ldots, r_n) : r_i \in \{0, 1, \ldots, k\} \text{ for all } i, r_1 + \cdots + r_n = k \right\} \tag{20}$$

We then have

$$P(T > t | Z_1 = z_1, \ldots, Z_k = z_k) \propto \int_0^\infty \cdots \int_0^\infty \Big( \sum_{i=1}^n \theta_i \Big)^k \Big( \prod_{i=1}^n \theta_i^{a_i - 1} \Big)$$
$$\exp\Big( - \sum_{i=1}^n ((b_i + z_1 + \cdots + z_k + t)\theta_i) \Big) d\theta_1 \ldots d\theta_n \propto$$
$$\Big( \prod_{i=1}^n 1/(b_i + z_1 + \cdots + z_k + t)^{a_i} \Big) \Big( \sum_{\mathbf{r} \in \mathcal{E}} \Big( \prod_{j=1}^n \Gamma(a_j + r_j)/(r_j!(b_j + z_1 + \cdots + z_k + t)^{r_j}) \Big) \Big) \tag{21}$$

Clearly this may be applied directly to the Marshall-Olkin model as well, since the modified system $(F, \psi)$ is also a series system.

**Example 2.** Let $(E, \phi)$ be a binary system with signed domination function $\delta$. Suppose that the distributions of the components and of the failure rates are as in the previous example. As in the first part of example 1, suppose that an experiment is performed giving time to failure $Z = z$. We then have

$$S(t|\boldsymbol{\theta}) = \textstyle\sum_A \delta(A) \exp(-(\sum_{i \in A} \theta_i)t) \text{ and } f(z|\boldsymbol{\theta}) = \sum_A \delta(A)(\sum_{i \in A} \theta_i) \exp(-(\sum_{i \in A} \theta_i)z).$$

Applying (18) this gives

$$P(T > t | Z = z) \propto \int_0^\infty \cdots \int_0^\infty \sum_{A_1, A_2} \delta(A_1)\delta(A_2)$$
$$\Big( \prod_{i=1}^n \theta_i^{a_i - 1} \Big) \Big( \sum_{i \in A_1} \theta_i \Big) \Big( \prod_{i=1}^n \exp(-(b_i + I(i \in A_1)z + I(i \in A_2)t)\theta_i) \Big) d\theta_1 \cdots d\theta_n \tag{22}$$
$$\propto \sum_{A_1, A_2} \delta(A_1)\delta(A_2) \Big( \prod_{i=1}^n 1/(b_i + I(i \in A_1)z + I(i \in A_2)t)^{a_i} \Big) \Big( \sum_{i \in A_1} a_i/(b_i + z + I(i \in A_2)t) \Big)$$

Again, the adjustment to a shock model is straightforward. In fact, the only changes needed in (22) is that the index $i$ must be replaced by an index $B$ running through $\mathcal{B}$ in the product and through the set $E_{A_1}$ in the sum, and that the indicator functions $I(i \in A_1), I(i \in A_2)$ must be replaced by $I(B \in E_{A_1}), I(B \in E_{A_2})$ respectively ($\delta$ still denotes the signed domination function of the basic system $(E, \phi)$ and $A_1, A_2$ are supposed to run through pairs of subsets of the basic component set $E$, cf. (8)).

# References


Barlow, R. E. and Proschan, F. (1975): Statistical theory of reliability and life testing. Probability models. Holt, Rinehart and Winston, New York.

Boyles, R. A. and Samaniego, F. J. (1984): Modelling and inference for multivariate binary data with positive dependence. JASA 79, pp. 188–193.

Egeland, T. and Huseby, A. B. (1991): On dependence and reliability computations. Networks 21, pp. 521–545.

Gåsemyr, J. and Natvig, B. (1991): Using expert opinions in Bayesian estimation of component life time in a shock model – a general predictive approach. Tech. Rep., University of Oslo.

Gåsemyr, J. and Natvig, B. (1992): Expert opinions in Bayesian estimation of system reliability in a shock model – the $MTP_2$ connection. Tech. Rep., University of Oslo.

Huseby, A. B. (1986): Combining experts' opinions – a retrospective approach. Tech. Rep., Center for industrial research, P. O. Box 350 Blindern, Oslo, Norway.

Huseby, A. B. (1988): Combining opinions in the predictive case. Bayesian Statistics 3. Editors J. M. Bernardo, M. H. De Groot, D. V. Lindley, A. F. M. Smith. Oxford University Press, pp. 641–651.

Lindqvist, B. (1991): The minimal path upper bound for the moments of a random reliability function. Submitted to Scand. J. Statist.

Marshall, A. W. and Olkin, I. (1967): A multivariate exponential distribution. JASA 62, pp. 30–44.

Natvig, B. and Eide, H. (1987): Bayesian estimation of system reliability. Scand. J. Statist. 14, pp. 19–27.