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1. Introduction

The theory of abstract computational complexity in ordinary recursion theory (ORT) was initiated by Rabin [7] and Blum [1]. Jacobs [5] generalized the notions to recursion theory on an admissible ordinal α (α-recursion theory) and proved in this setting certain main theorems such as Blum's theorem and the Compression theorem.

In ORT the notions of "finite" and "bounded" coincide. Thus when lifting a theorem from ORT to α-recursion theory, "finite" may be translated to either "bounded below α" or to the stronger and more natural "α-finite". Jacobs conjectured that "bounded below α" was the best possible in the theorems mentioned above. In this paper we shall prove that the stronger versions of these theorems are in fact true. Furthermore our constructions are uniform for all admissible α.

Rather than restricting ourselves to α-recursion theory we shall consider transitive rudimentarily closed structures $\underset{\sim}{M} = \langle M,\in,R\rangle$ which admit what we call an acceptable prewellordering. The special cases $\underset{\sim}{M} = \langle L_\alpha,\in\rangle$ and $\underset{\sim}{M} = \langle S_\beta,\in\rangle$ where α is admissible and β a limit ordinal constitute respectively α-recursion theory and β-recursion theory (see Friedman [4]).

We shall prove Blum's theorem and the Compression theorem for every weakly admissible structure $\underset{\sim}{M}$. This should be contrasted with the results in [9], where we showed that the structure of the tamely r.e. $\underset{\sim}{M}$-degrees is rich for every adequate such $\underset{\sim}{M}$, combined with Simpson's result [8] that (assuming AD) some additional hypothesis on $\underset{\sim}{M}$, such as adequacy, is necessary for a positive solution to Post's problem.

I would like to thank J.V. Tucker for discussions on the usefulness of a (generalized) analysis of computational complexity.

For a treatment of the rudimentary functions we refer to Devlin [2] where a proof of the following lemma can be found.

<u>Lemma 1.1.</u> $\models_{\underset{\sim}{M}}^{\Sigma_n} = \{\langle i, x_1, \ldots, x_m\rangle :$ The i:th Σ_n formula ϕ is m-ary and $\underset{\sim}{M} \models \phi(x_1, \ldots, x_m)\}$ is uniformly $\Sigma_n^{\underset{\sim}{M}}$ for transitive rudimentarily closed structures $\underset{\sim}{M}$.

The recursion theoretic notions are defined as follows: A set $A \subseteq M$ is <u>$\underset{\sim}{M}$-r.e.</u> if $A \in \Sigma_1(\underset{\sim}{M})$, i.e. A is definable over $\underset{\sim}{M}$ by a Σ_1 formula with parameters from M. A is <u>$\underset{\sim}{M}$-recursive</u> if A and $M - A$ are both $\underset{\sim}{M}$-r.e. If $A \in M$ then A is <u>$\underset{\sim}{M}$-finite</u>. A partial function is <u>$\underset{\sim}{M}$-recursive</u> if its graph is $\underset{\sim}{M}$-r.e.

<u>Definition 1.2.</u> $\underset{\sim}{M}$ admits an <u>acceptable</u> prewellordering if there is an $\underset{\sim}{M}$-recursive prewellordering $\precsim$ on M such that

(i) $L^x = \{y \in M : y \prec x\}$ is uniformly $\underset{\sim}{M}$-finite.

(ii) $|\precsim|$ = ordinal of $\precsim$ = limit ordinal and for each $\delta < |\precsim|$, $L^\delta \cap |\precsim| = \delta$.

(iii) $\Sigma_1\text{-cf}(|\precsim|)$ = limit ordinal, where $\Sigma_1\text{-cf}(|\precsim|)$ = least ordinal γ for which there is $\underset{\sim}{M}$-finite $x \subseteq L^\gamma$ and $\underset{\sim}{M}$-recursive function $f : x \to |\precsim|$, unbounded in $|\precsim|$.

(iv) If $x \in L^{\tau}$ then there is σ such that $x \subseteq L^{\sigma}$. If $x \in L^{\tau}$ and $\tau < \Sigma_1\text{-cf}(|\precsim|)$ then there is $\sigma < \Sigma_1\text{-cf}(|\precsim|)$ such that $x \subseteq L^{\sigma}$.

Henceforth $\underset{\sim}{M} = \langle M, \in, R \rangle$ will denote a transitive rudimentarily closed structure which admits an acceptable prewellordering $\prec$. $\Sigma_1\text{-cf}(|\precsim|)$ will be denoted by κ. The reader is referred to [9] for elementary recursion theoretic results such as the existence of $\underset{\sim}{M}$-recursive enumerations of the $\underset{\sim}{M}$-r.e. sets, the existence of an $\underset{\sim}{M}$-recursive selection operator and the ability to define $\underset{\sim}{M}$-recursive functions by recursion on κ.

In order to define the degree of admissibility of $\underset{\sim}{M}$ we need consider the following notion of projectum. Let $|\precsim|^* =$ least ordinal γ for which there is a partial $\underset{\sim}{M}$-recursive function $q : L^{\gamma} \xrightarrow{\text{onto}} M$. $\underset{\sim}{M}$ is said to be <u>admissible</u> if $\kappa = |\precsim|$, <u>weakly inadmissible</u> if $|\precsim|^* \leq \kappa < |\precsim|$ and <u>strongly inadmissible</u> otherwise. In case $|\precsim|^* \leq \kappa$, $\underset{\sim}{M}$ is said to be <u>weakly admissible</u>.

<u>Definition 1.3</u>. Let $D \subseteq M$ be $\underset{\sim}{M}$-recursive. Then $\Phi = \langle \{\phi_e : e \in M\}, \{\Phi_e : e \in M\} \rangle$ is a <u>D-complexity measure</u> for $\underset{\sim}{M}$ if $\{\phi_e : e \in M\}$ is an $\underset{\sim}{M}$-recursive enumeration of the partial $\underset{\sim}{M}$-recursive functions with domain a subset of D such that the s-m-n theorem holds, each Φ_e is a partial function with range a subset of κ, and the following conditions hold:

(i) For each $e \in M$, $\text{dom}\ \phi_e = \text{dom}\ \Phi_e$.

(ii) $\Phi_e(x) = \delta$ is an $\underset{\sim}{M}$-recursive relation of e, x and δ.

An M-complexity measure is simply called a <u>complexity measure</u>. We are, of course, mainly interested in complexity measures. The reason for introducing the parameter D is that it allows for a simple reduction of the weakly inadmissible case to the admissible case.

Using lemma 1.1 it is not difficult to see that there are $\underset{\sim}{M}$-recursive enumerations of the partial $\underset{\sim}{M}$-recursive functions such that the s-m-n theorem holds. Let $\{\phi_e : e \in M\}$ be such an enumeration and let $W = \{\langle e,x,y\rangle : \phi_e(x) = y\}$. Let $\lambda \sigma W^\sigma$ $(\sigma < \kappa)$ be an $\underset{\sim}{M}$-recursive enumeration of W. Define $\Phi_e(x) =$ least $\sigma[(\exists y \in \pi_2"W^\sigma)(\langle e,x,y\rangle \in W^\sigma)]$ where π_i is the i:th projection function. Then $\Phi = \langle\{\phi_e : e \in M\}, \{\Phi_e : e \in M\}\rangle$ is a complexity measure for $\underset{\sim}{M}$.

<u>Blum's theorem</u>. Let $\underset{\sim}{M}$ be a weakly admissible structure and let Φ be a complexity measure for $\underset{\sim}{M}$. Suppose g is a partial $\underset{\sim}{M}$-recursive function with range a subset of κ. Then there is a partial $\underset{\sim}{M}$-recursive function f such that $\text{dom } f = \text{dom } g$ and if $f = \phi_e$ then $\{x : \Phi_e(x) < g(x)\}$ is $\underset{\sim}{M}$-finite uniformly in e and an index for g.

In order to state the Compression theorem we need the following definition.

<u>Definition 1.4</u>. Let $D \subseteq M$ be $\underset{\sim}{M}$-recursive. Then $\Psi = \{\psi_e : e \in M\}$ is a <u>D-measured set</u> if $\text{dom } \psi_e \subseteq D$ and $\text{ran } \psi_e \subseteq \kappa$ for each e, and $\psi_e(x) = \delta$ is an $\underset{\sim}{M}$-recursive relation of e, x and δ.

An M-measured set is called a <u>measured set</u>. If Φ is a D-complexity measure then $\{\Phi_e : e \in M\}$ is a D-measured set.

<u>Compression theorem</u>. Let $\underset{\sim}{M}$ be a weakly admissible structure, Φ a complexity measure for $\underset{\sim}{M}$ and Ψ a measured set. Then there are total $\underset{\sim}{M}$-recursive functions h and s such that the following holds for each $e \in M$:

(i) $\text{dom } \phi_{s(e)} = \text{dom } \psi_e$.

(ii) If $\phi_\varepsilon = \phi_{s(e)}$ then $\{x : \Phi_\varepsilon(x) < \psi_e(x)\}$ is $\underset{\sim}{M}$-finite uniformly in ε and e.

(iii) $\{x : h(x, \psi_e(x)) < \Phi_{s(e)}(x)\}$ is $\underset{\sim}{M}$-finite uniformly in e.

2. Admissible $\underset{\sim}{M}$

In this section we prove slight generalizations of Blum's theorem and the Compression theorem for admissible structures $\underset{\sim}{M}$. It should be remarked that the notion of a complexity measure can naturally be formulated in the axiomatic framework of Moshovakis [6] and Fenstad [3] and that the constructions of this section can be carried out in that framework for any infinite computation theory.

Theorem 2.1. Let $\underset{\sim}{M}$ be an admissible structure and let Φ be a D-complexity measure for $\underset{\sim}{M}$. Suppose g is a partial $\underset{\sim}{M}$-recursive function with domain a subset of D and range a subset of $|\underset{\sim}{\leq}|$. Then there is a partial $\underset{\sim}{M}$-recursive function f such that dom f = dom g and if $f = \phi_e$ then $\{x : \Phi_e(x) < g(x)\}$ is $\underset{\sim}{M}$-finite.

Proof. The proof is a cancellation argument. We construct sets F and A in stages σ for $\sigma < |\underset{\sim}{\leq}|$. A is the set of indices cancelled during the construction and F is the graph of the function constructed. We use the notation A^σ to denote the $\underset{\sim}{M}$-finite part of A constructed by stage σ. $A^{<\sigma}$ denotes $\cup\{A^\tau : \tau < \sigma\}$.

Let $\lambda\sigma g^\sigma$ be an $\underset{\sim}{M}$-recursive approximation of g. To be precise, let θ be a $\Delta_0(\underset{\sim}{M})$ formula such that
$g(x) = \delta \iff \exists t\, \theta(x,\delta,t)$ and put $g^\sigma = \{\langle x,\delta\rangle \in L^\sigma \times \sigma : (\exists t \in L^\sigma)\theta(x,\delta,t)\}$.
Note that an index for an approximation of g can be obtained uniformly from an index for g.

The construction at stage σ : Let $H^\sigma = \{e \in L^\sigma : e \notin A^{<\sigma} \;\&\; (\exists x \in \pi_0''(g^\sigma - g^{<\sigma}))(\Phi_e(x) < g^\sigma(x))\}$. Here $\Phi_e(x) < g^\sigma(x)$ stands for $g^\sigma(x)\downarrow \;\&\; (\exists \delta < g^\sigma(x))(\Phi_e(x) = \delta)$, which is an $\underset{\sim}{M}$-recursive relation. Put $K_e^\sigma = \{x \in \pi_0''(g^\sigma - g^{<\sigma}) : \Phi_e(x) < g^\sigma(x)\}$ and let $h(\sigma)$ = some $\tau\,[(\forall e \in H^\sigma)(\forall x \in K_e^\sigma)(\phi_e(x) \in L^\tau)]$. Finally we put $A^\sigma = A^{<\sigma} \cup H^\sigma$ and $F^\sigma = F^{<\sigma} \cup \{\langle x, h(\sigma)\rangle : x \in \pi_0''(g^\sigma - g^{<\sigma})\}$.

To complete the construction put $A = \bigcup\{A^\sigma : \sigma < |\precsim|\}$ and $F = \bigcup\{F^\sigma : \sigma < |\precsim|\}$.

Let f be the partial $\underset{\sim}{M}$-recursive function whose graph is F. It is immediate from the construction that f is in fact a function, $\text{dom } f = \text{dom } g$ and $\text{ran } f \subseteq |\precsim|$.

Suppose $e \in A$. Let σ be such that $e \in A^\sigma - A^{<\sigma}$. Then $e \in H^\sigma$ and hence $K_e^\sigma \neq \emptyset$. Choose $x \in K_e^\sigma$. Then $f(x) = h(\sigma) \succ \phi_e(x)$ so $f \neq \phi_e$. Thus if $f = \phi_e$ then $e \notin A$.

To complete the proof we show $\{x : \Phi_e(x) < g(x)\}$ is $\underset{\sim}{M}$-finite whenever $f = \phi_e$. So suppose $f = \phi_e$ and choose τ such that $e \in L^\tau$. We claim $\{x : \Phi_e(x) < g(x)\} = \{x \in \pi_0''g^\tau : \Phi_e(x) < g^\tau(x)\}$, which suffices since $\Phi_e(x) < g^\tau(x)$ is an $\underset{\sim}{M}$-recursive relation of x. To prove the non-trivial inclusion suppose $\Phi_e(x) < g(x)$. Suppose $x \notin \pi_0''g^\tau$. Choose σ such that $x \in \pi_0''(g^\sigma - g^{<\sigma})$. Then $e \in A^{<\sigma}$ or $e \in H^\sigma$ since $\sigma > \tau$, contradicting the fact $e \notin A$. Thus $x \in \pi_0''g^\tau$ and the inclusion is proved. □

Remarks.

(i) The construction in the above proof is uniform in an index for g. To be precise, there is a partial $\underset{\sim}{M}$-recursive function $m(e,x)$ such that if e is an index for g then $\lambda x\, m(e,x)$ satisfies the conclusion of the theorem. Thus, using the s-m-n theorem for Φ, there is an $\underset{\sim}{M}$-recursive function s such that

$\lambda x\, m(e,x) = \phi_{s(e)}$ for such e.

(ii) The set $\{x : \Phi_{s(e)}(x) < g(x)\}$, where e is an index for g, is obtained uniformly from e by finding the least τ such that $s(e) \in L^{\tau}$.

(iii) In case $\underset{\sim}{M} = \langle L_{\alpha}, \in \rangle$ we choose the usual prewellordering of L_{α} which is definable without infinite parameters. Then the construction is uniform for all admissible α. This answers a question posed in Jacobs [5].

Theorem 2.2. Let $\underset{\sim}{M}$ be an admissible structure, Φ a D-complexity measure for $\underset{\sim}{M}$ and Ψ a D-measured set. Then there are total $\underset{\sim}{M}$-recursive functions h and s such that for each $e \in M$ the following holds:

(i) $\operatorname{dom} \phi_{s(e)} = \operatorname{dom} \psi_e$.

(ii) If $\phi_{\varepsilon} = \phi_{s(e)}$ then $\{x : \Phi_{\varepsilon}(x) < \psi_e(x)\}$ is $\underset{\sim}{M}$-finite uniformly in ε and e.

(iii) $\{x : h(x, \psi_e(x)) < \Phi_{s(e)}(x)\}$ is $\underset{\sim}{M}$-finite uniformly in e.

Proof. Let s be the $\underset{\sim}{M}$-recursive function mentioned in the remark following theorem 2.1. Then (i) and (ii) hold for each $e \in M$.

We define a set H, the graph of h, in stages σ as follows: Define

$$k(e,x,\beta) = \begin{cases} \Phi_{s(e)}(x) & \text{if } \Phi_{s(e)}(x) \geq \psi_e(x) = \beta \\ 0 & \text{otherwise} \end{cases}$$

and let $l(x,\beta,\sigma) = \text{some } \tau[(\forall e \in L^{\sigma})(k(e,x,\beta) < \tau)]$. Put

$H^\sigma = H^{<\sigma} \cup \{<x,\beta,l(x,\beta,\sigma)> : <x,\beta> \in L^\sigma \times \sigma - L^{<\sigma} \times L^{<\sigma}\}$ and let $H = \cup\{H^\sigma : \sigma < |\lesssim|\}$. Then h, the function with graph H, is $\underset{\sim}{M}$-recursive, single-valued and (can easily be extended to be) total.

To prove (iii) we fix e and choose σ_0 such that $e \in L^{\sigma_0}$. Suppose $h(x,\psi_e(x)) < \Phi_{s(e)}(x)$. Let τ be the least ordinal such that $<x,\psi_e(x)> \in L^\tau \times \tau$. Suppose $\tau \geq \sigma_0$. Then $\Phi_{s(e)}(x) > h(x,\psi_e(x)) = l(x,\psi_e(x),\tau) > k(e,x,\psi_e(x))$. It follows that $\Phi_{s(e)}(x) < \psi_e(x)$. By (ii) the set $K_e = \{x : \Phi_{s(e)}(x) < \psi_e(x)\}$ is $\underset{\sim}{M}$-finite uniformly in e. But then

$$\{x : h(x,\psi_e(x)) < \Phi_{s(e)}(x)\} = \{x \in K_e : h(x,\psi_e(x)) < \Phi_{s(e)}(x)\}$$
$$\cup \{x \in L^{\sigma_0} : (\exists \beta < \sigma_0)(\psi_e(x) = \beta \;\&\; h(x,\beta) < \Phi_{s(e)}(x))\},$$ which is $\underset{\sim}{M}$-finite uniformly in e. □

Note that theorem 2.2 has the following converse: Suppose $\Psi = \{\psi_e : e \in M\}$ is an $\underset{\sim}{M}$-recursive enumeration of $\underset{\sim}{M}$-recursive functions with domain a subset of D and range a subset of $|\lesssim|$. If the conclusions of the theorem hold then Ψ is a D-measured set.

To see this let $K_e = \{x : \Phi_{s(e)}(x) < \psi_e(x) \vee h(x,\psi_e(x)) < \Phi_{s(e)}(x)\}$ which is $\underset{\sim}{M}$-finite uniformly in e by our hypotheses. If $x \in K_e$ then $\psi_e(x)$ is defined so $\psi_e(x) = \delta$ can be decided $\underset{\sim}{M}$-recursively. If $x \notin K_e$ then $\psi_e(x) = \delta \Leftrightarrow \Phi_{s(e)}(x) \leq h(x,\delta) \;\&\; \psi_e(x) = \delta$ which again is $\underset{\sim}{M}$-recursively decidable.

3. $\underset{\sim}{M}$ weakly inadmissible

Let $\underset{\sim}{M}$ denote a weakly inadmissible structure. In [9] we associated to each such structure a transitive resolvable admissible structure $\mathfrak{A} = <N,\in,T>$, the admissible collapse of $\underset{\sim}{M}$, such that for each $A \subseteq N$, A is $\mathfrak{A}$-r.e. if and only if A is $\underset{\sim}{M}$-r.e.

Furthermore N has the following properties:

(i) There is a surjective $\underset{\sim}{M}$-recursive function $q:N \to M$.

(ii) If $x \in N$ and $A \subseteq M$ is $\underset{\sim}{M}$-recursive, then $x \cap A \in N$.

(iii) If $f:N \to M$ is $\underset{\sim}{M}$-recursive, then $g:N \to M$ defined by $g(x) = f''x$ is $\underset{\sim}{M}$-recursive (and in particular $g(x) \in M$).

The admissible collapse $\mathfrak{A}$ of a weakly inadmissible structure $\underset{\sim}{M}$ was used in [9] to reduce questions about the structure of the regular tamely r.e. $\underset{\sim}{M}$-degrees to questions about the structure of the regular $\mathfrak{A}$-r.e. degrees. Here we make a similar reduction by showing that Blum's theorem and the Compression theorem hold for $\underset{\sim}{M}$ since theorems 2.1 and 2.2 hold for $\mathfrak{A}$.

<u>Lemma 3.1</u> Let $\underset{\sim}{M}$ be weakly inadmissible and $\mathfrak{A} = \langle N, \in, T \rangle$ its admissible collapse. Then there is $\underset{\sim}{M}$-recursive (and hence $\mathfrak{A}$-recursive) $D \subseteq N$ and an $\underset{\sim}{M}$-recursive bijection $p:D \leftrightarrow M$.

<u>Proof</u>. Let $\lambda \sigma G^\sigma$ be an $\mathfrak{A}$-recursive resolution of N, i.e. $\lambda \sigma G^\sigma$ is an $\mathfrak{A}$-recursive function, $\sigma < \tau \Rightarrow G^\sigma \subseteq G^\tau$ and $N = \cup\{G^\sigma : \sigma < \kappa\}$ where $\kappa = \Sigma_1\text{-cf}(|\underset{\sim}{\mathfrak{A}}|) = {}_0(N)$. Let $q:N \to M$ be an $\underset{\sim}{M}$-recursive surjection and let $\lambda \sigma q^\sigma$ $(\sigma < \kappa)$ be an $\underset{\sim}{M}$-recursive approximation of (the graph of) q. Define $k(x) = \text{least } \sigma[(\exists y \in G^\sigma)(\langle y,x \rangle \in q^\sigma)]$ and put $r(x) = v \iff v = G^{k(x)} \cap \{y : \langle y,x \rangle \in q^{k(x)}\}$. Then $r:M \to N$ is a total $\underset{\sim}{M}$-recursive function such that $\forall x(r(x) \neq \emptyset)$ and $x_1 \neq x_2 \Rightarrow r(x_1) \cap r(x_2) = \emptyset$. Let $\lambda \sigma r^\sigma$ $(\sigma < \kappa)$ be an $\underset{\sim}{M}$-recursive approximation of r and let $D = \{\langle \sigma, v \rangle : v \in \pi_1''(r^\sigma - r^{<\sigma})\}$. Then $D \subseteq N$ is $\underset{\sim}{M}$-recursive. Define $p:D \to M$ by $p(\langle \sigma, v \rangle) =$ some $x[r(x) = v]$. It is easily verified that p is an $\underset{\sim}{M}$-recursive bijection. $\square$

Assume Φ is a complexity measure for a weakly inadmissible structure $\underset{\sim}{M}$. Let $\mathfrak{A}$ be the admissible collapse of $\underset{\sim}{M}$ and let $p:D \to M$ be as in lemma 3.1. We define a D-complexity measure Φ' for $\mathfrak{A}$ as follows:

$$\phi_e'(x) = y \iff \phi_{p(e)}(p(x)) = y \quad \& \quad y \in N .$$

$$\Phi_e'(x) = \delta \iff \Phi_{p(e)}(p(x)) = \delta .$$

Lemma 3.2 Φ' is a D-complexity measure for $\mathfrak{A}$.

Proof. As a sample we show that the s-m-n theorem holds. Suppose $m(e,x)$ is a partial $\mathfrak{A}$-recursive function. Define $l(e,z) = y \iff m(e,p^{-1}(z)) = y$. Then l is $\underset{\sim}{M}$-recursive so $l(e,z) = \phi_{s(e)}(z)$ for some $\underset{\sim}{M}$-recursive function s. Thus $m(e,x) = \phi_{s(e)}(p(x)) = \phi'_{p^{-1}(s(e))}(x)$ whenever $x \in D$. But $t(e) = p^{-1}(s(e))$ is $\underset{\sim}{M}$-recursive and hence $\mathfrak{A}$-recursive, so the s-m-n theorem holds for $\mathfrak{A}$-recursive enumerations of functions with domain a subset of D. □

Proof of Blum's theorem. It remains to prove the theorem for weakly inadmissible $\underset{\sim}{M}$. Let Φ be a complexity measure for $\underset{\sim}{M}$ and let g be a partial $\underset{\sim}{M}$-recursive function such that $\operatorname{ran} g \subseteq \kappa$. Using the notation above, define $g'(x) = \delta \iff g(p(x)) = \delta$. Then g' is partial $\mathfrak{A}$-recursive and $\operatorname{dom} g' \subseteq D$. Φ' is a D-complexity measure for $\mathfrak{A}$ by lemma 3.2, so by theorem 2.1 there is a partial $\mathfrak{A}$-recursive function f' such that $\operatorname{dom} f' = \operatorname{dom} g'$ and if $f' = \phi_e'$ then $\{x : \Phi_e'(x) < g'(x)\}$ is $\mathfrak{A}$-finite. Let $f(x) = f'(p^{-1}(x))$. Then $\operatorname{dom} f = \operatorname{dom} g$. Furthermore if $f = \phi_e$ then $f' = \phi'_{p^{-1}(e)}$ so $\{x : \Phi_e(x) < g(x)\} = p''\{x : \Phi'_{p^{-1}(e)}(x) < g'(x)\}$ is $\underset{\sim}{M}$-finite. The uniformity is easily checked. □

Proof of the Compression theorem. Again we assume $\underset{\sim}{M}$ is weakly inadmissible. Suppose Ψ is a measured set for $\underset{\sim}{M}$. Define $\Psi' = \{\psi'_e : e \in N\}$ by $\psi'_e(x) = \delta \iff e, x \in D$ & $\psi_{p(e)}(p(x)) = \delta$. Then Ψ' is a D-measured set for $\mathfrak{A}$. Let h' and s' be the functions obtained from theorem 2.2. Define $h(x,\delta) = h'(p^{-1}(x),\delta)$ and $s(e) = p(s'(p^{-1}(e)))$. To prove (iii) note that

$\{x : h(x,\psi_e(x)) < \Phi_{s(e)}(x)\} = p''\{x : h'(x,\psi'_{p^{-1}(e)}(x)) < \Phi'_{p^{-1}(s(e))}(x)\}$

$= p''\{x : h'(x,\psi'_{p^{-1}(e)}(x)) < \Phi'_{s'(p^{-1}(e))}(x)\}$. But

$\{x : h'(x,\psi'_{p^{-1}(e)}(x)) < \Phi'_{s'(p^{-1}(e))}(x)\}$ is $\mathfrak{A}$-finite by theorem 2.2 so $\{x : h(x,\psi_e(x)) < \Phi_{s(e)}(x)\}$ is $\underset{\sim}{M}$-finite. (i) and (ii) are proved similarly. □

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