A Loeb-measure Approach to Theorems

by Prohorov, Sazonov, and Gross

by

Tom L. Lindstrøm, Oslo.

I. Introduction.

It has been clear for some time now that the Loeb-measure
of nonstandard analysis [11] may be useful in the construction
of different kinds of limit measures. Indeed, Anderson's nonstan-
dard construction of a Brownian motion [1], may be regarded as a
direct construction of a weak limit measure (compare Billingsley
[4]). Work on weak convergence from a nonstandard point of view
have been carried on by Anderson and Rashid [3], and Loeb [12].
In another direction, Helms and Loeb [7], Hurd [9], and Helms [6]
have used the Loeb-measure to obtain limit equilibrium measures
in statistical mechanics.

In this paper we shall work with other - but related -
concepts of limit measures, and we hope to show the efficiency
of the Loeb-measure approach by giving simple proofs of three
classical theorems.

The first of these theorems is due to Prohorov [14]: Given
a projective system $\langle X_i, \tau_i, \pi_{ij} \rangle_{i,j \in I}$ of Hausdorff spaces en-
dowed with a cylindrical measure $\{\mu_i\}_{i \in I}$ of Radon measures on
the X_i, it gives a characterization of when there is a limit
Radon measure on the projective limit. The idea of the proof is

to construct the limit measure from the Loeb-measure of μ_i for an infinite $i \in {}^*I$, using a standard part map $\theta_i : X_i \longrightarrow X$.

The second theorem is due to Sazonov [16]; it characterizes the functions that are Fourier-transforms of probability measures on Hilbert spaces. The idea of the proof is that even when the measure does not exist on the Hilbert space H, measures exist on the hyperfinite dimensional subspaces of ${}^*H$ and we can perform the necessary calculations on these spaces.

The last theorem is by Gross [5] and is concerned with the extension of cylindrical measures on Hilbert spaces to measures on Banach spaces where the Hilbert space is densely embedded. Again the idea is the same; we use the Loeb-measures on hyper-finite dimensional subspaces and the standard part map.

Throughout this paper we shall work with polysaturated models for nonstandard analysis (see Stroyan and Luxemburg, [18]). For an introduction to the theory of the Loeb-measure and some of its applications, the reader should consult Loeb [13].

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II. Measures on inductive limits: Prohorov's theorem.

Let $\langle X_i, \tau_i, \pi_{ij} \rangle_{i,j \in I}$ be a projective system of Hausdorff spaces; i.e. I is a directed set; for each $i \in I$, $\langle X_i, \tau_i \rangle$ is a Hausdorff space; and for all $i,j,k \in I$, $i \leq j \leq k$, $\pi_{ij} : X_j \longrightarrow X_i$, $\pi_{jk} : X_k \longrightarrow X_j$, and $\pi_{ik} : X_k \longrightarrow X_i$ are continuous, surjective maps satisfying $\pi_{ik} = \pi_{ij} \circ \pi_{jk}$. By the projective limit of such

a system, we shall mean the Hausdorff space $\langle X, \tau \rangle$, where X consists of all elements $x \in \prod_{i \in I} X_i$ such that for all $i, j \in I$, $i \le j$, $x_i = \pi_{ij}(x_j)$; and where τ is the weakest topology making all the maps $\pi_i : X \to X_i$-defined by $\pi_i(x) = x_i$-continuous. Obviously $\pi_i = \pi_{ij} \circ \pi_j$ for $i \le j$. The set X may be empty even when all the X_i are non-empty; and the π_i need not be surjective.

We shall denote the $*$-version $^*\{\langle X_i, \tau_i, \pi_{ij} \rangle_{i,j \in I}\}$ by $\langle \widetilde{X}_i, \widetilde{\tau}_i, \widetilde{\pi}_{ij} \rangle_{i, j \in {}^*I}$. Let $st : {}^*X \to X$ be the standard part map in the limit topology, and for each $i \in I$ let st_i be the standard part map $st_i : \widetilde{X}_i \to X_i$. It is easy to see that if $st(x)$ is defined, then $st_i(\widetilde{\pi}_i(x))$ is defined for all $i \in I$, and $st(x) = \{st_i(\widetilde{\pi}_i(x))\}_{i \in I}$. On the other hand, if $st_i(\widetilde{\pi}_i(x))$ is defined for all $i \in I$, then $st_i(\widetilde{\pi}_i(x)) = \pi_{ij}(st_j(\widetilde{\pi}_j(x)))$ for $j \ge i$, and $st(x)$ is defined and equal to $\{st_i(\widetilde{\pi}_i(x))\}_{i \in I}$.

Let us recall a few simple facts of nonstandard topology: If X is a Hausdorff space and $st : {}^*X \to X$ the standard part map, then $^*G \supset st^{-1}(G)$ for all open sets G, and $^*K \subset st^{-1}(K)$ for all compact sets K. We are working with polysaturated models; and if A is an internal subset of $^*X$ in such a model, then $st(A)$ is closed (see e.g. Stroyan and Luxemburg [18], page 201).

Having now completed our survey of the topological prerequisites, we turn to measure theory: By a <u>Radon measure</u> μ on a topological space $\langle Y, \sigma \rangle$, we shall mean a Borel probability measure on Y such that for all Borel sets B and all $\varepsilon > 0$, there is a compact set $K \subset B$ and an open set $G \supset B$ such that

$$\mu(G) - \varepsilon < \mu(B) < \mu(K) + \varepsilon.$$

By a _cylindrical measure_ $\{\mu_i\}_{i \in I}$ on a projective system $\langle X_i, \tau_i, \pi_{ij} \rangle_{i,j \in I}$ of Hausdorff-spaces, we mean a net $\{\mu_i\}_{i \in I}$ of Radon measures on the X_i such that $\pi_{ij}(\mu_j) = \mu_i$. We write $\{\tilde{\mu}_i\}_{i \in *I}$ for the *-version $*(\{\mu_i\}_{i \in I})$. A _cylinder set_ in X is a set of the form $\pi_i^{-1}(B)$ where B is a Borel set in X_i, $i \in I$.

The problem we consider in this section is the following: A cylindrical measure obviously induces a finitely additive measure on the cylinder sets, and using the Kolmogorov Extension Theorem this may be extended to a σ-additive measure. But can it be extended to a Radon measure on X? This is far from obvious since the σ-algebra generated by the cylinder sets is much smaller in general than the Borel algebra on X. However, Prohorov's theorem gives a simple characterization of when such an extension exists.

Before we turn to the proof of the theorem, we must know how Radon measures arise from Loeb-measures; the study of such problems was initiated by Anderson [2], and has since been studied by Henson [3] and Loeb [12]. We shall only need the simple

Lemma 1: Let $\langle X, \mathcal{B}, \mu \rangle$ be a Borel probability space on a Hausdorff space. Assume that for each $\varepsilon > 0$ there is a compact K_ε with $\mu(K_\varepsilon) > 1 - \varepsilon$. Let $\langle Z, L(G), L(P) \rangle$ be the Loeb-space of a nonstandard probability space $\langle Z, G, P \rangle$, and let $\theta : Z \to X$ be a partial mapping such that $\theta^{-1}(B) \in L(G)$ and $L(P)(\theta^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}$. Assume further that $\theta(A)$ is closed for all $A \in G$. Then μ is a Radon measure.

Proof: Let $B \in \mathcal{B}$ be given; then $\theta^{-1}(B) \in L(G)$ and for each $\varepsilon > 0$ there is an $A \in G$, $A \subset \theta^{-1}(B)$ such that

$L(P)(\theta^{-1}(B)) - L(P)(A) < \frac{\epsilon}{2}$. But $\theta(A) \subset \theta\theta^{-1}(B) = B$ is closed,
and $\mu(\theta(A)) = L(P)(\theta^{-1}(\theta(A))) \geq L(P)(A)$. Hence $\theta(A) \cap K_{\epsilon/2}$ is
a compact set contained in B with

$$\mu(B) - \mu(\theta(A) \cap K_{\epsilon/2}) < \epsilon .$$

The outer regularity follows by applying this result to $X - B$,
and hence the lemma is proved.

We may now prove Prohorov's theorem:

Theorem 2 (Prohorov): Let $\{\mu_i\}_{i \in I}$ be a cylindrical measure
on a projective system $\langle X_i, \tau_i, \pi_{ij} \rangle$ of Hausdorff spaces. The
following is a necessary and sufficient condition that there exists
a Radon measure μ on X such that $\pi_i(\mu) = \mu_i$ for all $i \in I$:

(*) For all $\epsilon > 0$, there is a compact $K_\epsilon \subset X$ such that

$$\mu_i(\pi_i(K_\epsilon)) \geq 1 - \epsilon \quad \text{for all} \quad i \in I.$$

Proof: That (*) is necessary is almost obvious: Let $\epsilon > 0$
be given; since μ is Radon there exists a compact $K \subset X$ such
that $\mu(K) \geq 1 - \epsilon$. Since K is compact, so is $\pi_i(K)$, and hence
$\pi_i(K)$ is μ_i-measurable. But then

$$\mu_i(\pi_i(K)) = \mu(\pi_i^{-1}\pi(K)) \geq \mu(K) = 1 - \epsilon$$

which proves the necessity of (*).

We now turn to the sufficiency, and divide the proof into
four steps.

1. Construction of μ: Let $\omega \in {}^*I$ be larger than all the
elements in I - such an element exists by polysaturation - and
define $\theta : \widetilde{X}_\omega \rightarrow X$ by $\pi_i(\theta(x)) = \mathrm{st}_i(\widetilde{\pi}_{i\omega}(x))$. Obviously

$\theta \circ \widetilde{\pi}_\omega = st$, and hence θ maps internal sets on closed sets.

Let $\langle \widetilde{X}_\omega, L(\widetilde{\mathscr{B}}_\omega), L(\widetilde{\mu}_\omega) \rangle$ be the Loeb-space of $\langle \widetilde{X}_\omega, \widetilde{\mathscr{B}}_\omega, \widetilde{\mu}_\omega \rangle$, where $\widetilde{\mathscr{B}}_\omega$ is the *-Borel-algebra of $\widetilde{X}_\omega$. Define

$$\mu = \theta(L(\widetilde{\mu}_\omega)).$$

2. $\underline{\mu}$ $\underline{is}$ $\underline{a}$ $\underline{Borel\text{-}measure}$: We first show that if $K \subset X$ is compact, then $\theta^{-1}(K) \in L(\widetilde{\mathscr{B}}_\omega)$: The family $\{\pi_i^{-1}\pi_i(K)\}$ is decreasing, and so is $\mu_i(\pi_i(K))$ since $\mu_j(\pi_j(K)) \leq \mu_j(\pi_{ij}^{-1}\pi_i(K)) = \mu_i(\pi_i(K))$. Let

$$\beta = \lim_{i \in I} \mu_i(\pi_i(K)),$$

then

$$L(\widetilde{\mu}_\omega)(\widetilde{\pi}_\omega(*K)) = \beta,$$

since ω is infinitely large.

Since K is compact,

$$\theta^{-1}(K) = \widetilde{\pi}_\omega st^{-1}(K) \supseteq \widetilde{\pi}_\omega(*K),$$

and thus $\theta^{-1}(K)$ contains a set of measure β.

Let $\epsilon > 0$ be given, and choose $i \in I$ such that $\mu_i(\pi_i(K)) < \beta + \frac{\epsilon}{2}$. Since μ_i is Radon there is a $G_i \in \tau_i$ such that $\pi_i(K) \subset G_i$ and $\mu_i(G_i) < \beta + \epsilon$. Hence

$$L(\widetilde{\mu}_\omega)(\widetilde{\pi}_{i\omega}^{-1}(*G_i)) = {}^\circ\widetilde{\mu}_\omega(\widetilde{\pi}_{i\omega}^{-1}(*G_i)) = {}^\circ\widetilde{\mu}_i(*G_i) < \beta + \epsilon.$$

Since G_i is open, we also have

$$\theta^{-1}(K) = \widetilde{\pi}_\omega st^{-1}(K) \subseteq \widetilde{\pi}_\omega st^{-1}(\pi_i^{-1}\pi_i(K)) \subseteq \widetilde{\pi}_\omega st^{-1}(\pi_i^{-1}(G_i)) \subseteq \widetilde{\pi}_\omega *(\pi_i^{-1}G_i) = \widetilde{\pi}_\omega \widetilde{\pi}_i^{-1} *G_i = \widetilde{\pi}_{i\omega}^{-1}(*G_i).$$

This proves that for all $\epsilon > 0$, $\theta^{-1}(K)$ is contained in a

set of measure less than $\beta + \epsilon$. Since $\langle \widetilde{X}_\omega, L(\widetilde{\mathcal{B}}_\omega), L(\widetilde{\mu}_\omega) \rangle$ is complete, it follows that $\theta^{-1}(K) \in L(\widetilde{\mathcal{B}}_\omega)$, and that $L(\widetilde{\mu}_\omega)(\theta^{-1}(K)) = \lim_{i \in I} \mu_i(\pi_i K)$.

If now $F \subset X$ is closed, $F \cap K_{1/n}$ is compact, and hence $\theta^{-1}(\cup(F \cap K_{1/n})) \in L(\widetilde{\mathcal{B}}_\omega)$. The set $\theta^{-1}(F - \underset{n \in \mathbb{N}}{\cup}(F \cap K_{1/n}))$ is a subset of the null-set $\widetilde{X}_\omega - \theta^{-1}(\cup K_{1/n})$, and is consequently in $L(\widetilde{\mathcal{B}}_\omega)$. Hence $\theta^{-1}(F) \in L(\widetilde{\mathcal{B}}_\omega)$ for all closed F, and it follows that μ is a Borel-measure.

3. $\underline{\mu}$ is a Radon measure: Since θ maps internal sets on closed sets, this follows from Lemma 1, and $\underline{2}$ above.

4. Consistency conditions: It remains to show that if $A \in \mathcal{B}_i$, then $\mu(\pi_i^{-1}(A)) = \mu_i(A)$: Let $\epsilon > 0$ be given, and choose a neighbourhood G of A in τ_i such that $\mu_i(G) \leq \mu_i(A) + \epsilon$. Since $\pi_i^{-1}(G)$ is open: $\mu(\pi_i^{-1}(A)) = L(\widetilde{\mu}_\omega)(\theta^{-1}(\pi_i^{-1}(A))) = L(\widetilde{\mu}_\omega)(\widetilde{\pi}_\omega \mathrm{st}^{-1}(\pi_i^{-1}(A))) \leq L(\widetilde{\mu}_\omega)(\widetilde{\pi}_\omega \mathrm{st}^{-1}(\pi_i^{-1}(G))) \leq L(\widetilde{\mu}_\omega)(\widetilde{\pi}_\omega \widetilde{\pi}_i^{-1}(^*G)) = L(\widetilde{\mu}_\omega)(\widetilde{\pi}_{i\omega}^{-1}(^*G)) = \mu_i(G) \leq \mu_i(A) + \epsilon$.

This implies that $\mu(\pi_i^{-1}(A)) \leq \mu_i(A)$, but since the same must apply to the complement of A, we have $\mu(\pi_i^{-1}(A)) = \mu_i(A)$. This proves the theorem.

The above proof gives a straightforward construction of the limit measure as the "standard part" of the Loeb-measure of an infinitely large element in $\{\widetilde{\mu}_i\}_{i \in {}^*I}$. For applications of Prohorov's theorem, the reader should consult Schwartz [17]; we shall only give one which will be useful in the next section. We shall need the following version of the theorem:

Corollary 3: Let $\{\mu_i\}_{i \in I}$ be a cylindrical measure on $\langle X_i, \tau_i, \pi_{ij} \rangle$. Let Y be a set, and assume that for each $i \in I$, P_i is a mapping of Y on X_i such that $P_i = \pi_{ij} \circ P_j$, when $i < j$. Let σ be the weakest topology on Y making all the P_i continuous. Then there is a Radon measure ν on Y such that $\nu(P_i^{-i}(A)) = \mu_i(A)$ for all $i \in I$ and $A \in \mathcal{B}_i$ if and only if:

(**)   For all $\epsilon > 0$ there is a compact set $K_\epsilon \subset Y$ such that

$$\mu_i(P_i(K_\epsilon)) > 1 - \epsilon \text{ for all } i \in I.$$

Proof: The necessity of (**) is proved as in the proof of the theorem.

To prove the sufficiency, define $S : Y \rightarrow X$ by $S(y) = \{P_i(y)\}_{i \in I}$. S is continuous, and by construction $P_i = \pi_i \circ S$. Thus $S(K_\epsilon)$ is compact and $\mu_i(\pi_i(S(K_\epsilon))) = \mu_i(P_i(K_\epsilon)) > 1 - \epsilon$ for all $i \in I$. Using the theorem, we have a Radon measure μ on X such that $\mu_i(\pi_i^{-1}(A)) = \mu_i(A)$. Defining $\nu(S^{-1}(A)) = \mu(A)$, we prove the corollary.

We turn to our application: Let H be a real separable Hilbert-space, and let I be the set of all finite dimensional subspaces ordered by inclusion. If $E, F \in I$, $E \subset F$, let $P_E : H \rightarrow E$ and $P_{E,F} : F \rightarrow E$ be the projections. $\langle E, \tau_E, P_{E,F} \rangle$ is obviously a projective system of Hausdorff spaces, when τ_E is the norm topology.

Corollary 3 should now tell us when a cylindrical measure $\{\mu_E\}_{E \in I}$ on $\langle E, \tau_e, P_{E,F} \rangle$ can be extended to a measure on H. We introduce the following notation:

$$E_r = \{x \in H : \|x\| \le r\}$$

is the closed ball of radius r around the origin. It is easy to check that these balls are in the σ-algebra generated by the cylinder-sets. Define

$$\alpha_r^E = \mu_E(P_E(B_r)) \quad \text{for} \quad E \in I.$$

For fixed r, the set $\{\alpha_r^E\}_{E \in I}$ is decreasing, and we let

$$\alpha_r = \inf_E \alpha_r^E .$$

The function $r \to \alpha_r$ is increasing, and we may define

$$\alpha = \lim_{r \to \infty} \alpha_r.$$

Corollary 4: A cylindrical measure $\{\mu_E\}$ on H has a σ-additive extension ν if and only if $\alpha = 1$. The extension is a Radon measure.

Proof: Assume the extension ν exists. Then $B_r \subset P_E^{-1} P_E(B_r)$ for all $E \in I$, and hence $\nu(B_r) \le \alpha_r$. Since $\nu(H) = \nu(\bigcup_{n \in \mathbb{N}} B_n) = 1$, $\alpha_r \to 1$, and hence $\alpha = 1$.

Assume now that $\alpha = 1$. The weakest topology on H making all the finite dimensional projections continuous, is the weak topology. But on H the weak topology and the weak-* topology coincide, and hence by the Banach-Alaoglu Theorem (see e.g. Reed and Simon [15], page 115) all the balls B_r are compact in this topology. Given $\epsilon > 0$, choose r_ϵ such that $\alpha_{r_\epsilon} > 1-\epsilon$. Then $\mu_E(P_E(B_{r_\epsilon})) > 1-\epsilon$ for all $E \in I$. Applying Corollary 3 with $K_\epsilon = B_{r_\epsilon}$, we see that an extension ν exists, and is a Radon measure with respect to the weak topology. But since the balls

are ν-measurable, ν is also a Radon measure with respect to the norm topology.

From the nonstandard point of view, Corollary 4 has the following interpretation: Let $\{e_n\}$ be an orthonormal basis for H, and let $\{\tilde{e}_n\}_{n\in {}^*\mathbb{N}}$ be its $*$-version. Let E be the hyperfinite dimensional subspace of $*H$ generated by $\tilde{e}_1,\ldots,\tilde{e}_\eta$ for some $\eta \in {}^*\mathbb{N}\setminus\mathbb{N}$. It is then easy to see that $\alpha_s \leq {}^\circ\alpha_r^E \leq \alpha_r$ for $s \in \mathbb{R}$, $s < r$, and thus ${}^\circ\alpha_r^E \to \alpha$ as $r \to \infty$. Hence μ has a σ-additive extension if and only if $L(\tilde{\mu}_E)$ has support on the elements of finite norm in E; i.e. the near-standard elements in the weak topology. We may perhaps say that a limit measure $L(\tilde{\mu}_E)$ always exists; the problem is whether it has near-standard support so that we can push it down to H. In the following two sections we shall try to elucidate this theme.

III. Characteristic functionals on Hilbert spaces: Sazonov's theorem.

A famous theorem by S. Bochner characterizes the class of Fourier-transforms of probability measures on $\mathbb{R}^n$ as those functions $\varphi : \mathbb{R}^n \to C$ that take the value one in the origin, are positive definite and continuous. It is natural to guess that this theorem has a generalization to Hilbert spaces, and since most Hilbert space topologies coincide on finite dimensional spaces, one might conjecture that the task is "only" to find the right notion of continuity. That norm-continuity is not the right notion is easy to see; but in 1958 V.V. Sazonov [16] proved that the right topology is the one generated by the

Hilbert-Schmidt operators.

We shall now give a nonstandard proof of this theorem:

Let H be a real separable Hilbert space; a <u>Hilbert-Schmidt operator</u> $A : H \rightarrow H$ is a linear operator such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$ for any orthonormal basis $\{e_n\}$ in H. It is easy to see that the sum $\Sigma \|Ae_n\|^2$ is independent of which orthonormal basis we choose.

The sets of the form $\{x \in H : \|A_1 x\| < \delta_1, \ldots, \|A_m x\| < \delta_m\}$ where $A_1, \ldots, A_m$ are Hilbert-Schmidt operators, and $\delta_1, \ldots, \delta_m$ are positive real numbers, form a basis for the <u>Hilbert-Schmidt topology</u>. By polar decomposition, any Hilbert-Schmidt operator A can be written as the product UT of an partial isometry U, and a symmetric, positive Hilbert-Schmidt operator T. Hence we may assume that $A_1, \ldots, A_m$ above are symmetric. Moreover, if A_1, A_2 are Hilbert-Schmidt operators, then there exists a Hilbert-Schmidt operator $B = \sqrt{A_1^* A_1 + A_2^* A_2}$ such that $\|Bx\| < \delta$ implies $\|A_1 x\| < \delta$ and $\|A_2 x\| < \delta$. Thus we may replace the sequence $A_1, \ldots, A_m$ above with a single, symmetric H.-S. operator A and still have a basis for the topology.

Let $\varphi : H \rightarrow C$ with $\varphi(0) = 1$. Then φ is continuous in 0 if and only if for all $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists a symmetric Hilbert-Schmidt operator A and a $\delta \in \mathbb{R}$, $\delta > 0$, such that if $\|Ax\| < \delta$ then $|\varphi(x) - 1| < \varepsilon$. Taking the *-transfer of this, we get

$$\forall x \in {}^*H (\|{}^*Ax\| < \delta \rightarrow |{}^*\varphi(x) - 1| < \varepsilon).$$

By the internal definition principle this is equivalent to that for all $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there is a symmetric H.-S. operator A such that

$$\forall x \in {}^*H (\|{}^*Ax\| \approx 0 \rightarrow |{}^*\varphi(x) - 1| < \varepsilon).$$

To prove that φ is discontinuous in 0, it is thus enough to find a positive $\epsilon \in \mathbb{R}$, such that for all symmetric H.-S.-operators A there is an $x \in {}^*H$ with $\|{}^*Ax\| \approx 0$ and $|{}^*\varphi(x)-1| \geq \epsilon$.

To prove the theorem, we shall need the following notions: Let $\eta \in {}^*\mathbb{N} \setminus \mathbb{N}$, and define $\Omega = \{-1,1\}^\eta$, and let P be the uniform *-probability measure on Ω; $P\{\omega\} = (\frac{1}{2})^\eta$. By a symmetric binary martingale $\chi : \Omega \to {}^*\mathbb{R}$ we shall mean a *-stochastic variable χ defined by $\chi(\omega) = \sum_{i=1}^{\eta} \omega_i y_i$, where $\{y_i\}_{i \leq \eta}$ is an internal sequence of hyperreal numbers. The quadratic variation of χ is defined to be $[\chi] = \sum_{i=1}^{\eta} y_i^2$. (This should look familiar to readers acquainted with Anderson's Brownian motion [1]).

Lemma 5: Let χ be a symmetric binary martingale, and put $\gamma^2 = [\chi]$. If γ is finite, χ is finite a.e. in the Loeb-measure. If γ is infinite there is a set of measure $\geq \frac{1}{2}$ where $|\chi| \geq \sqrt{\gamma}$.

Proof: Assume that γ is finite. Using that the ω_i are independent with mean zero and variance one, we get

$$E(\chi^2) = E((\sum_{i=1}^{\eta} \omega_i y_i)^2) = \sum_{i,j=1}^{\eta} y_i y_j E(\omega_i \omega_j) = \sum_{i=1}^{\eta} y_i^2 = \gamma^2.$$

Consequently χ is finite $L(P)$-a.e.

Assume so that γ is infinite. We first consider the case where there is a $j \in \{1,2,\ldots,\eta\}$ with $|y_j| \geq \sqrt{\gamma}$, say $y_j \geq \sqrt{\gamma}$. Then either $\sum_{i \neq j} \omega_i y_i \geq 0$ or $\sum_{i \neq j} (-\omega_i) y_i \geq 0$, so for each ω such that $\sum_{i=1}^{\eta} \omega_i y_i < \sqrt{\gamma}$ there is a $\omega' = (-\omega_1, -\omega_2, \ldots, \omega_j, \ldots, -\omega_\eta)$ such that $\sum_{i=1}^{\eta} \omega_i' y_i \geq \sqrt{\gamma}$. Hence $\chi \geq \sqrt{\gamma}$ with probability $\geq \frac{1}{2}$. A similar argument applies for $y_j \leq -\sqrt{\gamma}$.

Assume so that γ is infinite, but $|y_j| < \sqrt{\gamma}$ for all j. Then $[\frac{\chi}{\gamma}] = 1$, and by the first part of the proof χ/γ is finite a.e.. Moreover $y_i/\gamma \approx 0$ for all i. We calculate the characteristic function of the random variable $^{\circ}(\chi/\gamma)$:

$$\int e^{i\,^{\circ}(\chi/\gamma)z}\, dL(P) = ^{\circ}\!\int e^{i(\chi/\gamma)z}\, dP = ^{\circ}\!\int e^{i\sum_{j=1}^{\eta} \frac{\omega_j y_j z}{\gamma}}\, dP$$

$$= ^{\circ}\!\int \prod_{j=1}^{\eta} e^{i\frac{\omega_j y_j z}{\gamma}}\, dP = ^{\circ}(\prod_{j=1}^{\eta} \int e^{i\frac{\omega_j y_j z}{\gamma}}\, dP) = ^{\circ}\prod_{j=1}^{\eta} \frac{e^{i\frac{y_j z}{\gamma}} + e^{-i\frac{y_j z}{\gamma}}}{2}$$

$$= ^{\circ}\prod_{j=1}^{\eta} \cos(\frac{y_j z}{\gamma}) = ^{\circ}e^{\sum_{j=1}^{\eta} \ln(\cos\frac{y_j z}{\gamma})} = ^{\circ}e^{(-\sum_{j=1}^{\eta} \frac{1}{2}(\frac{y_j z}{\gamma})^2 + o(\frac{y_j}{\gamma})^2)} = e^{-\frac{z^2}{2}}$$

where we have used the independence of ω_i and ω_j for $i \neq j$, and the Taylor-expansion of $\ln(\cos x)$.

This tells us that $^{\circ}(\chi/\gamma)$ is gaussian distributed with variance 1. Hence the probability that $|\frac{\chi}{\gamma}| \geq \frac{1}{\sqrt{\gamma}}$ is larger than $\frac{1}{2}$, and so is the probability that $|\chi| \geq \sqrt{\gamma}$. This proves the lemma.

The proof of part (b) of the following proposition contains the heart of the argument:

Proposition 6: Let $\{\mu_E\}$ be a cylindrical measure on a real, separable Hilbert space H. Then (recall the α of Corollary 4):

(a) If $\alpha = 1$, then for each positive $\epsilon \in \mathbb{R}$ there is a symmetric Hilbert–Schmidt operator T on H, and a $r_\epsilon \in \mathbb{R}$ such that $\nu(B_{r_\epsilon}) > 1-\epsilon$, and for $x \in {}^*H$: $\|{}^*Tx\| \approx 0 \Rightarrow \langle x,y \rangle \approx 0$ for $L({}^*\nu)$-almost all $y \in {}^*B_{r_\epsilon}$.

(b) If $\alpha < 1$, there is for each symmetric Hilbert-Schmidt operator T on H a set B with $^*\mu(B) > \frac{1-\alpha}{4}$, a $\gamma \in {}^*\mathbb{N} \setminus \mathbb{N}$, and an $x \in {}^*H$ such that

$$\|^*Tx\| \approx 0 \quad \text{and} \quad |\langle x,y \rangle| > \gamma^{\frac{1}{2}} \quad \text{for} \quad y \in B.$$

Proof: (a) This is the standard argument, adapted from Kuo [10]: Let r_ε be so large that $\nu(B_{r_\varepsilon}) > 1-\varepsilon$, and consider the bilinear form $\int_{B_{r_\varepsilon}} \langle x,z \rangle \langle y,z \rangle d\nu(z)$. It is continuous since

$$\left| \int_{B_{r_\varepsilon}} \langle x,z \rangle \langle y,z \rangle d\nu(z) \right| \leq \|x\| \|y\| \int_{B_{r_\varepsilon}} |z|^2 d\nu(z) \leq \|x\| \|y\| \|r_\varepsilon\|^2.$$

Consequently there is a symmetric, continuous, positive operator S such that

$$\langle Sx,y \rangle = \int_{B_{r_\varepsilon}} \langle x,z \rangle \langle y,z \rangle d\nu(z)$$

Now

$$\sum_{n=1}^{\infty} \langle Se_n, e_n \rangle = \sum_{n=1}^{\infty} \int_{B_{r_\varepsilon}} \langle x, e_n \rangle^2 d\nu(x) = \int_{B_{r_\varepsilon}} \sum_{n=1}^{\infty} \langle x, e_n \rangle^2 d\nu(x) = \int_{B_{r_\varepsilon}} \|x\|^2 d\nu(x) \leq r_\varepsilon^2 ,$$

showing that S is a trace-class operator. Hence $S = T \circ T$ for some symmetric Hilbert-Schmidt operator T, and we get

$$\|Tx\|^2 = \int_{B_{r_\varepsilon}} \langle x,z \rangle^2 d\nu(z).$$

(a) now follows from the *-version of this.

(b) Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of eigenvectors for T, and let $\{\tilde{e}_n\}_{n \leq \eta}$ be a hyperfinite initial segment of $^*(\{e_n\}_{n \in \mathbb{N}})$. Assume $^*T\tilde{e}_n = \beta_n \tilde{e}_n$, then $\sum_{n=1}^{\infty} \beta_n^2 < \infty$. Let E_T be

the hyperfinite dimensional subspace of $^*H$ generated by $\{\tilde{e}\}_{n \leq \eta}$.

Choose $\gamma \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that

$$\tilde{\mu}_{E_T} \{y \in E_T : \|y\| \geq \gamma^{5/4}\} \geq \frac{1-\alpha}{2};$$

such a γ exists by definition of α and the internal definition

principle.

Our element x shall be of the form $x = \sum\limits_{n=1}^{\eta} \frac{\omega_n}{\gamma^{1/4}} \tilde{e}_n$, where

$\omega_n = \pm 1$ for all n. Obviously $\|{}^*Tx\|^2 = \sum\limits_{n=1}^{\eta} \frac{\beta_n^2}{\gamma^{\frac{1}{2}}} \approx 0$. We now use

Lemma 5 to choose the sequence $\{\omega_n\}$ such that $\langle x, y \rangle$ is infi-

nite on a set of y's of large enough measure:

If y is such that $\|y\|_{E_T}^2 = \sum\limits_{n=1}^{\eta} y_n^2 \geq \gamma^{5/2}$, then $\sum\limits_{n=1}^{\eta} (\frac{\omega_n y_n}{\gamma^{1/4}})^2 \geq \gamma^2$.

By Lemma 5 is then $|\sum \frac{\omega_n y_n}{\gamma^{1/4}}| \geq \gamma^{\frac{1}{2}}$ for all ω in a set of measure

$\geq \frac{1}{2}$. Let now

$$A = \{(\omega, y) : \sum\limits_{n=1}^{\eta} \frac{\omega_n y_n}{\gamma^{1/4}} \geq \gamma^{\frac{1}{2}}\},$$

and apply Fubini's Theorem to the characteristic function K_A:

$$\int [\int K_A(\omega, y) d\tilde{\mu}_{E_T}(y)] dP(\omega) = \int [\int K_A(\omega, y) dP(\omega)] d\tilde{\mu}_{E_T}(y) \geq$$

$$\geq \frac{1}{2} \tilde{\mu}_{E_T} \{y \in E_T : \|y\| \geq \gamma^{5/4}\} \geq \frac{1-\alpha}{4}.$$

Hence there is a $\tilde{\omega} \in \Omega$ such that $\tilde{\mu}_{E_T} \{y : |\sum\limits_{n=1}^{\eta} \frac{\tilde{\omega}_n y_n}{\gamma^{1/4}} \geq \gamma^{\frac{1}{2}}\} \geq \frac{1-\alpha}{4}$.

Choose $x = \sum\limits_{n=1}^{\eta} \frac{\tilde{\omega}_n}{\gamma^{1/4}} \tilde{e}_n$. Then

$$^*\mu \{y \in {}^*H : |\langle x, y \rangle| \geq \gamma^{\frac{1}{2}}\} = \tilde{\mu}_{E_T} \{y \in E_T : |\sum\limits_{n=1}^{\eta} \frac{\tilde{\omega}_n}{\gamma^{1/4}} y_n| \geq \gamma^{\frac{1}{2}}\} \geq \frac{1-\alpha}{4},$$

which proves (b).

It is now easy to prove

Theorem 7 (Sazonov): Let $\varphi : H \to C$ be a complex-valued function on a real, separable Hilbert space. Then φ is the Fourier-transform of a Borel probability measure on H if and only if : $\varphi(0) = 1$, φ is positive definite and continuous in the Hilbert-Schmidt topology.

Proof: Assume $\varphi(x) = \int e^{i\langle x,y \rangle} d\mu(y)$, where μ is a probability measure on H; then $\varphi(0) = 1$ and φ is positive definite. Let $\epsilon > 0$ be given. Since $\alpha = 1$, we may apply Proposition 6(a) to get a symmetric Hilbert-Schmidt operator T and a set $B_{r_{\epsilon/3}}$ with $\mu(B_{r_{\epsilon/3}}) > 1 - \epsilon/3$ such that $\|{}^*Tx\| \approx 0 \Rightarrow \langle x,y \rangle \approx 0$ for almost all $y \in {}^*B_{r_{\epsilon/3}}$. Consequently $e^{i\langle x,y \rangle} \approx 1$ on a set of measure $\geq 1 - \epsilon/3$, and hence $|{}^*\varphi(x) - 1| < \epsilon$. By our nonstandard characterization of H.-S. continuity, it follows that φ is continuous in 0. Since φ is positive definite this implies that φ is H.-S.-continuous everywhere.

Assume now that φ is a positive definite, H.-S.-continuous function with $\varphi(0) = 1$. Using Bochner's Theorem on the finite dimensional subspaces of H, we see that φ is the Fourier-transform of a cylindrical measure $\{\mu_E\}$ on H. We shall show that $\alpha = 1$ by proving that if $\alpha < 1$, then φ can not be H.-S.-continuous.

So assume $\alpha < 1$: Given a symmetric H.-S.-operator T, there exist by Proposition 6(b) a set B with ${}^*\mu(B) > \frac{1-\alpha}{4}$, a $\gamma \in {}^*\mathbb{N} \smallsetminus \mathbb{N}$, and an $x \in {}^*H$ such that

$$\|{}^*Tx\| \approx 0 \quad \text{and} \quad |\langle x,y \rangle| \geq \gamma^{\frac{1}{2}} \quad \text{for } y \in B .$$

Let λ be the Lebesgue measure on $\mathbb{R}$. It is easy to see that for infinite z

$$^*\lambda\{a \in {}^*[1,2] : az \in \bigcup_{k \in {}^*Z} \{^*[2k\pi + \tfrac{\pi}{2}, 2k\pi + \tfrac{3\pi}{2}]\}\} \approx \tfrac{1}{2} > \tfrac{1}{3}.$$

Define

$$A = \{(a,y) : a\langle x,y \rangle \in \bigcup_{k \in {}^*Z} \{^*[2k\pi + \tfrac{\pi}{2}, 2k\pi + \tfrac{3\pi}{2}]\}\}.$$

By Fubini's Theorem

$$\int_1^2 [\int_B K_A(a,y)d^*\mu(y)]d^*\lambda(a) = \int_B^2[\int_1 K_A(a,y)d^*\lambda(a)]d^*\mu(y) \geq \tfrac{1}{3} \, {}^*\mu(B) \geq \tfrac{1-\alpha}{12}.$$

Hence there exist an a such that

$$^*\mu\{y : a\langle x,y \rangle \in \bigcup\{^*[2k\pi + \tfrac{\pi}{2}, 2k\pi + \tfrac{3\pi}{2}]\}\} \geq \tfrac{1-\alpha}{12}.$$

For such an a, $\mathrm{Re}(e^{i\langle ax,y \rangle}) = \cos(\langle ax,y \rangle) \leq 0$ for all y in a set of measure $> \tfrac{1-\alpha}{12}$, and consequently

$$\left| \int e^{i\langle ax,y \rangle} d^*\mu(y) - 1 \right| \geq \tfrac{1-\alpha}{12}.$$

Since $\|^*T(ax)\| \approx 0$, this implies that φ is not H.-S.-continuous in 0, and the theorem is proved.

What I would like to point out about the proof above, is the following: In this case the hard thing is to find out what happens when the limit measure does <u>not</u> exist. But in the nonstandard universe we do have a kind of limit measure $L(\tilde{\mu}_{E_T})$, which lives on a space E_T with all the algebraic and topological structure of a linear space. We can thus perform all the necessary calculations of Proposition 6 and Theorem 7 on this space. The argument could probably be carried through in a standard way working on the finite dimensional subspace of H, by picking suitable se-

quences of elements and the right notion of convergence of measures, but the clarity of the argument would then most likely be lost in a haze of convergence results.

IV. Cylindrical measures extended to Banach spaces: Gross' theorem.

In the first section we inquired when a cylindrical measure $\{\mu_E\}$ on a Hilbert space can be extended to a countably additive measure μ on H; and we saw - indirectly - how such a measure could be obtained from $L(\widetilde{\mu}_E)$ on a hyperfinite dimensional sub-space E of $^*H$ using a standard part map. We may ask our-selves in cases where such an extension is not possible, whether we may find some other reasonable limit measure, perhaps living on a larger space. One way of doing this in a nonstandard setting would be to use a standard part map connected with a weaker topo-logy; this would make more points near-standard. We shall now apply this strategy to prove a theorem of Gross [5] on the exten-sion of cylindrical measures to Banach spaces.

Let H be a real, separable Hilbert space with an inner product $\langle \circ, \cdot \rangle$ generating a norm $\| \circ \|$, and let $| \circ |$ be another norm on H. An element $y \in {}^*H$ is called $| \circ |$-near-standard if for all $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there is an $x \in H$ with $|x-y| < \epsilon$. If E is a hyperfinite dimensional subspace of $^*H$, define an equiva-lence relation $\sim_E$ on E by $x \sim_E y \iff |x-y| \approx 0$. Let $\text{Ns}_{|\,|}(E)$ denote the $| \circ |$-near-standard elements of E, and let $^\circ E = \text{Ns}_{|\,|}(E)/\sim_E$. Let $^\circ | \circ |$ be the norm defined on $^\circ E$ by $^\circ |x| = \text{st}|\widetilde{x}|$, when $\widetilde{x}$ is the equivalence class of x.

Lemma 8: Let $E \in {}^*I$ be such that $H \subset E$. Then $({}^{\circ}E, {}^{\circ}|\cdot|)$ is the completion of H in $|\cdot|$-norm.

Proof: If $x \in H$, identify it with its equivalence class $\tilde{x}$ under $\sim_E$. By definition of $|\cdot|$-near-standard, H is dense in ${}^{\circ}E$, and it only remains to prove that ${}^{\circ}|\cdot|$ is complete: Let $\{\tilde{x}_n\}_{n \in \mathbb{N}}$ be a Cauchy-sequence in ${}^{\circ}E$, and let for each $n \in \mathbb{N}$ $x_n \in \tilde{x}_n$. Then $\{x_n\}_{n \in \mathbb{N}}$ can be extended to an internal sequence $\{x_n\}_{n \in {}^*\mathbb{N}}$ of elements from E such that $|x_n - x_m| \approx 0$ for all $n, m \in {}^*\mathbb{N} \setminus \mathbb{N}$. Let $\gamma \in {}^*\mathbb{N} \setminus \mathbb{N}$, and let $\tilde{x}_\gamma$ be the equivalence class of x_γ. Since all x_n are $|\cdot|$-near-standard, so is x_γ, and the sequence $\{\tilde{x}_n\}$ converges to $\tilde{x}_\gamma$. The lemma is proved.

From now on we write B for ${}^{\circ}E$ and $|\cdot|$ for ${}^{\circ}|\ |$; and we have hence shown the existence of a standard part map
$$\text{st}_{|\,|} : \text{Ns}_{|\,|}(E) \to B.$$
Let $\mathcal{F}$ be the set of finite dimensional projections in H. If $\mu = \{\mu_F\}_{F \in I}$ is a cylindrical measure on H, the norm $|\ |$ is called μ-measurable if for all $\varepsilon \in \mathbb{R}_+$ there is a $P_0 \in \mathcal{F}$ such that
$$\mu\{|Px| > \varepsilon\} < \varepsilon \quad \text{for all } P \in \mathcal{F} \ , \ P \perp P_0 .$$

Gross' theorem says that if $|\ |$ is μ-measurable, then μ has an extension to Borel-measure on B. The key observation is the following:

Lemma 9: If $|\cdot|$ is μ-measurable, then $L(\tilde{\mu}_E)(\text{Ns}_{|\,|}(E)) = 1$ (E is as in Lemma 8.)

Proof: Let
$$A_m = \{x \in E : \exists v \in H(|v-x| < \tfrac{1}{m})\} .$$

If $P \in \mathcal{F}$, let $P_E^{\perp}$ be the orthogonal projection to $^*P$ in E.

For $n > m$, find a $P \in \mathcal{F}$ such that

$$\tilde{\mu}_E \{|P_E^{\perp}(x)| > \tfrac{1}{n}\} < \tfrac{1}{n},$$

by μ-measurability of $|\cdot|$. Then the set

$$\{x \in E : P(x) \in Ns_{||} (E)\} \cap \{x : |P_E^{\perp}(x)| \leq \tfrac{1}{n}\}$$

is contained in A_m, and has Loeb-measure $\geq 1 - \tfrac{1}{n}$. [*] Since this

holds for all $n > m$, $L(\tilde{\mu}_E)(A_m) = 1$. But $Ns_{||}(E) = \bigcap_m A_m$, and hence

$L(\tilde{\mu}_E)(Ns_{||}(E)) = 1$, which proves the lemma.

To prove the theorem, it only remains to push $L(\tilde{\mu}_E)$ down
to B using the standard part map. However, we must first agree
on what it means for a measure on B to extend a cylindrical
measure on H: Let B^* be the dual of B. If $y_1, \ldots, y_n \in B^*$
and A is a Borel-set in $\mathbb{R}^n$, then the set
$\{x \in B : (y_1(x), \ldots, y_n(x)) \in A\}$ is called a <u>cylinder set</u> in B. We
define a finitely additive measure $\hat{\mu}$ on the cylinder sets by

$$\hat{\mu}\{x \in B : (y_1(x), \ldots, y_n(x)) \in A\} = \mu\{x \in H : (\langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle) \in A\}$$

where we have identified H and H^* and embedded B^* in H^*.

Theorem 10 (Gross): Let $| \ |$ be μ-measurable. Then $\hat{\mu}$
has an extension to a σ-additive Borel-measure on B.

Proof: Let E be as in Lemmas 8 and 9. Define a measure ν
on B by $\nu = st_{||}(L(\tilde{\mu}_E))$; this is well-defined probability
measure since $L(\tilde{\mu}_E(Ns_{||}(E)) = 1$ by Lemma 9. To prove that ν
is a Borel-measure, it is enough to prove that all balls

$$B(a,r) = \{x \in B : |x-a| < r\}, \ a \in H,$$

_____

[*] $L(\tilde{\mu}_E)\{x \in E : P(x) \in Ns_{||}(E)\} = 0$ since $P \in \mathcal{F}$ (and not only $^*\mathcal{F}$)

are ν-measurable, since B is separable. But

$$st_{||}^{-1}(B(a,r)) = \bigcup_n \{x \in E : |x-a| < r - \frac{1}{n}\} \cap N_{||}(E)$$

which is $L(\tilde{\mu}_E)$-measurable, and hence ν is a Borel-measure.

It remains to prove that ν and $\hat{\mu}$ agree on the cylinder sets: Since

$$st_{||}^{-1}\{x \in B : (y_1(x), \dots, y_n(x)) \in A\} = \{x \in E : (y_1(x), \dots, y_n(x)) \in st^{-1}(A)\},$$

we have

$$\nu\{x \in B : (y_1(x), \dots, y_n(x)) \in A\} = L(\tilde{\mu}_E)\{x \in E : (y_1(x), \dots, y_n(x)) \in st^{-1}(A)\}.$$

But $E \subset {}^*H$, and hence $y_j(x) = \langle x, y_j \rangle$ and so

$$\nu\{x \in B : (y_1(x), \dots, y_n(x)) \in A\} = L(\tilde{\mu}_E)\{x \in E : (\langle x, y_1 \rangle, \dots, \langle x, y_n \rangle) \in st^{-1}(A)\}.$$

On the other hand

$$\hat{\mu}\{x \in B : (y_1(x), \dots, y_n(x)) \in A\} = L(\tilde{\mu}_E)\{x \in E : (y_1(x), \dots, y_n(x)) \in {}^*A\}$$

$$= L(\tilde{\mu}_E)\{x \in E : (\langle x, y_1 \rangle, \dots, \langle x, y_n \rangle) \in {}^*A\}.$$

Keeping $y_1, \dots, y_n$ fixed and letting A run through the Borel sets,

$$\tilde{\mu}_E\{x \in E : (\langle x, y_1 \rangle, \dots, \langle x, y_n \rangle) \in {}^*A\} = \mu_F\{x \in F : (\langle x, y_1 \rangle, \dots, \langle x, y_n \rangle) \in A\}$$

(where F is the subspace of H generated by $y_1, \dots, y_n$) defines a Radon measure on $\mathbb{R}^n$. Given an $\epsilon > 0$, there is thus an open set $G \supset A$, and a compact set $C \subset A$, such that

$$\mu_E\{x \in E : (\langle x, y_1 \rangle, \dots, \langle x, y_n \rangle) \in {}^*C\} + \epsilon \geq \mu_E\{x \in E : (\langle x, y_1 \rangle, \dots, \langle x, y_n \rangle) \in {}^*A\}$$

$$\geq \mu_E\{x \in E : (\langle x, y_1 \rangle, \dots, \langle x, y_n \rangle) \in {}^*G\} - \epsilon.$$

Now ${}^*C \subset st^{-1}(C)$ and ${}^*G \supset st^{-1}(G)$, and hence

$$\{x \in E : (\langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle) \in {}^*C\} \subseteq \{x \in E : \langle\langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle\rangle \in st^{-1}(A)\} \subseteq$$

$$\subseteq \{x \in E : (\langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle) \in {}^*G\}.$$

Since ϵ is arbitrary, this implies that ν and $\tilde{\mu}$ agree on the cylinder sets, and the theorem is proved.

The theorem above is more general than the versions in e.g. Gross [5], or Kuo [10] as they treat only the case where μ is normally distributed. I have found no references to the actual theorem in the literature, but it certainly ought to be wellknown. For applications the reader should consult Kuo [10].

The pattern of the proof is again the same; a limit measure exists naturally on an infinite element E, and using a suitable standard part map it is pushed down to a standard space. That we have been able to prove three central theorems in the theory of limit measures using this simple idea, should indicate that the Loeb-measure approach is both natural and promising; and I certainly hope that some original research may be carried on along these lines. Promising subjects should include stochastic integration in Hilbert spaces, and partial stochastic differential equations.

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