A REGULAR SET THEOREM FOR INFINITE COMPUTATION THEORIES

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A subset B of the domain of a recursion theory is said to be regular if $B \cap K$ is "finite" (in the theory) whenever the set K is "finite". Of course, in ordinary recursion theory every set is regular, so there the concept is not considered. However, when moving up to recursion theory on an admissible ordinal α, non-regular α-r.e. sets exist whenever $\alpha^* < \alpha$. In case α is in-admissible then there are non-regular α-recursive sets.

When studying α-r.e. degrees for an admissible ordinal α the obstacle of the existence of non-regular α-r.e. sets is circumvented by the following theorem due to Sacks.

Theorem 1 ([3]). Suppose α is an admissible ordinal. Then every α-r.e. set is of the same α-degree as a regular α-r.e. set.

Maass [1] has recently obtained a uniform version of theorem 1.

Let Θ be an infinite computation theory as defined in [6]. In this paper we prove the following analogue of theorem 1. (A weaker but for most degree theoretic purposes sufficient version is proved in [7].)

<u>Theorem 2</u>. Suppose Θ is an adequate infinite computation theory. Then every Θ-s.c. set B is of the same degree as a regular Θ-s.c. set D. Furthermore D may be chosen such that $\forall x(\forall y \sim x)\ (x \in D \Rightarrow y \in D)$.

<u>Remark</u>: Suppose Θ is the infinite computation theory over an adequate resolvable admissible set $\mathcal{A}$ with urelements. Then the theorem asserts that every $\mathcal{A}$-r.e. set is of the same $\mathcal{A}$-degree as a regular $\mathcal{A}$-r.e. subset of $_{o}(\mathcal{A})$, the ordinal of $\mathcal{A}$.

Thus we have that adequacy is a sufficient condition on Θ for the regular set theorem to hold. However, it is shown in [2] that the condition is not necessary. On the other hand, assuming AD, Simpson [4] has shown there is a Θ such that every regular Θ-s.c. set is Θ-computable.

The proof of theorem 1 was simplified by Simpson [5]. He utilized the wellordering of the domain in the form that every α-r.e. non-α-finite set has a 1-1 α-recursive enumeration. The analogous property is false for arbitrary adequate computation theories. Thus our proof of theorem 2 is modelled after Sacks' original proof of theorem 1.

For definitions and notation the reader is referred to [6].

<u>Proof of theorem 2</u>: Let B be a Θ-s.c. non-Θ-computable set. We are to find a regular Θ-s.c. set D such that $D \equiv B$. Let $B^* = \{\xi : K_\xi \cap B \neq \emptyset\}$ where $\lambda\xi K_\xi$ is a fixed enumeration of Θ-finite sets. We have $K_\xi \cap B^* = \emptyset \iff \bigcup\{K_\eta : \eta \in K_\xi\} \cap B = \emptyset$ and

$$K_\xi \cap B = \emptyset \iff \xi \notin B^*. \quad \text{Thus } B^* \equiv B.$$

Let $\pi : U \to L^{|\precsim|^*}$ be a projection such that $\pi(x) \to y_1$ & $\pi(x) \to y_2 \Rightarrow y_1 \sim y_2$. Then

(1) $K_\gamma \cap B^* = \emptyset \iff \bigcup\{K_\eta : \eta \in K_\gamma\} \cap B = \emptyset \iff H_\gamma \cap B^* = \emptyset$

where $\lambda\gamma H_\gamma$ is a Θ-computable mapping whose values are (canonical Θ-indices for) Θ-finite sets such that $\forall\gamma\ (H_\gamma \neq \emptyset)$,
$\xi \in H_\gamma \Rightarrow K_\xi = \bigcup\{K_\eta : \eta \in K_\gamma\}$ and
$\xi_1, \xi_2 \in H_\gamma \Rightarrow \xi_1 \sim \xi_2$ & $\pi(\xi_1) \sim \pi(\xi_2)$. Because of (1) it is convenient to work with B^* instead of B.

Let $\lambda\sigma B^\sigma$ be a <u>disjoint</u> ($\precsim$)-enumeration of B^* such that $\forall\sigma\ (B^\sigma \neq \emptyset)$ and $\forall\sigma,x,y\ (x \in B^\sigma$ & $y \in B^\sigma \Rightarrow x \sim y$ & $\pi(x) \sim \pi(y))$. Define

$$D^2 = \{\sigma : (\exists\tau \succ \sigma)(B^\tau \prec B^\sigma \ \&\ \pi(B^\tau) \prec \pi(B^\sigma))\}.$$

Note that expressions like $\pi(B^\tau) \prec \pi(B^\sigma)$ make sense and are Θ-computable. Clearly D^2 is Θ-s.c. and $U - D^2$ is unbounded.

<u>Claim 1</u>: D^2 is regular.

<u>Proof</u>: Given σ_0 we show $D^2 \cap L^{\sigma_0}$ is Θ-finite. Having defined $\sigma_0, \ldots, \sigma_n$ we choose, if possible, σ_{n+1} such that $\sigma_{n+1} \succ \sigma_n$ and $(\forall j \leq n)\ (B^{\sigma_{n+1}} \prec B^{\sigma_j} \vee \pi(B^{\sigma_{n+1}}) \prec \pi(B^{\sigma_j}))$. By the well-foundedness of $\prec$ the defined sequence is finite. Let σ_n be the last. Then

$$D^2 \cap L^{\sigma_0} = \{\sigma \prec \sigma_0 : (\exists\tau \precsim \sigma_n)(B^\tau \prec B^\sigma \ \&\ \pi(B^\tau) \prec \pi(B^\sigma) \ \&\ \tau \succ \sigma)\}.$$

One inclusion is obvious. So suppose $\sigma \in D^2 \cap L^{\sigma_0}$. Choose $\tau \succ \sigma$ such that $B^\tau \prec B^\sigma$ & $\pi(B^\tau) \prec \pi(B^\sigma)$. If $\tau \precsim \sigma_n$ then all is well. If $\tau \succ \sigma_n$ then by the choice of σ_n there is $j \leq n$ such that $B^{\sigma_j} \precsim B^\tau$ & $\pi(B^{\sigma_j}) \precsim \pi(B^\tau)$. But then $B^{\sigma_j} \prec B^\sigma$ & $\pi(B^{\sigma_j}) \prec \pi(B^\sigma)$ and $\sigma \prec \sigma_0 \precsim \sigma_j$. Thus the inclusion from left to right holds. ⊠

Claim 2: $D^2 \leq B^*$.

Proof: First we show

(2) $\sigma \notin D^2 \iff \pi^{-1}[\pi(L^{B^\sigma}) \cap (L^{\pi(B^\sigma)} - \bigcup_{\tau \prec \sigma} \pi(B^\tau))] \cap B^* = \emptyset$.

Suppose the right hand side is false for a given σ. Then there are x and τ such that

$x \in \pi^{-1}[\pi(L^{B^\sigma}) \cap (L^{\pi(B^\sigma)} - \bigcup_{\sigma' \prec \sigma} \pi(B^{\sigma'}))] \cap B^\tau$. In particular $\pi(x) \cap \pi(L^{B^\sigma}) \neq \emptyset$ so $x \in L^{B^\sigma}$ (since π is a projection) and hence $B^\tau \prec B^\sigma$. Furthermore $\pi(x) \cap (L^{\pi(B^\sigma)} - \bigcup_{\sigma' \prec \sigma} \pi(B^{\sigma'})) \neq \emptyset$ so $\pi(B^\tau) \prec \pi(B^\sigma)$ and $\tau \succ \sigma$. Thus $\sigma \in D^2$.

The converse of (2) follows by a similar argument. Using (2) we have

$K \cap D^2 = \emptyset \iff \bigcup_{\sigma \in K} \pi^{-1}[\pi(L^{B^\sigma}) \cap (L^{\pi(B^\sigma)} - \bigcup_{\tau \prec \sigma} \pi(B^\tau))] \cap B^* = \emptyset$,

so $D^2 \leq B^*$. ☒

We now make an assumption and show that if the assumption holds then $B^* \leq D^2$. On the other hand if the assumption is false, we find σ such that $B^* \equiv B^* \cap L^\sigma$. It is then easy to find a regular Θ-s.c. set D such that $B^* \cap L^\sigma \equiv D$.

Define

$k_1(\gamma) = \mu\sigma[H_\gamma \prec \sigma \;\&\; \pi(H_\gamma) \prec \min \pi(\{y : y \sim \sigma\})]$.

k_1 is Θ-computable and total (by adequacy). Let

$k(\gamma) = \mu\sigma[k_1(\gamma) \precsim B^\sigma \;\&\; \min \pi(\{y : y \sim k_1(\gamma)\}) \precsim \pi(B^\sigma) \;\&\; \sigma \notin D^2]$.

Note that $k \leq_w D^2$.

Claim 3: If k is total then $B^* \leq D^2$.

<u>Proof</u>: Note that $H_\gamma \cap B^* \neq \emptyset \iff H_\gamma \subseteq B^*$. We show $H_\gamma \subseteq B^* \iff H_\gamma \subseteq \bigcup\{B^\tau : \tau \prec k(\gamma)\}$. It then follows from (1) that $B^* \leq D^2$. So let $\xi \in H_\gamma \subseteq B^*$, say $\xi \in B^\tau$. We want to show $\tau \prec k(\gamma)$. $B^\tau \prec k_1(\gamma) \precsim B^{k(\gamma)}$ so $\tau \not\sim k(\gamma)$. Suppose $\tau \succ k(\gamma)$. Then since $k(\gamma) \notin D^2$ it must be that $\pi(B^{k(\gamma)}) \precsim \pi(B^\tau)$. But then $\pi(B^\tau) \sim \pi(H_\gamma) \prec \min \pi(\{y : y \sim k_1(\gamma)\}) \precsim \pi(B^{k(\gamma)}) \precsim \pi(B^\tau)$, a contradiction. Thus $\tau \prec k(\gamma)$. ☒

Now we assume k is not total. Choose γ such that

$$\forall \sigma [B^\sigma \prec k_1(\gamma) \vee \pi(B^\sigma) \prec \min \pi(\{y : y \sim k_1(\gamma)\}) \vee \sigma \in D^2].$$

Let $B^*_\gamma = B^* \cap L^{k_1(\gamma)}$. We will show $B^*_\gamma \equiv B^*$. Clearly $B^*_\gamma \leq B^*$. By adequacy we can choose σ_0 such that $\tau \succ \sigma_0 \Rightarrow \pi(B^\tau) \succ \min \pi(\{y : y \sim k_1(\gamma)\})$. Thus

(3) $\qquad \forall \tau \succ \sigma_0\ (B^\tau \prec k_1(\gamma) \vee \tau \in D^2)$.

Let $B' = B^* - (L^{k_1(\gamma)} \cup \bigcup\{B^\tau : \tau \precsim \sigma_0\})$. Since clearly $B^* - B' \leq B^*_\gamma$, it suffices to show $B' \leq B^*_\gamma$ in order to show $B^* \equiv B^*_\gamma$.

<u>Claim 4</u>: $B' \leq B^*_\gamma$.

<u>Proof</u>: We first show

(4) $\quad \xi \in B' \iff \exists \sigma, \tau [\sigma_0 \prec \sigma \prec \tau\ \&\ \xi \in B^\sigma\ \&\ B^\tau \prec k_1(\gamma) \precsim B^\sigma$
$\qquad\qquad\qquad \&\ \pi(B^\tau) \prec \pi(B^\sigma)]$.

The if direction is obvious. So suppose $\xi \in B'$. Then there is $\sigma \succ \sigma_0$ such that $\xi \in B^\sigma$ and, by (3) and the definition of B', $\sigma \in D^2$. Thus there is $\tau_1 \succ \sigma$ such that $B^{\tau_1} \prec B^\sigma$ and $\pi(B^{\tau_1}) \prec \pi(B^\sigma)$. If $B^{\tau_1} \prec k_1(\gamma)$ then we are done. If not, then $B^{\tau_1} \succsim k_1(\gamma)$ so $\tau_1 \in D^2$ by (3). Thus there is $\tau_2 \succ \tau_1$ such that

$B^{\tau_2} \prec B^{\tau_1}$ & $\pi(B^{\tau_2}) \prec \pi(B^{\tau_1})$. The sequence $\tau_1, \tau_2, \ldots$ must be finite so eventually we obtain τ_m such that $B^{\tau_m} \prec k_1(\gamma)$. This proves (4).

Now suppose we have chosen the enumeration of ®-finite sets $\lambda\xi K_\xi$ to be repetitive in the following sense: Given any x then every ®-finite set has an index in $U - L^x$. Then we can find a ®-computable mapping $\lambda\eta G_\eta$ whose values are ®-finite sets such that

$$(5)\ K_\eta \cap B' = \emptyset \iff (K_\eta - (L^{k_1(\gamma)} \cup \bigcup\{B^\tau : \tau \precsim \sigma_0\})) \cap B^* = \emptyset$$
$$\iff G_\eta \cap B' = \emptyset .$$

Furthermore $\lambda\eta G_\eta$ can be chosen to have the following properties: $\forall\eta(G_\eta \neq \emptyset)$, $G_\eta \cap B' \neq \emptyset \iff G_\eta \subseteq B'$, $G_\eta \subseteq B^* \iff G_\eta \subseteq B'$, and $\xi_1, \xi_2 \in G_\eta \Rightarrow \xi_1 \sim \xi_2$ & $\pi(\xi_1) \sim \pi(\xi_2)$.

Let $F_\eta = \{x \in L^{k_1(\gamma)} : \pi(x) \prec \pi(G_\eta)\}$, and let $l(\eta) = \mu\tau[(F_\eta - \bigcup\{B^\sigma : \sigma \prec \tau\}) \cap B^*_\gamma = \emptyset]$. Then l is total by adequacy and $l \leq_w B^*_\gamma$. Clearly $l(\eta)$ is a strict least upper bound for $\{\tau : B^\tau \subseteq F_\eta\}$. We show $G_\eta \cap B' = \emptyset \iff G_\eta \cap \bigcup\{B^\tau : \tau \prec l(\eta)\} = \emptyset$. Combining this with (5) we then have $B' \leq B^*_\gamma$. So suppose $\xi \in G_\eta \subseteq B'$. By (4) there is σ and τ such that $\sigma_0 \prec \sigma \prec \tau$, $\xi \in B^\sigma$, $B^\tau \prec k_1(\gamma)$ and $\pi(B^\tau) \prec \pi(B^\sigma)$. If $\sigma \succsim l(\eta)$ then $\tau \succ l(\eta)$ so $\pi(G_\eta) \precsim \pi(B^\tau)$. But $\pi(B^\tau) \prec \pi(B^\sigma) \sim \pi(G_\eta)$ so we have a contradiction. This shows $\sigma \prec l(\eta)$, which was all that remained to prove the claim. ⊠

Let $C = \bigcup\{\pi(x) : x \in B^*_\gamma\}$. It is easily seen that $C \equiv B^*_\gamma$ since B^*_γ is bounded. Let $\lambda\sigma C^\sigma$ be a disjoint $(\precsim)$-enumeration of C such that $\forall\sigma(C^\sigma \neq \emptyset)$ and $x, y \in C^\sigma \Rightarrow x \sim y$. Let $D = \{\sigma : (\exists\tau \succ \sigma)(C^\tau \prec C^\sigma)\}$, the deficiency set of C. D is clearly regular and $U - D$ is unbounded. We show $D \equiv B^*_\gamma$ thus completing the proof of the theorem.

We have $\sigma \notin D \iff (L^{C^\sigma} - \cup\{C^\tau : \tau \prec \sigma\}) \cap C = \emptyset$ so $K \cap D = \emptyset \iff \bigcup_{\sigma \in K} (L^{C^\sigma} - \cup\{C^\tau : \tau \prec \sigma\}) \cap C = \emptyset$. Thus $D \leq C \equiv B^*_\gamma$.

For the converse reducibility note that

(6) $K_\eta \cap B^*_\gamma = \emptyset \iff \cup\{K_\xi : \xi \in K_\eta \cap L^{k_1(\gamma)}\} \cap B = \emptyset \iff N_\eta \cap B' = \emptyset$

where $\lambda\eta N_\eta$ is a $\circledR$-computable mapping having properties similar to those of $\lambda\eta G_\eta$. Let $f(\eta) = \mu\tau[C^\tau \gtrsim \pi(N_\eta)\ \&\ \tau \notin D]$. f is total by adequacy and $f \leq_w D$. Let
$g(\eta) = \mu\tau[\pi^{-1}(\cup\{C^\sigma : \sigma \prec f(\eta)\}) - \cup\{B^\sigma : \sigma \prec \tau\} = \emptyset]$. Then g is total and $g \leq_w D$. We show $N_\eta \subseteq B' \iff N_\eta \subseteq \cup\{B^\tau : \tau \prec g(\eta)\}$. This together with (6) shows $B^*_\gamma \leq D$. So suppose $\xi \in N_\eta \subseteq B'$. By (4) there are σ, τ such that $\xi \in B^\sigma$, $\sigma \prec \tau$, $B^\tau \prec k_1(\gamma) \precsim B^\sigma$ and $\pi(B^\tau) \prec \pi(B^\sigma) \sim \pi(N_\eta)$. Thus $B^\tau \subseteq B^*_\gamma$ since $B^\tau \prec k_1(\gamma)$. Furthermore $B^\tau \subseteq \pi^{-1}(\cup\{C^{\tau'} : \tau' \prec f(\eta)\})$ since $\pi(B^\tau) \prec \pi(N_\eta)$ and D is a deficiency set for C. But then $\tau \prec g(\eta)$ so $\sigma \prec g(\eta)$.

As a final remark we note that the regular set produced is either D^2 or D. Both of these satisfy the last statement of the theorem.

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