Automorphisms and equivalence in von Neumann algebras

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1. Introduction. In his studies in ergodic theory Hopf [3] introduced an equivalence relation, which in the language of von Neumann algebras, is an equivalence relation on the projections in an abelian von Neumann algebra acted upon by a group of *-automorphisms. He then showed that, with some extra assumptions, "finiteness" of the partial ordering defined by the equivalence was equivalent to the existence of an invariant normal state. Later on the "semi-finite" case was taken care of by Kawada [6] in a well ignored paper, and then independently by Halmos [2]. In the theory of von Neumann algebras, Murray and von Neumann introduced their celebrated equivalence relation on the projections in [8] and again showed (at least for factors) the equivalence of finiteness (resp. semi-finiteness) and the existence of normal finite (resp. semi-finite) traces. It is the purpose of the present paper to introduce and study an equivalence relation which includes in the countably decomposable case the one by Hopf and in the general case the one by Murray and von Neumann. It is defined as follows. Let $\mathcal{R}$ be a von Neumann algebra acting on a

Hilbert space $\mathcal{H}$. Let G be a group and let $t \to U_t$ be a unitary representation of G on $\mathcal{H}$ such that $U_t^* \mathcal{R} U_t = \mathcal{R}$ for all $t \in G$. Then we say two projections E and F in $\mathcal{R}$ are G-equivalent if there is for each $t \in G$ an operator $T_t \in \mathcal{R}$ such that $E = \sum_{t \in G} T_t T_t^*$, $F = \sum_{t \in G} U_t^* T_t^* T_t U_t$.

Our main results now state that this relation is indeed an equivalence relation (Thm.1), that "semi-finiteness" is equivalent to the existence of a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$ (Thm.2), and that "finiteness" together with countable decomposability of $\mathcal{R}$ is equivalent to the existence of a faithful normal finite G-invariant trace on $\mathcal{R}$ (Thm.3), In the proofs we shall not follow the apparently natural approach of developing a comparison theory for the projections in $\mathcal{R}$ and then to construct the traces. We shall instead consider the cross product $\mathcal{R} \times G$, and then show that the canonical imbedding of $\mathcal{R}$ into the von Neumann algebra $\mathcal{R} \times G$ is close to being an isomorphism of $\mathcal{R}$ with the structure of G-equivalence into $\mathcal{R} \times G$ with the equivalence relation of Murray and von Neumann. In the last two sections of the paper we shall study the relation of G-equivalence to G-finite von Neumann algebras, and to the equivalence relation of Hopf.

We refer the reader to the book of Dixmier [1] for the theory of von Neumann algebras.

2. Statements of results.

In the present section we state the main results and definitions. The proofs will be given in section 3.

Theorem 1. Let $\mathcal{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let G be a group and $t \to U_t$ a unitary representation of G on $\mathcal{H}$ such that $U_t^* \mathcal{R} U_t = \mathcal{R}$ for all $t \in G$. If E and F are projections in $\mathcal{R}$ we write $E \underset{G}{\sim} F$ if for each $t \in G$ there is an operator $T_t \in \mathcal{R}$ such that

$$E = \sum_{t \in G} T_t T_t^* , \quad F = \sum_{t \in G} U_t^* T_t^* T_t U_t .$$

Then $\underset{G}{\sim}$ is an equivalence relation on the projections in $\mathcal{R}$.

Remark 1. If G is the one element group then the equivalence relation $\underset{G}{\sim}$ is the same as the usual equivalence relation $\sim$ for projections in a von Neumann algebra.

Remark 2. If G is the additive group of $\mathcal{R}$ and the representation $t \to U_t$ is the trivial representation, so $U_t = I$ for $t \in G$, then the equivalence relation $\underset{G}{\sim}$ is the one defined by Kadison and Pedersen [4, Def.A].

Remark 3. If $\mathcal{R}$ is abelian and countably decomposable the equivalence relation $\underset{G}{\sim}$ coincides with the one defined by Hopf [3] in ergodic theory. For this see Theorem 5.

Remark 4. If E and F are equivalent projections in $\mathcal{R}$, i.e. there is a partial isometry $V \in \mathcal{R}$ such that $E = VV^*$, $F = V^*V$, then $E \underset{G}{\sim} F$. This is clear from the definition of $\underset{G}{\sim}$, putting $T_e = V$, $T_t = 0$ for $t \neq e$.

Definition 1. With notation as in Theorem 1 we say two projections E and F in $\mathcal{R}$ are G-equivalent if $E \underset{G}{\sim} F$. We write $E \underset{G}{\prec} F$ if $E \underset{G}{\sim} F_0 \leq F$. A projection F is said to be $\underset{G}{\sim}$-finite if $E \leq F$ and $E \underset{G}{\sim} F$ implies $E = F$. $\mathcal{R}$ is said to be $\underset{G}{\sim}$-finite if the identity operator I is $\underset{G}{\sim}$-finite. $\mathcal{R}$ is said to be $\underset{G}{\sim}$-semi-finite if every non-zero projection in $\mathcal{R}$ majorizes a non-zero $\underset{G}{\sim}$-finite projection.

Theorem 2. With notation as in Theorem 1 there exists a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$ if and only if $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite.

Theorem 3. With notation as in Theorem 1 there exists a faithful finite G-invariant trace on $\mathcal{R}$ if and only if $\mathcal{R}$ is $\underset{G}{\sim}$-finite and countably decomposable.

3. Proofs. We first introduce some notation and follow [1,ChI, § 9] closely. Following the notation in Theorem 1 $\mathcal{R}$ acts on a Hilbert space $\mathcal{H}$, G is a group, considered as a discrete group, and $t \to U_t$ is a unitary representation of G on $\mathcal{H}$ such that $U_t^* \mathcal{R} U_t = \mathcal{R}$ for all $t \in G$. For $t \in G$ let $\mathcal{H}_t$ be a Hilbert space of the same dimension as $\mathcal{H}$ and J_t an isometry of $\mathcal{H}$ onto $\mathcal{H}_t$. Let $\tilde{\mathcal{H}} = \sum_{t \in G} \oplus \mathcal{H}_t$. We write an operator $R \in \mathcal{B}(\tilde{\mathcal{H}})$ - the bounded operators on $\tilde{\mathcal{H}}$ - as a matrix $(R_{s,t})_{s,t \in G}$, where $R_{s,t} = J_s^* R J_t \in \mathcal{B}(\mathcal{H})$. For each $T \in \mathcal{R}$ let $\Phi(T)$ denote the element in $\mathcal{B}(\tilde{\mathcal{H}})$ with matrix $(R_{s,t})$, where $R_{s,t} = 0$ if $s \neq t$, and $R_{s,s} = T$ for all $s \in G$. Then Φ is a *-isomorphism of $\mathcal{R}$ onto a von Neumann subalgebra $\tilde{\mathcal{R}}$ of $\mathcal{B}(\tilde{\mathcal{H}})$. For $y \in G$ let $\tilde{U}_y$ be the operator in $\mathcal{B}(\tilde{\mathcal{H}})$ with matrix $(R_{s,t})$, where $R_{s,t} = 0$ if $st^{-1} \neq y$, $R_{yt,t} = U_y$ for all $t \in G$. Then (see [1,Ch.I, § 9]) $y \to \tilde{U}_y$ is a unitary representation of G on $\tilde{\mathcal{H}}$ such that

$$\tilde{U}_y^* \Phi(T) \tilde{U}_y = \Phi(U_y^* T U_y), \qquad y \in G, \quad T \in \mathcal{R}.$$

If $\mathcal{B}$ denotes the von Neumann algebra generated by $\tilde{\mathcal{R}}$ and the $\tilde{U}_y, y \in G$, then each operator in $\mathcal{B}$ is represented by a matrix $(R_{s,t})$ where $R_{s,t} = T_{st^{-1}} U_{st^{-1}}$, $T_{st^{-1}} \in \mathcal{R}$.

We denote by $\mathcal{R}^G$ the von Neumann subalgebra of $\mathcal{R}$ consisting of the G-invariant operators in $\mathcal{R}$. $\mathcal{C}$ shall denote the center of $\mathcal{R}$, and $\mathcal{D}$ shall denote $\mathcal{C} \cap \mathcal{R}^G$. Whenever we write $P \sim Q$ for two projections in $\mathcal{B}$ we shall mean they are equivalent as operators in $\mathcal{B}$, i.e. there is a partial isometry $V \in \mathcal{B}$ such that $VV^* = P$, $V^*V = Q$, and we shall not consider P and Q as eqiuvalent in a von Neumann subalgebra of $\mathcal{B}$. The next lemma includes Theorem 1 and shows more, namely that $\underset{G}{\sim}$-equivalence is the same as equivalence in $\mathcal{B}$.

<u>Lemma 1</u>. Let E and F be projections in $\mathcal{R}$. Then $E \underset{G}{\sim} F$ if and only if $\Phi(E) \sim \Phi(F)$. Hence $\underset{G}{\sim}$ is an equivalence relation on the projections in $\mathcal{R}$.

<u>Proof:</u> Suppose $E \underset{G}{\sim} F$. Then for each $t \in G$ there is $T_t \in \mathcal{R}$ such that

$$E = \sum_{t \in G} T_t T_t^* , \quad F = \sum_{t \in G} U_t^* T_t^* T_t U_t .$$

Then we have

$$\begin{aligned}\Phi(E) &= \Sigma\, \Phi(T_t T_t^*) = \Sigma\, \Phi(T_t)\Phi(T_t)^* \\ &= \Sigma\, (\Phi(T_t)\tilde{U}_t)(\Phi(T_t)\tilde{U}_t)^* ,\end{aligned}$$

and

$$\begin{aligned}\Phi(F) &= \Sigma\, \Phi(U_t^* T_t^* T_t U_t) = \Sigma\, \tilde{U}_t^* \Phi(T_t^* T_t)\tilde{U}_t \\ &= \Sigma\, (\Phi(T_t)\tilde{U}_t)^*(\Phi(T_t)\tilde{U}_t) .\end{aligned}$$

Thus by a result of Kadison and Pedersen [4,Thm.4.1] $\Phi(E) \sim \Phi(F)$.

Conversely assume $\Phi(E) \sim \Phi(F)$. Then there is a partial isometry $V \in \mathcal{B}$ such that $VV^* = \Phi(E)$, $V^*V = \Phi(F)$. Say $V = (T_{st^{-1}}U_{st^{-1}})$. Then an easy calculation shows

$$E = \sum_{t\in G} T_t T_t^* \ , \quad F = \sum_{t\in G} U_t^* T_t^* T_t U_t \ ,$$

hence $E \underset{G}{\sim} F$. The proof is complete.

<u>Lemma 2</u>. Let $S = (T_{st^{-1}}U_{st^{-1}})$ belong to the center of $\mathcal{B}$. Then for each $s \in G$ we have

i) $TT_s = T_s U_s T U_s^*$ for all $T \in \mathcal{R}$,

ii) $T_{sy} = U_y^* T_{ys} U_y$ for all $y \in G$.

In particular $T_e \in \mathcal{D}$. Furthermore, if $R \in \mathcal{D}$ then $\Phi(R)$ belongs to the center of $\mathcal{B}$.

<u>Proof</u>. Let $T \in \mathcal{R}$. Then

$(TT_{st^{-1}}U_{st^{-1}}) = \Phi(T)S = S\Phi(T) = (T_{st^{-1}}U_{st^{-1}}\, T U_{ts^{-1}}\, U_{st^{-1}})$

and i) follows. Let $y \in G$. Then an easy computation shows

$$(T_{st^{-1}y^{-1}}U_{st^{-1}}) = S\tilde{U}_y = \tilde{U}_y S = (U_y T_{y^{-1}st^{-1}} U_y^* U_{st^{-1}}) \ .$$

Replacing y by y^{-1} and letting $t = e$, ii) follows. By i) $T_e T = TT_e$, so $T_e \in \mathcal{E}$. By ii) if $s = y^{-1}$ we find $T_e = U_y^* T_e U_y$, so $T_e \in \mathcal{R}^G$, hence $T_e \in \mathcal{D}$.

Finally let $R \in \mathcal{D}$, and let $S' = (S_{st^{-1}}U_{st^{-1}}) \in \mathcal{B}$. Then we have

$$\Phi(R)S' = (RS_{st^{-1}}U_{st^{-1}}) = (S_{st^{-1}}RU_{st^{-1}})$$
$$= (S_{st^{-1}}U_{st^{-1}}R) = S'\Phi(R) \ ,$$

hence $\Phi(R)$ belongs to the center of $\mathcal{B}$. The proof is complete.

Lemma 3. Let E be a projection in $\mathcal{R}$. Let D_E be the smallest operator in $\mathcal{D}$ majorizing E. Then D_E is a projection, and $\Phi(D_E)$ is the central carrier of $\Phi(E)$ in $\mathcal{B}$.

Proof. Since $\mathcal{D}$ is an abelian von Neumann algebra its positive operators form a complete lattice under infs and sups. Thus $D_E = \text{g.l.b.}\{A \in \mathcal{D} : E \leq A \leq I\}$, and D_E is well defined. Since $E \leq D_E$ and both operators commute we have $E = E^2 \leq D_E^2$. But $D_E \leq I$, so $D_E^2 \leq D_E$. Hence by minimality of D_E, $D_E = D_E^2$, so it is a projection. By Lemma 2 $\Phi(D_E)$ is a central projection in $\mathcal{B}$, hence if $C_{\Phi(E)}$ denotes the central carrier of $\Phi(E)$ in $\mathcal{B}$, then $\Phi(D_E) \geq C_{\Phi(E)}$. Now let $C_{\Phi(E)} = (T_{st^{-1}}U_{st^{-1}})$. By Lemma 2 $T_e \in \mathcal{D}$, and since $C_{\Phi(E)} \geq \Phi(E)$, $T_e \geq E$. By definition of D_E $T_e \geq D_E$. But $\Phi(D_E) \geq C_{\Phi(E)}$, so $D_E \geq T_e$, hence $T_e = D_E$. The operator $\Phi(D_E) - C_{\Phi(E)}$ is positive and has zeros on the main diagonal. Therefore it is 0, and $\Phi(D_E) = C_{\Phi(E)}$ as asserted.

Lemma 4. Let E be a projection in $\mathcal{R}$. Let C_E be its central carrier in $\mathcal{R}$, and let D_E be as in Lemma 4. Then $D_E = D_{C_E}$.

Proof. Since $E \leq C_E$, $D_E \leq D_{C_E}$. But $D_E \in \mathcal{C}$ and $D_E \geq E$, hence $D_E \geq C_E$. Therefore by definition of D_{C_E}, $D_E \geq D_{C_E}$, and they are equal.

Lemma 5. Let E be a countably decomposable projection in $\mathcal{R}$. Then $\Phi(E)$ is countably decomposable in $\mathcal{B}$.

Proof. Let x be a vector in $E\mathcal{H}$. Then x considered as a vector in $\sum_{t \in G} \oplus \mathcal{H}_t$ belongs to $\mathcal{H}_e$. Let F be the support of ω_x in $E\mathcal{R}E$. Then F is countably decomposable, and ω_x is a faithful normal state of $F\mathcal{R}F$. Let $\{F_\alpha\}_{\alpha\in J}$ be an orthogonal family of projections in $\mathcal{B}$ such that $\sum_{\alpha\in J} F_\alpha = \Phi(F)$. Let $F_\alpha = (T^\alpha_{st^{-1}} U_{st^{-1}})$. Then $F_\alpha \leq \Phi(F)$, so $T^\alpha_e \leq F$, hence $T^\alpha_e \in F\mathcal{R}F$. Furthermore, since $x \in \mathcal{H}_e$, $\omega_x(T_\alpha)=\omega_x(T^\alpha_e)$. Thus we have

$$1 = \omega_x(F) = \omega_x(\Phi(F)) = \Sigma\,\omega_x(F_\alpha) = \Sigma\,\omega_x(T^\alpha_e) .$$

Therefore $\omega_x(T^\alpha_e) = 0$ except for a countable number of $\alpha \in J$. But then $T^\alpha_e = 0$ and hence $F_\alpha = 0$ except for a countable number of $\alpha \in J$. Thus $\Phi(F)$ is countably decomposable in $\mathcal{B}$. Now E is a countable sum of orthogonal cyclic projections, hence $\Phi(E)$ is a countable sum of orthogonal countably decomposable projections. Hence $\Phi(E)$ is countably decomposable.
The proof is complete.

<u>Definition 2</u>. We say a projection E in $\mathcal{R}$ is $\underset{G}{\sim}$-<u>abelian</u> if $E\mathcal{R}E = E\mathcal{D}$.

Clearly a $\underset{G}{\sim}$-abelian projection is abelian.

<u>Lemma 6</u>. There is a projection $P \in \mathcal{D}$ such that there exists a $\underset{G}{\sim}$-abelian projection $E \leq P$ with $D_E = P$, and $I - P$ has no non-zero $\underset{G}{\sim}$-abelian subprojection.

<u>Proof</u>. Partially order the $\underset{G}{\sim}$-abelian projections in $\mathcal{R}$ by $E << F$ if $E \leq F$ and $D_{F-E} \leq I - D_E$. Then in particular $D_E F = E$. Let $\{E_\alpha\}$ be a totally ordered set of $\underset{G}{\sim}$-abelian projections, and let $E = \sup E_\alpha$, so $E_\alpha \to E$ strongly. Then

$$D_{E_\alpha} E = D_{E_\alpha} \lim_{\beta>\alpha} E_\beta = \lim_{\beta>\alpha} D_{E_\alpha} E_\beta = E_\alpha ,$$

hence if $A \in \mathcal{R}$ then

$$E A E D_{E_\alpha} = E_\alpha A E_\alpha = A_\alpha E_\alpha ,$$

where $A_\alpha \in \mathcal{D} D_{E_\alpha}$. Now it is well known that if Q_α is an increasing net of projections, and $Q_\alpha \to Q$ strongly, then $C_{Q_\alpha} \to C_Q$ strongly. Thus

$$\Phi(D_{E_\alpha}) = C_{\Phi(E_\alpha)} \to C_{\Phi(E)} = \Phi(D_E)$$

by Lemma 3, hence $D_{E_\alpha} \to D_E$ strongly. The same argument also shows

$$D_{E-E_\alpha} = \lim_{\beta>\alpha} D_{E_\beta - E_\alpha} \leq I - D_{E_\alpha} .$$

Thus $E = E(I - D_{E_\alpha}) + E_\alpha$, and since $A_\alpha = A_\alpha D_{E_\alpha}$ we have $E A E D_{E_\alpha} = A_\alpha E \in E\mathcal{D}$. Since $D_{E_\alpha} \to D_E$ it follows that $E A E = \lim_\alpha E A E D_{E_\alpha} \in E \mathcal{D}$. Therefore E is $\underset{G}{\sim}$-abelian. Now let E be a maximal $\underset{G}{\sim}$-abelian projection in $\mathcal{R}$. Let $P = D_E$. Suppose F is a $\underset{G}{\sim}$-abelian subprojection of $I - P$. Then $E + F$ is $\underset{G}{\sim}$-abelian. Indeed, if $A \in \mathcal{R}$ then there are $A_E \in D_E\mathcal{D}$ and $A_F \in D_F\mathcal{D}$ such that

$$\begin{aligned}(E+F)A(E+F) &= E A E + F A F = E A_E + F A_F \\ &= (E+F)(A_E + A_F) \in (E+F)\mathcal{D} .\end{aligned}$$

Thus $E + F$ is $\underset{G}{\sim}$- abelian. Since $E << E + F$, the maximality of E implies $F = 0$. The proof is complete.

Thus in order to prove theorems 2 and 3 we may consider two cases separately, namely the case when $\mathcal{R}$ has a $\underset{G}{\sim}$-abelian projection E with $D_E = I$, and the case when $\mathcal{R}$ has no non-zero $\underset{G}{\sim}$- abelian projection. We first treat the case with a $\underset{G}{\sim}$-abelian projection.

<u>Lemma 7</u>. Let E be a $\underset{G}{\sim}$-abelian projection in $\mathcal{R}$. Then C_E is not G-equivalent to a proper central projection. Furthermore

if Q is a central projection such that $Q \leq C_E$ then $Q = D_Q C_E$.

Proof. Let Q be as in the statement of the lemma. Since E is $\underset{G}{\sim}$-abelian there is an operator $D \in \mathcal{D}$ such that $QE = DE$, hence, since $E\mathcal{C} \cong C_E\mathcal{C}$, $Q = QC_E = DC_E$, and $D \geq Q$. By definition of D_Q, $D \geq D_Q$. But $D_Q \geq Q$, so $Q = QC_E \leq D_Q C_E \leq DC_E = Q$, so that $Q = D_Q C_E$. Now suppose P is a projection in $\mathcal{C}$ such that $P \leq C_E$ and $P \underset{G}{\sim} C_E$. Then in particular by Lemma 1 $\Phi(P) \sim \Phi(C_E)$, so they have the same central carrier in $\mathcal{B}$, hence $D_P = D_{C_E} = D_E$ by Lemma 4. By the preceding, $P = D_P C_E = C_E$. The proof is complete.

Lemma 8. Let E be a $\underset{G}{\sim}$-abelian projection in $\mathcal{R}$. Let $s \in G$ and let Q be a central projection orthogonal to C_E. Then if C_E and $C_E + Q$ are G-equivalent relative to $\mathcal{C}$, i.e. the operators T_t defining the equivalence belong to $\mathcal{C}$, then $Q = 0$.

Proof. Let $P = C_E$ and assume $P \underset{G}{\sim} P + Q$ relative to $\mathcal{C}$. Then since $\mathcal{C}$ is abelian, for each $t \in G$ there is $A_t \in \mathcal{C}^+$ such that $P = \sum_{t \in G} A_t$, $P + Q = \sum_{t \in G} U_t^* A_t U_t$. Since $E\mathcal{C} = E\mathcal{D}$ and $P\mathcal{C} \cong E\mathcal{C}$, we have $P\mathcal{C} = P\mathcal{D}$. Since $A_t \leq P$ there is $D_t \in \mathcal{D}^+$ such that $A_t = PD_t$. Thus we have

$$\Sigma PD_t = P = P(P+Q) = \Sigma PU_t^* A_t U_t$$

$$= \Sigma PU_t^* PD_t U_t = \Sigma PD_t U_t^* PU_t .$$

Now $PD_t U_t^* PU_t \leq PD_t$ for all t, hence we have $PD_t U_t^* PU_t = PD_t$ for all t. Let E_t denote the range projection of D_t. Then $E_t \in \mathcal{D}$. Since $U_t^* PU_t PD_t = PD_t$, $U_t^* P U_t PE_t = PE_t$. Thus $U_t^* PU_t \geq PE_t$, and thus $U_t^* PE_t U_t = U_t^* PU_t E_t \geq PE_t$. Consequently $PE_t \geq U_t PE_t U_t^*$. By Lemma 7 C_E is $\underset{G}{\sim}$-finite relative to $\mathcal{C}$, hence so is PE_t.

Therefore $PE_t = U_t PE_t U_t^*$, and $U_t^* PE_t U_t = PE_t$. Therefore we have

$$U_t^* A_t U_t = U_t^* PD_t U_t = U_t^* PE_t U_t D_t = PE_t D_t = PD_t = A_t ,$$

and $P = P + Q$, so that $Q = 0$. The proof is complete.

Lemma 9. Suppose E is a $\underset{G}{\sim}$-abelian projection in $\mathcal{R}$ with $D_E = I$. Then $\mathcal{R}$ is of type I , and there exists a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$.

Proof. Since E is abelian $C_E\mathcal{R}$ is of type I . Since every *-automorphism of $\mathcal{R}$ preserves the type I portion of $\mathcal{R}$, and $D_E = I$, $\mathcal{R}$ is of type I .

E is a sum of orthogonal cyclic projections E_α . If we can show the lemma for each E_α then it holds for E . Therefore we may assume E is cyclic, say $E = [\mathcal{R}'x]$. Then ω_x is faithful on $E\mathcal{R}E$, hence faithful on $E\mathcal{C}$. If $A \geq 0$ belongs to $C_E\mathcal{C}$ and $\omega_x(A) = 0$, then $0 = \omega_x(EA)$, so $EA = 0$. Hence $A = AC_E = 0$. Thus ω_x is faithful on $C_E\mathcal{C}$, so C_E is a countably decomposable projection in $\mathcal{C}$.

We shall now apply the previous theory to $\mathfrak{A} = \mathcal{C} \times G$ instead of $\mathcal{B} = \mathcal{R} \times G$. We use the same notation as before. By Lemma 7 C_E is $\underset{G}{\sim}$-finite. If $C_E = D_E = I$ then by Lemma 7 $\mathcal{C} = \mathcal{D}$, and it is trivial that there exists a faithful normal semi-finite G-invariant trace on $\mathcal{C}^+$. Assume $C_E \neq I$. Then there is $s \in G$ such that $U_s^* C_E U_s \neq C_E$. Since by Lemma 7 C_E is $\underset{G}{\sim}$-finite, and $U_s^* C_E U_s \underset{G}{\sim} C_E$, $U_s^* C_E U_s$ is not a subprojection of C_E . Thus $Q = U_s^* C_E U_s (I - C_E) \neq 0$. Since C_E is countably decomposable, so is Q , and hence $C_E + Q$. By Lemma 5 $\Phi(C_E + Q)$ is countably decomposable in $\mathfrak{A}$. Since $I = D_E \leq D_{C_E+Q}$, the central carriers of

$\Phi(C_E)$ and $\Phi(C_E+Q)$ are by Lemma 3 equal to I. If $\Phi(C_E)$ is properly infinite then by [1,Ch.III,§8,Cor.5] $\Phi(C_E) \sim \Phi(C_E+Q)$, so by Lemma 1 $C_E \underset{G}{\sim} C_E+Q$, contradicting Lemma 8. Thus $\Phi(C_E)$ is not properly infinite, and there is a non-zero central projection P in $\mathfrak{A}$ such that $P\Phi(C_E)$ is non-zero and finite. Since the central carrier of $\Phi(C_E)$ is I, $P\mathfrak{A}$ is semi-finite. Let φ be a normal semi-finite trace on $\mathfrak{A}^+$ with support P. For $A \in \mathcal{B}^+$ define $\tau(A) = \varphi(\Phi(A))$. Then τ is a normal G-invariant trace because $\tau(U_s^*AU_s) = \varphi(\tilde{U}_s^*\Phi(A)\tilde{U}_s) = \varphi(\Phi(A)) = \tau(A)$. Since $\tau(C_E) < \infty$ and $D_{C_E} = I$, τ is semi-finite, hence τ is a normal semi-finite G-invariant trace on $\mathcal{B}^+$. Let D be the support of τ. Then $0 \neq D \in \mathfrak{D}$. Now apply the preceding to $(I-D)\mathcal{B}$ and $E(I-D)$, and use Zorn's lemma to obtain a family D_α of orthogonal projections in $\mathfrak{D}$ with sum I, and a normal semi-finite G-invariant trace τ_α of $\mathcal{B}^+$ with support D_α. Let $\tau = \Sigma\tau_\alpha$. Then τ is a faithful normal semi-finite G-invariant trace on $\mathcal{B}^+$.

Now since $\mathcal{R}$ is of type I there is a faithful normal center valued trace ψ on $\mathcal{R}^+$ such that $U_s^*\psi(U_sAU_s^*)U_s = \psi(A)$ for each $s \in G$, $A \in \mathcal{R}^+$, see [11,p.3]. Then $\tau\circ\psi$ is a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$, see [1,Ch.III, §4,Prop.2]. The proof is complete.

<u>Lemma 10</u>. Suppose $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite and there are no non-zero $\underset{G}{\sim}$-abelian projections in $\mathcal{R}$. Then there is a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$.

<u>Proof</u>. Let E be a non-zero countably decomposable $\underset{G}{\sim}$-finite projection in $\mathcal{R}$. Since E is not $\underset{G}{\sim}$-abelian there is a projection $H \in E\mathcal{R}E$ such that $H \neq ED_H$. Let $F = H+(I-D_H)E$. Th

Then $F \leq E$, $F \neq E$, and $D_F = D_H + (I - D_H)D_E = D_E$. $\Phi(F)$ is not properly infinite in $\mathcal{B}$. Indeed, if it were, then since $\Phi(E)$ is countably decomposable by Lemma 5, [1,Ch.III,§8,Cor.5] would imply $\Phi(F) \sim \Phi(E)$, hence by Lemma 1 , $F \underset{G}{\sim} E$, contradicting the $\underset{G}{\sim}$-finiteness of E . Therefore there is a non-zero central projection P in $\mathcal{B}$ such that $P\Phi(F)$ is finite and non-zero. Thus $P\Phi(D_E)\mathcal{B} = P\Phi(D_F)\mathcal{B}$ is semi-finite and non-zero. Let φ be a normal semi-finite trace on $\mathcal{B}$ with support $P\Phi(D_E)$. For $A \in \mathcal{R}^+$ define $\tau(A) = \varphi(\Phi(A))$. As in the proof of Lemma 9 τ is a normal G-invariant trace on $\mathcal{R}^+$. Since $\tau(F) < \infty$ there is a non-zero central projection Q in $\mathcal{R}$ such that τ is faithful and semi-finite on $Q\mathcal{R}$ [1,Ch.I,§6,Cor.2]. Since τ is G-invariant $Q \in \mathcal{D}$. Now a Zorn's Lemma argument completes the proof just as in Lemma 9.

<u>Proof of Theorem 2</u>. By Lemma 6 there is a projection $P \in \mathcal{D}$ such that there exists a $\underset{G}{\sim}$-abelian projection $E \in P\mathcal{R}$ with $D_E = P$, and $I - P$ has no non-zero $\underset{G}{\sim}$-abelian subprojection. By Lemma 9 there is a faithful normal semi-finite G-invariant trace τ_1 on $P\mathcal{R}^+$. If $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite then by Lemma 10 there is a faithful normal semi-finite G-invariant trace τ_2 on $(I - P)\mathcal{R}^+$. Thus $\tau = \tau_1 + \tau_2$ is a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$.

Conversely assume there exists a faithful normal semi-finite G-invariant trace τ on $\mathcal{R}^+$. Suppose E is a projection in $\mathcal{R}$ such that $\tau(E) < \infty$. Since $E \underset{G}{\sim} F$ implies $\tau(E) = \tau(F)$ it is clear that E is $\underset{G}{\sim}$-finite. Thus $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite. The proof is complete.

Lemma 11. Suppose $\mathcal{C}$ is countably decomposable and $\mathcal{R}$ is $\underset{G}{\sim}$-finite. Then there is a faithful normal finite G-invariant trace on $\mathcal{R}$.

Proof. Since $\mathcal{R}$ is $\underset{G}{\sim}$-finite $\mathcal{R}$ is in particular finite. By [1,Ch.III,§4,Thm.3] there is a unique center valued trace ψ on $\mathcal{R}$ which is the identity on $\mathcal{C}$. By uniqueness ψ is G-invariant, so if τ is a faithful normal finite G-invariant trace on $\mathcal{C}$, then $\tau \circ \psi$ is one on $\mathcal{R}$. Therefore we may assume $\mathcal{R} = \mathcal{C}$. Now there exists a projection $P \in \mathcal{D}$ such that $P\mathcal{C} = P\mathcal{D}$, and G is freely acting on $(I-P)\mathcal{C}$, i.e. for each projection $E \neq 0$ in $(I-P)\mathcal{C}$ there is a non-zero subprojection F of E and $s \in G$ such that $U_s^* F U_s \leq I - F$, see e.g. [5]. Since I is countably decomposable, so is P, and there is a faithful normal state on $P\mathcal{C}$, hence a faithful normal finite G-invariant trace on $P\mathcal{C}$. We may thus assume G is freely acting. Let F be a non-zero projection in $\mathcal{C}$ and s an element in G such that $U_s^* F U_s \leq I - F$. Let $E = I - F$. Then $D_E = I$, and $F \underset{G}{\precsim} E$. As in the proof of Lemma 10 $\Phi(E)$ is not properly infinite, so we can choose a central projection $P \neq 0$ in $\mathcal{B}$ such that $P\Phi(E)$ is finite. Since $F \prec_G E$, $\Phi(F) \prec \Phi(E)$, by Lemma 1, hence $P\Phi(F) \prec P\Phi(E)$, so $P\Phi(F)$ is finite. Thus $P = P\Phi(E) + P\Phi(F)$ is finite in $\mathcal{B}$, and $P\mathcal{B}$ is finite. Since I is countably decomposable in $\mathcal{C}\,(=\mathcal{R})$ $\Phi(I)$ is countably decomposable in $\mathcal{B}$ by Lemma 5, hence so is P. Therefore by [1,Ch.I,§6,Prop.9] there is a faithful normal finite trace φ on $P\mathcal{B}$. Then τ defined by $\tau(A) = \varphi(\Phi(A))$ is a normal finite G-invariant trace on $\mathcal{C}$ with support $D \neq 0$ in $\mathcal{D}$. A Zorn's Lemma argument now gives a family τ_α of normal finite G-invariant traces on $\mathcal{C}$ with orthogonal supports D_α in $\mathcal{D}$. Since I is countably decomposable the family $\{\tau_\alpha\}$ is countable, and by multiplying each τ_α by a convenient positive scalar we may assume $\Sigma\tau_\alpha(D_\alpha) = 1$. Thus if $\tau = \Sigma\tau_\alpha$, then τ is a faithful normal finite G-invariant trace on $\mathcal{C}$. The proof is oomplete.

<u>Proof of Theorem 3</u>. Suppose there is a faithful normal finite G-invariant trace τ on $\mathcal{R}$. Then I is $\underset{G}{\sim}$-finite, for if E is a projection in $\mathcal{R}$ which is G-equivalent to I then $\tau(E) = \tau(I)$, hence $\tau(I - E) = 0$, hence $I - E = 0$, since τ is faithful. Thus $\mathcal{R}$ is $\underset{G}{\sim}$-finite. Again since τ is faithful, its support I is countably decomposable, i.e. $\mathcal{R}$ is countably decomposable. The converse follows from Lemma 11.

<u>Corollary</u>. If $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite then $\mathcal{B}$ is semi-finite. If $\mathcal{R}$ is $\underset{G}{\sim}$-finite and there is an orthogonal family of countably decomposable projections in $\mathcal{D}$ with sum I, then $\mathcal{B}$ is finite.

<u>Proof</u>. If $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite, then by Theorem 2 there is a faithful normal semi-finite G-invariant trace on $\mathcal{R}$. Thus there is a faithful normal semi-finite trace on $\mathcal{B}$ by [1,Ch.I, §9,Prop.1], hence $\mathcal{B}$ is semi-finite. If P is a projection in $\mathcal{D}$ then by Lemma 2 $\Phi(P)$ is a central projection in $\mathcal{B}$. Thus in order to show the last part of the corollary we may assume I is countably decomposable. Then by Theorem 3 there is a faithful normal finite G-invariant trace on $\mathcal{R}$, hence by [1,Ch.I,§9, Prop.1] there is a normal finite trace on $\mathcal{B}$, so $\mathcal{B}$ is finite. The proof is complete.

4. G-finite von Neumann algebras. Let notation be as in Theorem 1. Following [7] we say $\mathcal{R}$ is G-finite if there is a family $\mathcal{F}$ of normal G-invariant states which separate $\mathcal{R}^+$, i.e. if $A \in \mathcal{R}^+$, and $\omega(A) = 0$ for all $\omega \in \mathcal{F}$, then $A = 0$. For semi-finite von Neumann algebras it would be natural to compare this concept with those of $\underset{G}{\sim}$-finite and $\underset{G}{\sim}$-semi-finite. Since a $\underset{G}{\sim}$-finite von Neumann algebra is necessarily finite we cannot expect a G-finite semi-finite von Neumann algebra to be $\underset{G}{\sim}$-finite. We say G acts ergodically on $\mathcal{C}$ if $\mathcal{D}$ $(= \mathcal{C} \cap \mathcal{R}^G)$ is the scalars.

Theorem 4. Let $\mathcal{R}$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let G be a group and $t \to U_t$ a unitary representation of G on $\mathcal{H}$ such that $U_t^* \mathcal{R} U_t = \mathcal{R}$ for all $t \in G$. Assume either that G acts ergodically on the center of $\mathcal{R}$ or the center is elementwise fixed under G. Then $\mathcal{R}$ is G-finite if and only if $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite and there is an orthogonal family of finite G-invariant projections in $\mathcal{R}$ with sum I.

Proof. Assume $\mathcal{R}$ is G-finite. Suppose first that G acts ergodically on the center $\mathcal{C}$ of $\mathcal{R}$, and suppose ω is a faithful normal G-invariant state on $\mathcal{R}$. Then by [11] there is a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$, hence by Theorem 2 $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite. In general, by Zorn's Lemma there is a family $\{\omega_\alpha\}$ of normal G-invariant states with orthogonal supports E_α such that $\Sigma E_\alpha = I$. Then each E_α is G-invariant, and by the first part of the proof $E_\alpha \mathcal{R} E_\alpha$ is $\underset{G}{\sim}$-semi-finite. In particular, E_α is the sup of an increasing net of $\underset{G}{\sim}$-finite projections. Let F be a projection in $\mathcal{R}$. We show F has a non-zero $\underset{G}{\sim}$finite subprojection. By the above considerations there is E_α and a $\underset{G}{\sim}$-finite subprojection F_α of E_α such that $C_{F_\alpha} F \neq 0$. Let $F_1 = C_{F_\alpha} F$. Then there is a non-zero subprojection F_o of F_1 such that $F_o \precsim F_\alpha$. Say $F_o \sim G_\alpha \leq F_\alpha$. Since F_α is $\underset{G}{\sim}$-finite, so is G_α . Indeed, if $G_\alpha \underset{G}{\sim} H \leq G_\alpha$ then by Lemma 1 $\Phi(G_\alpha) \sim \Phi(H)$, hence $\Phi(F_\alpha) = \Phi(G_\alpha) + \Phi(F_\alpha - G_\alpha) \sim \Phi(H) + \Phi(F_\alpha - G_\alpha)$, so again by Lemma 1, $F_\alpha \underset{G}{\sim} H + F_\alpha - G_\alpha$, so that $H = G_\alpha$ by finiteness of F_α . Thus G_α is $\underset{G}{\sim}$-finite. Since G_α is in particular finite there is by [1,Ch.III,§2,Prop.6] a unitary operator $U \in \mathcal{R}$ such that $UF_oU^{-1} = G_\alpha$. But then F_o is $\underset{G}{\sim}$-finite, for if $F_o \underset{G}{\sim} F_2 \leq F_o$ then $UF_2U^{-1} \sim F_2 \underset{G}{\sim} G_\alpha$, so by transitivity $UF_2U^{-1} \underset{G}{\sim} G_\alpha$. Since $UF_2U^{-1} \leq G_\alpha$, they are equal by finiteness of G_α , so $F_2 = F_o$, and F_o is $\underset{G}{\sim}$-finite. Therefore the projection F has a non-zero $\underset{G}{\sim}$-finite subprojection F_o , and $\mathcal{R}$ is $\underset{G}{\sim}$-semi-finite.

Next assume $\mathcal{C} = \mathcal{D}$. Then every normal semi-finite trace on $\mathcal{R}^+$ is G-invariant [10,Cor.2.2], so there exists a faithful normal semi-finte G-invariant trace on $\mathcal{R}^+$, hence by Theorem 2, $\mathcal{R}$ is $\underset{G}{\sim}$- semi-finite.

Let τ be a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$. Let $\{\omega_\alpha\}$ be as before with orthogonal supports $\{E_\alpha\}$. Then there is a positive self-adjoint operator $H_\alpha \in L^1(\mathcal{R},\tau)$ affiliated with $\mathcal{R}^G$ such that $\omega_\alpha(T) = \tau(H_\alpha T)$ for $T \in \mathcal{R}$, see e.g. [1,Ch.I,§6,no.10]. Let E be a finite spectral projection of H_α . Then E is G-invariant. A Zorn's Lemma argument now gives an orthogonal family of finite G-invariant projections in $\mathcal{R}$ with sum I .

Conversely assume $\mathcal{R}$ is $\underset{G}{\sim}$-simi-finite and having an orthogonal family $\{E_\alpha\}$ of finite non-zero G-invariant projections with sum I . Let by Theorem 2 τ be a faithful normal semi-finite G-invariant trace on $\mathcal{R}^+$. Let $c_\alpha = \tau(E_\alpha)^{-1}$, and let

$\omega_\alpha(T) = c_\alpha \tau(E_\alpha T)$. Then $\{\omega_\alpha\}$ is a separating family of normal G-invariant states on $\mathcal{R}$, hence $\mathcal{R}$ is G-finite. The proof is complete.

The above theorem is probably true without the assumptions of the action of G on $\mathcal{C}$. A direct proof of this would be quite interesting.

<u>5. Abelian von Neumann algebras</u>. Assume $\mathcal{R}$ is an abelian von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let G be a group and suppose $t \to U_t$ is a unitary representation of G on $\mathcal{H}$ such that $U_t^* \mathcal{R} U_t = \mathcal{R}$ for all $t \in G$. We say two projections E and F in $\mathcal{R}$ are <u>equivalent in the sense of Hopf</u> and write $E \underset{H}{\sim} F$ if there is an orthogonal family of projections $\{E_\alpha\}_{\alpha \in J}$ in $\mathcal{R}$ and $t_\alpha \in G$, for $\alpha \in J$, such that $E = \Sigma E_\alpha$, $F = \Sigma U_{t_\alpha}^* E_\alpha U_{t_\alpha}$. Since each $U_{t_\alpha}^* E_\alpha U_{t_\alpha}$ is a projection, and their sum is a projection, they are all mutually orthogonal. Since we can collect the E_α's for which t_α coincide the definition of equivalence in the sense of Hopf is equivalent to the existence of an orthogonal family of projections $\{E_t\}_{t \in G}$ in $\mathcal{R}$ such that $E = \sum_{t \in G} E_t$, $F = \sum_{t \in G} U_t^* E_t U_t$. This ordering was introduced by Hopf [3]. Just as for $\underset{G}{\sim}$ we define $\underset{H}{\sim}$-finite, $\underset{H}{\sim}$-semi-finite, and $\prec_H$. Note that if $E \underset{H}{\sim} F$ as above, if we let $T_t = E_t$, then $E = \Sigma T_t T_t^*$, $F = \Sigma U_t^* T_t^* T_t U_t$, so $E \underset{G}{\sim} F$. It is plausible that the converse is true too. If we assume $\mathcal{R}$ is countably decomposable, we can prove this via a proof which makes use of the known results on invariant measures if $\mathcal{R}$ is $\underset{H}{\sim}$-finite and $\underset{H}{\sim}$-semi-finite. A direct proof would be much more desirable.

Theorem 5. Assume $\mathcal{R}$ is countably decomposable, and let notation be as above. Then two projections E and F in $\mathcal{R}$ are G-equivalent if and only if they are equivalent in the sense of Hopf.

Outline of proof. It remains to be shown that if $E \underset{G}{\sim} F$ then $E \underset{H}{\sim} F$. Assume $E \underset{G}{\sim} F$. By Lemma 1 $\Phi(E) \sim \Phi(F)$, so they have the same central carrier C . By Lemma 3 $\Phi(D_E) = C = \Phi(D_F)$, so $D_E = D_F$. Suppose first E and F are such that EP and FP are $\underset{H}{\sim}$-infinite for all non-zero projections $P \in \mathcal{D}$. In a von Neumann algebra two properly infinite countably decomposable projections with the same central carriers are equivalent [1,Ch.III, §8,Cor.5]. Using the comparison theory for $\mathcal{R}$ with the Hopf ordering $\prec_H$, as developed in [6], see also [9], we can modify the proof of the quoted result for von Neumann algebras, to show $E \underset{H}{\sim} F$. If E is $\underset{H}{\sim}$-finite then since $D_E = D_F$, we may assume $\mathcal{R}$ is $\underset{H}{\sim}$-semi-finite, so by [6] there is a faithful normal semi-finite G-invariant trace τ on $\mathcal{R}^+$. From the comparison theorem on $\mathcal{R}$ [6, Lem.16], or [9,Lem.2.7], there exist two orthogonal projections P and Q in $\mathcal{D}$ with sum I such that $PE \prec_H PF$ and $QF \prec_H QE$. Since $PE \underset{G}{\sim} PF$ we have $\tau(PE) = \tau(PF)$. But if a proper subprojection F_o of PF is such that $PE \underset{H}{\sim} F_o$ then $\tau(PE) = \tau(F_o) < \tau(PF) = \tau(PE)$, a contradiction. Thus $PE \underset{H}{\sim} PF$, and similary $QE \underset{H}{\sim} QF$. Thus $E \underset{H}{\sim} F$, and the proof is complete.

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