ON ABSOLUTELY MEASURABLE SETS

by

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Dedicated to Professor Andrzej Mostowski
on the occasion of his sixtieth birthday.

Let E be an analytic space, i.e. E is a Hausdorff space which is the continuous image of the Baire-space N^N, where N is the set of natural Numbers. Let $\mathcal{E}$ be the class of absolutely measurable sets in E, i.e. the sets which are μ-measurable with respect to every σ-finite, complete and regular Borel measure μ on E.

Problem: Give a description of the class $\mathcal{E}$.

As usual we shall "describe" a set by its definability characteristics, hence the precise version of the problem is: Which sets in the projective hiearchy over E are absolutely measurable.

It turns out that the answer depends on the underlying set theoretic axioms (Gödel [7], Soloway [23]). Classically,, i.e. on the basis of ordinary Zermelo-Fraenkel set theory, one can show that every analytic (Suslin, $\underset{\sim}{\Sigma}^1_1$:) set is absolutely measurable. But there are consistent extensions ZF_1 and ZF_2 of ordinary ZF set theory such that in ZF_1 one can prove the existence of a non-absolutely measurable set of class $\underset{\sim}{\Sigma}^1_2$ (i.e. the sets which are projections of co-analytic sets), whereas in ZF_2 one can prove that every $\underset{\sim}{\Sigma}^1_2$-set is absolutely measurable.

The problem is thus undecidable in ZF. And at present there is no universally accepted extension of ZF which allows us to decide the problem.

Our main concern in this paper is to use some recent results in set theory to push the classical results a bit further. This part is based on the cand.real. Thesis of D. Normann [17]. We shall also include a brief section giving a survey of some known results relating various set theoretic axioms to questions of measurability.

1. PRELIMINARIES.

We shall need various results from set theory, topology and measure theory. The backbround required in topology and measure theory will be very standard, and we shall be brief on this point. The background needed from set theory may not be so familiar to the "ordinary" mathematicians, which is our excuse for being somewhat more expansive here.

1.1. Background from set theory.

By "ordinary set theory" we mean the Zermelo-Fraenkel-system. (For a brief introduction to ZF, see Cohen [4], chapter II, Jech [10].) It is well known that in ZF one can obtain the universe of all sets by starting from the empty set and iterating the power-set operation along the ordinals. Let V be the universe of all sets. Let $V_0 = \emptyset$ and $V_\alpha = \mathcal{P}(\bigcup_{\beta<\alpha} V_\beta)$, where $\mathcal{P}$ is the powerset operation, and α, β are ordinals. Symbolically, one then can write $V = \bigcup V_\alpha$, where α runs through the ordinals. (More precisely the following statement is provable in ZF : $\forall x \exists \alpha (x \in V_\alpha)$.) This seems to give a description of the universe, but the "description" is only apparent, one of the main defects being that the power set operation is left unanalyzed. And it is precisely this fact which has ramifications for questions of measurability.

1.1.1. Constructability. A very restricted notion of the power-set operation was used by Gödel (Gödel [7], [8], see also Cohen [4] ch. III and Jech [10]) as a technical device for proving the consistency of the axiom of choice and the generalized continuum hypothesis with ZF.

Let $\langle M, R_1,\ldots,R_n\rangle$ be a first order structure (i.e. M is a non-empty set and $R_1,\ldots,R_n$ are relations over M). By Def $(\langle M,R_1,\ldots,R_n\rangle)$ we understand the collection of subsets X of M which are first order definable over M, i.e. there is some formula Φ and elements $y_1,\ldots,y_m \in M$ such that $X = \{x \in M; M \models \Phi(x,y_1\ldots y_m)\}$, where $M \models \Phi(x,y_1\ldots y_m)$ means that the formula Φ is satisfied by the elements $x,y_1\ldots y_m$ in the structure M.

The constructible universe, L, is now defined by the following transfinite induction (α and λ are ordinal):

$$L_0 = \emptyset$$

$$L_{\alpha+1} = \mathrm{Def}\,(\langle L_\alpha, \varepsilon_\alpha\rangle) \quad \text{where } \varepsilon_\alpha \text{ is the membership-relation restricted to } L_\alpha.$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \quad \text{when } \lambda \text{ is a limit ordinal.}$$

$$L = \bigcup_{\alpha} L_\alpha \quad \text{where } \alpha \text{ runs through all ordinals.}$$

If we compare the definitions of V and L, we notice that in the latter case we have introduced a spesific meaning to the powerset operation, viz. the powerset consists of those subsets of the given set which are ordinal definable over the given set, with the ε-relation as the only relation.

1.1.2. <u>Forcing</u>. We begin with a brief remark on models for ZF: A model for ZF is a non-empty set M and a binary relation E on M such that $\langle M,E\rangle$ satisfies the axioms of ZF. M is the "universe" of the model, the elements of M are the sets of the model, and E is the membership-relation of the model. The model M is standard if E is the usual $\in$-relation and M is transitive, i.e. $x \in y \in M$ implies $x \in M$. We know that if ZF is consistent. there is an (even countable model for ZF, but not necessarily a standard model.

Let M be a countable standard model for ZF. The forcing method, as introduced by P.J. Cohen [4], (see also Jech [10] is a technique for adding new sets to M. We explain this:

A partially ordered set $\mathbb{P} = \langle \mathbb{P}, \leq \rangle$ is called a set of <u>conditions</u> if every $p \in \mathbb{P}$ has a proper extension in $\mathbb{P}$ (i.e. there is a $q \in \mathbb{P}$ such that $q \leq p$, $q \neq p$) and every $p \in \mathbb{P}$ has incompatible extensions in $\mathbb{P}$.

Each $\mathbb{P}$ determines a Boolean algebra in the following way. For $X \subseteq \mathbb{P}$, set $\neg X = \{q \mid \forall r \leq q;\ r \notin X\}$. Define

$$\mathbb{B} = BA(\mathbb{P}) = \{X \subseteq \mathbb{P};\ \neg\neg X = x\}.$$

$\mathbb{B}$ with obvious operations will be a complete Boolean algebra (the algebra of "regular open sets").

Given a complete Boolean algebra $\mathbb{B}$, we can extend the notion of first order structure to the notion of a $\mathbb{B}$-valued structure. Normally a relation can be viewed as a map from the domain into the Boolean algebra {false, true}, ($\{0, 1\}$). In the $\mathbb{B}$-valued case the relations are maps from the domain into $\mathbb{B}$. And given any such maps we can define for any sentence ϕ in the language appropriate to the structure, the truth-value $[\![\phi]\!]$ in a manner which completely generalizes the classical case (where $[\![\phi]\!]$ is either $\mathbb{0}$ (= false) or $\mathbb{1}$ (= true)).

If $\mathbb{B} = BA(\mathbb{P})$ we shall define a certain $\mathbb{B}$-valued model of ZF, $V^{\mathbb{P}}$, consisting of <u>the heriditary $\mathbb{P}$-sets</u>. Let x be any set. The domain and range of x is defined as follows: $y \in dom(x)$ iff $\langle y,z \rangle \in x$ for some z, and $y \in rng(x)$ iff $\langle z,y \rangle \in x$ for some z. Set $h(x) = x \cup dom(x) \cup dom(dom(x)) \cup \ldots$. $h(x)$ is called the heriditary domain of x. x is called a heriditary $\mathbb{P}$-set if $h(x)$ is a relation and $rng(h(x)) \subseteq \mathbb{P}$. As a very simple example of a heriditary $\mathbb{P}$-set we mention $x = \{\langle\emptyset,p\rangle \; ; \; p \in \mathbb{P}\}$. $V^{\mathbb{P}}$ shall be the class of all heriditary $\mathbb{P}$-sets.

On $V^{\mathbb{P}}$ one can, by recursion, define $\mathbb{B}$-valued relations I (identity) and E (membership) such that

$$\underset{\cdot}{V}^{\mathbb{P}} = \langle V^{\mathbb{P}}, I, E\rangle$$

is a $\mathbb{B}$-valued model for ZF (i.e. $[\![\phi]\!] = 1\!\!1$ for all ZF-axioms ϕ).

Once more, let M be a countable standard model for ZF and let $\mathbb{P} \in M$ be a set of conditions in M. A subset $\Delta \subseteq \mathbb{P}$ is called <u>dense</u> if $\forall p \in \mathbb{P} \; \exists q \leq p (q \in \Delta)$. $G \subseteq \mathbb{P}$ is called <u>$\mathbb{P}$-generic over M</u> if the following three conditions are satisfied:

i. $\forall p,q \, (p \in G \wedge p \leq q \rightarrow q \in G)$

ii. $\forall p,q \in G$ (p,q are compatible, i.e. have a common extension)

iii. If $\Delta \in M$ and Δ is dense in $\mathbb{P}$, then $G \cap \Delta \neq \emptyset$.

A simple but basic result is that for every $p \in \mathbb{P}$ there exists a $\mathbb{P}$-generic G over M such that $p \in G$. (It is essential that M is countable.)

By induction we can now define a map $G^* : V^{\mathbb{P}} \rightarrow V$ by

$$G^*(x) = \{G^*(y) \mid \exists p \in G \; (\langle y,p\rangle \in x)\}.$$

Let N be the G^*-image of $V^{\mathbb{P}} \cap M$. One may now show that N is an "ordinary" (i.e. two-valued) model for ZF, that $M \subseteq N$, that M and N have the same ordinals, that $G \in N - M$, and that N is the constructible closure of $M \cup \{G\}$ along the ordinals of M; which we write $N = M[G]$.

<u>Example</u>. Let M be a countable standard model for ZF. Let $\mathbb{P}$ consists of all finite 0-1 sequences, each $p \in \mathbb{P}$ is a finite

bit of a characteristic function of a subset of ω. Let G be $\mathbb{P}$-generic over M. Then $\bigcup G$ is (the characteristic function of) a subset of ω, which belong to $N = M[G]$ but not to M. (If $G \in M$, then $\mathbb{P} - G$ is dense, hence should meet G by iii above.)

We need one more notion. If $\mathbb{B} = BA(\mathbb{P})$ and M is a $\mathbb{B}$-valued structure we define the <u>forcing relation</u> between a condition $p \in \mathbb{P}$ and a sentence Φ in the language appropriate to the structure M by

$$p \Vdash \Phi \quad \text{iff} \quad p \in [\![\Phi]\!].$$

We have now the following relationship ("completeness theorem for forcing") between forcing and validity in the model $N = M[G]$.

$$N \models \Phi \quad \text{iff} \quad \exists p \in G - (p \Vdash \Phi),$$

i.e. a sentence Φ is valid in N iff it is forced by some condition in G. (The expert will note that we have been somewhat inexact with respect to the various languages involved.)

1.1.3. <u>The projective hierarchy</u>. The purpose of this section is to fix notation. Let $X_n = (N^N)^n$ and $X = \bigcup_n \mathcal{P}(X_n)$. The <u>projective sets</u> is the least subclass of X which contains all Borelsets and is closed under the following operations:

i. Projection of a set in a space X_n down to a space X_m of lower dimension.

ii. Complementation within a space X_n.

Since X_n and X_m are homeomorphic for all n and m we can always assume that projections lower dimension by one. Thus a set A in X_n is projective iff for some m and some Borelset $B \subseteq X_{n+m}$.

$$\langle g_1,\dots,g_n\rangle \in A \quad \text{iff} \quad Q_1f_1\dots Q_mf_m(\langle f_1,\dots,f_m,\ g_1,\dots,g_n\rangle \in B),$$

where $Q_1, \ldots Q_m$ is an alternating string of quantifiers $\forall, \exists$. A belongs to class $\underset{\sim}{\Pi}_m^1$ if $Q_1 = \forall$, and to class $\underset{\sim}{\Sigma}_m^1$ if $Q_1 = \exists$. $\underset{\sim}{\Delta}_m^1$ is defined as $\underset{\sim}{\Pi}_m^1 \cap \underset{\sim}{\Sigma}_m^1$.

A subset $A \subseteq (N^N)^k \times N^n$ is called <u>recursive</u> if we can effectively decide if an element belongs to A or not. We shall not need a more precise definition in this paper. A set A is recursive relative to some $f \in N^N$ if there is a recursive set B such that $x \in A$ iff $\langle x,f \rangle \in B$.

For every Borelset A there are sets S and R recursive relative to some function f such that

$$x \in A \text{ iff } \forall f \exists n (\langle x,f,n \rangle \in R) \text{ iff } \exists g \forall m (\langle x,g,m \rangle \in S).$$

This can be used to give more specific form to the defining condition for a set in the projective hierarchy.

If we build the hierarchy with absolutely recursive sets, we get the socalled "analytic" hierarchy of recursion theory. (For further information on this topic see Addison [9], or Shoenfield [19], chapter 7).

1.2. <u>Background from topology and measure theory</u>. As a general reference we mention J. Hoffmann Jørgensen, <u>The theory of analytic spaces</u> [9]. Here we recall a few basic definitions and results.

A <u>Polish space</u> is a Hausdorff space which admits a complete metric and which has a countable, dense subset.

An <u>analytic space</u> is a Hausdorff space, which is the continuous image of a Polish space. Such spaces need not be metrizeable.

Any Polish space is homeomorphic to a G_δ set in I^N, where I is the unit interval in $\mathbb{R}$. And any analytic space is the continuous image of Baire space N^N. Further, any compact subspace of an analytic space is Polish.

An <u>analytic measure space</u> is a pair $\langle E,\mu\rangle$, where E is an analytic space and μ is a measure on E satisfying:

i. μ is σ-finite, i.e. E is a countable union of sets with finite μ-measure.

ii. μ is a Borelmeasure, i.e. every Borelset in E is μ-measurable.

iii. μ is regular, i.e. $\mu(A) = \sup\{\mu(K);\ K \text{ compact} \wedge K \subseteq A\}$, for all μ-measurable A.

iv. μ is complete, i.e. every subset of a set with μ-measure 0 is μ-measurable.

We refer the reader to Hoffmann Jørgensen [9] for the importance of analytic measure spaces. Perhaps it suffices to say that the class is large enough to include the most important examples of topological measure spaces, but small (or "nice") enough to exclude various pathologies which may occur in general measure spaces. Analytic measure spaces was first introduced by P. Cartier generalizing more specialized examples considered in the context of probability theory of D. Blackwell and A.Kolmogoroff (see [9] for exact references.)

2. <u>SURVEY</u>. We restrict ourselves to question about Lebesgue-measurability over the real line $\mathbb{R}$. As will be seen from the next section both positive and negative results can be lifted to arbitrary analytic measure spaces. The classical result is that every $\underset{\sim}{\Sigma}^1_1$ set is measurable. And equally classical is the result that there are non-measurable sets.

2.1. The possibilities. This is a question of consistency. The basic results are due to K. Gödel and R. Solovay.

2.1.1. Negative results. Already Gödel [7] in 1938 knew that if one adds the assumption that every set is constructible (which we write symbolically as $V = L$) to ZF, then there are $\underset{\sim}{\Delta}^1_2$ subsets of $\mathbb{R}$ which are not Lebesgue-measurable. Gödel showed that if ZF is consistent, so is $ZF + V = L$ (i.e. ZF with the axiom of constructibility added). And in the theory $ZF + V = L$ there is a $\underset{\sim}{\Delta}^1_2$ well ordering of the reals. If one goes back to section 1.1.1 one easily sees from the construction of $L_{\alpha+1}$ from L_α that there is a well-ordering of L. A more delicate analysis is needed to see that it is $\underset{\sim}{\Delta}^1_2$ when restricted to the reals. (The details were published by Addison [2].) And applying the $\underset{\sim}{\Delta}^1_2$ well-ordering in the usual construction of a non-measurable set at once gives a $\underset{\sim}{\Delta}^1_2$ counterexample.

2.1.2. Positive results. Using Cohen forcing technique R.Solovay in [23] (for an exposition, see also Jech [10]) proved the following result: Assume that there is a standard model for ZF pluss the assumption that there exists a strongly inaccessible cardinal number. Then there is a standard model for ZF (with the axiom of choice weakened to the axiom of dependent choice) in which every subset of $\mathbb{R}$ is Lebesgue-measurable. Notice that whereas the (uncountable) axiom of choice is needed for getting a non-measurable set of $\mathbb{R}$ in ZF alone, the axiom of dependent choice, DC , which says that if R is a binary relation on a set A such that $\forall x \exists y \langle x,y\rangle \in R$, then there is a map $f : N \to A$ such that $\forall n \langle f(n),f(n+1)\rangle \in R)$ is sufficient to obtain the usual "positive" results of measure theory.

2.2. Axiomatic extensions. This is a question of truth. The assumption is that there is a real universe of sets, and in this universe which exists independent of our attempts at describing it through some particular axioms, every individual set of reals is Lebesgue-measurable or not. What we so far can conclude from 2.1 is that the current attempts to describe this universe (e.g. through the ZF axioms) has only been partially successful.

This need not be a scandal if our insight into the "true" universe of set theory leads us to evident principles extending ZF and which allows us to decide e.g. the problem of which sets of reals are Lebesgue-measurable.

So let us briefly review some attempts at pinning down the fine structure of the continuum by axiomatic extensions.

2.2.1. Constructibility. This was already discussed in section 1.1.1 and 2.1.1. In the theory $ZF + V = L$ there exists a $\underset{\sim}{\Delta}^1_2$ well-ordering of the reals which enables us to answer almost all questions about the projective hierarchy (see Addison [1], [2]).

2.2.2. Measurable cardinals. Axioms asserting the existence of certain very large cardinals are natural candidates for extending ZF set theory. There is one such axiom which has remarkable consequences for the projective hierarchy.

A cardinal K is called measurable if there exists a less than K-additive 0-1 valued measure on the powerset of K which takes the value 0 on singletons and value 1 on K. It is not perhaps immediate that a measurable cardinal is a large cardinal. However, it is known that if K is an uncountable measurable cardinal, then there are K strongly inaccessible cardinals less than K. (For an introduction to this topic, see Shoenfield [20].)

Let MC mean that there is an uncountable measurable cardinal. In the theory ZF + MC one can show that every $\underset{\sim}{\Sigma}^1_2$ set is Lebesgue-measurable. This result is due to Solovay (unpublished, a proof can be extracted from [23], see also section 3.1 below).

This seems to be the optimal result in ZF + MC. The axioms V = L and MC contradicts each other. But the ideas of measurability and constructibility can be joined in the following way. Let L_μ be the sets relatively constructible from a measure μ on K. Then if ZF + MC is consistent, then ZF + V = L_μ is consistent (where V = L_μ says that every set belongs to L_μ). And in ZF + V = L_μ one can prove that there is a $\underset{\sim}{\Delta}^1_3$ well-ordering of the reals, which at once gives a $\underset{\sim}{\Delta}^1_3$ non-Lebesgue-measurable set of reals. These results are due to Silver [21].

REMARK. We should perhaps add a technical comment: In the theory ZF + MC one can prove that $\forall a \subset \omega\ (\omega_1^{L[a]} < \omega_1)$, i.e. for all subsets $a \subset \omega$ the first uncountable cardinal in the partial universe L[a] (the sets relatively constructible from a) is countable in the "true" universe. This assumption suffices to obtain all consequences for the projective hierarchy which follows MC (see Solovay [22] for added information). And this is a consistent extension of ZF.

2.2.3. Determinateness. Associated with every subset $A \subset N^N$ we have the following game. There are two players who alternatively chooses elements from the set N. In the end they produce an element $x \in N^N$. If $x \in A$ player I has won, if $x \in N^N - A$ player II is the winner.

The axiom of determinateness, AD, asserts that for each $A \subseteq N^N$ there is a winning strategy for one of the players.

PD, the axiom of projective determinateness, makes the similar statement for all projective subsets of N^N.

It may not be obvious why AD should have any ramifications for problems of measurability. On a very general level we can make the following remark. Let A be a subset of N^N, and let players I and II follow strategies σ and τ, respectively. Let $\sigma * \tau$ denote the element of N^N produced by the play. The game associated with A is determinate if either I or II has a winning strategy. This can be written as $\exists\sigma\forall\tau(\sigma*\tau \in A) \vee \exists\tau\forall\sigma(\sigma*\tau \notin A)$. An equivalent form is the following

$$\forall\tau\exists\sigma(\sigma*\tau \in A) \rightarrow \exists\sigma\forall\tau(\sigma*\tau \in A) ,$$

i.e. the existence of "local" counterstrategies gives the existence of a "global", i.e. winning, strategy. Usually implications of the type $\forall\exists \rightarrow \exists\forall$ require some assumptions of finiteness, compactness, uniformly boundedness, or the like. AD gives it for free. (For an introduction to this topic see Mycielski [15], Fenstad [6].)

AD contradicts the axiom of choice, but is hopefully consistent with the axiom of dependent choice, DC. (see section 2.1.2). Assuming AD Mycielski and Swierczkowski [16] showed that every subset of the real line is Lebesgue-measurable. A closer look at their proof shows that in the theory ZF + PD every projective set is measurable. (This is pointed out e.g. in [6].)

In general one may say that the axiom PD enables one to lift various regularity properties which classically was known for only the first few levels of the projective hierarchy to all levels of the projective hierarchy (In addition to measurability one noteworthy result is the extension of the uniformization principle due to Y. Moschovakis, see his forthcoming monograph [14] for full information.)

REMARK. We add a comment on the relationship between MC and AD. It was proved by D. Martin [11] that in ZF + MC it follows that every $\underset{\sim}{\Pi}_1^1$ game is determinate. Conversely, it has been proved by R. Solovay (unpublished) that if we add the assumption that every $\underset{\sim}{\Delta}_2^1$ game is determinate, then there exists inner models with (many!) measurable cardinals.

2.3. Do any of the axioms discussed in section 2.2 represent a "true" insight into the universe of all sets? Constructibility is a consistent extension of ZF, but consistency need not imply truth. Many people tend to believe that the notion of power set build into the idea of constructibility is too restrictive. And besides, the existence of a $\underset{\sim}{\Delta}_2^1$ well ordering in ZF + V = L blocks the extension to higher levels of the projective hierarchy of various "true" properties which are true at the first few levels.

Judged from its consequences the axiom MC and even to a larger degree the axiom PD is desirable. But no one has so far claimed any insight into the "true" universe of set theory compelling us to accept either MC or PD as a true extension of ZF.

3. EXTENSIONS. This section is divided into three parts. First, we shall introduce the notions of absolute and provable $\underset{\sim}{\Delta}_2^1$ sets of reals and show that every such set is absolutely measurable. Second, we shall lift various results about measurability to arbitrary analytic measure spaces. Finally, we shall add some remarks on how complex an absolutely measurable set of reals can be.

3.1. Absolute $\underset{\sim}{\Delta}^1_2$ sets. We introduce the basic definitions.

Definition. A set $a \subseteq N^N$ is provably $\underset{\sim}{\Delta}^1_2$ if there are Σ^1_2 and Π^1_2 formulas Ψ and Φ, respectively, and a parameter $y \in N^N$ such that

i) $x \in A$ iff $\Psi(x,y)$ iff $\Phi(x,y)$

ii) $ZF \vdash \forall x \forall y(\Psi(x,y) \leftrightarrow \Phi(x,y))$.

(Here $ZF \vdash \Phi$ means that Φ is provable in ZF.)

A set $A \subseteq N^N$ is absolute $\underset{\sim}{\Delta}^1_2$ if there are Σ^1_2 and Π^1_2 formulas Ψ and Φ, respectively, and a parameter $y \in N^N$ such that

iii) $x \in A$ iff $\Psi(x,y)$ iff $\Phi(x,y)$,

and such that for all (countable) standard models M of ZF such that $x,y \in M$

iv) $\Phi(x,y)$ iff $M \models \Phi(x,y)$ and $\Psi(x,y)$ iff $M \models \Phi(x,y)$.

($\models$ is the satisfaction relation, see section 1.1.1.)

REMARK. By an absoluteness argument one immediately sees that if A is provably $\underset{\sim}{\Delta}^1_2$ then A is absolute $\underset{\sim}{\Delta}^1_2$.

LEMMA.

a. The class of absolute $\underset{\sim}{\Delta}^1_2$ sets is a σ-algebra.

b. There are absolute $\underset{\sim}{\Delta}^1_2$ sets which are not in the σ-algebra generated by the $\underset{\sim}{\Sigma}^1_1$ sets.

The proof of a follows immediately from the fact that arbitrary countable unions can be coded by a single parameter from N^N. For the proof of b let $\sigma(\underset{\sim}{\Pi}^1_1)$ denote the σ-algebra generated by the $\underset{\sim}{\Pi}^1_1$ sets. The elements of $\sigma(\underset{\sim}{\Pi}^1_1)$ can be coded in the following way. First observe that there is a Π^1_1 set $A \subset N \times N^N \times N^N$ such that every $B \in \underset{\sim}{\Pi}^1_1$ can be written in the form

$$B = \{x \; ; \; \langle n,y,x\rangle \in A\} \; ,$$

for some $n \in N$ and $y \in N^N$. We now set

i) For all $x \in N^N$, $n \in N$ $y = \langle 1,\langle n,x\rangle\rangle$ is a code and
$B_y = \{z \; ; \; \langle n,x,z\rangle \in A\}$.

ii) If x is a code, then $y = \langle 2,x\rangle$ is a code and
$B_y = N^N \setminus B_x$.

iii) Let $x = \langle x_i\rangle_{i \in N}$ and assume that x_i is a code for all i, then $y = \langle 3,x\rangle$ is a code, and
$B_y = \bigcup_{i \in N} B_{x_i}$.

We can now prove in ZF that the relations "y is a code for a $\sigma(\underset{\sim}{\Pi}^1_1)$-set" and "$x \in B_y$" both are Δ^1_2, hence the set $\{x \; ; \; x \notin B_x\}$ gives the required counterexample for b. (We omit the somewhat messy details of the proof.)

REMARK. The notion of an absolute $\underset{\sim}{\Delta}^1_2$ set may be too "meta-mathematical" for the taste of an "ordinary" mathematician. It would be interesting to get some alternative description of this class. The notion of absolute $\underset{\sim}{\Delta}^1_2$ is certainly not original with the present authors, but we are unable to find a suitable reference in the literature.

We now come to the main result of this section.

THEOREM. If for all $x \in N^N$ there is a countable standard model M of ZF such that $x \in M$, then every absolute $\underset{\sim}{\Delta}^1_2$ set is absolutely measurable.

Proof. Let μ be an atomless, finite, positive Borel-measure (note that this represents no essential restriction) and M a countable standard model of ZF in which μ can be defined (- note that a measure is determined by its values on the base elements and thus may be represented by a sequence of reals). We call x random over M if x belongs to no Borel-set of μ-measure 0 which is codeable over M (-note that similar to the coding of the σ-algebra $\sigma(\underset{\sim}{\Pi}_1^1)$ introduced above, one can introduce a coding for the Borel-sets; a Borel-set is then codeable over M if the code for the set belongs to M).

We now observe, since M is countable, that the set of non-random elements has μ-measure 0, and that no element of M is random, since $\{x\}$ is codable over M whenever $x \in M$.

We shall now make use of the forcing technique (see section 1.1.2). First we introduce a set of conditions. Let B_1 and B_2 be Borel-sets codeable over M and define $B_1 \simeq B_2$ if $\mu\ (B_1 \Delta B_2) = 0$. $\simeq$ is an equivalence relation and let $\mathcal{B}$ be the set of equivalence classes $[B]$. Set $\mathbb{P} = \mathcal{B} \setminus \{[\emptyset]\}$. Define an ordering $\leq$ by $[B_1] \leq [B_2]$ if $\mu(B_2 \setminus B_2) = 0$. It is easily verified that $\langle \mathbb{P}, \leq \rangle$ is a set of conditions.

LEMMA. Let x be random over M. Then the set

$$G_x = \{[B]\ ;\ x \in B,\ B \text{ is codable over } M,\ \mu(B) > 0\}$$

is $\mathbb{P}$-generic over M.

We content overselves by verifying condition iii in the definition of a $\mathbb{P}$-generic set. So let $\Delta \in M$ be a dense subset of $\mathbb{P}$, we first verify that in M

$$(*) \qquad \mu\Big(\bigcup_{[B] \in \Delta} B\Big) = \mu(N^N).$$

For the proof of (*) assume that in M we have $\mu_*(\bigcup_{[B]\in\Delta} B) = r$ and that $\mu(N^N) > r$ (here μ_* is the inner measure associated with μ). Let C be a Borel-set in M such that $C \subseteq \bigcup_{[B]\in\Delta} B$ and such that $\mu(C) = r$. Consider $D = N^N \setminus C$. By density [D] must have an extension $[E] \in \Delta$. Then $\mu(C \cup E) > r$, but $C \cup E \subseteq \bigcup_{[B]\in\Delta} B$ - a contradiction.

From (*) it now follows that if $x \notin \bigcup_{[B]\in\Delta} B$, then x would belong to some Borel-set C which is codable over M and which has μ-measure 0. But this contradicts the fact that x is random over M. Hence $x \in \bigcup_{[B]\in\Delta} B$, which shows that $\Delta \cap G_x \neq \emptyset$. This end the proof of the lemma.

<u>REMARK</u>. Since $\{x\} = \bigcap_{[B]\in G_x} B$, we see that $x \in M[G_x]$, for every x which is random over M.

Let now Φ be a formula which is absolute with respect to all forcing extensions of M. Define

$$\Psi(G) = \forall y(y \in \bigcap_{[B]\in G} B \to \Phi(y))$$

We then see that $\Psi(G_x) \leftrightarrow \Phi(x)$, for all x. Define

$$E = \bigcup\{B \ ; \ [B] \Vdash \Psi(G)\},$$

and let x be random over M. Using the completeness theorem for forcing we obtain

$$\begin{aligned} x \in E \quad &\text{iff} \quad \exists B([B] \Vdash \Psi(G) \wedge x \in B) \\ &\text{iff} \quad \exists [B] \in G_x([B] \Vdash \Psi(G)) \\ &\text{iff} \quad M[G_x] \models \Psi(G_x) \\ &\text{iff} \quad M[G_x] \models \Phi(x). \end{aligned}$$

(In this part we again beg the expert to overlook some looseness with respect to the languages involved.)

The proof is now finished: Let A be absolute $\underset{\sim}{\Delta}_2^1$ in the parameter y and let M be a countable standardmodel containing y. Let Φ be the defining formula for A, we then see that for x random over M, $x \in A$ iff $x \in E$, where E is the set above. Since the non-random elements over M has measure 0, it follows that $A \Delta E$ is a subset of a set with measure 0. And since E obviously is a Borel-set, we conclude that A is μ-measurable.

REMARKS. 1. The argument above is an analysis of the appropriate part of Solovay [23]. His purpose, as explained in section 2.1.2, was to obtain a consistency result, but our result is quite easy to read off from his proof. Thus we at most claim some novelty in the way we have presented the theorem (e.g. in the use of absolute $\underset{\sim}{\Delta}_2^1$ sets).

2. Adding the Shoenfield absoluteness theorem [18] to the above argument gives the result about $\underset{\sim}{\Sigma}_2^1$ sets mentioned in section 2.2.2. This is due to Solovay (unpublished).

3. Restricting ourselves to provable $\underset{\sim}{\Delta}_2^1$ sets we do not need the assumption about inner models, hence the result is a pure ZF result. The reason is that since the proof, only uses a finite part of the axioms, we can use a Skolem-Löwenheim argument to obtain an "inner model".

4. Our assumption about inner models is stronger than ZF + con(ZF). Is this assumption a reasonable adition to ZF (i.e. can it be accepted as a true statement)? At least one of the authors are inclined to believe so.

3.2. Analytic measure spaces. Let E be an analytic space (see section 1.2 for definitions and a list of basic properties). There are two ways of defining the projective hierarchy on E:

i. Starting with the Borel-sets in E^k, $k = 1,2,\ldots,$ we generate the projective hierarchy by the method of section 1.1.3.

ii. Let π be a continuous and surjective map $\pi : N^N \to E$, we let $A \subseteq E^k$ belong to the class Π_n iff $\pi^1(A) \in \underset{\sim}{\Pi}_n^1$.

As we shall later see the two possibilities are equivalent for Polish spaces, but in general ii. defines a larger class than i. Since we will use condition ii in lifting results from $\mathbb{R}$ to arbitrary analytic measure spaces, we make the following definition.

Definition. Let E be an analytic space, $\pi : N^N \to E$ a continuous and surjective mapping, $n \geq 1$, and $A \subseteq E$:

$$A \text{ belongs to class } \Pi_n(\Sigma_n, \Delta_n) \text{ iff } \pi^{-1}(A) \in \underset{\sim}{\Pi}_n^1(\underset{\sim}{\Sigma}_n^1, \underset{\sim}{\Delta}_n^1).$$

LEMMA.

a. Every Borel-set in E is of class Δ_1.

b. Π_n is closed under countable intersection and unions, and Δ_n is a σ-algebra.

c. Let $A \subseteq E^{k+1}$ be of class Π_n and let B be the projection of A on E^k. Then B is of class Σ_{n+1}.

d. Let $\pi_1, \pi_2 : N^N \to E$ be two Borel-continuous maps ($\pi : F \to E$ is called Borel continuous if Y Borel in E implies $\pi^{-1}(Y)$ Borel in F), and let $A \subseteq E$. Then

$$\pi_1^{-1}(A) \in \underset{\sim}{\Pi}_n^1(\underset{\sim}{\Sigma}_n^1) \text{ iff } \pi_2^{-1}(A) \in \underset{\sim}{\Pi}_n^1(\underset{\sim}{\Sigma}_n^1).$$

We omit the proofs. (For the proof of d. note that the set $\{\langle x,y\rangle ; \pi_1(x) = \pi_2(y)\}$ is Borel.) The lemma shows that method ii includes method i, and it shows that the definition is independent of the particular mapping $\pi = N^N \to E$.

Definition. A set $A \subseteq E$ is called absolute Δ_2 if $\pi^{-1}(A)$ is absolute $\underset{\sim}{\Delta}{}_2^1$. (Part d. of the lemma above holds equally well for absolute $\underset{\sim}{\Delta}{}_2^1$ set, hence the definition of absolute Δ_2 is independent of the map π.)

We now come to the main result of this section. This was proved by D. Normann in [17]. Let $\Gamma(E)$ denote any of the classes $\Pi_n, \Sigma_n, \Delta_n$, absolute Δ_2 in the analytic space E.

THEOREM. The following three conditions are equivalent:

i. Every set in $\Gamma(N^N)$ is absolutely measurable.

ii. Let $\langle P,\mu\rangle$ be a Polish measure space: Every $\Gamma(P)$ set is μ-measurable.

iii. Let $\langle E,\mu\rangle$ be an analytic measure space: Every $\Gamma(E)$ set is μ-measurable.

Proof. It suffices to prove ii $\to$ iii and i $\to$ ii.

ii $\to$ iii. Recall from section 1.2 that every compact subset of an analytic space is Polish. Further recall that the measures involved are regular and σ-finite, i.e. there exists a increasing sequence of compacts $\langle K_n\rangle_{n\in N}$ such that

$$\mu(E \setminus \bigcup_{n\in N} K_n) = 0 \ , \ \mu(K_n) < \infty \ , \text{ all } n.$$

Let $A \in \Gamma(E)$, it suffices to show that $A \cap K_n$ is μ-measurable for all n. If $\pi_1 = N^N \to E$ is surjective and continuous, it follows that $\pi^{-1}(K_n)$ is closed in N^N, hence $\pi_1^{-1}(A \cap K_n)$ is of class $\Gamma(N^N)$. Since K_n is Polish, there is a continuous surjection $\pi_2 : N^N \to K_n$. As in d. of the lemma it follows that $\pi_2^{-1}(A \cap K_n)$ is of class $\Gamma(N^N)$, which means that $A \cap K_n$ is of class $\Gamma(K_n)$ in K_n. By ii, this means that $A \cap K_n$ is μ-measurable.

i $\to$ ii. Recall from section 1.2 that a Polish space is homomorphic to a G_δ set in I^N. From this it follows that if $\langle P,\mu\rangle$ is a Polish measure space there is a Borel-set $Q \subseteq N^N$ and a Borel isomorphism $\pi : Q \to P$. Let $A \in \Gamma(P)$; in order to show that A is μ-measurable, we define a measure μ' on N^N by

$$\mu'(Y) = \mu\pi(Y \cap Q),$$

whenever the latter is defined. It is a matter of routine to verify that μ' is a complete and σ-finite Borel-measure on N^N. Since $\pi^{-1}(A)$ is of class $\Gamma(N^N)$, it follows by i that $\mu'(\pi^{-1}(A))$ is defined, which means that $\mu(A) = \mu\pi(\pi^{-1}(A) \cap Q)$ is defined, i.e. A is μ-measurable.

REMARKS. 1. The Borel isomorphism Π between P and a Borel-set $Q \subseteq N^N$ is precisely what is needed to verify that the two methods mentioned in the beginning of this section leads to the same hierarchies over P.

2. We have a strong negative result. All uncountable analytic spaces E includes a Cantorlike subspace [9, p.118]; hence E includes a Borel-set homeomorphic to N^N. Then, if there is a set in $\Gamma(N^N)$ which is not absolutely measurable, there will be a set in $\Gamma(E)$ which is not absolutely measurable. On the other

hand: If there is a non μ-measurable set $A \in \Gamma(E)$ we must, by our theorem have a set $B \in \Gamma(N^N)$ such that B is not absolutely measurable. Thus, given Γ as above, the following two statements are equivalent:

a. There is an uncountable analytic space E in which some $\Gamma(E)$-set is not absolutely measurable.

b. In all uncountable analytic spaces E, some $\Gamma(E)$-set is not absolutely measurable.

3. In Normann [17] several other results are generalized from N^N to arbitrary Polish and analytic spaces. E.g. one may show that every analytic space E is the continuous injective image of some $\underset{\sim}{\Pi}{}^1_1$ set in N^N, which suffices to show the following result: Let $\underset{\sim}{\Pi}{}^1_1 \subseteq \Gamma(N^N)$ and assume that every uncountable $\Gamma(N^N)$ set includes a perfect subset. Then the same is true for sets of class $\Gamma(E)$, where E is an arbitrary analytic space.

3.3. On the complexity of absolutely measurable sets. Our results so far go in one direction: every "nice" set is absolutely measurable. Is there a converse, i.e. are absolutely measurable sets necessarily nice? We give a lemma due to D. Normann [17] which suffices to answer the question in the negative in most cases.

LEMMA. There is a set of reals A of cardinality ω_1 such that $\mu(A) = 0$ for all atomless Borel-measures μ.

Proof. Let A consist of one code for each countable ordinal. Let M be a countable standard model for ZF such that μ is definable over M. Then the set $A_0 = \{x \in A;$ the ordinal coded by x belongs to $M\}$ is countable, since M is countable, hence A_o is μ-measurable and $\mu(A_o) = 0$.

From section 3.1 we know that $\mu(\{x ; x$ is non-random over $M\}) = 0$. The proof will be concluded if we can show that if $x \subseteq A$ and the ordinal coded by x does not belong to M, then x is non-random over M. Suppose not, then $x \in M[G_x]$ (see the remark following the lemma of section 3.1). But from the general theory of forcing (see section 1.1.2) we know that M and $M[G_x]$ have the same ordinals. Now if $x \in M(G_x]$, then the ordinal coded by x also belongs to $M[G_x]$, hence to M, contradicting our assumption.

<u>REMARK</u>. 1. Here too, we only need a finite part of ZF, so the assumption about inner models can be eliminated.

2. A cannot contain a perfect subset. (This is well-known, see e.g. [22].)

Some consequences of the lemma and the remark are:

1. If $2^{\omega_1} > 2^{\omega_o}$, then there are absolutely measurable sets which are not in the projective hierarchy.
2. Let A be the set of the lemma. We can prove in the theory $ZF + \forall a \subset \omega(\omega_1^{L[a]} < \omega_1)$ that A is no $\underset{\sim}{\Sigma}^1_2$.
3. Let A be the set of the lemma. We can prove in the theory $ZF + PD$ that A is not projective.

To prove 1 notice that all subsets of A are absolutely measurable, then use a cardinality argument. To prove 2 and 3 notice that in $ZF + \forall a \subset \omega(\omega_1^{L[a]} < \omega_1)$ every uncountable $\underset{\sim}{\Sigma}^1_2$ set contains a perfect subset, Solovay [22], and in $ZF + PD$ every projective set contains a perfect set, see e.g. [15].

REMARK. The lemma does not answer the question about the complexity of absolutely measurable sets in every case. It has been proved consistent by Martin and Solovay [12] that every $\underset{\sim}{\Sigma}_2^1$ set is absolutely measurable and that every set of cardinality ω_1 is $\underset{\sim}{\Pi}_1^1$.

4. A CONCLUDING REMARK. A "main problem" in set theory is to analyse the notion of powerset operator. This is not only an "internal" problem for set theory, but is a problem which interacts with other parts of mathematics. In this paper we have tried to show its ramifications for the problem of measurability. In one direction we tried to push the classical results a bit further without introducing an analysis going beyond ZF. In another we commented on various axiomatic extensions of this analysis. But the main problem of "grasping" the power set operation, - if it ever can be grasped - is still there.

REMARK. It has been claimed that the idea of an absolute powerset, which underlies the set theoretic approach, is not a clear and consistent notion and must be abandoned. For various predicative or constructive approaches see e.g. Bishop [9], Feferman [5], and Martin-Løf [13].

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