

FRACTIONAL BROWNIAN FIELDS AS INTEGRALS OF WHITE NOISE

by

Tom Lindstrøm*
 Department of Mathematics
 University of Oslo, Norway

I. Introduction

A *Fractional Brownian Field* (FBF) of index p is a random field $L_p : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

- (i) $L_p(0) = 0$ a.s.
- (ii) For all $x_1, x_2, \dots, x_n \in \mathbb{R}^d$, the random vector $(L_p(x_1), L_p(x_2), \dots, L_p(x_n))$ is gaussian with mean zero
- (iii) For all $x, y \in \mathbb{R}^d$, $E((L_p(x) - L_p(y))^2) = \|x - y\|^p$
- (iv) $x \rightarrow L_p(x, \omega)$ is continuous for almost all ω .

L_p exists for $0 < p < 2$ (and even for $p = 2$ when the dimension d is one).

Note that the covariance of L_p can be calculated from (i)-(iii):

$$E(L_p(x)L_p(y)) = \frac{1}{2}E(L_p^2(x) + L_p^2(y) - (L_p(x) - L_p(y))^2) = \frac{1}{2}(\|x\|^p + \|y\|^p - \|x - y\|^p)$$

Since L_p is gaussian, this tells us that its law is uniquely determined by (i)-(iv). When $p = 1$, L_p is Lévy-Brownian motion - one of several natural generalizations of Brownian motion to the multiparameter case (the Brownian sheet is another one).

FBFs are statistically self-similar in the following sense. If $x_0 \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}_+$, then the random field \hat{L}_p defined by

$$\hat{L}_p(x) = \alpha^{p/2} L_p\left(\frac{x - x_0}{\alpha}\right)$$

is also an FBF of index p . In the last twenty years, this fact has been used extensively to generate pictures of "fractal landscapes" and also to model fractal phenomena in natural and social sciences (see, e.g., [2], [5], [6], [7] and [10]). My own interest in FBFs grew out of a wish to understand some recent applications of these techniques to oil reservoir modeling [3], [4], [8].

The purpose of the present note is to point out that FBFs can be conveniently represented as integrals of white noise. As a matter of fact, all that is required is a slight extension of Andreas Stoll's representation theorem for Levy-Brownian motion (see [1] and [9]). One way of phrasing Stoll's result is to say that for $d > 1$,

* Acknowledgement: This research is supported in part by a grant from the VISTA program.

$$L_1(x) = c_d \int_{\mathbb{R}^d} \left(\frac{1}{\|x-y\|^{(d-1)/2}} - \frac{1}{\|y\|^{(d-1)/2}} \right) dW(y),$$

where W is white noise on \mathbb{R}^d and c_d is a constant depending only on the dimension d . In this paper I shall show that

$$L_p(x) = k_{p,d} \int_{\mathbb{R}^d} \left(\frac{1}{\|x-y\|^{(d-1)/2}} - \frac{1}{\|y\|^{(d-p)/2}} \right) dW(y),$$

where the constant $k_{p,d}$ depends only on the index p and the dimension d .

Before I begin, I should perhaps point out that in one respect my notation is a little unconventional - most other authors use $H=p/2$ to indicate the index of an FBF.

II. The main theorem.

For the convenience of the reader, I'll start by recalling briefly how one integrates with respect to white noise.

Let m be Lebesgue measure on \mathbb{R}^d , and let \mathcal{M}_F be the Lebesgue measurable sets with finite measure. A *white noise* on \mathbb{R}^d is a function $W : \Omega \times \mathcal{M}_F \rightarrow \mathbb{R}$ satisfying:

- (i) For each $A \in \mathcal{M}_F$ the random variable $W(\cdot, A)$ is gaussian with mean zero and variance $m(A)$.
- (ii) If $A, B \in \mathcal{M}_F$ are disjoint, then $W(\cdot, A)$ and $W(\cdot, B)$ are independent, and $W(\cdot, A \cup B) = W(\cdot, A) + W(\cdot, B)$ almost surely.

Assume that $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ is a simple function where A_1, A_2, \dots, A_n are disjoint elements of \mathcal{M}_F . Then the integral of f with respect to W is defined to be the random variable

$$\int f dW = \sum_{i=1}^n \alpha_i W(A_i).$$

A trivial calculation gives the Itô-isometry

$$E\left(\left(\int f dW\right)^2\right) = \int f^2 dm. \quad (1)$$

Using (1) and the fact that simple functions are dense in $L^2(m)$, we can define $\int f dW$ for an arbitrary $f \in L^2(m)$ by

$$\int f dW = \lim_{n \rightarrow \infty} \int f_n dW \quad (\text{in } L^2(\Omega)),$$

where $\{f_n\}$ is any sequence of simple functions converging to f in $L^2(m)$ -norm. Note that the Itô-isometry (1) extends to arbitrary $f \in L^2(m)$.

The following lemma shows that the integrands we are going to work with really belong to $L^2(m)$.

Lemma 1.

Assume that $0 < p < 2$. Then for each $x \in \mathbb{R}^d$, the function

$$y \rightarrow \frac{1}{\|x - y\|^{(d-p)/2}} - \frac{1}{\|y\|^{(d-p)/2}}$$

belongs to $L^2(m)$.

Proof:

Since the measurability is obvious, we only have to show that

$$\int \left(\frac{1}{\|x - y\|^{(d-p)/2}} - \frac{1}{\|y\|^{(d-p)/2}} \right)^2 dy < \infty \quad (2)$$

Hence we have to check the singularities at $y = x$ and $y = 0$, and the behavior at infinity.

Beginning with the singularity at $y = x$, we note that close to this point, the integrand is of order of magnitude

$$\frac{1}{\|x - y\|^{d-p}}.$$

Introducing polar coordinates centered at x , we see that our integral converges at $y = x$ if and only if the integral

$$\int_0^a \frac{1}{r^{1-p}} dr \quad (a > 0)$$

converges at the origin, i.e. if and only if $p > 0$. The singularity at $y = 0$ is of the same type, and again we get convergence for $p > 0$.

It remains to check the behaviour at infinity. Note that the integrand in (2) equals

$$\frac{(\|y\|^{(d-p)/2} - \|x - y\|^{(d-p)/2})^2}{\|x - y\|^{d-p} \|y\|^{d-p}} \leq \frac{(\|y\|^{(d-p)/2} - (\|x\| + \|y\|)^{(d-p)/2})^2}{\|x - y\|^{d-p} \|y\|^{d-p}}$$

By the Intermediate Value Theorem, the numerator in the last expression is of order of magnitude

$$(\|y\|^{(d-p)/2-1})^2 = \|y\|^{d-p-2},$$

and hence the expression itself is of order of magnitude

$$\frac{\|y\|^{d-p-2}}{\|x-y\|^{d-p}\|y\|^{d-p}} \sim \frac{1}{\|y\|^{d-p+2}}$$

when y is large. Changing to polar coordinates again, we see that our integral converges at infinity if the one dimensional integral

$$\int_a^\infty \frac{dr}{r^{3-p}} \quad (a > 0)$$

converges, i.e. if $p < 2$.

□

To prove that our processes are continuous we shall use the following fact (see, e.g., [1, theorem 4.8.4] or [9]).

Lemma 2.

Let $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function such that $y \rightarrow F(x, y)$ is in $L^2(m)$ for all $x \in \mathbb{R}^d$. Assume also that there are positive constants C and α such that

$$\int |F(x, y) - F(z, y)|^2 dy \leq C \|x - z\|^\alpha$$

for all $x, z \in \mathbb{R}^d$. Then the stochastic integral

$$x \rightarrow \int F(x, y) dW(y)$$

has a continuous version.

To exploit this lemma in our present setting, we need the following observation:

Lemma 3.

Assume that $0 < p < 2$. Then

$$\int_{\mathbb{R}^d} \left(\frac{1}{\|x-y\|^{(d-p)/2}} - \frac{1}{\|z-y\|^{(d-p)/2}} \right)^2 dy \leq C_{p,d} \|x-z\|^p, \quad (3)$$

where

$$C_{p,d} = \int_{\mathbb{R}^d} \left(\frac{1}{\|w-e\|^{(d-p)/2}} - \frac{1}{\|w\|^{(d-p)/2}} \right)^2 dw$$

and e is any unit vector in \mathbb{R}^d .

Proof:

The convergence of the integrals follows from Lemma 1. Let us introduce a new variable w by

$$w = \frac{y - z}{\|x - z\|}.$$

Note that

$$\|z - y\| = \|x - z\| \|w\|$$

and

$$\|x - y\| = \|x - z - \|x - z\|w\| = \|x - z\| \|w - e\|,$$

where e is the unit vector $\frac{x-z}{\|x-z\|}$. Substituting all this into the left hand side of (3), we get

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\frac{1}{\|x - y\|^{(d-p)/2}} - \frac{1}{\|z - y\|^{(d-p)/2}} \right)^2 dy \\ &= \int_{\mathbb{R}^d} \left(\frac{1}{(\|x - z\| \|w - e\|)^{(d-p)/2}} - \frac{1}{(\|x - z\| \|w\|)^{(d-p)/2}} \right)^2 \|x - z\|^d dw \\ &= \|x - z\|^p \int_{\mathbb{R}^d} \left(\frac{1}{\|w - e\|^{(d-p)/2}} - \frac{1}{\|w\|^{(d-p)/2}} \right)^2 dw, \end{aligned}$$

where the value of the last integral is clearly independent of which unit vector e we are using. □

If p and d are not both equal to one, $C_{p,d}$ is positive, and we can define

$$k_{p,d} = C_{p,d}^{-\frac{1}{2}} = \left[\int_{\mathbb{R}^d} \left(\frac{1}{\|w - e\|^{(p-d)/2}} - \frac{1}{\|w\|^{(p-d)/2}} \right)^2 dw \right]^{-\frac{1}{2}}$$

(if $p = d = 1$, $C_{p,d} = 0$ and the whole construction breaks down - see, however, the remark at the end of this section). Combining the three lemmas above, we can now prove the main result.

Theorem.

Assume that $0 < p < 2$ and that not both p and d are equal to one. Then the continuous version of the integral

$$L_p(x) = k_{p,d} \int \left(\frac{1}{\|x - y\|^{(d-p)/2}} - \frac{1}{\|y\|^{(d-p)/2}} \right) dW(y)$$

is an FBF of index p .

Proof:

By Lemma 1 the integral exists when $0 < p < 2$, and by Lemma 2 and Lemma 3 it has a continuous version. It remains to check the conditions (i)-(iv) in the definition of an FBF.

Condition (i) is obviously satisfied, and condition (ii) follows easily from the definition of a white noise integral. By the Itô-isometry (1)

$$\begin{aligned} & E((L_p(x) - L_p(z))^2) \\ &= k_{p,d}^2 \int_{\mathbb{R}^d} \left(\frac{1}{\|x-y\|^{(d-p)/2}} - \frac{1}{\|z-y\|^{(d-p)/2}} \right)^2 dy \\ &= k_{p,d}^2 C_{p,d} \|x-z\|^p = \|x-z\|^p, \end{aligned}$$

where we have also used Lemma 3 and the definition of $k_{p,d}$. This proves (iii), and since we have already checked (iv), the proof is complete. □

Remark:

In the excluded case $p = d = 1$, the FBF is (essentially) a one-dimensional Brownian motion. If we are willing to make our formulas a little more complicated, it is possible to find a representation which works even when $p = d = 1$, namely

$$L_p(x) = K_{p,d} \int \left(\frac{x-y}{\|x-y\|^{(d-p-2)/2}} - \frac{y}{\|y\|^{(d-p-2)/2}} \right) d\vec{W}(y),$$

where \vec{W} is an \mathbb{R}^d -valued white noise and $K_{p,d}$ is a normalizing constant. When $p = 1$, this representation was used by Stoll [9].

III. An invariance principle

Most applications of FBFs are of a numerical nature, and in practice one often finds oneself working not in all of \mathbb{R}^d but on an approximating lattice. It may therefore be of some interest to have an invariance principle for FBFs which tells us how they can be approximated by discrete objects. The invariance principle I shall describe in this section is very close to the one in Stoll's paper [9] (see also [1]), and for that reason I shall only give a brief sketch of the proof.

For each $n \in \mathbb{N}$, let

$$\Gamma_n = \left\{ \left(\frac{k_1}{2^n}, \frac{k_2}{2^n}, \dots, \frac{k_d}{2^n} \right) : k_1, k_2, \dots, k_d \in \mathbb{Z} \right\}$$

be a lattice in \mathbb{R}^d , and define

$$\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n.$$

To approximate white noise in \mathbb{R}^d , introduce a family $\{w_y\}_{y \in \Gamma}$ of i.i.d. random variables with mean zero and variance one. Note that when n is large,

$$W_n(A) = \sum_{y \in \Gamma_n \cap A} w_y \cdot 2^{-nd/2}$$

is a good approximation to white noise. Our goal is to show that the random field $l_p^{(n)} : \Omega \times \Gamma_n \rightarrow \mathbb{R}$ defined by

$$l_p^{(n)}(x) = k_{p,d} \sum_{y \in \Gamma_n} \left(\frac{1}{\|x-y\|^{(d-p)/2}} - \frac{1}{\|y\|^{(d-p)/2}} \right) 2^{-nd/2} w_y,$$

converges to L_p as n goes to infinity.

To make this precise, we first decide on an interpolation method which turns functions $f : \Gamma_n \rightarrow \mathbb{R}$ into continuous functions $\bar{f} : \mathbb{R}^d \rightarrow \mathbb{R}$. It doesn't matter much which one we choose, but let us agree that on each hypercube

$$\prod_{i=1}^d \left[\frac{k_i}{2^n}, \frac{k_i+1}{2^n} \right], \quad k_1, k_2, \dots, k_d \in \mathbb{Z}$$

\bar{f} should achieve its minimum and maximum at corners. Let $C(\mathbb{R}^d, \mathbb{R})$ be the space of all continuous functions from \mathbb{R}^d to \mathbb{R} with the topology of uniform convergence on compacts.

Invariance Principle.

For all continuous functions $\phi : C(\mathbb{R}^d, \mathbb{R}) \rightarrow \mathbb{R}$,

$$\int \phi(\bar{l}_p^{(n)}) dp \rightarrow \int \phi(L_p) dp \quad \text{as } n \rightarrow \infty,$$

i.e. the law of $\bar{l}_p^{(n)}$ converges weakly to the law of L_p .

Proof:

Since this argument is almost identical to the ones given in [1] and [9] for the case where $p = 1$, I'll only sketch the main idea and leave the details to the reader.

Let $\{w_y\}_{y \in \Gamma^*}$ be the nonstandard version of the family $\{w_y\}_{y \in \Gamma}$, and fix an infinitely large, nonstandard integer N . As in [1] and [9] one easily checks that the standard part of $\bar{l}_p^{(N)}$ is an FBF of index p , and hence

$$\int \phi(L_p) dP \approx \int^* \phi(\bar{l}_p^{(N)}) d^*P$$

for all $\phi \in C(\mathbb{R}^d, \mathbb{R})$. By the nonstandard characterization of convergence, this immediately implies the theorem. □

References

1. S. Albeverio, J.E. Fenstad, R. Høegh-Krohn & T. Lindstrøm: *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, 1986.
2. J. Feder: *Fractals*, Plenum Publishing Corporation, 1988.
3. T.A. Hewett: Fractal Distributions of Reservoir Heterogeneity and their Influence on Fluid Transport, SPE 15386, 1986 Annual SPE Meeting, New Orleans.
4. T.A. Hewett & R.A. Behrens: Conditional Simulation of Reservoir Heterogeneity with Fractals, SPE 18326, 1988 Annual SPE Meeting, Houston, Texas, 645-659.
5. B.B. Mandelbrot: *The Fractal Geometry of Nature*, W.H. Freeman and Co, 1982.
6. B.B. Mandelbrot & J.W. van Ness: Fractional Brownian Motion, Fractional Noises and Applications, SIAM Reviw 10 (1968), 422-437.
7. B.B. Mandelbrot & J.R. Wallis: Some Long-run Properties of Geophysical Records, Water Resources Research 5 (1969), 321-340.
8. J.L. Mathews, A.S. Emanuel & K.A. Edwards: A Modeling Study of the Mitsue Stage 1 Miscible Flood Using Fractal Geostatistics, SPE 18327, 1988 Annual SPE Meeting, Houston Texas, 661-674.
9. A. Stoll: A Nonstandard Construction of Lévy Brownian Motion, Probab. Th. Rel. Fields 71 (1986), 321-334.
10. R. Voss: Random Fractal Forgeries, in *Fundamental Algorithms for Computer Graphics* (R.A. Earnshaw, ed.), Springer-Verlag 1985, 805-835.