On a class of complex function spaces.

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In [3] and [4] E. Effros proposed and investigated the complex analogue of the preduals of real L^1-spaces, also termed Lindenstrauss spaces.

The aim of the present note is to characterize those complex function spaces which are complex Lindenstrauss spaces, in terms of orthogonal measures on the Choquet-boundary. The main result is the following:

Theorem. Let X be a compact Hausdorff space and $A \subseteq C_{\mathbb{C}}(X)$ a closed linear subspace, separating the points of X and containing the constant functions. Let S denote the state space of A. Then the following statements are equivalent:

(i) A is a complex Lindenstrauss space.

(ii) $\mu \in A^{\perp} \cap M(\partial_A X) \Longrightarrow \mu = 0$.

(iii) $Z = \text{conv}(S \cup -iS)$ is a Choquet-Simplex.

(iv) A is self-adjoint and $\text{Re}\, A$ is a real Lindenstrauss space.

As a consequence we shall see that no uniform algebra is a complex Lindenstrauss space unless it is $C_{\mathbb{C}}(X)$.

Preliminaries and notation.

Let X be a compact Hausdorff space and let $A \subseteq C_{\mathbb{C}}(X)$ be a closed linear subspace, separating the points of X and containing the constant functions.
The state space of A i.e.

$$S = \{p \in A^* \mid p(\mathbb{1}) = \|p\| = 1\}$$

is a w^*-closed face of the closed unit ball K of A^*.

Define

$$Z = \operatorname{conv}(S \cup -iS)$$

and let $\theta\colon A \to A(Z)$ be defined as

$$\theta a(z) = \operatorname{Re} z(a) , \qquad \forall z \in Z .$$

Then θ is a bicontinuous real-linear isomorphism of A onto the space $A(Z)$ of continuous affine functions on Z, cf. [2].

We note that S is a closed face of Z with complementary face $S' = -iS$. Moreover, the barycentric coefficient in the decomposition after S and S' is uniquely determined i.e. S is a parallel face of Z. For details we refer to [1].

Let φ denote the canonical embedding of X into S i.e.

$$\varphi(x)(a) = a(x) , \qquad \forall a \in A .$$

Also let T denote the unit circle and define $\Phi\colon T \times X \to K$ by

$$\Phi(\lambda, x) = \lambda\varphi(x)$$

and $L\colon C_{\mathbb{C}}(X) \to C_{\mathbb{C}}(T \times X)$ by

$$Lf(\lambda, x) = \lambda f(x) \qquad \forall (\lambda, x) \in T \times X$$

It follows from [5] and [6] that $L^* \circ \Phi^{-1}$ maps maximal probability measures on K into complex boundary measures on X;

i.e. $\mu \in M_1^+(\partial_e K)$ implies $L^*(\Phi^{-1}\mu) \in M(\partial_A X)$.

Following Effros [4] we define for $f \in C_{\mathbb{C}}(K)$ the function

$$\mathrm{inv}_T f(p) = \int_T f(\alpha p) d\alpha \quad , \quad \forall p \in K ,$$

where $d\alpha$ is the normalized Haar measure on T . It is easily verified that inv_T is a norm-decreasing projection in $C_{\mathbb{C}}(K)$.

Similarly, we write

$$\mathrm{hom}_T f(p) = \int_T \alpha^{-1} f(\alpha p) d\alpha \quad , \quad \forall p \in K$$

and observe that hom_T is a norm-decreasing projection in $C_{\mathbb{C}}(K)$. Taking adjoints of these projections we obtain the following norm-decreasing w*-continuous projections in $M(K)$:

$$\mathrm{inv}_T \mu = \mu \circ \mathrm{inv}_T$$

$$\mathrm{hom}_T \mu = \mu \circ \mathrm{hom}_T .$$

In [4] Effros proved that complex Banach spaces V with V^* isometrically isomorphic to $L^1(Y,\mathcal{B},\mu)$ for some measure space $(Y,\mathcal{B},\mu)$ can be characterized by the following condition on the closed unit ball K in V^*

(*) $\mu,\nu \in M_1^+(\partial_e K)$ with $r(\mu) = r(\nu) \Longrightarrow \mathrm{hom}_T \mu = \mathrm{hom}_T \nu$

Here $r\colon M_1^+(K) \to K$ denotes the barycentric map.
Such Banach spaces are called <u>complex Lindenstrauss spaces</u>.

<u>Lemma 1</u>. Let $\mu \in M_1^+(\partial_e K)$. Then the measures $\mathrm{hom}_T \mu$ and $\mathrm{inv}_T \mu$ are boundary measures on K .

<u>Proof</u>: [4, Lemma 4.2].

If $\nu \in M(X)$, then we denote by $\varphi(\nu)$ the direct image of ν under φ .

Lemma 2. Let $\mu \in M_1^+(\partial_e K)$. Then the measure $\nu = L^*(\Phi^{-1}\mu)$ is a complex boundary measure on X such that

$$\hom_T \mu = \hom_T(\varphi\nu)$$

Proof: By [5] ν is a complex boundary measure on X .

Let $f \in C_{\mathbb{C}}(K)$. Then

$$L(\hom_T f\circ\varphi)(\lambda,x) = \hom_T f(\lambda\varphi(x))$$

and hence,

$$\hom_T(\varphi\nu)(f) = \int_X \hom_T f\circ\varphi \, d\nu = \int_X \hom_T f\circ\varphi \, d\, L^*(\Phi^{-1}\mu)$$

$$= \int_{T\times X} L(\hom_T f\circ\varphi) \, d\Phi^{-1}\mu = \int_{\Phi(T\times X)} (L(\hom_T f\circ\varphi))\circ\Phi^{-1} d\mu$$

$$= \int_K \hom_T f \, d\mu = \hom_T\mu(f)$$

and the lemma is proved.

We shall need the following fact on the embedding of S in Z.

Lemma 3. S is a split face of Z if and only if A is closed under complex conjugation.

Proof: Assume S is a split face of Z . Let $a \in A$ and decompose $a = a_1 + ia_2$.

Define $b_1 \in A(S)$ and $b_2 \in A(-iS)$ by

$$b_1(p) = \theta a(p) \ , \quad b_2(-ip) = -\theta a(-ip) \ , \quad \forall p \in S$$

Since S is a split face of Z and $S' = iS$ is closed, it follows from [1, Prop II.6.19] that there exists $h \in A$ such that

$$\theta h|_S = b_1 \quad , \quad \theta h|_{S'} = b_2$$

Thus for $x \in X$ we shall have

$$a_1(x) - ia_2(x) = \theta a(x) - i\theta a(-ix) = \theta h(x) + i\theta h(-ix) = h(x)$$

Hence $\bar{a} \in A$.

Conversely, we assume that A is closed under complex conjugation. Consider convex combinations

$$\lambda p_1 + (1-\lambda)(-iq_1) = \lambda p_2 + (1-\lambda)(-iq_2)$$

where $p_i, q_i \in S$ and $0 < \lambda < 1$ for $i = 1,2$.

If $p_1 \neq p_2$, then it follows from the Hahn-Banach theorem and the assumption on A that we can find $a = \bar{a} \in A$ such that $\theta(a)(p_1) \neq \theta(a)(p_2)$. Moreover,

$$\lambda\theta a(p_1) + (1-\lambda)\theta a(-iq_1) = \lambda\theta a(p_2) + (1-\lambda)\theta a(-iq_2) .$$

Since $\theta a|_{S'} = 0$, we shall have

$$\lambda\theta a(p_1) = \lambda\theta a(p_2)$$

which is a contradiction, and the lemma is proved.

Proof of theorem.

i) ==> ii) . Let $\mu \in A^{\perp} \cap M(\partial_A X)$ and decompose μ as

$$\mu = \lambda_1\mu_1 - \lambda_2\mu_2 + i\lambda_3\mu_3 - i\lambda_4\mu_4 ,$$

where $\lambda_i \geq 0$ and $\mu_i \in M_1^+(\partial_A X)$ for $i = 1,2,3,4$.

Let $p_i = r(\varphi(\mu_i))$ for $i = 1,2,3,4$. Then

$$0 = \lambda_1 p_1 - \lambda_2 p_2 + i\lambda_3 p_3 - i\lambda_4 p_4 ,$$

or equivalently

$$\lambda_1 p_1 + \lambda_4(-ip_4) = \lambda_2 p_2 + \lambda_3(-ip_3) .$$

Let z be the common value of the left and right hand sides of this equation.

Since $\mathbb{1} \in A$ we conclude that $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. Hence we may assume that $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = 1$. Specifically, $z \in Z$.

If $\psi: K \to K$ is defined by $\psi(p) = -ip \quad \forall p \in K$, then the measures

$\nu_1 = \lambda_1(\varphi(\mu_1)) + \lambda_4 \psi(\varphi(\mu_4))$,

$\nu_2 = \lambda_2(\varphi(\mu_2)) + \lambda_3 \psi(\varphi(\mu_3))$,

are maximal probability measures representing z.

Since A is a complex Lindenstrauss space, it satisfies the condition (*). Hence:

$$\hom_T \nu_1 = \hom_T \nu_2$$

Let $f \in C_{\mathbb{C}}(X)$, define $\bar{f}$ on $\Phi(T \times X)$ by

$$\bar{f}(\lambda\varphi(x)) = \lambda f(x) ,$$

and extend $\bar{f}$ to $\bar{f} \in C_{\mathbb{C}}(K)$ (Tietze). Then

$$\hom_T \bar{f}(\varphi(x)) = f(x) \qquad \forall x \in X .$$

Moreover,

$$\begin{aligned} \hom_T \nu_1(\bar{f}) &= \lambda_1 \int_X \hom_T \bar{f} \circ \varphi \, d\mu_1 + \lambda_4 \int_X \hom_T \bar{f} \circ \psi \circ \varphi \, d\mu_4 \\ &= \lambda_1 \int_X f \, d\mu_1 - i \lambda_4 \int_X f \, d\mu_4 \end{aligned}$$

Similarly,

$$\hom_T \nu_2(\bar{f}) = \lambda_2 \int_X f \, d\mu_2 - i \lambda_3 \int_X d \, d\mu_3 ,$$

and hence

$$0 = \lambda_1\mu_1(f) - \lambda_2\mu_2(f) + i\lambda_3\mu_3(f) - i\lambda_4\mu_4(f) = \mu(f)$$

i.e. $\mu = 0$ and (ii) follows.

ii) ⟹ i). The condition (*) is seen to be an immediate consequence of Lemma 2.

ii) ⟹ iii). First we observe that S is a Choquet-Simplex since (ii) asserts that there is no real annihilating boundary measures. Hence it suffices to prove that S is a split face of Z or equivalently that A is selfadjoint.

To see this we assume that $a \in A$ and $\bar{a} \notin A$. Then there exsits a measure $\mu \in A^{\perp}$ such that $\mu(\bar{a}) \neq 0$. Decompose μ into real and imaginary parts i.e. $\mu = \mu_1 + i\mu_2$ and choose real boundary measures $\nu_i \in M(\partial_A X)$ such that

$$\mu_i - \nu_i \in A^{\perp} \qquad \text{for} \quad i = 1,2 .$$

Define $\nu = \nu_1 + i\nu_2$, then $\nu \in A^{\perp} \cap M(\partial_A X)$ and from (ii) we conclude that $\nu = 0$ and hence $\nu_1 = 0 = \nu_2$. In particular $\mu_i \in A^{\perp}$ for $i = 1,2$ and hence $\mu(\bar{a}) = 0$, and we have obtained a contradiction.

iii) ⟹ ii). Let $\mu \in A^{\perp} \cap M(\partial_A X)$ and decompose μ as $\mu = \mu_1 + i\mu_2$. Since A is selfadjoint, we shall have that $\mu_i \in A^{\perp}$ for $i = 1,2$, and since S is a Choquet-Simplex, we conclude that $\mu_i = 0$ for $i = 1,2$ and hence $\mu = 0$.

iii) ⟺ iv). Trivial.

This completes the proof of the theorem.

Remark. In order to prove that (ii) implies (i) we could have used the fact that the space of complex boundary measures $M(\partial_A X)$ is an L^1-space [4], and since every $p \in A^*$ can be represented by a complex boundary measure $\mu_p \in M(\partial_A X)$ with $\|p\| = \|\mu_p\|$, condition (ii) asserts that A^* is isometrically isomorphic to

$M(\partial_A X)$, and (i) follows.

Specializing to uniform algebras, we obtain the following:

Corollary. Let A be a uniform algebra. Then A is a complex Lindenstrauss space if and only if $A = C_{\mathbb{C}}(X)$.

Proof: If A is a complex Lindenstrauss space, then A is a selfadjoint, and now the Stone-Weirstrass theorem yields $A = C_{\mathbb{C}}(X)$.

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