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DUALITY THEORY FOR LOCALLY COMPACT GROUPS WITH PRECOMPACT CONJUGACY CLASSES II
THE DUAL SPACE

by

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Abstract. The present paper is concerned with the dual space $\hat{G}$ consisting of all unitary equivalence classes of continuous irreducible unitary representations of separable $[FC]^-$ groups (i.e. groups with precompact conjugacy classes). The main purpose of the paper is to extend certain results from the duality theory of abelian groups and $[Z]$ groups to the larger class of $[FC]^-$ groups. In addition, we deal briefly with square-integrability for representations of $[FC]^-$ groups. Most of our results are proved for type I groups. Our key result is that $\hat{G}$ may be written as a disjoint union of abelian topological T_4 groups, which are all open in $\hat{G}$.

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Introduction. In [23] we studied the character space $\mathcal{X}(G)$ of locally compact groups G possessing precompact conjugacy classes ($[FC]^-$ groups), and we were able to extend much of the classical duality theory for abelian groups [2)] to this larger class of groups using $\mathcal{X}(G)$ as dual object [3)]. The present paper is concerned with the dual space $\hat{G}$ of all unitary equivalence classes of continuous irreducible unitary Hilbert space representations of separable $[FC]^-$ groups G. $\hat{G}$ is given the Fell topology, obtained from the hull kernel topology on the dual space (specter) of the group C^*-algebra, [4].

For an arbitrary locally compact group G insight in the structure of $\hat{G}$ often tells more about the group itself than the corresponding knowledge of $\mathcal{X}(G)$. This is of course partly due to the fact that the elements of $\hat{G}$ always separate the points of G, whereas this property [4)] holds for $\mathcal{X}(G)$ if $G \in [SIN]$ but not generally. Even in the case of a type I $[FC]^-$ group the character space may constitute only a small part of the dual. For $\mathcal{X}(G)$ is then isomorphic to the dual space $(G/C)^\wedge$ of the $[FC]^- \cap [SIN]$ quotient group G/C, where C denotes the intersection of all closed invariant neighborhoods of e in G, [18] Theorem 5.12. We shall prove, e.g., that $\hat{G}$ is connected if and only if G is aperiodic (i.e., G contains no nontrivial compact subgroup). For $\mathcal{X}(G)$ this result can hold only when

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2) Such a generalization was carried out in [9] for the class of locally compact groups G such that G modulus its center is compact.

3) $\mathcal{X}(G)$ consists of the nonzero extreme points of the set of all continuous positive definite G-invariant functions f on G such that $f(e) \leqq 1$; and $\mathcal{X}(G)$ was provided with the topology of uniform convergence on compacta.

4) If $x \in G$, $x \neq e$, then there is a character $\lambda \in \mathcal{X}(G)$ such that $\lambda(x) \neq 1$.

G is replaced by G/C , see [23] (2.12).

On the other hand the character space $\mathfrak{X}(G)$ is a locally compact Hausdorff space for $G \in [FC]^-$, [12] , whereas $\hat{G}$ need not even be a T_0 space (but $\hat{G}$ is always locally compact, [4] 3.3.8). In fact $\hat{G}$ is T_0 iff G is typeI iff $\hat{G}$ is T_4 (1.3). Whenever our method of proof utilizes topological separation properties of $\hat{G}$ it is therefore necessary (and sufficient) to assume that G is type I. Under this hypothesis we can show that $\hat{G}$ is locally Euclidean iff G is compactly generated, and $\hat{G}$ is locally connected iff G possesses property (L).

In addition to extending classical duality results like the ones mentioned above, this paper also deals briefly with square-integrable representations of $[FC]^-$ groups, and we provide simple necessary and sufficient conditions for $[\pi] \in \hat{G}$ to be square-integrable.

For convenience we shall list the classes of groups with which we are dealing.

- [SIN] - class of locally compact groups possessing a fundamental system of invariant neighborhoods of e.
- $[FC]^-$ - class of locally compact groups possessing precompact conjugacy classes.
- $[FIA]^-$ - locally compact groups with precompact inner automorphism group. One has $[FIA]^- = [SIN] \cap [FC]^-$, see [10] .
- $[FD]^-$ - locally compact groups with precompact commutator group.

The following inclusions hold: $[FIA]^- \subseteq [FC]^-$ and $[FD]^- \subseteq [FC]^-$, [10]. Of course, all the classes of groups listed above contain the abelian groups and the compact groups.

If π is a unitary representation of a closed subgroup H of the locally compact group G we shall let π^G denote the unitary representation of G induced from π. If ρ is a representation of G, ρ_H will denote the representation of the subgroup H given by restriction, $\rho_H(h) = \rho(h)$, all $h \in H$. Let α and β be unitary representations of G. We shall write $\alpha \simeq \beta$ if they are unitarily equivalent. If the topological groups G and L are isomorphic, we shall use the notation $G \approx L$.

If $\pi \in \mathrm{Irr}(G)$ ([4] 18.5.1) we let $[\pi]$ denote the equivalence class of π in $\hat{G}$. For $S \subset \mathrm{Irr}(G)$ we let $[S]$ be the image of S in $\hat{G}$ under the canonical map $\pi \mapsto [\pi]$. Finally, we let ρ' denote the representation of G given by $\rho'(x) = \rho(xN)$ whenever ρ is a representation of a factor group G/N.

We would like to mention at this point that E. Kaniuth recently obtained results related to the present ones under somewhat different conditions on the groups. (Topology in Duals of SIN -groups, Math. Z. 134 (1973) 67-80.)

1. Groups with abelian quotients. Let G be a separable locally compact group of type I and suppose that N is a closed normal subgroup of G such that G/N is abelian. Assume also that N is typeI and regularly embedded in G. In this section we shall analyze the dual of such groups. We shall assume the reader is familiar with Mackey's little group method, [1] and [17]. The main reason for the above hypothesis on G is that $[FC]^-$ groups, which are our main concern in this paper, satisfies exact sequences of topological groups $e \to P \to G \to V \times A \to e$, where P is maximal periodic, A is abelian, discrete, and aperiodic, and V is isomorphic to R^n for some n, ([19] Theorem(3.16)). Some condition on G like type I -ness is necessary in the following result [5].

(1.1) Proposition. Let G and N be as above. If $[\pi] \in \hat{G}$ then the set $\hat{G}_{\pi,N} = \{ [\gamma] \in \hat{G} : \gamma_N$ is quasi -equivalent to $\pi_N \}$ equals $[(G/N)^\wedge \otimes \pi]$.

Proof. Suppose $[\gamma] \in \hat{G}_{\pi,N}$, and let $[\rho] \in \hat{N}$ be an element of the G-orbit θ_π in $\hat{N}$ which is determined by π ([1]). Let $K(\rho)$ be the stability group of ρ under G's action on $\hat{N}$ by inner automorphisms. By the Mackey theory there is a $[\sigma] \in K(\rho)^\wedge$ so that the induced representation σ^G is equivalent to π. Moreover,

5) See also Theorem (1.3), (1) $\Longleftrightarrow$ (2) .

σ may be constructed as follows: There is a multiplier ω on $K(\rho)/N$ such that ρ extends to an ω representation of $K(\rho)$. Let $\tilde{\rho}$ be any such extension of ρ; then there exists an irreducible $\bar{\omega}$ representation β of $K(\rho)/N$ such that $\beta' \otimes \tilde{\rho} \simeq \sigma$. Because of our hypothesis on G and N all primary $\bar{\omega}$ representations of $K(\rho)/N$ are type I ([17] Theorem 8.4), and it follows from [3] Theorem(3.3) together with [3] Theorem(3.1) that all the irreducible ω^{-1} representations of $K(\rho)/N$ are on the form $\beta \otimes \chi$ where χ is some character of $K(\rho)/N$. From this it follows that every $[\gamma] \in \hat{G}_{\pi,N}$ is induced from representations on the form $\chi' \otimes \beta' \otimes \tilde{\rho} = \chi' \otimes \sigma$. Observe now that

$$(\chi' \otimes \sigma)^G \simeq \tilde{\chi}' \otimes \sigma^G$$

where $\tilde{\chi}$ is any extension of the character χ to G/N and $\tilde{\chi}'$ denotes the lift of $\tilde{\chi}$ to G, ([7] Lemma 4.2). This completes our **proof.**[6)]

The following result is well-known and will be helpful when combined with (1.1).

<u>(1.2) Lemma.</u> Let G be a separable locally compact group and let K be a compact normal subgroup. Then $\hat{G}_{\pi,K} = \{[\gamma] \in \hat{G} : \gamma_K$ is quasi-equivalent to $\pi_K\}$ is an open and closed subspace of $\hat{G}$.

<u>Proof.</u> If $[\pi] \in \hat{G}$ let θ_π be the G-orbit in $\hat{K}$ associated to $[\pi]$; the map $p : [\pi] \mapsto \theta_\pi$, $\hat{G} \to \hat{K}/G$ is continuous, [6] Lemma 3 (the orbit space $\hat{K}/G$ is given the quotient topology induced from $\hat{K}$).

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6) A different proof of (1.1) was pointed out to the author by J.Brezin. Brezin's proof also uses the Mackey theory.

Since K is compact $\hat{K}$ and $\hat{K}/G$ are discrete. The result now follows from the fact that $\hat{G}_{\pi,K} = p^{-1}(\{\theta_\pi\})$. Q.E.D.

Let G be a locally compact separable type I group, and suppose that K is a compact normal subgroup of G such that G/K **is abelian**(7). Let $\pi \in \mathrm{Irr}(G)$ be fixed but arbitrary. There **is a natural** map f_π of $\widehat{G/K}$ onto $[\widehat{G/K} \otimes \pi]$ given by $\alpha \mapsto [\alpha' \otimes \pi]$. We let $M_\pi = [\widehat{G/K} \otimes \pi]$ and provide M_π with the topology induced from $\hat{G}$. We may define a natural groupstructure on M_π by letting

$$[\alpha' \otimes \pi] * [\beta' \otimes \pi] = [\alpha' \beta' \otimes \pi] \qquad , \alpha, \beta \in \widehat{G/K} .$$

It is easy to check that the product $*$ is well defined. Moreover, $\widehat{G/K}$ acts on M_π as a locally compact Hausdorff topological transformation group. Indeed, the map $(\alpha, [\beta' \otimes \pi]) \mapsto [\alpha' \beta' \otimes \pi]$ from $\widehat{G/K} \times M_\pi$ onto M_π is continuous, and the map $\alpha \mapsto ([\beta' \otimes \pi] \mapsto [\alpha' \beta' \otimes \pi])$ is a homomorphism of $\widehat{G/K}$ into the group of homeomorphisms of M_π , and the map $[\alpha' \otimes \pi] \longmapsto [\alpha_0' \alpha' \otimes \pi]$ is a homeomorphism for each $\alpha_0 \in \widehat{G/K}$. Since $M_\pi / \widehat{G/K}$ consists of one point, it is certainly T_0 and it follows from J. Glimm ([8] Theorem 1) that the map $\alpha \cdot G(\pi) \mapsto [\alpha' \otimes \pi]$ from $\widehat{G/K} / G(\pi)$ onto M_π is a homeomorphism, where $G(\pi) = \{ \alpha \in \widehat{G/K} : \alpha' \otimes \pi \simeq \pi \}$ is the stability group of $[\pi]$. Here the type I hypothesis was needed. Since the quotient map $\widehat{G/K} \longrightarrow \widehat{G/K} / G(\pi)$ is open, it follows that f_π is open and M_π is Hausdorff. It also

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(7) **The following arguments hold whenever $[\widehat{G/K} \otimes \pi]$ is closed in $\hat{G}$ since $[\widehat{G/K} \otimes \pi]$ is then locally compact. We shall only need to consider compact K .**

follows that M_π is a topological group with the product $*$ defined above, and M_π may be identified with the quotient group $\widehat{G/K} / G(\pi)$. Thus M_π is even T_4 by abelian theory [11]. We have almost proved the following theorem.

(1.3) Theorem. Let G be a separable locally compact group with a compact normal subgroup K such that the qoutient G/K is abelian. Then the following statements are equivalent.

(1) G is type I

(2) $\hat{G}_{\pi,K} = \{ [\gamma] \in \hat{G} ; \gamma_K$ is quasi-equivalent to $\pi_K \}$ equals $[\widehat{G/K} \otimes \pi]$ for all $[\pi] \in \hat{G}$.

(3) $\hat{G}$ is the disjoint union of locally compact abelian topological T_4 groups $[\widehat{G/K} \otimes \pi]$, which are open and closed in $\hat{G}$.

(4) $\hat{G}$ is T_4.

Proof. There is little left to prove. (1) $\Rightarrow$ (2) : If G is type I then $\hat{G}_{\pi,K}$ equals $[\widehat{G/K} \otimes \pi]$ for all $[\pi] \in \hat{G}$, by (1.1). (2) $\Rightarrow$ (3) follows from (1.1) and (1.2) together with the arguments above. (3) $\Rightarrow$ (4) $\Rightarrow$ (1) is obvious. Q.E.D.

Note: If G is a (separable) $[FC]^-$ group then G satisfies an exact sequence of topological groups $e \longrightarrow K \longrightarrow G \xrightarrow{p} A \longrightarrow e$, where K is compact, and $A = R^n \times D$ where D is a discrete $[FC]^-$ group [21]. If G is type I, so is D and it can be shown that the commutator subgroup D' of D is finite (S. Grosser and M. Moskowitz, Representation theory of central topological groups, Theorem 5.4, TAMS 129 (1967) 361-390 ; and [18] Theorem 5.12). Hence, replacing K with

$p^{-1}(D')$ we may assume that A is abelian in the above sequence. Conversely, if G is on the form $e \to K \to G \to A \to e$ where K is compact and A is abelian, it follows easily that G is an $[FC]^-$ group. From these remarks it is clear that Theorem (1.3) is a theorem about $[FC]^-$ groups.

In this connection we note that J. Liukkonen, [16] Theorem 3.6, proved that type I separable $[FC]^-$ groups have Hausdorff duals. We feel that our proof of this result is simpler than Liukkonens proof, but more important, Theorem (1.3) reduces the study of $\hat{G}$ for type I $[FC]^-$ groups to the study of abelian character groups. This will be needed in the proofs of our subsequent results.

The following refinement of Proposition (1.1) will also be useful.

(1.4) Proposition. Let G be a separable type I $[FC]^-$ group, and let N be a closed normal subgroup of G such that G/N is abelian. Assume N consists entirely of periodic elements. Then $\hat{G}_{\pi,N} = \{[\gamma] \in \hat{G} : \gamma_N$ is quasi-equivalent to $\pi_N\}$ equals $[\widehat{G/N} \otimes \pi]$, all $[\pi] \in \hat{G}$.

Proof. Let G' be the commutator subgroup of G. Since G is type I G satisfies an exact sequence of topological groups $e \to K \to G \to A \to e$ where K is compact and A is abelian. Hence $\overline{G'}$ is compact, and we may let $\overline{G'} = K$ in the above sequence. Moreover, N contains $\overline{G'}$ since G/N is abelian. Hence $\hat{G}_{\pi,N} \subset \hat{G}_{\pi,K} = [\widehat{G/K} \otimes \pi]$, all $[\pi] \in \hat{G}$, by (1.1). Now $N \subset P$ = the (maximal) periodic subgroup of G, and P is the union of all its compact G-invariant subgroups, so N also has this property. Let $[\alpha \otimes \pi] \in \bigcap_{K \subset N} [\widehat{G/K} \otimes \pi]$. Then $\alpha_K = 1$ for all G-invariant compact $K \subset N$. Hence $\alpha_N = 1$ and we have $\hat{G}_{\pi,N} \subset \bigcap_K [\widehat{G/K} \otimes \pi] \subset [\widehat{G/N} \otimes \pi]$. Clearly $[\widehat{G/N} \otimes \pi] \subset \hat{G}_{\pi,N}$. Q.E.D.

We turn now to the characterization of the connected components of $\hat{G}$ for type I $[FC]^-$ groups.

<u>(1.5) Proposition.</u> Let G be a separable $[FC]^-$ group of type I. Then the connected component $\mathcal{C}_\pi$ of $[\pi] \in \hat{G}$ is on the form $\hat{G}_{\pi,P} = [\widehat{G/P} \otimes \pi]$ where P denotes the periodic subgroup of G.

<u>Proof.</u> From (1.4) $\hat{G}_{\pi,P} = [\widehat{G/P} \otimes \pi]$. Also, $[\widehat{G/P} \otimes \pi]$ is connected, being the continuous image of the connected group $\widehat{G/P}$ under the canonical map $\alpha \mapsto [\alpha' \otimes \pi]$ ($\widehat{G/P}$ is connected by abelian theory since G/P is aperiodic, [19] Corollary2 to Theorem 2.5). Hence we have $\hat{G}_{\pi,P} \subset \mathcal{C}_\pi$. As we saw in the proof of (1.4) $\hat{G}_{\pi,P}$ equals the intersection of all the open and closed $\hat{G}_{\pi,K}$, K a G-invariant and compact subgroup of P. Thus $\hat{G}_{\pi,P} = \mathcal{C}_\pi$. Q.E.D.

<u>2. Duality theory for $[FC]^-$ groups.</u> We are now in a position to make an analysis, like we did for the character space $\mathcal{X}(G)$ in [23], of the dual space $\hat{G}$ of separable $[FC]^-$ groups. First we observe that if K is a compact normal subgroup of G and $[\pi] \in \hat{G}$, then the restriction π_K is equivalent to a direct sum $m \oplus \omega$ where m is some (at most countable) cardinal number and the sum is taken

over some G-orbit in $\hat{K}$ under the action of G on $\hat{K}$ by inner automorphisms. Observe also that $\hat{G}_{\pi,K}$ consists of all elements in G which lies over the same G-orbit as $[\pi]$. Using this together with Theorem (1.3) and the fact that $\mathcal{C}_\pi = [\widehat{G/P} \otimes \pi]$ for type I groups, we shall prove results analogous to the ones in Section 2 and Section 3 of [23]. We shall also discuss square-integrable representations and groups with compact duals.

<u>(2.1) Proposition.</u> Let G be a separable $[FC]^-$ group. Then the following results hold.

(1) $\hat{G}$ is connected iff G is aperiodic.

(2) Let G be type I. Then $\hat{G}$ is totally disconnected iff G is periodic.

(3) If G is type I and the periodic subgroup P is finite then $\hat{G}$ has finitely many connected components. If $G \in [FIA]^-$ and $\hat{G}$ has finitely many connected components, then the periodic subgroup P of G is finite.

(4) Let G be type I. $\hat{G}$ is locally Euclidean iff G is compactly generated.

<u>Proof.</u> (1) If $\hat{G}$ is connected then the component $\mathcal{C}_1$ of the trivial representation 1 constitutes all of $\hat{G}$. But $\mathcal{C}_1$ is easily seen to equal $\widehat{G/P}$ even in the non type I case. By the Gelfand-Raikov theorem $P = (e)$ and G is aperiodic. Conversely, if G is aperiodic then G is abelian [10]. Hence $\hat{G}$ is connected by abelian theory.
(2) Assume $\hat{G}$ is totally disconnected and let P be the periodic subgroup of G. Then $\widehat{G/P} = \mathcal{C}_1$ is trivial, hence $G = P$. Conversely, let $G = P$. With notation as in (1.2) $\hat{G}_{\pi,K}$ is open and closed in $\hat{G}$ for any $[\pi] \in \hat{G}$, and K compact. Since $G = P$ is the union

of all compact normal subgroups K we have $\bigcap_K \hat{G}_{\pi,K} = \hat{G}_{\pi,P} = [\pi]$ (as in the proof of (1.4)). Hence $\mathcal{C}_\pi = [\pi]$ and $\hat{G}$ is totally disconnected.

(3) Let G be type I and suppose P is finite. Then there is only a finite number of G-orbits in $\hat{P}$, and hence there is only a finite number of different connected components $[\widehat{G/P} \otimes \pi] = \hat{G}_{\pi,P}$.

Conversely, let $G \in [FIA]^-$ and suppose that $\hat{G}$ has only a finite number of connected components $\mathcal{C}_\pi$. Then each $\mathcal{C}_\pi$ is open and closed in $\hat{G}$. Let $\# : [\pi] \mapsto \pi^\#$ be the canonical map of $\hat{G}$ onto $\mathcal{X}(G)$ ([1] 5.1 and 5.8). $\#$ is an open and continuous map, hence it maps the open $\mathcal{C}_\pi$ onto the connected component $\mathcal{C}_{\pi^\#}$ of the character $\pi^\#$ in $\mathcal{X}(G)$. Therefore $\mathcal{X}(G)$ has only a finite number of connected components, and it follows from [23] Proposition(2.10) that the periodic subgroup P of G is finite.

(4) Let G be typeI and fix a compact normal subgroup K of G such that G/K is abelian. In view of Theorem (1.3) $\hat{G}$ is locally Euclidean iff $\widehat{G/K}$ is locally Euclidean. By abelian theory ([19] Corollary 1 to Thm.2.5) the last assertion holds iff G/K is compactly generated, which is equivalent to G be compactly generated. Q.E.D. (8)

We say that G is an (L) group if every compact subset F of G is contained in an open compactly generated normal subgroup N of G such that G/N is aperiodic [20] .

(8) The author recently proved (3) without any typeI or $[FIA]^-$ assumtion on the group (On the dual topology of $[FC]^-$ groups, forthcoming in Math. Scand.)

(2.2) Proposition. (1) Let $G \in [FC]^-$ be separable and type I. Then $\hat{G}$ is locally connected if G is an (L) group.

(2) Let G be an $[FD]^-$ group. Then G is an (L) group if $\hat{G}$ is locally connected.

Proof. (1) Suppose G is an (L) group. Arguing as in the proof of [23] Proposition(3.4) (with $\mathcal{B} = \mathcal{J}(G)$) we see that the periodic subgroup P is compact. If G is type I each connected component $\mathcal{C}_\pi$ is on the form $[\widehat{G/P} \otimes \pi] = \hat{G}_{\pi,P}$ and is open and closed in $\hat{G}$ since P is compact, (1.1) and (1.2). Also, the canonical map $\widehat{G/P} \longrightarrow [\widehat{G/P} \otimes \pi]$ is open and continuous so that it suffices to prove that $\widehat{G/P}$ is locally connected. Since our proposition holds for abelian groups (K. Fan [5]), the problem is reduced to showing that $G/P = \mathbb{R}^n \times A$ (where A is discrete abelian and aperiodic) is an (L) group if G is an (L) group, and this was proved in the last part of [23] Proposition(3.3).

(2) Conversely, suppose $G \in [FD]^-$ and $\hat{G}$ is locally connected. Let C denote the intersection of all compact invariant neighborhoods of e in G. It is easy to check that G is an (L) group whenever G/C is so (see [23], the proof of Proposition (3.5)).

Let us show that G/C is an (L) group. Since C is compact

$\widehat{G/C}$ is naturally embedded in $\hat{G}$ as an open subspace, hence $\widehat{G/C}$ is locally connected. Now $G/C \in [FIA]^-$ so we may apply the open and continuous surjection $\# : \widehat{G/C} \longrightarrow \mathfrak{X}(G/C)$ (see the proof of (2.1)). Thus $\mathfrak{X}(G/C)$ is locally connected and it follows from [23] Proposition(3.5) that G/C is an (L) group.
Q.E.D.

It has been known for some time that an $[FIA]^-$ group G is discrete iff its character space $\mathfrak{X}(G)$ is compact ([12] and [18]). An easy consequence of this is that G is discrete if its dual $\hat{G}$ is compact, indeed, $\mathfrak{X}(G)$ is a continuous image of $\hat{G}$ [18] . It is natural to ask if $[FC]^-$ groups having compact duals are discrete. We turn now to an example which shows that this fails already for type I groups.

<u>(2.3) Example.</u> Let G be the semi direct product of the real line $\mathbb{R}$ and the integers $\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}$ by $\beta(n)x = 2^n x$, all $n \in \mathbb{Z}$, $x \in \mathbb{R}$. Let L be the subgroup of $\mathbb{R}$ consisting of the integers. L is normal in G since $\beta(n)k \in L$, all $n \in \mathbb{Z}$, $k \in L$. Let $H = G/L$, then H satisfies an exact sequence of topological groups
$e \longrightarrow \mathbb{T} \longrightarrow H \longrightarrow \mathbb{Z} \longrightarrow e$, where $\mathbb{T}$ denotes the circle group. Hence H is an $[FC]^-$ group. Arguing word by word as in [2] , the proof of Theorem 4.7 , we see that $\hat{G}$ is compact. Hence $\hat{H}$ is compact, being homeomorphic to a closed subspace of $\hat{G}$.

Finally, we turn to the question of square-integrability for representations of $[FC]^-$ groups. We say that $[\pi] \in \hat{G}$ is square-integrable if all the coordinate functions $x \longmapsto (\pi(x)u, u)$ of π are in $\mathcal{L}^2(G)$ w.r.t. left Haar measure on G. If G is unimodular $[\pi]$ is square-integrable iff π occurs in the left regular representation of G as a subrepresentation, [4]. It is known that $[FC]^-$ groups are unimodular.

(2.4) Theorem. Let G be a separable and type I $[FC]^-$ group, and fix $[\pi] \in \hat{G}$. Then the following three statements are equivalent.

(1) $[\pi]$ is square-integrable.

(2) There is a compact normal subgroup K of G such that G/K is abelian and $[\widehat{G/K} \otimes \pi] = [\pi]$.

(3) $[\pi]$ is open in $\hat{G}$.

Each of the above statements implies the following.

(4) $[\widehat{G/P} \otimes \pi] = [\pi]$, where P is the periodic subgroup of G.

Proof. $(1) \Longrightarrow (2)$: Let $[\pi] \in \hat{G}$ be square-integrable, and let ρ be an irreducible subrepresentation of π_K. If $G(\rho)$ is the stability group of ρ under G's action by inner automorphisms, we know from the Mackey theory that there is a multiplier $\omega : G(\rho)/K \times G(\rho)/K \to \mathbb{T}$ such that ρ extends to an ω-representation $\tilde{\rho}$ of $G(\rho)$ [17]; moreover there is an ω^{-1} representation σ' of $G(\rho)$ (which is the identity operator on K) such that π is induced from the ordinary representation $\tilde{\rho} \otimes \sigma'$. Let σ be the representation of $G(\rho)/K$ given by σ', and set $S_\omega = \{ x \in G(\rho) : \omega(x,y) = \omega(y,x)$, all $y \in G(\rho) \}$. We have $\sigma = \alpha \otimes \gamma'$ where α is a character of $G(\rho)/K$ and γ is an ω^{-1} representation of $G(\rho)/S_\omega$ ([3] Theorem 3.1)

(Replace ω by a similar multiplier, if necessary).
Hence σ' equals α times the identity operator on S_ω. Using [14] Corollary 11.1 we see that σ' is a square-integrable ω^{-1} representation, and since $\sigma'(s) = \alpha(s)I$ on S_ω it is clear that S_ω is compact. S_ω is also normal in G, since G/S_ω is abelian. Moreover, the type I hypothesis on G implies that ω is a type I multiplier. Hence $G(\rho)/S_\omega$ has the unique irreducible ω^{-1} representation γ, [3] Theorem 3.3. Hence

$$(\tilde{\rho} \otimes \sigma') \otimes \beta = \tilde{\rho} \otimes (\alpha \otimes \gamma')' \otimes \beta \simeq \tilde{\rho} \otimes \sigma' ,$$

for all characters β of $G(\rho)$ which are one on S_ω. Using the fact that $(\tilde{\rho} \otimes \sigma')^G \otimes \tilde{\chi} \simeq (\tilde{\rho} \otimes \sigma' \otimes \chi)^G$ where χ is any character of $G(\rho)$ equal to 1 on K and $\tilde{\chi}$ is any extension of χ to G ([7] Lemma 4.2), we now have

$$\pi \otimes \tilde{\chi} \simeq (\tilde{\rho} \otimes \sigma' \otimes \chi)^G \simeq (\tilde{\rho} \otimes \sigma')^G \simeq \pi ,$$

for any character $\tilde{\chi}$ of G which is 1 on S_ω (and χ its restriction to $G(\rho)$). Hence $[\widehat{G/S_\omega} \otimes \pi] = [\pi]$, and (2) follows.
(2) $\Longrightarrow$ (3): If $[\widehat{G/K} \otimes \pi] = [\pi]$ for some compact K then $[\pi]$ is open in $\hat{G}$ by (1.1) and (1.2). (3) $\Longrightarrow$ (1): If $[\pi]$ is open in $\hat{G}$ then $[\pi]$ is square-integrable by [4] 18.8.5, since $\mu([\pi]) > 0$, where μ denotes the Plancherel measure for G.(9)
(2) $\Longrightarrow$ (4): Suppose there is a compact normal subgroup K of G such that G/K is abelian and $[\widehat{G/K} \otimes \pi] = [\pi]$. Then $K \subset P$ and we have

$$[\pi] \in [\widehat{G/P} \otimes \pi] \subset [\widehat{G/K} \otimes \pi] = [\pi]$$

so that $[\pi] = [\widehat{G/P} \otimes \pi]$. Q.E.D.(10)

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(9) It is known that $[FC]^-$ groups are amenable ([15]), i.e. $\hat{G} = \hat{G}_r$. This fact was needed at this point to ensure that $[\pi] \in \hat{G}_r$. $\hat{G}_r$ denotes the reduced dual of G ([4]).

(10) The author recently proved that a separable $[FC]^-$ group possessing a square-integrable irreducible representation is automatically typeI (On the dual topology of $[FC]^-$ groups, to appear in Math. Scand.).

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