Automorphisms for locally compact groups

1. Let G be a locally compact group and $\mathrm{Aut}(G)$ the group of all its topological automorphisms with the Birkhoff topology. A neighborhood basis of the identity automorphism consists of sets $N(C,V) = \{\theta \in \mathrm{Aut}(G): \theta(x) \in Vx \text{ and } \theta^{-1}(x) \in Vx, \text{ all } x \in C\}$, where C is compact and V is a neighborhood of the identity e of G. As is well known, $\mathrm{Aut}(G)$ is a Hausdorff topological group but not generally locally compact [1;p.57]. In this article we are mainly concerned with the topological properties of $\mathrm{Aut}(G)$ and its subgroup $\mathrm{Int}(G)$ of inner automorphisms. We prove that for G arbitrary locally compact $\mathrm{Aut}(G)$ is a complete topological group. In particular, if G is also separable $\mathrm{Aut}(G)$ is a <u>Polish group</u>. As far as we can determine this result is new; and of course, this fact will be useful for the further study of $\mathrm{Aut}(G)$. Furthermore, we give two new characterizations of the topology for $\mathrm{Aut}(G)$, (1.1. and 1.6.). In Section 2 we turn to the question of when certain subgroups (among them $\mathrm{Int}(G)$) are closed in $\mathrm{Aut}(G)$, and several equivalent conditions are given; for instance, $\mathrm{Int}(G)$ is closed iff G admits no nontrivial central sequences (2.2). Applications to more special classes of groups are also given, as well as to the question of unimodularity of $\mathrm{Int}(G)$, (2.5). We remark that there is no separability assumption on the groups before 1.11.

1.1. <u>Lemma</u>. The sets $W_{\varphi_1,\ldots,\varphi_n;\varepsilon}$

$$= \{\tau \in \mathrm{Aut}(G): \|\varphi_j \circ \tau - \varphi_j\|_\infty < \varepsilon \text{ and } \|\varphi_j \circ \tau^{-1} - \varphi_j\|_\infty < \varepsilon \quad 1 \le j \le n\}$$

where $\varphi_j \in C_c(G)$ and $\varepsilon > 0$, form a basis for the neighborhoods of the identity in $\mathrm{Aut}(G)$.

<u>Proof</u>. Let $\varphi_1,\dots,\varphi_n \in C_c(G)$ and $\varepsilon > 0$ be given. Set $F = \bigcup_{i=1}^{n}$ support (φ_i), and let W be a symmetric nbh. of e in G such that $|\varphi_i(x) - \varphi_i(wx)| < \varepsilon$ for all $x \in G$, $w \in W$, $1 \leq i \leq n$. We claim $N(F,W) \subseteq W_{\varphi_1,\dots,\varphi_n;\varepsilon}$. Let $\tau \in N(F,W)$. Then for $x \in F$, $\tau(x)x^{-1} \in W$ and $\tau^{-1}(x)x^{-1} \in W$, so

(*) $|\varphi_i(x) - \varphi_i(\tau(x))| < \varepsilon$ and $|\varphi_i(x) - \varphi_i(\tau^{-1}(x))| < \varepsilon$, $1 \leq i \leq n$.

If $\tau(x) \in F$, then $\tau^{-1}(\tau(x))\tau(x)^{-1} \in W$, i.e. $x\tau(x)^{-1} \in W$, so $|\varphi_i(x) - \varphi_i(\tau(x))| < \varepsilon$, $1 \leq i \leq n$. Similarly if $\tau^{-1}(x) \in F$ then $|\varphi_i(x) - \varphi_i(\tau^{-1}(x))| < \varepsilon$, $1 \leq i \leq n$. Clearly if $x \notin F$ and $\tau(x) \notin F$, then $|\varphi_i(x) - \varphi_i(\tau(x))| < \varepsilon$, since in this case $\varphi_i(x) = \varphi_i(\tau(x)) = 0$, $1 \leq i \leq n$. Thus, for $x \notin F$, we have the following subcases:

(a) $\tau(x) \in F$ and $\tau^{-1}(x) \in F$

(b) $\tau(x) \in F$ and $\tau^{-1}(x) \notin F$

(c) $\tau^{-1}(x) \in F$ and $\tau(x) \notin F$

(d) $\tau(x) \notin F$ and $\tau^{-1}(x) \notin F$.

In each case (*) is satisfied. Thus $\tau \in N(F,W)$ implies $\|\varphi_i - \varphi_i \circ \tau\|_\infty < \varepsilon$ and $\|\varphi_i - \varphi_i \circ \tau^{-1}\|_\infty < \varepsilon$, i.e., $\tau \in W_{\varphi_1,\dots,\varphi_n;\varepsilon}$.

Conversely, let $F \subseteq G$ be compact and W a neighborhood of e in G. Let U be a compact neighborhood of e in G such that $U^2 \cdot U^{-1} \subset W$. Let $\psi \in C_c(G)$ be such that $0 \leq \psi \leq 1$, support $(\psi) \subset U^2$, and $\psi(u) \geq \frac{1}{2}$, $\forall\, u \in U$. (The existence of such a ψ is clear.) Let $\{x_1,\dots,x_n\}$ be a finite subset of F such that $\{Ux_i : 1 \leq i \leq n\}$ covers F. Define $\psi_i \in C_c(G)$ by $\psi_i(x) = \psi(yx_i^{-1})$ $1 \leq i \leq n$. We claim $W_{\psi_1,\dots,\psi_n;\frac{1}{2}} \subset N(F,W)$.

Indeed, suppose $\tau \in W_{\psi_1,\dots,\psi_n;\frac{1}{2}}$ and let $x \in F$. Then $x \in Ux_j$ for some j, and

$$|\psi_j(x) - \psi_j(\tau(x))| < \tfrac{1}{2} \text{ implies } \tau(x) \in U^2 x_j .$$

But then $\tau(x)x^{-1} \in U^2 x_j x^{-1} \subset U^2 U^{-1} \subset W$. Similarly, $|\psi_j(x) - \psi_j(\tau^{-1}(x))| < \frac{1}{2}$ implies $\tau^{-1}(x)x^{-1} \in W$. This proves the claim. □

1.2. By Braconnier [1] there is a continuous (modular) homomorphism $\Delta : \mathrm{Aut}(G) \to \mathbb{R}^+$ with the property

$$\Delta(\alpha)^{-1} \int_G f \circ \alpha^{-1}(x)dx = \int_G f(x)dx , \quad \text{for } f \in C_c(G) ,$$

where dx is a fixed Haar measure. Defining

$$\tilde{\theta}(f) = \Delta(\theta)^{-1} f \circ \theta^{-1} , \quad f \in L^1(G) , \quad \theta \in \mathrm{Aut}(G) ,$$

it is easy to see that $\tilde{\theta}$ becomes an automorphism of the group algebra $L^1(G)$. Denote by λ the left regular representation of G as well as the left regular representation of $L^1(G)$ on $L^2(G)$. Viewing $\tilde{\theta}$, $\theta \in \mathrm{Aut}(G)$, as an automorphism of $\lambda(L^1(G))$, we show that $\tilde{\theta}$ can be extended to an automorphism of the von Neumann algebra of the left regular representation, $\mathcal{R}(G) = \lambda(L^1(G))'' = \lambda(G)''$. We define a unitary operator U^θ, $\theta \in \mathrm{Aut}(G)$, by

$$U^\theta g = \Delta(\theta)^{-\frac{1}{2}} g \circ \theta^{-1} , \quad g \in L^2(G) .$$

A straight forward calculation shows

$$\lambda(\tilde{\theta}(f)) = U^\theta \lambda(f) U^{\theta^{-1}} .$$

The unitary implementation $\theta \longmapsto U^\theta$ allows us to define $\tilde{\theta}(T)$ for $T \in \mathcal{R}(G)$ by

$$\tilde{\theta}(T) = U^\theta T U^{\theta^{-1}} .$$

1.3. <u>Lemma</u>. The map $\alpha \in \mathrm{Aut}(G) \longmapsto U^\alpha g \in L^2(G)$ is continuous $(g \in L^2(G))$.

Proof. Let $G \in C_c(G)$ and $\varepsilon > 0$ be given. Fix a compact neighborhood U_1 of e in G and set $K = U_1 \cdot \text{support}(g)$. By Lemma 1.1. there is a neighborhood $N(C, U)$ in $\text{Aut}(G)$ so that $\alpha \in N(C,U)$ implies

$$\|g \circ \alpha^{-1} - g\|_\infty < \varepsilon / 2\mu(K)^{\frac{1}{2}},$$

where μ is a left Haar measure on G. We can assume support $(g) \subset C$ and $U = U^{-1} \subset U_1$. If $\alpha \in N(C, U)$ and $x \in \text{support}(g \circ \alpha^{-1})$, then $x \in U \cdot \text{support}(g) \subset K$. By continuity of Δ there is a neighborhood N_1 of the identity $\iota \in \text{Aut}(G)$ so that for $\alpha \in N_1$,

$$|\Delta(\alpha)^{-\frac{1}{2}} - 1| < \varepsilon / 2\|g\|_\infty \mu(K)^{\frac{1}{2}}.$$

Set $N = N_1 \cap N(C,U)$. Then if $\alpha \in N$,

$$\| U^\alpha g - g\|_\infty = \|\Delta(\alpha)^{-\frac{1}{2}} g \circ \alpha^{-1} - g\|_\infty < \varepsilon / \mu(K))^{\frac{1}{2}}.$$

Since support $(U^\alpha g - g) \subset K$ we have

$$\| U^\alpha g - g\|_2^2 \leq \int_K \| U^\alpha g - g\|_\infty^2 \, d\mu(x) \leq \| U^\alpha g - g\|_\infty \mu(K) < \varepsilon^2.$$

If $h \in L^2(G)$ is arbitrary, $\varepsilon > 0$, let $g \in C_c(G)$ with $\|g-h\|_2 < \varepsilon$. If $\| U^\alpha g - g\|_2 < \varepsilon$, $\alpha \in N$, then

$$\| U^\alpha h - h\|_2 \leq \| U^\alpha h - U^\alpha g\|_2 + \| U^\alpha g - g\|_2 + \|g - h\|_2 < 3\varepsilon. \quad \square$$

1.4. Our next aim is to study $\text{Aut}(G)$ by embedding it in $\text{Aut}(\mathcal{R}(G))$, and we shall prove that the embedding is topological if $\text{Aut}(\mathcal{R}(G))$ is provided with the appropriate topology, **namely the uniform-weak topology, and a neighborhood** base at the identity $\iota \in \text{Aut}(\mathcal{R}(G))$ is given by

$$\{\alpha \in \text{Aut}(\mathcal{R}(G)) : |\langle(\alpha - \iota)\mathcal{R}_1, \varphi_i\rangle| < \varepsilon, \varphi_i \in \mathcal{R}(G)_*, 1 \leq i \leq n\}, \quad \varepsilon > 0,$$

where $\mathcal{R}_1$ denotes the unit ball in $\mathcal{R}(G)$. Recall that the predual, $\mathcal{R}(G)_*$, is the Fourier algebra $A(G)$, [3]. Let

$$W_{\varphi_1,\dots,\varphi_n;\epsilon} = \{\alpha \in \mathrm{Aut}(G) : \|\varphi_i - \varphi_i \circ \alpha\| < \epsilon,\ 1 \le i \le n\}\ ,\quad \varphi_i \in A(G)\ ,$$

where $\|\cdot\|$ denotes the norm in $A(G)$.

1.5. Lemma. $W_{\varphi_1,\dots,\varphi_n;\epsilon} = \{\alpha \in \mathrm{Aut}(G) : |\langle(\tilde{\alpha}-\iota)\mathcal{R}_1, \varphi_i\rangle| < \epsilon,\ 1 \le i \le n\}$.

Proof. First note $\langle\tilde{\alpha}(T),\varphi\rangle = \langle T,\varphi\circ\alpha\rangle$, $T \in \mathcal{R}(G)$, $\varphi \in A(G)$, $\alpha \in \mathrm{Aut}(G)$; i.e., $\tilde{\alpha}^t(\varphi) = \varphi\circ\alpha$: If $T = \lambda(f)$, $f \in L^1(G)$, we have

$$\langle\tilde{\alpha}(\lambda(f)),\varphi\rangle = \Delta(\alpha)^{-1}\int_G f\circ\alpha^{-1}(x)\,\varphi(x)\,dx = \langle\lambda(f),\varphi\circ\alpha\rangle .$$

Since $\{\lambda(f) : f \in L^1(G)\}$ is dense in $\mathcal{R}(G)$, the claim follows. Now $\langle(\tilde{\alpha}-\iota)T,\varphi\rangle = \langle T,\varphi\circ\alpha-\varphi\rangle$, $T \in \mathcal{R}_1$. Taking the supremum over all $T \in \mathcal{R}_1$ we get

$$\langle(\tilde{\alpha}-\iota)\mathcal{R}_1,\varphi\rangle \doteq \|\varphi\circ\alpha - \alpha\|\ ,\quad \varphi \in A(G)\ ,$$

and the lemma follows. □

1.6. Proposition. The sets $W_{\varphi_1,\dots,\varphi_n;\epsilon}$, $\varphi_i \in A(G)$ and $\epsilon > 0$, form a base at the identity $\iota \in \mathrm{Aut}(G)$ for the Birkhoff topology. Hence the embedding $\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\mathcal{R}(G))$ is topological.

Proof. We show first that the topology generated by the sets $W_{\varphi_1,\dots,\varphi_n;\epsilon}$ is weaker than that of $\mathrm{Aut}(G)$. By Lemma 1.5, for $\varphi \in A(G)$, $\alpha \in \mathrm{Aut}(G)$,

$$\|\varphi - \varphi\circ\alpha\| = \sup_{T\in\mathcal{R}_1}|\langle T - \tilde{\alpha}(T)\ ,\ \varphi\rangle| .$$

Writing $\varphi = (f * g^{\sim})^{\vee}$, $f, g \in L^2(G)$, we have

$$\|\varphi - \varphi\circ\alpha\| = \sup_{T\in\mathcal{R}_1} |\langle(T - \tilde{\alpha}(T))f,g\rangle|$$

$$= \sup_{T\in\mathcal{R}_1} |\langle(T - U^{\alpha}\,T\,U^{\alpha^{-1}})f,g\rangle|$$

$$= \sup_{T\in\mathcal{R}_1} |\langle (U^{\alpha^{-1}}T - TU^{\alpha^{-1}})f, U^{\alpha^{-1}}g\rangle|$$

$$\leq \sup_{T\in\mathcal{R}_1} |\langle (U^{\alpha^{-1}}T - T)f, U^{\alpha^{-1}}g\rangle| + \sup_{T\in\mathcal{R}_1} |\langle (T - U^{\alpha^{-1}}T)f, U^{\alpha^{-1}}g\rangle|$$

Now

$$|\langle (T - TU^{\alpha^{-1}})f, U^{\alpha^{-1}}g\rangle| \leq \|T(f - U^{\alpha^{-1}}f)\|_2 \|U^{\alpha^{-1}}g\|_2$$

$$\leq \|f - U^{\alpha^{-1}}f\|_2 \|g\|_2 \quad , \quad \text{all} \quad T \in \mathcal{R}_1 .$$

By continuity of the map $\alpha \mapsto \alpha^{-1} \mapsto U^{\alpha^{-1}}f$ (Lemma 1.3) given $\epsilon > 0$, there is a neighborhood N_1 of $\iota \in \mathrm{Aut}(G)$ so that $\|f - U^{\alpha^{-1}}f\|_2\|g\|_2 < \frac{\epsilon}{2}$, all $\alpha \in N_1$. Furthermore,

$$|\langle (U^{\alpha^{-1}}T - T)f, U^{\alpha^{-1}}g\rangle$$

$$= |\langle U^{\alpha^{-1}}Tf, U^{\alpha^{-1}}g\rangle - \langle Tf, U^{\alpha^{-1}}g\rangle|$$

$$= |\langle Tf, g\rangle - \langle Tf, U^{\alpha^{-1}}g\rangle| = |\langle Tf, g - U^{\alpha^{-1}}g\rangle|$$

$$\leq \|Tf\|_2 \|g - U^{\alpha^{-1}}g\|_2 \leq \|f\|_2 \|g - U^{\alpha^{-1}}g\|_2 ,$$

all $T \in \mathcal{R}_1$.

Again there is a neighborhood N_2 of $\iota \in \mathrm{Aut}(G)$ so that $\|f\|_2\|g - U^{\alpha^{-1}}g\|_2 < \epsilon/2$. Letting $N = N_1 \cap N_2$, we get $\|\varphi - \varphi\circ\alpha\| < \epsilon$.

Conversely, let $F \subset G$ be compact and W a neighborhood of e in G. Let U be a compact neighborhood of e such that $U^2 \cdot U^{-1} \subset W$.

Since $A(G)$ is a regular algebra, there exists $\psi \in A(G)$ with $0 \leq \psi \leq 1$, $\psi(u) = 1$ for $u \in U$, and support $(\psi) \subset U^2$ [3; Lemma 3.2]. Let $\{x_1,\ldots,x_n\} \subset F$ be so that $\{Ux_i : 1 \leq i \leq n\}$ covers F. Define $\psi_i(y) = \psi(yx_i^{-1})$, $1 \leq i \leq n$. We claim $W_{\varphi_1,\ldots,\varphi_n;1} \subset N(F,W)$. Indeed, suppose $\tau \in W_{\varphi_1,\ldots,\varphi_n;1}$ and let $x \in F$. Then $x \in Ux_j$

for some j. Now $\|\psi_j \circ \tau - \psi_j\| < 1$ implies $\|\psi_j \circ \tau - \psi_j\|_\infty < 1$, so that $|\psi_j \circ \tau(x) - \psi_j(x)| < 1$. But for $x \in Ux_j$, $\psi_j(x) = \psi_j(ux_j) = \psi(u) = 1$, where $x = ux_j$, $u \in U$. Hence $\tau(x) \in$ support (ψ_j), or $\tau(x) \in U^2 x_j$. But then

$$\tau(x)x^{-1} \in U^2 x_j x^{-1} \in U^2 U^{-1} \subset W .$$

In addition

$$\|\psi_j \circ \tau^{-1} - \psi_j\| = \|\psi_j \circ \tau - \psi_j\| < 1 ,$$

so the same argument as above yields $\tau^{-1}(x) \in Wx$. □

1.7. Corollary. Suppose G has small neighborhoods of the identity, invariant under inner automorphisms (i.e., $G \in [SIN]$). Then viewing the group $Int(G)$ as a subgroup of $Aut(\mathcal{R}(G))$, the pointwise-weak and uniform-weak topologies coincide on $Int(G)$.

Proof. As is well known, $G \in [SIN]$ if and only if $\mathcal{R}(G)$ is a finite von Neumann algebra. The conclusion follows from [7; Proposition 3.7]. □

Note that the above can just as well be stated for $[SIN]_B$-groups where $B \subset Aut(G)$ is a subgroup. Also, the corollary is not too surprising in view of the fact that for $[SIN]$-groups the point-open and Birkhoff topologies of $Aut(G)$ agree on $Int(G)$ [6; Satz 1.6].

1.8. We say that G is an $[FIA]_B^-$-group if B is a relatively compact subgroup of $Aut(G)$ (see [5]). It is now a trivial consequence of 1.6 that $G \in [FIA]_B^-$ if and only if B, viewed as a subgroup of $Aut(\mathcal{R}(G))$ endowed with the uniform-weak topology, is relatively compact. Cf. [4; Theorem 2.4]. By [4; Corollary 1.6], the pointwise-weak topology may be substituted for the uniform-weak topology.

1.9. Next we show in an elementary way that for an arbitrary locally compact group G, $\mathrm{Aut}(G)$ is a complete topological group (in its two-sided uniformity).

<u>Theorem</u>. Let G be a locally compact group, then $\mathrm{Aut}(G)$ is complete with respect to its two-sided uniformity.

<u>Proof</u>. Let (α_ν) be a Cauchy net in $\mathrm{Aut}(G)$. Since $\alpha \mapsto U^\alpha$, $\mathrm{Aut}(G) \to \mathcal{L}(L^2(G))$ is continuous in the strong operator topology, it is also weakly continuous. Now $U^\alpha \in \mathcal{L}(L^2(G))_1$ (= unit ball of $\mathcal{L}(L^2(G))$); also the weak and ultraweak topology coincide on $\mathcal{L}(L^2(G))_1$ and $\mathcal{L}(L^2(G))_1$ is compact in this topology. Thus (U^{α_ν}) has a point of accumulation $U \in \mathcal{L}(L^2(G))_1$; let (α_μ) be a subnet such that $U^{\alpha_\mu} \xrightarrow[\mu]{} U$ weakly. Then for $f,g \in L^2(G)$

$$\langle (U^{\alpha_\nu} - U)f, g\rangle = \langle (U^{\alpha_\nu} - U^{\alpha_\mu})f, g\rangle + \langle (U^{\alpha_\mu} - U)f, g\rangle$$

$$= \langle f - U^{\alpha_\nu^{-1}\alpha_\mu} f, U^{\alpha_\nu^{-1}} g\rangle + \langle (U^{\alpha_\mu} - U)f, g\rangle$$

$$\leq \|f - U^{\alpha_\nu^{-1}\alpha_\mu} f\|_2 \|g\|_2 + \langle (U^{\alpha_\mu} - U)f, g\rangle \xrightarrow[(\mu,\nu)]{} 0$$

since $\alpha_\nu^{-1}\alpha_\mu \xrightarrow[(\nu,\mu)]{} \iota$ in $\mathrm{Aut}(G)$. Thus $U^{\alpha_\nu} \xrightarrow[\nu]{} U$ in the weak operator topology. Similarly $U^{\alpha_\nu^{-1}}$ converges weakly to some $V \in \mathcal{L}(L^2(G))_1$. We claim $V = U^{-1}$. Let $f,g \in L^2(G)$, $\varepsilon > 0$. Let ν_0 be such that for $\nu > \nu_0$

$$|\langle U^{\alpha_\nu} Vf - UVf, g\rangle| < \varepsilon, \text{ and } \|U^{\alpha_\nu^{-1}} g - U^{\alpha_{\nu_0}^{-1}} g\|_2 < \frac{\varepsilon}{2\|f\|_2}.$$

Choose ν_1 such that $\nu \succ \nu_1$ implies

$$|\langle U^{\alpha_\nu^{-1}} f - Vf, U^{\alpha_{\nu_o}^{-1}} g\rangle| < \epsilon .$$

Then for $\nu,\mu \succ \nu_o$ and ν_1, we have

$$|\langle U^{\alpha_\mu} U^{\alpha_\nu^{-1}} f - UVf, g\rangle|$$

$$\leq |\langle U^{\alpha_\mu} U^{\alpha_\nu^{-1}} f - U^{\alpha_\mu} Vf, g\rangle| + |\langle U^{\alpha_\mu} Vf - UVf, g\rangle| ,$$

where $|\langle U^{\alpha_\mu} Vf, g\rangle| < \epsilon$. Also

$$|\langle U^{\alpha_\nu^{-1}} f - U^{\alpha_\mu} Vf, g\rangle| = |\langle U^{\alpha_\nu^{-1}} f - Vf, U^{\alpha_\mu^{-1}} g\rangle|$$

$$\leq |\langle U^{\alpha_\nu^{-1}} f - Vf, U^{\alpha_{\nu_o}^{-1}} g\rangle| + |\langle U^{\alpha_\nu^{-1}} f - Vf, U^{\alpha_\mu^{-1}} g - U^{\alpha_{\nu_o}^{-1}} g\rangle|$$

$$< \epsilon + \|U^{\alpha_\nu^{-1}} f - Vf\|_2 \|U^{\alpha_\mu^{-1}} g - U^{\alpha_{\nu_o}^{-1}} g\|_2$$

$$< \epsilon + 2\|f\|_2 \|U^{\alpha_\mu^{-1}} g - U^{\alpha_{\nu_o}^{-1}} g\|_2 < 2\epsilon ,$$

so that

$$|\langle U^{\alpha_\mu} U^{\alpha_\nu^{-1}} f - UVf, g\rangle| < 3\epsilon .$$

But

$$\langle U^{\alpha_\mu} U^{\alpha_\nu^{-1}} f, g\rangle = \langle U^{\alpha_\mu \alpha_\nu^{-1}} f, g\rangle \xrightarrow[(\mu,\nu)]{} \langle f,g\rangle ,$$

hence

$$\langle UVf, g\rangle = \langle f,g\rangle \quad , \quad \text{all } f,g \in L^2(G) ;$$

thus $V = U^{-1}$. In addition,

$$\langle Uf,g\rangle = \lim_\nu \langle U^{\alpha_\nu} f, g\rangle = \lim_\nu \langle f, U^{\alpha_\nu^{-1}} g\rangle = \langle f, Vg\rangle ,$$

so $V = U^*$, and we have $U^{-1} = U^*$, so U is unitary. A standard argument now shows U^{α_ν} converges strongly to U:

$$\|U^{\alpha_\nu} f - Uf\|_2^2 = \langle U^{\alpha_\nu} f, U^{\alpha_\nu} f\rangle - \langle Uf, U^{\alpha_\nu} f\rangle$$

$$- \langle U^{\alpha_\nu} f, Uf\rangle + \langle Uf, Uf\rangle = 2\langle f,f\rangle - \langle Uf, U^{\alpha_\nu} f\rangle - \langle U^{\alpha_\nu} f, Uf\rangle \xrightarrow[\nu]{} 0 .$$

It remains to show that $\lambda(x) \mapsto U\lambda(x)U^{-1}$ defines an automorphism of $\lambda(G)$ (and thus of G). Fix $x \in G$, clearly $(\alpha_\nu(x))$ is a Cauchy net in G and (since G is complete) converges to an element, say $\alpha(x) \in G$. Then

$$U^{\alpha_\nu}\lambda(x)U^{\alpha_\nu^{-1}} = \lambda(\alpha_\nu(x)) \underset{\nu}{\longrightarrow} \lambda(\alpha(x)) \quad \text{weakly},$$

and

$$U^{\alpha_\nu}\lambda(x)U^{\alpha_\nu^{-1}} \underset{\nu}{\longrightarrow} U\lambda(x)U^{-1} \quad \text{weakly}.$$

I.e. $\lambda(\alpha(x)) = U\lambda(x)U^{-1}$. To prove α is a homomorphism, $\lambda(\alpha(xy)) = U\lambda(xy)U^{-1} = (U\lambda(x)U^{-1})(U\lambda(y)U^{-1}) = \lambda(\alpha(x))\lambda(\alpha(y)) = \lambda(\alpha(x)\alpha(y))$; so $\alpha(xy) = \alpha(x)\alpha(y)$. Also $\lambda(\alpha(x^{-1})) = U\lambda(x^{-1})U^{-1} = U\lambda(x)^{-1}U^{-1} = (U\lambda(x)U^{-1})^{-1} = \lambda(\alpha(x))^{-1} = \lambda(\alpha(x)^{-1})$, i.e. $\alpha(x^{-1}) = \alpha(x)^{-1}$.

To prove continuity of α, let $(x_\mu) \to x_o$ in G. Then

$$\lambda(\alpha(x_\mu)) = U\lambda(x_\mu)U^{-1} \underset{\mu}{\longrightarrow} U\lambda(x_o)U^{-1} = \lambda(\alpha(x_o))$$

in the weak operator topology. But $x \mapsto \lambda(x)$ is a homeomorphism of G into $\lambda(G)$, where $\lambda(G) \subset \mathcal{L}(L^2(G))$ carries the weak topology ([4; Lemma 2.2]). Thus $\alpha(x_\mu) \to \alpha(x_o)$. Similarly, α^{-1} is continuous, and we have $\alpha \in \mathrm{Aut}(G)$, so that $\mathrm{Aut}(G)$ is complete. □

1.10. <u>Remark</u>. Since by 1.6 $\mathrm{Aut}(G)$ is topologically embedded in the complete group $\mathrm{Aut}(\mathcal{R}(G))$, [7; Proposition 3.5], it would be natural to prove completeness of $\mathrm{Aut}(G)$ by showing it is closed in $\mathrm{Aut}(\mathcal{R}(G))$. Actually, such a proof can be given, utilizing the profound duality theory in [9]. We sketch the argument. Consider a net (α_ν) in $\mathrm{Aut}(G)$ such that $\tilde{\alpha}_\nu \to \gamma \in \mathrm{Aut}(\mathcal{R}(G))$ in uniform weak topology. By duality theory $\mathcal{R}(G)$ is a Hopf - von Neumann algebra with comultiplication $\delta : \mathcal{R}(G) \to \mathcal{R}(G) \otimes \mathcal{R}(G)$ which is a

σ - weakly continuous isomorphism given by $\delta(T) = W^{-1}(T \otimes 1)W$, $T \in \mathcal{R}(G)$, where $Wk(s,t) = k(s,st)$, $k \in L^2(G \times G)$, $s,t \in G$, [9; Section 4]. Furthermore, one has

$$\{T \in \mathcal{R}(G) : \delta(T) = T \otimes T\} - \{0\} = \{T \in \mathcal{R}(G) : T = \lambda(s), \text{ for some } s \in G\}.$$

Notice that $\mathrm{Aut}(G)$ corresponds to the subgroup
$\{\alpha \in \mathrm{Aut}(\mathcal{R}(G)) : \delta(\alpha\,\lambda(s)) = \alpha\,\lambda(s) \otimes \alpha\,\lambda(s), \text{ all } s \in G\}$.
Since $\tilde{\alpha}_\nu \to \gamma \in \mathrm{Aut}(\mathcal{R}(G))$ and $\delta(\tilde{\alpha}_\nu\,\lambda(s)) = \tilde{\alpha}_\nu\,\lambda(s) \otimes \tilde{\alpha}_\nu\,\lambda(s)$, all $s \in G$; continuity of δ gives

$$\delta(\gamma(\lambda(s))) = \gamma(\lambda(s)) \otimes \gamma(\lambda(s)), \quad \text{all } s \in G.$$

Thus $\gamma = \tilde{\alpha}$ for some $\alpha \in \mathrm{Aut}(G)$. □

1.11. <u>Corollary</u>. If G is a separable locally compact group, then $\mathrm{Aut}(G)$ is a <u>Polish topological group</u>.

<u>Proof</u>. Indeed, if $G = \bigcup_{n=1}^{\infty} F_n$, F_n compact, and if $\{U_m\}_{m \in \mathbb{N}}$ is a neighborhood base at $e \in G$, then $\{N(F_n, U_m)\}_{n,m}$ is a neighborhood base at $\iota \in \mathrm{Aut}(G)$, so that $\mathrm{Aut}(G)$ is metrizable [8], and by 1.9. it is complete. □

2. We proceed now to applications of the Theorem in 1.9. First we turn to the question of when certain subgroups of $\mathrm{Aut}(G)$ are closed. The following result contains a group theoretical analog. to [2; Theorem 3.1].

2.1. <u>Proposition</u>. Let G be a separable locally compact group, and B a subgroup of $\mathrm{Aut}(G)$. Suppose there is a Polish group H and a continuous surjective homomorphism $\omega : H \to B$. Then the following are equivalent.

a) B is closed in $\mathrm{Aut}(G)$.

b) $\omega : H \to B$ is open onto its range B.

c) For any neighborhood V of the identity in H there exist $\varphi_1, \dots, \varphi_n \in C_c(G)$ and $\varepsilon > 0$ such that, for all $h \in H$,

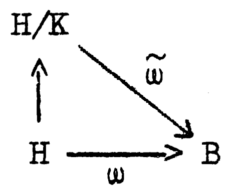
$$\|\varphi_i \circ \omega(h) - \varphi_i\|_\infty < \varepsilon \quad \text{and}$$

$$\|\varphi_i \circ \omega(h^{-1}) - \varphi_i\|_\infty < \varepsilon, \quad 1 \le i \le n, \quad \Rightarrow h \in V\cdot(\ker \omega)$$

d) Same statement as c) with $C_c(G)$ replaced by the Fourier algebra $A(G)$ (and its norm $\|\cdot\|$).

<u>Proof.</u> a) => b): If B is closed in $\mathrm{Aut}(G)$ then H and B are both Polish. Observe then that a continuous homomorphism between two Polish groups is open [2; Lemma 3.4]

b) => c): Put $K = \ker \omega$. Since ω is open it follows from Lemma 1.1 that given a neighborhood V of the identity in H there are functions $\varphi_1, \dots, \varphi_n \in C_c(G)$ and $\varepsilon > 0$ so that $W_{\varphi_1, \dots, \varphi_n; \varepsilon} \cap B \subset \omega(V)$. Now ω can be lifted to a map $\tilde{\omega}$ of $H/K \to B$, so that the diagram commutes and $\tilde{\omega}$ is a homeomorphism.



Thus $\omega(h) \in W_{\varphi_1, \dots, \varphi_n; \varepsilon}$ implies $\omega(h) \in \omega(V) = \tilde{\omega}(VK)$, hence $\tilde{\omega}(hK) \in \tilde{\omega}(VK)$, so that $h \in hK \subset VK$.

c) <==> d) is clear in view of Proposition 1.6.

d) => a): By 1.6 there is a sequence (φ_n) from $A(G)$ such that the sets $W_n = W_{\varphi_1, \dots, \varphi_n; 1/n}$ form a base for the identity in $\mathrm{Aut}(G)$. Let $\{V_n\}$ be a countable base for the identity in H. By d), given n there is an $m(n)$ so that $\omega(h) \in W_{m(n)}$ implies $h \in V_n K$. Let $\theta \in B^-$ and choose a sequence (α_n) from B so

a) B is closed in $\mathrm{Aut}(G)$.

b) $\omega : H \to B$ is open onto its range B.

c) For any neighborhood V of the identity in H there exist $\varphi_1, \ldots, \varphi_n \in C_c(G)$ and $\varepsilon > 0$ such that, for all $h \in H$,

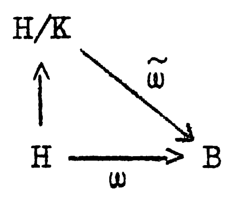
$$\|\varphi_i \circ \omega(h) - \varphi_i\|_\infty < \varepsilon \quad \text{and}$$

$$\|\varphi_i \circ \omega(h^{-1}) - \varphi_i\|_\infty < \varepsilon, \quad 1 \leq i \leq n, \quad \Rightarrow h \in V \cdot (\ker \omega)$$

d) Same statement as c) with $C_c(G)$ replaced by the Fourier algebra $A(G)$ (and its norm $\|\cdot\|$).

<u>Proof.</u> a) $\Rightarrow$ b): If B is closed in $\mathrm{Aut}(G)$ then H and B are both Polish. Observe then that a continuous homomorphism between two Polish groups is open [2; Lemma 3.4]

b) $\Rightarrow$ c): Put $K = \ker \omega$. Since ω is open it follows from Lemma 1.1 that given a neighborhood V of the identity in H there are functions $\varphi_1, \ldots, \varphi_n \in C_c(G)$ and $\varepsilon > 0$ so that $W_{\varphi_1, \ldots, \varphi_n; \varepsilon} \cap B \subset \omega(V)$. Now ω can be lifted to a map $\tilde{\omega}$ of $H/K \to B$, so that the diagram commutes and $\tilde{\omega}$ is a homeomorphism.



Thus $\omega(h) \in W_{\varphi_1, \ldots, \varphi_n; \varepsilon}$ implies $\omega(h) \in \omega(V) = \tilde{\omega}(VK)$, hence $\tilde{\omega}(hK) \in \tilde{\omega}(VK)$, so that $h \in hK \subset VK$.

c) $\Leftrightarrow$ d) is clear in view of Proposition 1.6.

d) $\Rightarrow$ a): By 1.6 there is a sequence (φ_n) from $A(G)$ such that the sets $W_n = W_{\varphi_1, \ldots, \varphi_n; 1/n}$ form a base for the identity in $\mathrm{Aut}(G)$. Let $\{V_n\}$ be a countable base for the identity in H. By d), given n there is an $m(n)$ so that $\omega(h) \in W_{m(n)}$ implies $h \in V_n K$. Let $\theta \in B^-$ and choose a sequence (α_n) from B so that $\alpha_n \to \theta$ and $\alpha_{n+j}^{-1}\alpha_n \in W_{m(n)}$ for $j \geq 0$. Setting $\tilde{\omega}^{-1}(\alpha_n) = h_n K$, we have $h_{n+j}^{-1}h_n \circ K \subset V_n K$, $j \geq 0$. This is to say that $(h_n K)$ is Cauchy in the left uniformity of H/K. Since H/K is locally compact, it is complete, and $h_n K \underset{n}{\Rightarrow} hK \in H/K$, hence $\omega(h) = \tilde{\omega}(hK) = \theta$ by continuity of $\tilde{\omega}$, and thus $\theta \in B$. □

2.2. A sequence (x_n) from G is said to be <u>central</u> if $\mathrm{ad}(x_n) \underset{n}{\Rightarrow} \iota$ in $\mathrm{Aut}(G)$. (x_n) is <u>trivial</u> if there is a sequence (z_n) from the center $Z(G)$ of G such that $x_n z_n^{-1} \underset{n}{\Rightarrow} e$.

<u>Corollary</u>. Suppose G is separable, then $\mathrm{Int}(G)$ is closed $\Longleftrightarrow$ all central sequences in G are trivial.

<u>Proof</u>. If $\mathrm{Int}(G)$ is closed, let $(x_n$ be a central sequence, and let $\{V_n\}$ be a base for the identity in G. By d) of 2.1. we can find, for each $n \in \mathbb{N}$, $\varphi_n \in A(G)$ and $\epsilon_n > 0$ so that

$$\|\varphi_n \circ \omega(x) - \varphi_n\| < \epsilon_n \quad (x \in G) \implies x \in V_n \cdot Z(G).$$

Choosing a sequence (k_j) from $\mathbb{N}$ such that

$$k \geq k_j \implies \|\varphi_j \circ \mathrm{Ad}(x_{k_j}) - \varphi_j\| < \epsilon_j ,$$

we have $x_{k_j} \in V_j \cdot Z(G)$, hence $x_{k_j} c_{k_j}^{-1} \in V_j$ for some $c_{k_j} \in Z(G)$. Let $c_n = c_{k_j}$ for $k_j \leq n < k_{j+1}$; $j = 1,2,3,\ldots$. Then $x_n c_n^{-1} \underset{n}{\Rightarrow} e$, and (x_n) is trivial.
The converse is shown in the same way as d) => a) in 2.1. □

2.3. We remark that the class of groups for which $\mathrm{Aut}(G)$ is locally compact includes the compactly generated Lie groups [6; Sats 2.2]. For $\mathrm{Int}(G)$ we have the following

<u>Corollary</u>. Let G be separable and locally compact. Then $\mathrm{Int}(G)$ is locally compact $\Longleftrightarrow$ $\mathrm{Int}(G)$ is closed.

Proof. If $Int(G)$ is locally compact, it is necessarily closed [8; Theorem 5.11]. On the other hand if $Int(G)$ is closed, take $G = H$ and $\omega = Ad$, where $Ad(g)x = gxg^{-1}$, $x, g \in G$, in 2.1. Then by continuity of Ad, $Int(G)$ is homeomorphic with $G/Z(G)$. $Z(G)$ = center of G. □

2.4. Let G_F be the closed normal subgroup of elements x in G having relatively compact conjugacy classes $\{gxg^{-1} : g \in G\}$. If $G \in [SIN]$, G_F is open since any compact $Int(G)$-invariant neighborhood of e is contained in G_F. Let $\omega : G \to Aut(G_F)$ be the continuous homomorphism $\omega(g) = Ad(g)|_{G_F}$, [8], and let B be the subgroup $\omega(G) \subset Aut(G_F)$. Clearly G_F is an $[SIN]_B$ group, and we have

Corollary. Let G be separable. Then, with notation as above, B is closed $\Longleftrightarrow$ B is compact $\Longleftrightarrow$ $G/\ker\omega$ is compact.

Proof. The first equivalence is proved in [5]. If B is closed, B is homeomorphic with $G/\ker\omega$ (The Proposition in 2.1, a) $\Rightarrow$ b)) so by compactness of B, $G/\ker\omega$ must be compact. Conversely, if $G/\ker\omega$ is compact then so is $B = \tilde{\omega}(G/\ker\omega)$ by continuity of the lifted map $\tilde{\omega}$. □

Specializing the preceeding corollary even further we obtain

2.5. Corollary. Let G be a locally compact group and suppose $Int(G)^-$ is compact. Then $Int(G)$ is closed $\Longleftrightarrow$ $G/Z(G)$ is compact ($Z(G)$ = the center of G).

Proof. This follows immediately from the Corollary in 2.4. if G is separable. From [6] $Int(G)$ is closed $\Longleftrightarrow$ $Int(G)$ is compact.

But $\mathrm{Int}(G)$ compact implies $\mathrm{Ad} : G \to \mathrm{Int}(G)$ is open [8; Theorem 5.29], hence $\mathrm{Int}(G) \cong G/Z(G)$, and so $G/Z(G)$ is compact. Conversely if $G/Z(G)$ is compact, lifting Ad to a continuous map $G/Z(G) \to \mathrm{Int}(G)$ we see that $\mathrm{Int}(G)$ is compact, hence closed.

2.6. Corollary. Let G be a separable locally compact group. Then $\mathrm{Int}(G)$ is unimodular $\iff$ G is unimodular and $\mathrm{Int}(G)$ is closed.

Proof. If $\mathrm{Int}(G)$ is unimodular, in particular it is closed, so by the Proposition in 2.1 it is topologically isomorphic with $G/Z(G)$, so that the latter is unimodular. It is then easy to see G is unimodular, we give a proof for completeness. Let dz and $d\dot{x}$ be Haar measures on $Z(G)$ and $G/Z(G)$ respectively, and $x \mapsto \dot{x}$, $G \to G/Z(G)$ the canonical map. Let

$$\mu(\varphi) = \int_{G/Z(G)} \int_{Z(G)} \varphi(xz)\,dz\,d\dot{x}\ , \qquad \varphi \in C_c(G)\,.$$

By the Weil integration formula μ is a left Haar measure on G. Using right invariance of $d\dot{x}$ and the fact that $Z(G)$ is the center, one verifies easily that μ is even right invariant. Thus G is unimodular. Conversely, if G is unimodular and $\mathrm{Int}(G)$ is closed we show that $G/Z(G)$ is unimodular. It will then follow that $\mathrm{Int}(G)$ is unimodular. since $\mathrm{Int}(G) \cong G/Z(G)$.

Define μ as above. By assumption μ is right-invariant. Since the mapping $C_c(G) \to C_c(G/Z(G))$, $\varphi \mapsto \tilde{\varphi}$, $\tilde{\varphi}(\dot{x}) = \int_{Z(G)} \varphi(xz)dz$ is surjective [8, Theorem 15.21]; $\mu(\varphi) = \mu(\varphi_y)$ for all $\varphi \in C_c(G)$, $y \in G$, then implies $d\dot{x}$ is right-invariant:

$$\int_{G/Z(G)} \tilde{\varphi}_{\dot{y}}(\dot{x})d\dot{x} = \mu(\varphi_{\dot{y}}) = \mu(\varphi) = \int_{G/Z(G)} \tilde{\varphi}(\dot{x})d\dot{x}\,,$$

(here $\varphi_y(x) = \varphi(yx)$). Thus $\mathrm{Int}(G)$ is unimodular. $\square$

2.7. <u>Corollary</u>. Let G be separable and locally compact and suppose $\mathrm{Int}(G)$ is closed. Each of the following properties for G implies the same property for $\mathrm{Int}(G)$.

a) There is a compact neighborhood of the identity element invariant under inner automorphisms ([IN] - property).
b) There is a neighborhood basis of the identity consisting of compact sets which are invariant under inner automorphisms ([SIN] - property).
c) All the conjugacy classes of G are precompact ($[FC]^-$-property).

<u>Proof</u>. We need only notice that in each case $G/Z(G)$ has the required property. The easy details will be omitted. □

Next we give an example of a group G for which $\mathrm{Int}(G) \neq \mathrm{Int}(G)^-$, using Corollary 2.5.

2.8. <u>Example</u>. Fix an irrational number λ, and let $\omega_\lambda : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$; $\omega_\lambda((m_1,m_2),n_1,n_2)) = e^{\pi i\, \lambda m_1 n_2}$, where $\mathbb{T}$ is the circle group. Let $G = \mathbb{Z}^2 \times \mathbb{T}$ with the product topology and group composition

$$(m_1,m_2,s)\cdot(n_1,n_2,t) = (m_1+n_1, m_2+n_2, e^{\pi i\, \lambda m_1 n_2} st).$$

G is a topological group with center $((0,0)) \times \mathbb{T}$, and since $((0,0)) \times \mathbb{T}$ is open, G has small $\mathrm{Int}(G)$-invariant neighborhoods of the identity. Moreover, all the conjugacy classes of G are precompact, so by the Ascoli theorem for groups [6, Satz 1.7], $\mathrm{Int}(G)^-$ is compact. However $G/Z(G)$ is non-compact, being infinite discrete; and hence $\mathrm{Int}(G)$ is not closed (Corollary 2.5). To see this directly, choose a sequence (k_n) from $\mathbb{Z}$ such that $e^{\pi i\, \lambda k_n} \xrightarrow[n]{} e^{\pi i/2}$, and put $\alpha_\lambda(m_1,m_2,e^{\pi i t}) = (m_1,m_2,e^{\pi i\, m_2 t/2})$. Then $\alpha_\lambda \in \mathrm{Aut}(G)$ and is not inner. A routine calculation shows that $\mathrm{Ad}(0,k_n,1) \xrightarrow[n]{} \alpha_\lambda$ (by [6; Satz 1.6] it suffices to check $\mathrm{Ad}(0,k_n,1) \xrightarrow[n]{} \alpha_\lambda$ pointwise).

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