

On the Bohr compactification of a transformation group¹⁾

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The purpose of this note is to extend the concept of Bohr compactifications to transformation groups (G, Z) , where G is a topological group acting on a topological space Z . The Bohr compactification will be a transformation group (\hat{G}, \hat{Z}) where \hat{G} is the Bohr compactification of the group G , and \hat{Z} is the Hausdorff completion of Z with respect to a certain uniform structure \underline{u}_B on Z . \underline{u}_B is obtained by considerations very similar to those given by E. M. Alfsen and P. Holm in [1] for topological groups. The almost periodic functions on Z will be exactly the \underline{u}_B -uniformly continuous functions. Further (\hat{G}, \hat{Z}) will have the following factorization property: Any transformation group homomorphism between (G, Z) and (H, X) , where X is compact and the action of H on X is uniformly equicontinuous, can be lifted to (\hat{G}, \hat{Z}) .

In the case when Z is a homogenous space, $Z = G/H$ for some subgroup H of G , we shall see that as in the group case, \underline{u}_B is uniquely determined by the related topology on Z , and this topology can be characterized exactly as in the group case. If further the action of G on G/H is amenable, we are able to generalize a result in [6] giving a particular simple

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characterization of this topology. In the last section these results are applied to give a characterization of the Bohr compactification of a semi-direct product.

1. Bohr compactification of a transformation group

We shall start with some definitions, notation concerning uniform spaces will be as in [5]. By a transformation group is meant a pair $(G, \underline{\mathcal{J}}; Z, \underline{\mathcal{Z}})$, abbreviated (G, Z) when no confusion arises, where $(G, \underline{\mathcal{J}})$ is a topological group, $(Z, \underline{\mathcal{Z}})$ a topological space, together with an action $G \times Z \rightarrow Z$ (written $(g, z) \rightarrow gz$) such that $ez = z$, $g(hz) = (gh)z$ for $g, h \in G$, $z \in Z$, and the map $\pi_z: G \rightarrow Z$ given by $\pi_z(g) = gz$ is continuous for all $z \in Z$.

If $g \in G$, let π^g be the map $Z \rightarrow Z$ given by $\pi^g(z) = gz$, if f is a function on Z and $g \in G$, ${}_g f$ denotes the function ${}_g f(x) = f(gx)$. If A is a subset of G and B is a subset of Z ,

$$A \cdot B = \{ab: a \in A, b \in B\}, \text{ and}$$

if U is a subset of $Z \times Z$, let

$$GU = \{(gx, gy): g \in G, (x, y) \in U\}.$$

U is called G -invariant if $GU = U$. If $\underline{\mathcal{U}}$ is a uniform structure on Z , it is not difficult to see that the family $\{\pi^g: g \in G\}$ is $\underline{\mathcal{U}}$ -uniformly equicontinuous iff $\underline{\mathcal{U}}$ admits a basis $\underline{\mathcal{B}}$ with

each $U \in \mathfrak{B}$ G -invariant. In this case we shall call (G, Z) \underline{u} -uniformly equicontinuous. If (Z, \underline{Z}) is compact, there is a unique uniform structure on Z with \underline{Z} as its topology, and unless otherwise specified uniform continuity is with respect to this uniform structure.

The next lemma, which will not be needed, shows that in most of the situations later considered the action $G \times Z \rightarrow Z$ will be jointly continuous.

Lemma 1 Suppose Z has a uniform structure \underline{u} such that (G, Z) is \underline{u} -uniformly equicontinuous and $\pi_z: G \rightarrow Z$ is continuous for each $z \in Z$. Then the action $G \times Z \rightarrow Z$ will be jointly continuous.

Proof: Given $z_0 \in Z$ and a neighborhood $U[z_0]$ of z_0 for some $U \in \underline{u}$. Then there is $V \in \underline{u}$ such that $V \circ V \subset U$, and further a $W \in \underline{u}$ with $GW \subset V$. Further there is a neighborhood O of e in G such that $Oz_0 \subset W[z_0]$. If now $z \in W[z_0]$ and $g \in O$, we have $(z_0, z) \in W$, so $(gz_0, gz) \in V$. Further $(z_0, gz_0) \in W$, so $(z_0, gz) \in W \circ V \subset V \circ V \subset U$, and $gz \in U[z_0]$. This shows jointly continuity at each point (e, z_0) , but since each π^g will be continuous, we will have jointly continuity at each point $(g_0, z_0) \in G \times Z$.

π^g can be considered as an element of Z^Z , and if we give Z^Z the product topology the map from G into Z^Z given by $g \rightarrow \pi^g$ will be continuous, since π_z is continuous for each fixed $z \in Z$. If (Z, \underline{Z}) is compact, the closure $E(G, Z)$ of $\{\pi^g : g \in G\}$ in Z^Z is compact. $E(G, Z)$ will be a topological group in this topology iff (G, Z) is uniformly equicontinuous. (See [2, Prop. 4.4])

Some more notations: a subset U of $Z \times Z$ is called cofinite if there is a finite subset A of Z such that $Z = U[A]$. It is well known that a uniform structure \underline{u} on Z is totally bounded iff each $U \in \underline{u}$ is cofinite. A subset B of Z is called relatively dense if there is a finite set A in G such that $Z \cong A \cdot B$. The diagonal in $Z \times Z$ will be denoted Δ .

Theorem 1. Let \underline{u}_B be the collection of all subsets U of $Z \times Z$ admitting a sequence $\{U_n\}_{n=0}^{\infty}$ of subsets such that

- (i) each U_n is a symmetric $\underline{Z} \times \underline{Z}$ -neighborhood of Δ in $Z \times Z$, $U_{n+1} \circ U_{n+1} \subset U_n$ for $n = 0, 1, 2, \dots$,
 $U_0 \subset U$,
- (ii) each U_n is cofinite, and
- (iii) each U_n is G -invariant.

Then \underline{u}_B defines a uniform structure on Z which is the finest uniform structure \underline{u} on Z satisfying

- (a) \underline{u} defines a topology on Z coarser than \underline{Z} ,
- (b) \underline{u} is totally bounded, and
- (c) (G, Z) is \underline{u} -uniformly equicontinuous.

Proof: To prove that \underline{u}_B is a uniform structure, the only non trivial fact to verify is that \underline{u}_B is closed under finite intersections. If $U, V \in \underline{u}_B$ with corresponding sequences $\{U_n\}$ and $\{V_n\}$, let $W = U \cap V$ and $W_n = U_n \cap V_n$. $\{W_n\}$ will then satisfy (i) and (iii) with respect to W , so we will have to show that also (ii) holds. For this it will suffice to prove that if U and V are cofinite, then so is $(U^{-1} \cdot U) \cap (V^{-1} \cdot V)$. The argument will be quite similar to the one given in [1, Prop. 3]. If $Z = U[a_1, \dots, a_n] = V[b_1, \dots, b_m]$, take an element c_{ij} from $U[a_i] \cap V[b_j]$ whenever this intersection is non-empty. Any $z \in Z$ is then in $U[a_i] \cap V[b_j]$ for some i, j . Thus $(a_i, z) \in U$ and $(a_i, c_{ij}) \in U$, so $(c_{ij}, z) \in U^{-1} \cdot U$. By a similar argument $(c_{ij}, z) \in V^{-1} \cdot V$, so $Z = \bigcup_{i,j} (U^{-1} \cdot U) \cap (V^{-1} \cdot V)[c_{ij}]$.

So \underline{u}_B is a uniform structure on Z . Now a uniform structure \underline{u} on Z satisfies (a) iff each $U \in \underline{u}$ is a $\underline{Z} \times \underline{Z}$ -neighborhood of Δ in $Z \times Z$. (b) holds iff each $U \in \underline{u}$ is cofinite, and (c) holds iff \underline{u} has a basis \mathfrak{B} with each $V \in \mathfrak{B}$ G -invariant. Thus \underline{u}_B satisfies (a), (b) and (c). Conversely

if \underline{u} is a uniform structure on Z satisfying (a), (b) and (c), and $U \in \underline{u}$, then U admits a sequence $\{U_n\}$ from \underline{u} satisfying (i), (ii) and (iii). Thus $U \in \underline{u}_B$, so \underline{u} is coarser than \underline{u}_B .

By reasons we soon shall see, \underline{u}_B will be called the Bohr-uniformity on Z , and \underline{Z}_B , the related topology on Z , is called the Bohr topology. Now take \hat{Z} to be the Hausdorff completion of Z with respect to \underline{u}_B , so \hat{Z} is compact, further let ρ denote the canonical map of Z into \hat{Z} . We shall next see that each homomorphism of (G, Z) into a transformation group (H, X) with X compact and (H, X) uniformly equicontinuous can be factorized via \hat{Z} , but first we must define what this means.

Definition. By a homomorphism between two transformation groups (G, Z) and (H, X) we shall mean a map f which maps Z continuously into X , and such that f is a continuous group homomorphism from G into H , and further f shall commute with the actions, i.e.

$$f(gx) = f(g) f(x) \text{ for } g \in G, x \in Z.$$

Let (\hat{G}, ρ) be the Bohr-compactification of G as defined in [1]. Each π^g is \underline{u}_B -uniformly continuous, so π^g can be extended to \hat{Z} , and it is not difficult to see that the family $\{\pi^g: g \in G\}$ also will be uniformly equicontinuous on \hat{Z} . By a previous remark $E(G, \hat{Z})$, the closure of $\{\pi^g\}$ in $\hat{Z}^{\hat{Z}}$

is a compact topological group, and the map $\pi: G \rightarrow E(G, \hat{Z})$ given by $g \rightarrow \pi^g$ is a continuous homomorphism, so π can be lifted to \hat{G} . This will define an action of \hat{G} on \hat{Z} such that $\rho: (G, Z) \rightarrow (\hat{G}, \hat{Z})$ is a homomorphism of transformation groups.

Proposition 1. Given any homomorphism $f: (G, Z) \rightarrow (H, X)$ between (G, Z) and a transformation group (H, X) with X compact and (H, X) uniformly equicontinuous, there is a unique homomorphism $\hat{f}: (\hat{G}, \hat{Z}) \rightarrow (H, X)$ such that $f = \hat{f} \circ \rho$.

Proof: We have earlier seen that the map $H \rightarrow E(H, X)$ given by $g \rightarrow \pi^g$ is a continuous homomorphism of H into a compact topological group. This means that f can be factorized via \hat{G} , i.e. there is a uniquely determined continuous function $\hat{f}: \hat{G} \rightarrow E(H, X)$ such that

$$f(g)x = \hat{f}(\rho(g))x \quad \text{for } g \in G, x \in X.$$

If \underline{u} is the uniquely determined uniform structure on X , the sets $(f \times f)^{-1}(U)$ with $U \in \underline{u}$ defines a uniform structure \underline{u}_f on Z satisfying (a), (b) and (c) in Theorem 1. Thus \underline{u}_f is coarser than \underline{u}_B , and f is \underline{u}_B -uniformly continuous. Therefore f has a unique extension \hat{f} to \hat{Z} with $f = \hat{f} \circ \rho$. \hat{f} will be a homomorphism of transformation groups, because if $x \in G$ and $z \in Z$

$$\hat{f}(\rho(x)\rho(z)) = \hat{f} \circ \rho(xz) = f(xz) = f(x)f(z) = \hat{f}(\rho(x))\hat{f}(\rho(z)),$$

hence $\hat{f}(xz) = \hat{f}(x)\hat{f}(z)$ for all $x \in \hat{G}$, $z \in \hat{Z}$, since $\rho(G)$ is dense in \hat{G} and $\rho(Z)$ is dense in \hat{Z} .

We shall conclude this section by examining the relation between \underline{u}_B and the almost periodic functions on Z . Denote by $CB(Z)$ the continuous, bounded, complex valued functions on Z . $f \in CB(Z)$ is called almost periodic if $\{g f: g \in G\}$ is relatively compact in $CB(Z)$, $CB(Z)$ has the sup-norm topology. Let $AP(Z)$ denote the almost periodic functions on Z . If Z is compact, it is not difficult to prove by using Ascoli's Theorem that $AP(Z) = CB(Z)$ iff (G, Z) is uniformly equicontinuous. (cf. [2, Prop. 4.15])

Proposition 2. The following are equivalent:

- (i) f is almost periodic
- (ii) f is \underline{u}_B -uniformly continuous
- (iii) There is $\hat{f} \in CB(\hat{Z})$ with $f = \hat{f} \circ \rho$

Proof: (ii) \Rightarrow (iii) is obvious, and by the remark just made $AP(\hat{Z}) = CB(\hat{Z})$, so (iii) \Rightarrow (i). To prove (i) \Rightarrow (ii), define a pseudo-metric d_f on Z by $d_f(x, y) = \sup_{g \in G} |f(gx) - f(gy)|$, we shall prove that d_f defines a uniform structure on Z

satisfying (a), (b) and (c) in Theorem 1. Since $d_f(gx, gy) = d_f(x, y)$ for $g \in G$, $x, y \in Z$, (c) holds. To prove (a), let $x \in Z$ and $\epsilon > 0$ be given. Let $g_1, \dots, g_n \in G$ be such that $G = \bigcup_{i=1}^n \{g: \|g f - g_i f\| < \epsilon/4\}$, and let $W = \{y \in Z: |g_i f(x) - g_i f(y)| < \epsilon/4 \text{ for } i = 1, \dots, n\}$, this is a Z -neighborhood of x in Z . If $y \in W$ and $g \in G$, choose g_i such that $\|g f - g_i f\| < \epsilon/4$, then

$$\begin{aligned} |f(gx) - f(gy)| &\leq |f(gx) - f(g_i x)| + |f(g_i x) - f(g_i y)| \\ &\quad + |f(g_i y) - f(gy)| < \epsilon/4 + \epsilon/4 + \epsilon/4 = 3\epsilon/4. \end{aligned}$$

So $d_f(x, y) \leq 3\epsilon/4 < \epsilon$, thus $W \subset \{y | d_f(x, y) < \epsilon\}$, this proves (a).

We now have to prove that d_f is a totally bounded metric.

So let $\epsilon > 0$ be given, take $g_1, \dots, g_n \in G$ such that for any $g \in G$ $\|g f - g_i f\| < \epsilon/4$ for some i . Let

$U_i = \{(x, y): x, y \in Z, |g_i f(x) - g_i f(y)| < \epsilon/8\}$, since f is bounded, U_i is cofinite, and an argument similar to the one

given in Theorem 1 shows that $U = \bigcap_{i=1}^n U_i^{-1} \circ U_i$ also is

cofinite. $U \subset \{(x, y): x, y \in Z, |f(g_i x) - f(g_i y)| < \epsilon/4 \text{ for}$

$i = 1, \dots, n\}$, so there is a finite set $\{y_1, \dots, y_m\}$

in Z such that any $x \in Z$ satisfies $|f(g_i x) - f(g_i y_j)| < \epsilon/4$

for $i = 1, \dots, n$ for some y_j . Further, if g is any element

in G , take g_i such that $\|g f - g_i f\| < \epsilon/4$. Then

$$\begin{aligned} |f(gx) - f(gy_j)| &\leq |f(gx) - f(g_i x)| + |f(g_i x) - f(g_i y_j)| + \\ &+ |f(g_i y_j) - f(gy_j)| < 3\epsilon/4, \end{aligned}$$

so $d_f(x, y_j) \leq 3\epsilon/4 < \epsilon$, and $Z = \bigcup_{j=1}^m \{x: d_f(x, y_j) < \epsilon\}$.

Thus d_f is a totally bounded metric, and (b) holds.

By Theorem 1 we now have that d_f defines a uniform structure coarser than \underline{u}_B , and since f obviously is d_f -uniformly continuous, f is also \underline{u}_B -uniformly continuous. So (i) \Rightarrow (ii), and the proof is complete.

2. The homogenous case

We shall now study more closely the case when Z is a homogenous space G/H , where H is a subgroup of G , Z has the quotient topology \underline{Z} , with the usual action of G on Z . It turns out that \underline{u}_B then can be described by subsets of Z , as in the case with $Z = G$.

π will denote the canonical map $G \rightarrow G/H$, and $O = \pi(e)$. If V is a neighborhood of O in Z , let $\tilde{V} = \{(gx, go): g \in G, x \in V\}$. \tilde{V} is then a $\underline{Z} \times \underline{Z}$ -neighborhood of the diagonal Δ in $Z \times Z$, and $G\tilde{V} = \tilde{V}$. If U and V are subsets of Z , we define

$$U \circ V = \pi^{-1}(U) \cdot V = \{gy: y \in V, \pi(g) \in U\} \text{ and}$$

$$U^{-1} = \pi(\pi^{-1}(U)^{-1}) = \{\pi(g) : \pi(g^{-1}) \in U\}.$$

We will use the notation U^2 for $U \circ U$.

Lemma 2. $(U \circ V)^\sim = \tilde{V} \circ \tilde{U}$ and $(U^{-1})^\sim = (\tilde{U})^{-1}$.

Proof: Trivial verifications.

Theorem 2. If $Z = G/H$, $\underline{\mu}_B$ has a basis consisting of all sets \tilde{V} , where V is a subset of Z admitting a sequence $\{V_n\}$ of neighborhoods of 0 in Z such that (i) $V_{n+1} \circ V_{n+1} \subset V_n$, $V_n^{-1} = V_n$, $V_0 \subset V$, and (ii) each V_n is relatively dense.

Proof: If V and $\{V_n\}$ is as above, we will find that \tilde{V}_n is symmetric, $\tilde{V}_{n+1} \circ \tilde{V}_{n+1} = (V_{n+1} \circ V_{n+1})^\sim \subset \tilde{V}_n$ and $\tilde{V}_0 \subset \tilde{V}$. \tilde{V}_n is a $\underline{Z} \times \underline{Z}$ -neighborhood of Δ , because $gV_{n+1} \times gV_{n+1}$ is a neighborhood of $(\pi(g), \pi(g))$ contained in \tilde{V}_n . \tilde{V}_n is G -invariant, and if A_n is a finite set with $Z = A_n V_n$, $A_n V_n \subset \tilde{V}_n[\pi(A_n)]$, so \tilde{V}_n is cofinite. We conclude that $\tilde{V} \in \underline{\mu}_B$.

Conversely, suppose $U \in \underline{\mu}_B$ is given with corresponding sequence $\{U_n\}$ satisfying (i), (ii) and (iii) in Theorem 1. Take $V_n = U_n[0]$, then V_n is a \underline{Z} -neighborhood of 0 in Z , and $U_0 = \tilde{V}_0$: $(x, y) \in \tilde{V}_0 \Leftrightarrow \exists g \in G, gx \in V_0, gy = 0 \Leftrightarrow \exists g, gy = 0 (0, gx) \in U_0 \Leftrightarrow \exists g (gx, gy) \in U_0 \Leftrightarrow (x, y) \in U_0$.

So it remains to prove that $\{V_n\}$ satisfies (i) and (ii) above. Using that U_n is G -invariant it is not difficult to prove that $V_{n+1} \circ V_{n+1} \subset V_n$ and $V_n^{-1} = V_n$. To show that V_n is relatively dense, take $g_1, \dots, g_m \in G$ such that $Z = U_n[\pi(g_1), \dots, \pi(g_m)]$. Then, if $z \in U_n[\pi(g_i)]$ $(\pi(g_i), z) \in U_n$ so $(0, g_i^{-1}z) \in U_n \Rightarrow g_i^{-1}z \in V_n$, so $Z = \bigcup_{i=1}^m g_i V_n$. Thus given $U \in \underline{u}_B$ there is a sequence $\{V_n\}$ of subsets of Z satisfying (i) and (ii) above with $\tilde{V}_0 \subset U$. This proves the theorem.

When $Z = G/H$, the relation between, \hat{Z} , \hat{G} , G and H is as follows: $\hat{Z} = \hat{G}/\overline{\rho(H)}$, ρ is the map from $G \rightarrow \hat{G}$ and $\overline{\rho(H)}$ is the closure in \hat{G} . It is also easy to see that $f \in AP(Z)$ iff $f \circ \pi \in AP(G)$. If \underline{Z}_B is the topology \underline{u}_B defined on Z , it should be clear that the sets V satisfying the conditions in Theorem 2 are exactly the \underline{Z}_B -neighborhoods of 0 in Z .

3. Amenable actions

If f is a bounded real valued function on Z , we define its upper mean value $\overline{M}(f)$ by

$$\overline{M}(f) = \inf\left\{\sup_{x \in Z} \sum_{i=1}^n \alpha_i f(a_i x) : a_i \in G, \alpha_i > 0, \sum_{i=1}^n \alpha_i = 1\right\}$$

The lower mean value of f is defined as $\underline{M}(f) = -\overline{M}(-f)$. It is then not difficult to show the following, cf. [6]:

$$\inf_{x \in Z} f(x) \leq \bar{M}(f) \leq \sup_{x \in Z} f(x)$$

$$\bar{M}(\lambda f) = \lambda \bar{M}(f) \quad \text{for } \lambda \geq 0$$

$$\bar{M}(g f) = \bar{M}(f), \quad \bar{M}(f - g f) \leq 0 \quad \text{for } g \in G$$

If the action of G on Z is abelian, i.e. $(gh)z = (hg)z$ for $g, h \in G, z \in Z$, we have

$$\bar{M}(f + g) \leq \bar{M}(f) + \bar{M}(g)$$

If A is a subset of Z with characteristic function χ_A it is easy to see that A is relatively dense iff $\underline{M}(\chi_A) > 0$.

Let E be a linear space of complex valued bounded functions on Z which contains the constants and is closed under complex conjugation and translations (i.e. $f \in E, g \in G \Rightarrow g f \in E$). A linear functional m on E is called an invariant mean if $m(\bar{f}) = \overline{m(f)}$, $\inf_{x \in Z} f(x) \leq m(f) \leq \sup_{x \in Z} f(x)$ if $f \in E$ is real-valued, and $m(g f) = m(f)$ for $f \in E, g \in G$. It is easy to see, cf. [6], that we will then have $\underline{M}(f) \leq m(f) \leq \bar{M}(f)$ for a real valued $f \in E$.

A function $f \in CB(Z)$ is called uniformly continuous if the map $g \rightarrow g f$ from G into $CB(Z)$ (with sup-norm) is continuous. The space of all uniformly continuous functions on Z is denoted $UCB(Z)$. The transformation group (G, Z) will be called

amenable if there is an invariant mean on $UCB(Z)$. (cf. [4] for equivalent definitions when G and Z are locally compact.)

If $Z = G/H$ is a homogenous space with (G, Z) amenable, we shall now prove that the characterization of \underline{Z}_B in Theorem 2 can be improved, the proof will follow the same pattern as that of Theorem 2 in [6].

Theorem 3. Let $Z = G/H$ where H is a subgroup of G , and suppose (G, Z) is amenable. Then a subset W of Z is a \underline{Z}_B -neighborhood of 0 iff there is a relatively dense subset E of Z and a \mathfrak{J} -neighborhood V of e in G such that $((VE) \circ (VE)^{-1})^2 \subset W$.

Proof: If W is a \underline{Z}_B -neighborhood of 0, take a sequence $\{V_n\}$ as in Theorem 2 with $V_0 \subset W$, let $E = V_3$ and $V = \pi^{-1}(V_3)$. $VE = V_3 \circ V_3$, so $((VE) \circ (VE)^{-1})^2 = V_3^8 \subset V_0 \subset W$.

Now suppose the subsets E and V are given, and let $h: G \rightarrow [0, 1]$ be a right uniformly continuous function on G (as defined in [5, p. 210]) with $h(e) = 1$ and $h(x) = 0$ for $x \notin V$. Then define a function $j: Z \rightarrow [0, 1]$ by $j(x) = \sup\{h(g): g \in G, x \in gE\}$. Then $j(x) = 1$ for $x \in E$, $j(x) = 0$ for $x \notin VE$, and $j \in UCB(Z)$ since h is right uniformly continuous.

Let m be an invariant mean on $UCB(Z)$, and define $\varphi: G \rightarrow [0, 1]$ by

$$\varphi(x) = m(\underset{x}{j} \cdot j) = m_{t \in Z} [j(xt) j(t)].$$

Then φ is a positive definite function, and the uniform continuity of j implies that φ is continuous. $\varphi(e) = m(j^2) \geq m(j)^2 \geq \underline{M}(\chi_E)^2 > 0$, since E is relatively dense. $\varphi(x) \neq 0 \Rightarrow (xVE) \cap (VE) \neq \emptyset \Rightarrow x \in \pi^{-1}(VE) \cdot \pi^{-1}(VE)^{-1} = F$, so $\varphi(x) = 0$ for $x \notin F$.

Let M be the unique invariant mean on the space of linear combinations of continuous, positive definite functions on G , and exactly as in [6] we find that

$$M(\varphi) \geq m(j)^2 > 0.$$

Define a function $\psi: G \rightarrow [0,1]$ by $\psi(x) = M_{t \in G} [\varphi(t)\varphi(t^{-1}x)]$. As in [6], ψ will be a positive definite and almost periodic function on G , $\psi(e) = M(|\varphi|^2) \geq |M(\varphi)|^2 > 0$, and $\psi(x) = 0$ for $x \notin F^2$. So $W_1 = \{x \in G: |\psi(x) - \psi(e)| < \psi(e)\}$ is a \underline{I}_B -neighborhood of e in G contained in F^2 . Thus $W_0 = \pi(W_1)$ is a \underline{Z}_B -neighborhood of 0 in Z , and $\pi(F) = VE \circ (VE)^{-1}$, so $W_0 \subset \pi(F^2) = F \cdot \pi(F) \subset \pi(F) \circ \pi(F) \subset (VE \circ (VE)^{-1})^2 \subset W$. Thus W is a \underline{Z}_B -neighborhood of 0 in Z , and the theorem is proved.

Remark. This theorem is an analogue of Theorem 2B in [6].

Relatively accumulating subsets E of Z can be defined exactly as in [6], they can be characterized by $\overline{M}(\chi_E) > 0$, so it is easy to state and prove an analogue of Theorem 2A too.

Corollary. W is a \underline{Z}_B -neighborhood of 0 in Z iff there is a symmetric, relatively dense \underline{Z} -neighborhood U of 0 such that $U^7 \subset W$.

Proof: In Theorem 3 take $E = U$ and V a symmetric neighborhood of e in G with $V^2 \subset \pi^{-1}(U)$.

Example. The following example shows that the finite chain characterization of \underline{Z}_B may also hold when (G, Z) is not amenable. Take $G = \text{SL}(2, \mathbb{R})$, the group of 2×2 real matrices with determinant 1, $H = \left\{ \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\}$, and $Z = G/H$. Then Z is compact, in fact Z is homeomorphic to the unit circle \mathbb{T} in \mathbb{C} by the map $\pi: G \rightarrow \mathbb{T}$

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a - ic}{\sqrt{a^2 + c^2}}$$

\hat{G} will be the trivial group, so \hat{Z} is a one-point space, thus the only \underline{Z}_B -neighborhood in Z is Z itself. H is amenable as a solvable Lie group, so if (G, Z) was amenable, G also would be. (cf. [3, Theorem 2.3.3], the proof is the same as

when H is normal.) Hence (G, Z) is not amenable.

The sets $V_\varepsilon = \{e^{i\theta} : |\theta| < \varepsilon\}$, $\varepsilon \in \langle 0, \pi \rangle$ is a basis for the topology of T at $1 = \pi(e)$. Want to prove

$$V_\varepsilon^{-1} = V_\pi$$

$$\text{Write } m(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad a(t) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$\text{Then } \pi^{-1}(V_\varepsilon) = \{m(\theta) a(t) n(x) : |\theta| < \varepsilon, t > 0, x \in \mathbb{R}\}$$

$$\text{So } V_\varepsilon^{-1} = \{\pi(n(x) a(t) m(\theta)) : |\theta| < \varepsilon, t > 0, x \in \mathbb{R}\}$$

$$\pi(n(x) m(\theta)) = \frac{r(x, \theta)}{|r(x, \theta)|} \quad \text{where } r(x, \theta) = e^{i\theta} - x \sin\theta.$$

So if (x, θ) varies over $\mathbb{R} \times \langle -\varepsilon, \varepsilon \rangle$ it is easy to see that $V_\pi \subset V_\varepsilon^{-1}$. Further it is not too difficult to see that $-1 \notin V_\varepsilon^{-1}$, so $V_\varepsilon^{-1} = V_\pi$. So the only symmetric neighborhoods of 0 in Z are V_π and Z itself.

Next we will show that $V_\varepsilon \circ V_\varepsilon^{-1} = Z$, and then it suffices to show $-1 \in V_\varepsilon \circ V_\varepsilon^{-1}$. Take any $\theta \in \langle 0, \varepsilon \rangle$, then

$$\pi \begin{pmatrix} \cos\theta & (\sin\theta)^{-1} \\ -\sin\theta & 0 \end{pmatrix} = \cos\theta + i \sin\theta \in V_\varepsilon, \text{ and}$$

$$\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \in V_\varepsilon^{-1} \text{ by what we just proved. So}$$

$$\pi \begin{pmatrix} \cos\theta & (\sin\theta)^{-1} \\ -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -1 \text{ is in } V_\varepsilon \circ V_\varepsilon^{-1}$$

The compactness of Z , means that any neighborhood of 0 in Z is relatively dense, so we therefore can draw the conclusion that W is a \underline{Z}_B -neighborhood in Z iff there is a symmetric neighborhood V of 0 in Z such that $V \circ V \subset W$.

This example answers for transformation groups a question asked in [6] for topological groups, the finite chain characterization of \underline{Z}_B is not equivalent to amenability of (G, Z) . The example also suggests that for topological groups the answer to the question is negative. One would also be interested in examples of transformation groups where the finite chain characterization of \underline{Z}_B does not hold, but none is known to the author.

4. The Bohr compactification of a semi-direct product.

In this section we shall see that Theorem 2 can be used to characterize the Bohr compactification of a semi-direct product $G = H \rtimes Z$. The result is that \hat{G} is a semi-direct product of two groups, \hat{H} and \hat{Z} , where \hat{H} is the Bohr compactification of H and \hat{Z} is a certain compactification of Z , depending on τ . This problem has been

studied by other methods (and in greater generality) in [7].

First recall the definition of a semi-direct product:

Let H and Z be topological groups with H acting on Z as a group of automorphisms, that is, there is a jointly continuous map $\tau: H \times Z \rightarrow Z$ satisfying

$$(*) \quad \tau(hk, x) = \tau(h, \tau(k, x)) \quad \text{and} \quad \tau(h, xy) = \tau(h, x)\tau(h, y)$$

for $h, k \in H$ and $x, y \in Z$. Let $G = H \times Z$ with product topology and group structure given by $(h, x)(k, y) = (hk, x\tau(h, y))$. G is then a topological group, called a semi-direct product of H and Z , written $G = H \hat{\tau} Z$.

Define an action of G on Z by $(h, x)y = x\tau(h, y)$, for $h \in H, x, y \in Z$. This is really an action, and the action is jointly continuous. Let \hat{Z} be the Bohr compactification of Z as G -space (as defined in Section 1), and \hat{H} the Bohr compactification of H (as a topological group). We shall see that \hat{Z} is a topological group, that τ can be lifted to a map $\hat{\tau}: \hat{H} \times \hat{Z} \rightarrow \hat{Z}$ also satisfying (*), and that the corresponding semi-direct product $\hat{H} \hat{\tau} \hat{Z}$ is the Bohr compactification of G .

Observe that the action of G on Z is transitive (i.e. $Gx = Z$ for each $x \in Z$), and the stability subgroup of $e \in Z$ is H (considered as a closed subgroup of G in

the obvious way), so as a G -space $Z \cong G/H$, and the isomorphism is also a homeomorphism.

Now let us find out what the topology Z_B of Theorem 2 looks like in this case. The projection $\pi: G \rightarrow G/H \cong Z$ is simply $\pi(h,x) = x$, so if U is a subset of Z , the subset U^{-1} defined in the beginning of Section 2 is the same as $\pi((H,U)^{-1}) = \tau(H,U^{-1})$, since $(h,u)^{-1} = (h^{-1}, \tau(h^{-1}, u^{-1}))$. So a subset U of Z is symmetric in the sense of Section 2 iff U is H -invariant (i.e. $\tau(H,U) = U$) and $U = U^{-1}$ in the group Z . Further, the set $U \circ V$ defined in Section 2 is the set $\pi^{-1}(U)V = (H,U)V = U \cdot \tau(H,V)$. So if V is H -invariant, $U \circ V = U \cdot V$ (multiplication in Z). Applying these observations on Theorem 2, we have proved the first half of

Proposition 3. A subset V of Z is a Z_B -neighborhood of e in Z iff there is a sequence $\{V_n\}$ of Z -neighborhoods of e such that

$$(i) \quad V_0 \subset V, \quad V_{n+1}^2 \subset V_n, \quad V_n^{-1} = V_n$$

(ii) each V_n is H -invariant and Z -relatively dense.

Further Z_B is a group topology on Z , and the group-operations are uniformly continuous with respect to Z_B .

Proof: Just note that a H -invariant subset of Z is

G -relatively dense iff it is Z -relatively dense. The uniform continuity of the group operations then follows from [6, Lemma 1].

Let $\tau^h: Z \rightarrow Z$ be the map defined by $\tau^h(x) = \tau(h, x)$. The family $\{\tau^h: h \in H\}$ is then Z_B -uniformly equicontinuous, so τ can be extended to a continuous map $\hat{\tau}: \hat{H} \times \hat{Z} \rightarrow \hat{Z}$. Let ρ denote the natural map of both H and Z into \hat{H} , respectively \hat{Z} . Since $\rho(H)$ is dense in \hat{H} and $\rho(Z)$ is dense in \hat{Z} , the equations in (*) also hold for $\hat{\tau}$. So we can form the semi-direct product $\hat{H} \hat{\tau} \hat{Z}$.

Theorem 4. With notation as above, $\hat{G} \cong \hat{H} \hat{\tau} \hat{Z}$.

Proof: We shall prove that $(\hat{H} \hat{\tau} \hat{Z}, \rho)$ has the unique factorization property defined in [1] with respect to $G = H \tau N$. So let φ be a continuous homomorphism from G into a compact group K . Define an action of G on K by $(h, x)k = \varphi(e, x)\varphi(h, e)k\varphi(h^{-1}, e)$. This is an action, and since K has a basis of neighborhoods invariant under inner automorphisms, it is easy to see that the action is uniformly equicontinuous.

Define $\psi: Z \rightarrow K$ by $\psi(x) = \varphi(e, x)$, then $\psi((h, x)y) = (h, x)\psi(y)$, so by Proposition 1 ψ can be uniquely

extended to a map $\hat{\psi}: \hat{H} \rightarrow K$. $\hat{\psi}$ is a homomorphism on $\rho(Z)$, so by density also on \hat{Z} . Define $\psi': H \rightarrow K$ by $\psi'(h) = \varphi(h, e)$, then ψ' can be uniquely extended to a homomorphism $\hat{\psi}': \hat{H} \rightarrow K$. Then define $\hat{\phi}: \hat{H} \hat{\tau} \hat{Z} \rightarrow K$ by $\hat{\phi}(h, x) = \hat{\psi}(x) \hat{\psi}'(h)$. If ρ also denotes the canonical map from $H \tau Z \rightarrow \hat{H} \hat{\tau} \hat{Z}$, we have $\varphi = \hat{\phi} \circ \rho$, $\hat{\phi}$ is unique, and by using the density of $\rho(H \tau Z)$ in $\hat{H} \hat{\tau} \hat{Z}$ it is not difficult to prove that $\hat{\phi}$ is a homomorphism.

Finally, let us also look at Z_B in the amenable situation. Let $UCB_G(Z)$ denote the space of uniformly continuous functions defined in Section 3, i.e. $f \in UCB_G(Z)$ iff to each $\epsilon > 0$ there is a neighborhood U of e in G such that $|f((h, x)y) - f(y)| = |f(x\tau(h, y)) - f(y)| < \epsilon$ for $(h, x) \in U$ and all $y \in Z$. So $UCB_G(Z)$ is a subspace of $UCB_r(Z)$ the ordinary right uniformly continuous functions on Z as defined in [5, p. 210]. Corresponding to the definition in Section 3 we shall say that Z is G -amenable if there is a G -invariant mean on $UCB_G(Z)$.

Lemma 3. G is amenable $\Leftrightarrow H$ is amenable and Z is G -amenable. If G is locally compact, then Z G -amenable $\Rightarrow Z$ amenable.

Proof: The first \Rightarrow is trivial, for the other way let m' be a left invariant mean on $UCB_r(H)$ and m'' a G -invariant on $UCB_G(Z)$. Then define m on $UCB_r(G)$ by $m(f) = m'_{h \in H} (m''_{x \in Z} (f(h,x)))$. Rather straightforward, although note quite trivial calculations show that m is well defined and that m will be a left invariant mean on $UCB_r(G)$.

That Z G -amenable $\Rightarrow Z$ amenable when G is locally compact follows from [4, Theorem 3.1]. The converse of the last implication seems unlikely to be true.

Going back to the characterization of Z_B , the corollary of Theorem 3 gives:

Corollary (to Proposition 3). If the action of G on Z is amenable, a subset V of Z is a Z_B -neighborhood of e iff there is a symmetric, H -invariant, relatively dense Z -neighborhood U of e with $U^7 \subset V$.

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