Discrete groups and simple C^*-algebras

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1 Introduction

Let G denote a discrete group and let us say that G is *C^*-simple* if the reduced group C^*-algebra associated with G is simple. We notice immediately that there is no interest in considering here the full group C^*-algebra associated with G, because it is simple if and only if G is trivial. Since Powers in 1975 ([26]) proved that all non-abelian free groups are C^*-simple, the class of C^*-simple groups has been considerably enlarged (see [1,2,6,7,12,13,14,16,24] as a sample!), and two important subclasses are the so-called weak Powers groups ([6,13]; see section 4 for definition and examples) and the groups of Akemann-Lee type ([1,2]), which are groups possessing a normal non-abelian free subgroup with trivial centralizer.

The problem of giving an intrisic characterization of C^*-simple groups is still open. It is known that a C^*-simple group has no normal amenable subgroup other than the trivial one ([24; proposition 1.6]) and is ICC (since the center of the associated reduced group C^*-algebra must be the scalars). One may of course wonder if the converse is true. On the other hand, most C^*-simple groups are known to have *a unique trace*, i.e. the canonical trace on the reduced group C^*-algebra is unique, which naturally raises the problem whether this is always true or not ([13; §2, question (2)]). These questions seem to be quite hard to answer, and more modestly, we will deal in this paper with the following three problems

(I) Let G denote a group possessing a normal C^*-simple subgroup with trivial centralizer. Is G C^*-simple? (cf. [13; §2, question 3], where normality is not assumed, but is necessary as remarked in [7; page 9]).

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(II) Let G denote a group obtained as the extension of a C^*-simple group by a C^*-simple group. Is G C^*-simple? (cf. [6; §2, question 2]).

(III) Let G denote a C^*-simple group and α an action of G on a simple C^*-algebra $\mathcal{A}$ with identity. Is the reduced crossed product $C^*_r(\mathcal{A}, G, \alpha)$ simple? (cf. [15; Problem in the introduction])

We will show that the answer to (I) is always positive. An amusing consequence is that the automorphism group and the holomorph of a C^*-simple group are C^*-simple. Very recently, Nitica and Todök have shown that the automorphism group of a non-amenable free product of groups is C^*-simple and has a unique trace (INCREST Preprint 1990). Now, let us say that a group is an *ultraweak Powers group* if it contains a normal weak Powers group with trivial centralizer. It is then true that an ultraweak Powers group is C^*-simple. On the other hand, all the known C^*-simple groups, with the possible exception of some of the matrix groups considered in [16], may be build up from ultraweak Powers groups.

In connection with this last remark and with (II), we will show that the extension of a C^*-simple group by an ultraweak Powers group is C^*-simple, thus generalizing a result of Boca and Nitica ([7]); note that they also consider unicity of the trace).

Extending some work of de la Harpe and Skandalis ([15]), the same Boca and Nitica have shown that the answer to (III) is positive whenever G is a weak Powers group ([6; Corollary 2.7]). We will show that this is also true for extensions of weak Powers groups by weak Powers groups, and moreover for extensions of ultraweak Powers groups by ultraweak Powers groups if we also assume the existence of a G-invariant faithful state on $\mathcal{A}$. A more complete answer to (II) and (III) feels to be out of reach with the tools used in this paper.

Our main idea to obtain all the above cited results is to consider reduced twisted crossed products of C^*-algebras by discrete groups ([22,27]). We will first repeat the necessary definitions and main properties in the next section. Then we extend the results of Kishimoto ([18; Theorem 3.1]) and of Boca and Nitica ([6; proposition 2.6]) to this setting and combine them with a decomposition theorem for such twisted products analogous to [22; Theorem 4.1] and [4; Theorem 1]. As a by-product of this approach, the simplicity of all reduced twisted group C^*-algebras associated to ultraweak

Powers groups (or extension of these) is determined. The same problem for nilpotent discrete groups has recently been studied in [21,23].

For general background information, we refer to [25,30]. A standing assumption throughout this paper will be that *all groups are considered as discrete, and all C^*-algebras are supposed to have an identity.* Or course. this does not yield for ideals in C^*-algebras, which, unless otherwise specified, are always two-sided closed ideals.

Our notation will be quite standard. For example, if $\mathcal{A}$ denotes a C^*-algebra, $\mathcal{H}$ a Hilbert space and G a group, then

$Aut(\mathcal{A})$	$=$	the group of $*$-automorphisms of $\mathcal{A}$,
$\mathcal{U}(\mathcal{A})$	$=$	the group of unitaries in $\mathcal{A}$,
$\mathcal{B}(\mathcal{H})$	$=$	the bounded linear operators acting on $\mathcal{H}$,
$l^2(G,\mathcal{H})$	$=$	the Hilbert space of $\mathcal{H}$-valued functions ξ on $\mathcal{H}$ such that $\sum_{g\in G}\|\xi(g)\|^2<+\infty$,
$Aut(G)$	$=$	the group of automorphisms of G,
I	$=$	the identity operator on $\mathcal{H}$,
e	$=$	the identity in G.

If $u\in\mathcal{U}(\mathcal{A})$, or if $\mathcal{A}$ acts on $\mathcal{H}$ and $u\in(\mathcal{B}(\mathcal{H}))$ is such that $u\mathcal{A}u^*=\mathcal{A}$, then $ad(u)$ denotes the $*$-automorphism of $\mathcal{A}$ implemented by u.

2 Reduced twisted crossed products

Let $\mathcal{A}$ be a C^*-algebra with identity I and G a discrete group with identity e, and suppose we are given a *cocycle crossed action* (α,u) of G on $\mathcal{A}$, by which we mean that (α,u) is a pair of maps $\alpha : G\to Aut(\mathcal{A})$ and $u : G\times G\to\mathcal{U}(\mathcal{A})$ satisfying

$$\alpha_r\alpha_s = ad(u(r,s))\alpha_{rs} \tag{1}$$

$$u(r,s)u(rs,t) = \alpha_r(u(s,t))u(r,st) \tag{2}$$

$$u(s,e) = u(e,t) = I \tag{3}$$

for all $r,s,t\in G$.

From an axiomatic point of view, condition (3) may be replaced in the above setting by: $u(e,e)=I$ and $\alpha_e=$ the identity automorphism.

We notice for later use that (2) may be equivalently formulated as

$$\alpha_r(u(s,t))^* u(r,s) = u(r,st)u(rs,t)^*, \tag{4}$$

or

$$u(r,s)^* \alpha_r(u(s,t)) = u(rs,t)u(r,st)^*. \tag{5}$$

Let now $\mathcal{A}$ be faithfully and non-degenerately represented as a C^*-algebra of operators on a Hilbert space $\mathcal{H}$. The *reduced twisted crossed product* ([22,27]) $C_r^*(\mathcal{A}, G, \alpha, u)$ may then be defined as the C^*-subalgebra of $\mathcal{B}(l^2(G,\mathcal{H}))$ generated by $\pi_\alpha(\mathcal{A})$ and $\lambda_u(G)$, where π_α is the faithful representation of $\mathcal{A}$ on $l^2(G,\mathcal{H})$ defined by

$$(\pi_\alpha(a)\xi)(h) = \alpha_{h^{-1}}(a)\xi(h),$$

while, for each $g \in G$, $\lambda_u(g)$ is the unitary operator on $l^2(G,\mathcal{H})$ defined by

$$(\lambda_u(g)\xi)(h) = u(h^{-1}, g)\xi(g^{-1}h)$$

$$(a \in \mathcal{A}, \xi \in l^2(G,\mathcal{H}), h \in G).$$

the pair (π_α, λ_u) is then a *covariant* representation of $(\mathcal{A}, G, \alpha, u)$, which means that

$$\pi_\alpha(\alpha_g(a)) = ad(\lambda_u(g))(\pi_\alpha(a)) \tag{6}$$

$$\lambda_u(g)\lambda_u(h) = \pi_\alpha(u(g,h))\lambda_u(gh) \tag{7}$$

$$(a \in \mathcal{A}, g, h \in G).$$

As remarked in [27; p. 552] (see also [22; remark 3.12]), $C_r^*(\mathcal{A}, G, \alpha, u)$ is independent, up to isomorphism, of the choice of $\mathcal{H}$. Hence, we may assume without loss of generality that each α_g is implemented by a unitary operator in $\mathcal{B}(\mathcal{H})$.

Remarks

1. Zeller-Meier considered in [32] reduced twisted crossed products under the extra assumption that the cocycle map u takes unitary values only in the center of $\mathcal{A}$.

2. If u is trivial, we obtain the ordinary reduced crossed product of $\mathcal{A}$ by α, denoted by $C_r^*(\mathcal{A}, G, \alpha)$.

3. If $\mathcal{A}$ reduces to the scalars, then α is the identity map and $\mathcal{U}(\mathcal{A})$ the circle group T, and we now obtain the C^*-algebra generated by the (left) u-regular (projective) representation of G on $l^2(G)$, denoted by $C_r^*(G,u)$. The cocycle map $u : G \times G \to T$ is sometimes called a multiplier in the literature. If it is trivial, we obtain $C_r^*(G)$, the reduced (left) group C^*-algebra of G.

For further use, we note first that it is easy to check that

$$\begin{aligned} \lambda_u(g)^* &= \lambda_u(g^{-1})\pi_\alpha(u(g,g^{-1}))^* \\ &= \pi_\alpha(u(g^{-1},g))^*\lambda_u(g^{-1}) \end{aligned} \tag{8}$$

for all $g \in G$.

Secondly, we define a map $\tilde{u} : G \times G \to \mathcal{U}(\mathcal{A})$ by

$$\tilde{u}(g,h) = u(g,h)u(ghg^{-1},g)^* \quad (g,h \in G).$$

Then we have

$$\lambda_u(g)\lambda_u(h)\lambda_u(g)^* = \pi_\alpha(\tilde{u}(g,h))\lambda_u(ghg^{-1}) \quad (g,h \in G). \tag{9}$$

Indeed, using (3), (4), (6), (7) and (8), we obtain:

$$\begin{aligned} \lambda_u(g)\lambda_u(h)\lambda_u(g)^* &= \pi_\alpha(u(g,h))\lambda_u(g,h)\lambda_u(g)^* \\ &= \pi_\alpha(u(g,h))\lambda_u(gh)\pi_\alpha(u(g^{-1},g))^*\lambda_u(g^{-1}) \\ &= \pi_\alpha(u(g,h))\pi_\alpha(\alpha_{gh}(u(g^{-1},g))^*)\lambda_u(gh)\lambda_u(g^{-1}) \\ &= \pi_\alpha(u(g,h))\pi_\alpha(\alpha_{gh}(u(g^{-1},g))^*)\pi_\alpha(u(gh,g^{-1}))\lambda_u(ghg^{-1}) \\ &= \pi_\alpha(u(g,h))\pi_\alpha(u(gh,g^{-1}g)u(ghg^{-1},g)^*)\lambda_u(ghg^{-1}) \\ &= \pi_\alpha(u(g,h)(u(ghg^{-1},g)^*)\lambda_u(ghg^{-1}) \\ &= \pi_\alpha(\tilde{u}(g,h))\lambda_u(ghg^{-1}). \end{aligned}$$

The following decomposition theorem is analogous to the one proved in [22; Theorem 4.1] for full twisted crossed products of C^*-algebras and the one proved in [4; Theorem 1] for regular extensions of von Neumann algebras.

Theorem 2.1: Suppose H is a normal subgroup of G and π denotes the canonical homomorphism from G onto the factor group $K = G/H$. Let (α', u') denote the restriction of (α, u) to H (we will just write (α, u) instead of (α', u') later). For each $g \in G$, there exists $\gamma_g \in Aut(C_r^*(\mathcal{A}, H, \alpha', u')$ such that

$$\gamma_g(\pi_{\alpha'}(a)) = \pi_{\alpha'}(\alpha_g(a)) \quad (a \in \mathcal{A}) \tag{10}$$

$$\gamma_g(\lambda_{u'}(h)) = \pi_{\alpha'}(\tilde{u}(g,h))\lambda_{u'}(ghg^{-1}) \quad (h \in H). \tag{11}$$

Further, let $n : K \to G$ be a section for π with $n(e) = e$ and define

$$\beta : K \to Aut(C_r^*(\mathcal{A}, H, \alpha', u') \quad \text{by} \quad \beta = \gamma \circ n,$$

$$m : K \times K \to H \qquad \text{by } m(k,l) = n(k)n(l)n(kl)^{-1},$$

and

$$v : K \times K \to \mathcal{U}((C_r^*(\mathcal{A}, H, \alpha', u')) \quad \text{by}$$

$$v(k,l) \quad = \pi_{\alpha'}(u(n(k), n(l))u(m(k,l), n(kl))^*)\lambda_{u'}(m(k,l))$$

Then $(\beta, v))$ is a cocycle crossed action of K on $C_r^*(\mathcal{A}, H, \alpha', u')$ such that

$$C_r^*(\mathcal{A}, G, \alpha, u) \simeq C_r^*(C_r^*(\mathcal{A}, H, \alpha', u'), K, \beta, v).$$

Proof: The proof given in [4: Theorem 1] adapts almost verbatim. For the convencience of the reader, we repeat here the main steps.

For each $g \in G$, let $a(g)$ denote the unitary operator which implements α_g on $\mathcal{H}$, and define $b(g) \in \mathcal{B}(l^2(H, \mathcal{H}))$ by

$$(b(g)\xi)(h) = u(h^{-1}, g)a(g^{-1})^* u(g^{-1}, h^{-1}g)\xi(g^{-1}hg)$$
$$(\xi \in l^2(H, \mathcal{H}), h \in H).$$

Then $b(g)$ is a unitary operator on $l^2(H, \mathcal{H})$, which is such that $ad(b(g))$ restricted to $C_r^*(\mathcal{A}, H, \alpha', u')$ has the required properties of γ_g.

Further, apart from some notational changes, the computations required to check that (β, v) is a cocycle crossed section on $C_r^*(\mathcal{A}, H, \alpha', u')$ are precisely those effectued in [22; p. 306–307].

At last, the unitary operator $\Lambda : l^2(K, l^2(H, \mathcal{H})) \to l^2(G, \mathcal{H})$ defined by

$$\Lambda\xi(g) = u(g^{-1}n(\pi(g^{-1}))^{-1}, n(\pi(g^{-1})))^*[\xi(\pi(g))(n(\pi(g^{-1}))g)]$$

satisfies

$$\begin{aligned}\Lambda\pi_\beta(\pi_{\alpha'}(a))\Lambda^* &= \pi_\alpha(a) \qquad (a\in\mathcal{A}),\\ \Lambda\pi_\beta(\lambda_{u'}(h))\Lambda^* &= \lambda_u(h) \qquad (h\in H),\\ \Lambda\lambda_v(k)\Lambda^* &= \lambda_u(n(k)) \qquad (k\in K),\end{aligned}$$

from which it follows that Λ implements a $*$-isomorphism from $C_r^*(C_r^*(\mathcal{A},H,\alpha',u'),K,\beta,v)$ onto $C_r^*(\mathcal{A},G,\alpha,u)$. □

When there is no danger of confusion, we will canonically identify $\mathcal{A}$ with $\pi_\alpha(\mathcal{A})$ via π_α, and write λ instead of λ_u. The next theorem is essentially well-known, at least in the Zeller-Meier case ([32; thèorème 4.12]).

Theorem 2.2: There exists a faithful conditional expectation ([28; §9]) E from $C_r^*(\mathcal{A},G,\alpha,u)$ onto $\mathcal{A}$ such that

$$E(\lambda(g)) = 0 \quad \text{for all } g\in G,\ g\neq e. \tag{12}$$

For each $x\in C_r^*(\mathcal{A},G,\alpha,u)$, define its Fourier-coefficient $x(g)\in\mathcal{A}$ at $g\in G$ by $x(g) = E(x\lambda(g)^*)$. Then the norm-bounded function $x(\cdot): G\to\mathcal{A}$ uniquely determines x as an element of $C_r^*(\mathcal{A},G,\alpha,u)$, and, for $x,y\in C_r^*(\mathcal{A},G,\alpha,u)$ we have

$$(xy)(g) = \sum_{h\in H} x(h)\alpha_h(y(h^{-1}g))u(h,h^{-1}g) \tag{13}$$

(the sum being taken in the strong operator topology) and

$$\begin{aligned} x^*(g) &= \alpha_g(x(g^{-1})u(g^{-1},g))^* \\ &= u(g,g^{-1})^*\alpha_g(x(g^{-1}))^* \end{aligned} \tag{14}$$

for all $g\in G$.

Proof: Instead of going through the machinery which leads to [32; thèorème 4.12], we will sketch the main lines of the proof following the arguments given in [3,20].

For each $g\in G$, let P_g denote the orthogonal projection from $l^2(G,\mathcal{H})$ onto $l^2(\{g\},\mathcal{H})$ (identified as a subspace of $l^2(G,\mathcal{H})$).

For each $y\in\mathcal{B}(l^2(G,\mathcal{H}))$, one then defines $Q(y) = \sum_{g\in G} P_g y P_g$, the sum taken in the strong-operator topology, and, as in [3; 6.1.3 (2)], one verifies that Q is a faithful normal conditional expectation from $\mathcal{B}(l^2(G,\mathcal{H}))$ onto $\{P_g; g \in G\}'$. Now, since $\mathcal{A} = \pi_\alpha(\mathcal{A}) \subset \{P_g; g \in G\}'$, $P_g\lambda(h)P_g = 0$ for all $g \in G$, all $h \in G$, $h \neq e$, and Q has norm one, one obtains easily that the restriction of Q to $C_r^*(\mathcal{A}, G, \alpha, u)$ has the desired properties of E.

Further, each α_g being unitarily implemented, the map α extends to a map $\tilde{\alpha} : G \to Aut(\mathcal{A}'')$ such that $(\tilde{\alpha}, u)$ becomes a cocycle crossed action of G on $\mathcal{A}''$. We may then form the regular extension $\mathcal{A}'' \times_{(\tilde{\alpha},u)} G$ ([4]), which is defined as $\{\pi_{\tilde{\alpha}}(\mathcal{A}), \lambda(G)\}''$, where $\pi_{\tilde{\alpha}}$ is defined analogously to π_α on $l^2(G,\mathcal{H})$. As above, one obtains that the restriction of Q to $\mathcal{A}''\times_{(\tilde{\alpha},u)} G$, say $\tilde{E}$, is a faithful normal conditional expectation from $\mathcal{A}'' \times_{(\tilde{\alpha},u)} G$ onto $\mathcal{A}''$ (identified with $\pi_{\tilde{\alpha}}(\mathcal{A}'')$), and that $\tilde{E} = E$ on $C_r^*(\mathcal{A}, G, \alpha, u)$.

If, for each $y \in \mathcal{A}\times_{(\tilde{\alpha},u)} G$, we define $y(g) = \tilde{E}(y\lambda(g)^*)$, $g \in G$, then we have that $\sum_{g\in G} y(g)\lambda(g)$ converges to y in the $\mathcal{A}''$-topology on $\mathcal{A}\times_{(\tilde{\alpha},u)} G$, by proceeding as in [20] (see also [8; lemma 1]). The $\mathcal{A}''$-topology on $\mathcal{A}\times_{(\tilde{\alpha},u)} G$ is the one defined by the pseudonorms $y \to \omega(\tilde{E}(y^*y)^{1/2})$, $\omega \in (\mathcal{A}'')_*$. The second part of the theorem folows now easily for all $x, y \in \mathcal{A}''\times_{(\tilde{\alpha},u)} G$, and therefore especially for all $x, y \in C_r^*(\mathcal{A}, G, \alpha, u)$. □

It should be noted that for $x \in C_r^*(\mathcal{A}, G, \alpha, u)$, the sum $\sum_{g\in G} x(g)\lambda(g)$ does not necessarily converge to x in norm (consider $C_r^*(\mathbf{Z}) \simeq C(\mathbf{T})$).

Notation: For $x \in C_r^*(\mathcal{A}, G, \alpha, u)$, we set

$$\text{supp}(x) = \{g \in G | x(g) \neq 0\}.$$

The next corollary is also nearly a classic.

Corollary 2.3: Let $x \in C_r^*(\mathcal{A}, G, \alpha, u)$. With E defined in theorem 2.2, we have

(a) $E(\lambda(g)x\lambda(g)^*) = \alpha_g(E(x))$, $g \in G$.

(b) $E(xx^*) = \sum_{g\in G} x(g)x(g)^*$ (in the strong operator topology)

(c) $E(x^*x) = \sum_{g\in G} \alpha_g^{-1}(x(g)^*x(g))$ (in the strong operator topology)

(d) if σ is a G-invariant trace σ on $\mathcal{A}$, then $\tau = \sigma \circ E$ is a trace on $C_r^*(\mathcal{A}, G, \alpha, u)$.

Proof: We show (a) and (b) and leave the proof of (c) and (d) to the reader.

(a)

$$\begin{aligned} E(\lambda(g)x\lambda(g)^*) &= (\lambda(g)x)(g) \\ &= \alpha_g(x(g^{-1}g))u(g,g^{-1}g) \\ &= \alpha_g(x(e)) = \alpha_g(E(x)) \end{aligned}$$

(by (3) and (13)).

(b)

$$\begin{aligned} \alpha_g(x^*(g^{-1})) &= \alpha_g(\alpha_{g^{-1}}(x(g)u(g,g^{-1}))^*) \\ &= u(g,g^{-1})(x(g),u(g,g^{-1}))^*u(g,g^{-1})^* \\ &= x(g)^*u(g,g^{-1})^* \end{aligned}$$

(by (1) and (14)). Hence,

$$\begin{aligned} E(xx^*) &= xx^*(e) \\ &= \sum_{g\in G} x(g)\alpha_g(x^*(g^{-1}))u(g,g^{-1}) \\ &= \sum_{g\in G} x(g)x(g)^* \end{aligned}$$

(by (13)). □

Regarding (d), we remind that a linear functional φ on $\mathcal{A}$ is called *G-invariant* if $\varphi(\alpha_g(a)) = \varphi(a)$ for all $g \in G$, $a \in \mathcal{A}$.

3 The trivial centralizer condition and Kishimoto's result

Let G denote a group possessing a normal subgroup H with trivial centralizer, which means that

$$\{g \in G | gh = hg \text{ for all } h \in H\} = \{e\}.$$

Problem (I), which is due to de la Harpe ([13]), asks whether G is C^*-simple whenever H is C^*-simple. Our approach to answer it positively is based on the following easy observation:

Let $g \in G$, $g \notin H$, and denote by σ_g the automorphism of H defined by $\sigma_g(h) = ghg^{-1}$, $h \in H$. Then σ_g is outer.

Indeed, if there exists a $p \in H$ such that $\sigma_g(h) = php^{-1}$ for all $h \in H$, then $p^{-1}g$ belongs to the centralizer of H. Hence, $p^{-1}g = e$, which is impossible since $g \notin H$.

Quite naturally, our attention is then drawn to a result of Kishimoto ([18; Theorem 3.1]), which says that the reduced crossed product of a simple C^*-algebra by a discrete group of outer automorphisms is simple. An inspection of his proof, which relies on arguments given by Elliott in [11], makes it clear that his result is also true in the twisted setting. For the reader's convenience, we present here a slightly modified proof, recalling first Kishimoto's key lemma.

Lemma 3.1 (cf. [18; lemma 3.2]): Let a be a positive element of simple C^*-algebra $\mathcal{A}$, $\{a_i; i = 1, \ldots, n\}$ elements of $\mathcal{A}$, $\{\alpha_i; i = 1, \ldots, n\}$ outer automorphisms of $\mathcal{A}$ and $\epsilon > 0$. Then there exists a positive $x \in \mathcal{A}$ with $\|x\| = 1$ such that $\|xax\| \geq \|a\| - \epsilon$, $\|xa_i\alpha_i(x)\| \leq \epsilon$, $i = 1, \ldots, n$.

Theorem 3.2: Let (α, u) be a cocycle crossed action of a group G on a simple C^*-algebra $\mathcal{A}$ such that each α_g is outer, $g \neq e$. Then $\mathcal{B} = C_r^*(\mathcal{A}, G, \alpha, u)$ is simple.

Proof: Let E denote the canonical conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$ obtained from Theorem 2.2, and let $J \neq \mathcal{B}$ be an ideal in $\mathcal{B}$. We must show that $J = \{0\}$. Now, since E is faithful, it is enough to show that $E(J) = \{0\}$.

For $x \in \mathcal{B}$, define $|||x||| = \inf\{\|x + j\|; j \in J\}$, (the C^*-seminorm on $\mathcal{B}$ induced by the norm on $\mathcal{B}/J$). Since $\mathcal{A}$ is simple, $\mathcal{A} \cap J = \{0\}$, and we see that the restriction to $\mathcal{A}$ of the canonical homomorphism from $\mathcal{B}$ onto $\mathcal{B}/J$ is injective. Hence, $|||x||| = \|x\|$ for all $x \in \mathcal{A}$.

We are going to show that $\|E(b)\| \leq |||b|||$ for all $b \in \mathcal{B}$. Since $|||j||| = 0$ for all $j \in J$, this will imply that $E(J) = \{0\}$ as desired.

By a density argument, we may suppose that $b \in \mathcal{B}$ is of the form $b = a + \sum_{g \in F} a_g \lambda(g)$, where $a \in \mathcal{A}$, F is a finite subset og $G \backslash \{e\}$ and $a_g \in \mathcal{A}$ $(g \in F)$. Suppose first that a is positive and let $\epsilon > 0$. By lemma 3.1, there exists a positive $x \in \mathcal{A}$ with $\|x\| = 1$ such that

$$\|xax\| \geq \|a\| - \epsilon, \quad \|xa_g\alpha_g(x)\| \leq \epsilon \ (g \in F).$$

Therefore, we have

$$\begin{aligned} |||x\left(\sum_{g\in F} a_g\lambda(g)\right)x||| &\leq \sum_{g\in F} |||xa_g\lambda(g)x||| \\ &= \sum_{g\in F} \|xa_g\alpha_g(x)\| \\ &\leq \operatorname{card}(F)\cdot\epsilon, \end{aligned}$$

which implies that

$$\begin{aligned} \|E(b)\| = \|a\| &\leq \|xax\| + \epsilon \\ &\leq |||xbx||| + |||x\left(\sum_{g\in F} a_g\lambda(g)\right)x||| + \epsilon \\ &\leq |||b||| + (\operatorname{card}(F)+1)\cdot\epsilon. \end{aligned}$$

This is true for all $\epsilon > 0$, so we have shown that $\|E(b)\| \leq |||b|||$. At last, if a is not positive, then $E(a^*b) = a^*E(b) = a^*a$ is positive, and

$$\|E(b)\|^2 = \|a^*a\| = \|E(a^*b)\| \leq |||a^*b||| \leq \|E(b)\| \, |||b|||,$$

which proves the desired inequality in this case too. □

We notice that theorem 3.2 is in fact true under some weaker hypothesis: it is enough to suppose that $\mathcal{A}$ is G-simple and that $\tilde{\Gamma}(\alpha_g) \neq \{1\}$ for all $g \in G$, $g \neq e$, where $\tilde{\Gamma}(\alpha_g)$ denotes the strong Connes spectrum of α_g (cf. [18]).

We next adapt some arguments of Behncke ([5]) and prove the following lemma:

Lemma 3.3: Let H denote a normal subgroup of a group G and (α, u) a cocycle crossed action of G on a C^*-algebra $\mathcal{A}$ which possess a faithful G-invariant state φ. Let $g \in G$ and denote by σ the automorphism of H defined by $\sigma(h) = ghg^{-1}$ $(h \in G)$. Suppose that $\{\sigma(h)ph^{-1}; h \in G\}$ is infinite for all $p \in H$. Then γ_g, as obtained from theorem 2.1, is an outer automorphism of $\mathcal{B} = C_r^*(\mathcal{A}, H, \alpha, u)$. In fact, γ_g is freely acting, which means that 0 is the only element b of $\mathcal{B}$ satisfying

$$\gamma_g(x)b = bx \quad \text{for all } x \in \mathcal{B}.$$

Proof: Let us first point out that it follows easily from the G-invariance of φ and the cocycle equation (1) that $\varphi(u(g_1,g_2)xu(g_1,g_2)^*)=\varphi(x)$ for all $g_1,g_2\in G$, $x\in\mathcal{A}$, which implies that $\varphi(\tilde{u}(g_1,g_2)x\tilde{u}(g_1,g_2)^*)=\varphi(x)$ for all $g_1,g_2\in G$, $x\in\mathcal{A}$. Further, by representing $\mathcal{A}$ via the GNS-construction for φ, we may assume that φ is a vector-state of $\mathcal{A}$.

Now, let $b\in\mathcal{B}$ and suppose

$$\gamma_g(x)b=bx \quad \text{for all } x\in\mathcal{B}.$$

Especially, we have

$$\gamma_g(\lambda(h))b=b\lambda(h) \text{ for all } h\in H,$$

i.e.

$$\tilde{u}(g,h)\lambda(\sigma(h))b=b\lambda(h) \text{ for all } h\in H,$$

so

$$b=\tilde{u}(g,h)\lambda(\sigma(h))b\lambda(h)^* \text{ for all } h\in H.$$

If E denotes the canonical conditional expectation from $\mathcal{B}$ onto $\mathcal{A}$ obtained from theorem 2.2, then we have for all $h,p\in H$

$$\begin{aligned}
&\tilde{u}(g,h)\alpha_{\sigma(h)}(b(p))\\
&\quad=\tilde{u}(g,h)\alpha_{\sigma(h)}(E(b\lambda(p)^*))\\
&\quad=\tilde{u}(g,h)E(\lambda(\sigma(h))b\lambda(p)^*\lambda(\sigma(h))^*)\\
&\quad=E(\tilde{u}(g,h)\lambda(\sigma(h))b\lambda(h)^*\lambda(h)\lambda(p)^*\lambda(\sigma(h))^*)\\
&\quad=E(b\lambda(h)\lambda(p)^*\lambda(\sigma(h))^*)\\
&\quad=E(b\lambda(\sigma(h)ph^{-1})^*\nu(p,h)^*)\\
&\quad=E(b\lambda(\sigma(h)ph^{-1})^*)\nu(p,h)^*\\
&\quad=b(\sigma(h)ph^{-1})\nu(p,h)^*
\end{aligned}$$

since $\nu(p,h)=u(\sigma(h),p)u(\sigma(h)ph^{-1},h)^*\in\mathcal{U}(\mathcal{A})$ satisfies

$$\begin{aligned}
&\lambda(\sigma(h))\lambda(p)\lambda(h)^*=u(\sigma(h),p)\lambda(\sigma(h)p)\lambda(h)^*\\
&\quad=u(\sigma(h),p)\lambda(\sigma(h)p)u(h^{-1},h)^*\lambda(h^{-1})\\
&\quad=u(\sigma(h),p)\alpha_{\sigma(h)p}(u(h^{-1},h))^*\lambda(\sigma(h)p)\lambda(h^{-1})\\
&\quad=u(\sigma(h),p)\alpha_{\sigma(h)p}(u(h^{-1},h))^*u(\sigma(h)p,h^{-1})\lambda(\sigma(h)ph^{-1})\\
&\quad=u(\sigma(h),p)u(\sigma(h)ph^{-1},h)^*\lambda(\sigma(h)ph^{-1}) \qquad \text{(by (5))}\\
&\quad=\nu(p,h)\lambda(\sigma(h)ph^{-1}).
\end{aligned}$$

Therefore, we have

$$b(\sigma(h)ph^{-1}) = \tilde{u}(g,h)\alpha_{\sigma(h)}(b(p))\nu(p,h) \quad \text{for all } h,p\in H.$$

Now, fix $p \in H$ and choose an infinite sequence $\{h_n\}$ in H such that $\{p_n\} = \{\sigma(h_n)ph_n^{-1}\}$ is an infinite sequence in H. By the above computation, we get

$$\begin{aligned}
&\varphi(b(p_n)b(p_n)^*) \\
&\quad = \varphi(\tilde{u}(g,h_n)\alpha_{\sigma(h_n)}(b(p)b(p)^*)\tilde{u}(g,h_n)) \\
&\quad = \varphi(\alpha_{\sigma(h_n)}(b(p)b(p)^*)) \\
&\quad = \varphi(b(p)b(p)^*)
\end{aligned}$$

for all $n \in \mathbf{N}$.

Since

$$\begin{aligned}
\sum_{n\in\mathbf{N}} \varphi(b(p_n)b(p_n)^*) &\leq \sum_{h\in H} \varphi(b(h)b(h)^*) \\
&= \varphi\left(\sum_{h\in H} b(h)b(h)^*\right) \\
&= \varphi(bb^*) < \infty\,,
\end{aligned}$$

(by strong continuity of φ and corollary 2.3.b)), this implies that $\varphi(b(p)b(p)^*) = 0$. Thus, $b(p) = 0$ by faithfulness of φ. Our choice of $p \in H$ being arbitrary, this implies that $b = 0$ as required. □

It is clear that the analog of lemma 3.3 for regular extensions of von Neumann algebras is also true. We next state a well-known result (cf. [5,17]) in a form suitable for our purpose:

Lemma 3.4: Let H be an ICC-group. Then $\sigma \in Aut(H)$ is outer if and only if $\{\sigma(h)ph^{-1} | h\in H\}$ is infinite for all $p \in H$.

Proof: Suppose there exists $p \in H$ such that $L = \{\sigma(h)ph^{-1} | h \in H\}$ is finite. Clearly, $p \in L$, and $L = \{p\}$ implies that σ is inner. Suppose that $L \neq \{p\}$. Then there exists $q \in L$ with $q \neq p$, and one verifies easily that $L = \{\sigma(h)qh^{-1} | h\in H\}$. This implies that

$$\begin{aligned}
C = \{hpq^{-1}h^{-1} | h\in H\} &= \{hph^{-1}(hqh^{-1})^{-1} | h\in H\} \\
&\subseteq L\cdot L^{-1},
\end{aligned}$$

so C is finite. Since H is ICC, we must have that $pq^{-1} = e$, which contradicts that $p \neq q$. Hence, $L = \{p\}$ and σ is inner. The converse part of the statement is trivial. □

Lemma 3.4 is also true for H being an infinite R-group ($h^n = p^n$ $(h, p \in H,\ n \in \mathbf{N})$ implies $h = p$) or a group with no normal subgroup of finite index (cf. [5; page 589]). The same remark concerning the following theorem is therefore valid, but will be of no use in this paper.

Theorem 3.5: Let H be a normal ICC subgroup of a group G with trivial centralizer, and (α, u) a cocycle crossed action of G on a C^*-algebra $\mathcal{A}$ which possess a faithful G-invariant state. Then $(C^*_r(\mathcal{A}, G, \alpha, u)$ is simple whenever $C^*_r(\mathcal{A}, H, \alpha, u)$ is simple.

Proof: Let $\pi : G \to K$ denote the canonical homomorphism from G onto the factor group $K = G/H$, and choose a section $n : K \to G$ for π with $n(e) = e$. By theorem 2.1, there exists a cocycle crossed action (β, ν) of K on $C^*_r(\mathcal{A}; H, \alpha, u)$ such that $C^*_r(\mathcal{A}, G, \alpha, u) \simeq C^*_r(C^*_r(\mathcal{A}, H, \alpha, u), K, \beta, \nu)$, where $\beta_k = \gamma_{n(k)}$ for each $k \in K$ as defined in theorem 2.1. Now, let σ_k denote the automorphism of H defined by $\sigma_k(h) = n(k)hn(k)^{-1}$ $(h \in H)$, for each $k \in K$. As observed at the beginning of this section, σ_k is outer for each $k \in K$, $k \neq e$. Lemmas 3.3 and 3.4 imply then that β_k is an outer automorphism of $C^*_r(\mathcal{A}, H, \alpha, u)$ for each $k \in K$, $k \neq e$. Hence, if $C^*_r(\mathcal{A}, H, \alpha, u)$ is simple, then $C^*_r(\mathcal{A}, G, \alpha, u)$ is simple too, as a consequence of theorem 3.2. □

We may now answer problem (I) positively:

Corollary 3.6: Let H denote a normal subgroup of a group G with trivial centralizer. Then G is C^*-simple whenever H is C^*-simple.

Proof: Recall that a C^*-simple group is ICC and apply theorem 3.5 with $\mathcal{A} = \mathbf{C}$ and (α, u) trivial. □

Corollary 3.7. Let H be a C^*-simple group and G denote its automorphism group. Then G is C^*-simple.

Proof: The center of H being trivial, we may identify H canonically as a normal subgroup of G with trivial centralizer, and the result follows from Corollary 3.6. □

For a group H, define $A_n(H)$ for $n \in \mathbf{N}$ recursively by

$$\begin{aligned} A_1(H) &= Aut(H), \\ A_n(H) &= Aut(A_{n-1}(H)) \quad (n \geq 2). \end{aligned}$$

If H has trivial center, one gets a normal tower

$$H \trianglelefteq A_1(H) \trianglelefteq A_2(H) \trianglelefteq \cdots$$

which satisfies the trivial centralizer condition at each step. By corollary 3.7, $A_n(H)$ is C^*-simple for all $n \in \mathbf{N}$ whenever H is C^*-simple. On the other hand, it is an open question in group-theory whether this series necessarily terminates (it does for H finite by a theorem of Wielandt ([31])).

Before we state our next corollary, we remind the well-known fact that the direct product of two C^*-simple groups is C^*-simple, as remarked in [6; page 192] and proved in [29; page 117].

Corollary 3.8. Let H denote a C^*-simple group and G its holomorph, i.e. G is the semi-direct product of H by its automorphism group $K = Aut(H)$ under the natural action. Then G is C^*-simple.

Proof: Let us write H' for the canonical copy of H in K. Then it is easy to check that the semi-direct product of H by H' is a normal subgroup of K with trivial centralizer, which is isomorphic to the direct product of H by H. Since $H \times H$ is C^*-simple from the above remark, G is then C^*-simple too by Corollary 3.6. □

This corollary makes it clearly possible to define another normal tower of C^*-simple groups starting from a C^*-simple group.

We will obtain some other corollaries to Theorem 3.5 in the next section. We conclude this section with the following

Problem: Suppose G contains a normal subgroup H wioth trivial centralizer. Then does G have a unique trace whenever H is C^*-simple and has a unique trace? (We know that G is C^*-simple.)

It should be noticed that Longo has constructed examples of simple C^*-algebras with several traces, obtained as crossed products of simple C^*-algebras admitting a unique trace ([19]). However, his construction cannot be used, at least directly, to produce an example answering the above question negatively.

4 Weak Powers groups and ultraweak Powers groups

We first recall that a *weak Powers group* ([6]) is a group G satisfying the following property:

Given any non-empty finite subset $F \subseteq G \setminus \{e\}$, *which is included into a conjugacy class*, and any integer $n \geq 1$, there exists a partition $G = D \coprod E$ and elements $g_1, \ldots, g_n \in G$ such that

(i) $fD \cap D = \emptyset$ for all $f \in F$,

(ii) $g_i E \cap g_j E = \emptyset$ for all $i, j, = 1, \ldots, n,\ i \neq j$.

Of course, (i) is then true for all $f \in F \cup F^{-1}$. In the original definition of a Powers group ([13]), F can be any non-empty finite subset og $G \setminus \{e\}$. The class of weak Powers groups includes a wide variety of groups within the categories of matrix groups ([13]), of free products with amalgamation and HNN-extensions ([13]), of fundamental groups of graphs of groups ([6]) and of hyperbolic groups ([14]). As a last example, let us mention the quotient of the pure braid group on k generators ($k \geq 3$) by its center ([12]).

In all this section (α, u) *will denote a cocycle crossed action of a group* G *on a* C^**-algebra* $\mathcal{A}$. We will say that $\mathcal{A}$ is *G-simple* if $\{0\}$ and $\mathcal{A}$ are the only ideals in $\mathcal{A}$ which are invariant under all $\alpha_g, g \in G$. A careful reading of [6] ensures one that the proofs of [6; propositions 2.3 and 2.6] may be adapted to yield the following two results:

Theorem 4.1: Let G be a weak Powers group. If $\mathcal{A}$ is G-simple (so especially if $\mathcal{A}$ is simple), then $\mathcal{B} = C_r^*(\mathcal{A}, G, \alpha, u)$ is simple.

Theorem 4.2: Let G be a weak Powers group. If τ is a trace on $\mathcal{B} = C_r^*(\mathcal{A}, G, \alpha, u)$ then $\sigma = \tau|_{\mathcal{A}}$ is a G-invariant trace on $\mathcal{A}$ such that $\tau = \sigma \circ E$, where E denotes the canonical conditional expectation of $\mathcal{B}$ onto $\mathcal{A}$. Therefore, if $\mathcal{A}$ has a unique G-invariant trace, then $\mathcal{B}$ has a unique trace.

Since there are several misprints in [6], making it difficult to follow the proofs, we will present a proof of these two results based on the ideas of [6,15]. At this point, it should also be noticed that the reduced twisted crossed products considered in [6] are of the Zeller-Meier type, and assumed to satisfy the normalizing condition:

$$u(g, g^{-1}) = I \quad \text{for all } g \in G.$$

As pointed out to us by I. Raeburn, this condition may be assumed without loss of generality whenever there exists a square root map $\sqrt{\ }$ on $\mathcal{A}$ satisfying $\sqrt{\alpha_g(u)} = \alpha_g(\sqrt{u})$ $(g \in G,\ u \in \mathcal{U}(\mathcal{A}))$. However, as we shall presently see, there is no need of assuming this condition in the sequel.

The key lemma here is a variation of Powers original argument due to de la Harpe and Skandalis, which proof is easily obtained from their proof of [15; lemma 1].

Lemma 4.3: Let x be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, and suppose there exists a projection p and unitaries u_1, u_2, u_3 in $\mathcal{B}(\mathcal{H})$ such that $pxp = 0$ and that the projections $u_i(I-p) = u_i^*$ are pairwise orthogonal $(i = 1, 2, 3)$. Then

$$\left\| \frac{1}{3} \sum_{i=1}^{3} u_i x u_i^* \right\| \leq d \|x\|,$$

for any number d satisfying $\frac{5}{6} + \frac{\sqrt{2}}{9} < d < 1$ (such as $d = 0.991$).

Let us next introduce some terminology.

A *simple G-averaging process on* $\mathcal{B} = C_r^*(\mathcal{A}, G, \alpha, u)$ will be a linear map $\phi : \mathcal{B} \to \mathcal{B}$ such that there exist $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in G$ satisfying that

$$\phi(b) = \frac{1}{n} \sum_{i=1}^{n} \lambda(s_i) b \lambda(s_i)^* \quad \text{for all } b \in \mathcal{B}.$$

Further, a *G-averaging process on $\mathcal{B}$* will be a linear map $\psi : \mathcal{B} \to \mathcal{B}$ such that there exist $m \in \mathbf{N}$ and $\phi_1, \ldots, \phi_m$ simple G-averaging processes on $\mathcal{B}$ with $\psi = \phi_m \circ \phi_{m-1} \circ \ldots \circ \phi_1$. It is clear that such a ψ is positive and bounded with $\|\psi\| = 1$.

For $g \in G$, we set

$$\zeta(g) = \{F' \subseteq G \mid \text{there exist a finite subset } F \text{ of the conjugacy class of } g \text{ with } F' \subseteq (F \cup F^{-1})\} .$$

At last, we let $\mathcal{B}_0$ denote the dense $*$-subalgebra of $\mathcal{B}$ generated by $\mathcal{A}$ and $\lambda(G)$. In other words,

$$\mathcal{B}_0 = \{x \in \mathcal{B} \mid \mathrm{supp}(x) \text{ is a finite subset of } G\}.$$

Lemma 4.4: Let x be a self-adjoint element of $\mathcal{B}_0$ with $\mathrm{supp}(x) \in \zeta(g)$ for some $g \in G \setminus \{e\}$, and let ψ be a G-averaging process on $\mathcal{B}$. Then $\psi(x)$ is a self-adjoint element of $\mathcal{B}_0$ with $\mathrm{supp}(\psi(x)) \in \zeta(g)$.

Proof: It is rather trivial to check that $\psi(x)$ is a self-adjoint element of $\mathcal{B}_0$. Next, for $a \in \mathcal{A}$, $s, h \in G$, we have that

$$\begin{aligned} \lambda(s)(a\lambda(h))\lambda(s)^* &= \alpha_s(a)\lambda(s)\lambda(h)\lambda(s)^* \\ &= \alpha_s(a)\tilde{u}(s,h)\lambda(shs^{-1}). \end{aligned}$$

From this observation, it follows easily that, if ϕ is a simple G-averaging process on $\mathcal{B}$, then $\phi(x) \in \zeta(g)$, and the same result for ψ follows then by induction. □

Lemma 4.5: Suppose G is a weak Powers group and let x be a self-adjoint element of $\mathcal{B}_0$ with $\mathrm{supp}(x) \in \zeta(g)$ for some $g \in G \setminus \{e\}$. Then, for any $\delta > 0$, there exists a G-averaging process ψ_g on $\mathcal{B}$ such that

$$\|\psi_g(x)\| < \delta .$$

Proof: Let D, E and g_1, g_2, g_3 be given from the definition of G being a weak Powers group (with $n = 3$). Let p be the projection of $l^2(G, \mathcal{H})$ onto $l^2(D, \mathcal{H})$ and set $u_i = \lambda(g_i)$ $(i = 1, 2, 3)$. A straightforward computation shows that $u_i(I - p)u_i^*$ is the projection of $l^2(G, \mathcal{H})$ onto $l^2(g_i E, \mathcal{H})$. It follows therefore from (i) and (ii) in the definition of a weak Powers group that $pxp = 0$ and that $u_i(I-p)u_i^*$ are pairwise orthogonal ($i = 1,2,3$). By lemma 4.3, $\phi_1(\cdot) = \frac{1}{3}\sum_{i=1}^{3} \lambda(g_i)\cdot\lambda(g_i)^*$ is a simple G-averaging process on $\mathcal{B}$ satisfying $\|\phi_1(x)\| \leq d\|x\|$ for $d = 0.991$. By lemma 4.4, we may proceed inductively and obtain that, for each $k \in \mathbf{N}$, there exists a simple G-averaging process ϕ_k on $\mathcal{B}$ such that, if $\psi_k = \phi_k \circ \phi_{k-1} \circ \ldots \circ \phi_1$, then $\|\psi_k(x)\| \leq d^k\|x\|$. Therefore, if $\delta > 0$ is given, there exists a ψ_δ as required.

□

Lemma 4.6: Suppose G is weak Powers group and let x be a self-adjoint element of $\mathcal{B}$ with $e \notin \mathrm{supp}(x)$. Then, for any $\epsilon > 0$, there exists a G-averaging process ψ_ϵ on $\mathcal{B}$ such that $\|\psi_\epsilon(x)\| < \epsilon$.

Proof: By a density argument, we may assume that the given x lies in $\mathcal{B}_0$. Then there exist $n \in \mathbf{N}$, $a_i \in \mathcal{A}\setminus\{0\}$ and $h_i \in G\setminus\{e\}$ ($i = 1,\ldots,n$) such that $x = x_1 + \ldots + x_n$, where $x_i = a_i\lambda(h_i) + u(h_i^{-1}, h_i)^*\alpha_{h_i}(a_i^*)\lambda(h_i^{-1})$ is a self-adjoint element of $\mathcal{B}_0$ satisfying

$$\mathrm{supp}(x_i) = \{h_i\} \cup \{h_i\}^{-1} \in \zeta(h_i) \quad (i=1,\ldots,n).$$

By lemma 4.5, there exists a G-averaging process ψ_1 on $\mathcal{B}$ such that $\|\psi_1(x_1)\| < \epsilon/n$. Set $\tilde{x}_1 = x_1$. If $n \geq 2$, we may proceed inductively, using repeatedly lemmas 4.4 and 4.5, in such a way that, for each $\tilde{x}_k = \psi_{k-1} \circ \ldots \circ \psi_1(x_k)$, there exists a G-averaging process ψ_k on $\mathcal{B}$ such that $\|\psi_k(\tilde{x}_k)\| < \epsilon/n$ ($k = 2,\ldots,n$). Then $\psi_\epsilon = \psi_n \circ \ldots \circ \psi_1$ is a G-averaging process on $\mathcal{B}$ such that

$$\|\psi_\epsilon(x_i)\| = \|\psi_n \circ \ldots \circ \psi_i(\tilde{x}_i)\| \leq \|\psi_i(\tilde{x}_i)\| < \epsilon/n \quad (i=1,\ldots,n),$$

so $\|\psi_\epsilon(x)\| \leq \sum_{i=1}^{n} \|\psi_\epsilon(x_i)\| < n\cdot\epsilon/n = \epsilon$, as desired. □

Lemma 4.7: Suppose $\mathcal{A}$ is G-simple and let a be a non-zero positive element of $\mathcal{A}$. Then there exist $n \in \mathbf{N}$, $a_1,\ldots,a_n \in \mathcal{A}$ and $h_1,\ldots,h_n \in G$ such that

$$\sum_{i=1}^{n} a_i\alpha_{h_i}(a)a_i^* \geq I\,.$$

Proof: Let J be the two-sided ideal of $\mathcal{A}$ algebraically generated by $\{\alpha_s(s) | s \in G\}$. Then J is G-invariant. Indeed, let $w \in J$ and $g \in G$. By definition of J, there exist $m \in \mathbf{N}$, $x_1, y_1, \ldots, x_m, y_m \in \mathcal{A}$ and $s_1, \ldots, s_m \in G$ such that

$$w = \sum_{i=1}^{m} x_i \alpha_{s_i}(a) y_i$$

Hence,

$$\begin{aligned}
\alpha_g(w) &= \sum_{i=1}^{m} \alpha_g(x_i)\alpha_g(\alpha_{s_i}(a))\alpha_g(y_i) \\
&= \sum_{i=1}^{m} (\alpha_g(x_i)u(g,s_i))\alpha_{gs_i}(a)(u(g,s_i)^*\alpha_g(y_i)) \\
&\in J
\end{aligned}$$

Since $J \neq \{0\}$ and $\mathcal{A}$ is G-simple, this implies that $\overline{J} = \mathcal{A}$. Thus $J = \mathcal{A}$, since $\mathcal{A}$ has an identity. Therefore, there exist $n \in \mathbf{N}$, $c_1, d_1, \ldots, c_n, d_n \in \mathcal{A}$ and $h_1, \ldots, h_n \in G$ such that

$$\sum_{i=1}^{n} c_i \alpha_{h_i}(a) d_i = \frac{I}{2}.$$

Now, set $a_i = c_i + d_i$, $i = 1, \ldots, n$. Then

$$\sum_{i=1}^{n} a_i \alpha_{h_i}(a) a_i^* = \sum_{i=1}^{n} c_i \alpha_{h_i}(a) c_i^* + \sum_{i=1}^{n} d_i \alpha_{h_i}(a) d_i^* + I \geq I.$$

□

Proof of theorem 4.1: Let J be a non-zero ideal in $\mathcal{B}$ and let y be a non-zero positive element of J. If E denotes the canonical conditional expectation from $\mathcal{B}$ onto $\mathcal{A}$, then $E(y)$ is a non-zero positive element of $\mathcal{A}$, and lemma 4.7 implies that there exist $n \in \mathbf{N}$, $a_1, \ldots, a_n \in \mathcal{A}$ and $h_1, \ldots, h_n \in G$ such that

$$\sum_{i=1}^{n} a_i \alpha_{h_i}(E(y)) a_i^* \geq I.$$

Since

$$\begin{aligned}
E\left(\sum_{i=1}^{n} a_i \lambda(h_i) y \lambda(h_i)^* a_i^*\right) &= \sum_{i=1}^{n} a_i E(\lambda(h_i) y \lambda(h_i)^*) a_i^* \\
&= \sum_{i=1}^{n} a_i \alpha_{h_i}(E(y)) a_i^* \qquad \text{(by Corollary 2.3b)),}
\end{aligned}$$

we may, by replacing y with $\sum_{i=1}^{n} a_i\lambda(h_i)y\lambda(h_i)^* a_i^*$ if necessary, suppose that $E(y) \geq I$. By lemma 4.6 (with $x = y - E(y)$ and $\epsilon = \frac{1}{2}$), we obtain that there exist a G-averaging process ψ on $\mathcal{B}$ such that

$$\|\psi(y - E(y))\| < \frac{1}{2},$$

hence

$$\|\psi(y) - \psi(E(y))\| < \frac{1}{2}.$$

Since $\psi(E(y)) \geq \psi(I) = I$ and $\psi(y)$ is positive, this implies that $\psi(y)$ is invertible. But clearly $\psi(y) \in J$, hence $J = \mathcal{B}$, and $\mathcal{B}$ is simple. □

Proof of theorem 4.2: Observe first that if τ is a trace on $\mathcal{B}$ and ψ is a G-averaging process on $\mathcal{B}$, then $\tau(\psi(b)) = \tau(b)$ for all $b \in \mathcal{B}$. It follows therefore easily from lemma 4.6 that $\tau(x - E(x)) = 0$ for all self-adjoint elements x in $\mathcal{B}$. Hence $\tau(x) = \tau(E(x)) = \sigma(E(x))$ for all such x, where $\sigma = \tau|_{\mathcal{A}}$. Consequently, $\tau = \sigma \circ E$ on $\mathcal{B}$. The last assertion follows then from corollary 2.3.d). □

We now obtain

Corollary 4.8: Suppose H is a normal subgroup of G such that the factor group $K = G/H$ is a weak Powers group. Then

(a) $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple whenever $C_r^*(\mathcal{A}, H, \alpha, u)$ is simple.

(b) $C_r^*(\mathcal{A}, G, \alpha, u)$ has a unique trace whenever $C_r^*(\mathcal{A}, H, \alpha, u)$ has a unique trace.

Especially, we have

(c) G is C^*-simple whenever H is C^*-simple.

(d) G has a unique trace whenever H has a unique trace.

Proof: (a) and (b) follows from theorem 4.1 and theorem 4.2, in combination with theorem 2.1, while (c) and (d) are special cases of (a) and (b). □

If H is also a weak Powers group, corollary 4.8 (c) and (d) is proved in [6; Proposition 2.10] (with a different proof). We will shortly give a stronger version of part (c) (corollary 4.12).

As a partial answer to problem (III) in the introduction, we have:

Corollary 4.9: Suppose G is an extension of a weak Powers group by a weak Powers group, and $\mathcal{A}$ is simple (resp. has a unique trace). Then $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple (resp. has a unique trace).

Proof: This follows from theorem 4.1 (resp. 4.2) combined with corollary 4.8(a) (resp. 4.8(b)). □

With the results of section 3 at hand together with those of this section, we are in position to prove the announced results about ultraweak Powers groups. We recall that an *ultraweak Powers group* is a group containing a weak Powers group with trivial centralizer. Examples are furnished by the automorphism group and the holomorph of any weak Powers group.

Corollary 4.10: Suppose G is an ultraweak Powers group and $\mathcal{A}$ is simple with a faithful G-invariant state. Then $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple. Especially G is C^*-simple and $C_r^*(G, \omega)$ is simple for any cocycle $\omega : G \times G \to \mathbf{T}$.

Proof: The first assertion follows from theorems 3.5 and 4.1, while the second is a consequence of the first. □

In connection with the problem raised at the end of section 3, one may ask whether an ultraweak Powers group necessarily has a unique trace.

Corollary 4.10 provides another partial answer to problem (III). In fact, one can push this game a little bit further:

Corollary 4.11: Suppose G is an extension of an ultraweak Powers group H by an ultraweak Powers group K and $\mathcal{A}$ is simple with a faithful G-invariant state φ. Then $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple. Especially G is C^*-simple and $C_r^*(G, \omega)$ is simple for any cocycle $\omega : G \times G \to \mathbf{T}$.

Proof: Decompose $C_r^*(\mathcal{A},G,\alpha,u)$ as in theorem 2.1: $C_r^*(\mathcal{A},G,\alpha,u) \simeq (C_r^*(C_r^*(\mathcal{A},H,\alpha,u),K,\beta,v)$ and denote by E' the canonical expectation from $C^*(\mathcal{A},H,\alpha,u)$ onto $\mathcal{A}$.

We have that $C_r^*(\mathcal{A},H,\alpha,u)$ is simple by corollary 4.10. Further, one checks easily that $\tilde{\varphi}=\varphi\circ E'$ is a faithful K-invariant state on $C_r^*(\mathcal{A},H,\alpha,u)$. Another application of corollary 4.10 gives the first assertion and therefore the second. □

If H in the above corollary contains a normal Powers subgroup with trivial centralizer, then it follows easily from [6; Proposition 1.5] that G is an ultraweak Power group too.

Our last corollary provides a quite general answer to problem (II).

Corollary 4.12: Suppose G is an extension of a C^*-simple group H by an ultraweak Powers group K. Then G is C^*-simple.

Proof: Decompose $C_r^*(G)\simeq C_r^*(C_r^*(H),K,\beta,v)$ as in theorem 2.1 and notice that the canonical trace on $C_r^*(H)$ is K-invariant. Apply then corollary 4.10. □

By an inductive argument, Corollary 4.12 remains true if K has a normal tower $K_1 \trianglelefteq K_2 \trianglelefteq \ldots \trianglelefteq K_n = K$ where K_i and K_{i+1}/K_i are ultraweak Powers groups $(i=1,\ldots,n-1)$.

We conclude this paper with a couple of remarks about the braid group B_n with n generators $(n\geq 3)$ ([10,12]). Denote by C_n its center (which is isomorphic to $\mathbf{Z}$) and set $\mathcal{B}_n = B_n/C_n$.

Let us first observe that $\mathcal{B}_n$ is an ultraweak Powers group. In fact, $\mathcal{B}_n$ is a group of Akemann-Lee type: Dyer and Grossmann show in the course of the proof of [10; Corollary 17] that $\mathcal{B}_n$ contains a normal copy of the free group on $n-1$ generators with trivial centralizer. Hence, $\mathcal{B}_n$ is C^*-simple (and has a unique trace). Dyer and Grossmann also show that $Aut(B_n)\simeq Aut(\mathcal{B}_n)$ ([10; Theorem 20]). It folows therefore from Corollary 3.6 that $Aut(B_n)$ is C^*-simple.

At last, we note that $Aut(B_n)$ $(n\geq 4)$ and $Aut(Aut(B_3))$ are complete ([10; Theorem 22 and Proposition 23]) so that the tower of automorphisms groups ends very quickly in this case, as it does in the case of free groups ([9]), cf. our comments following corollary 3.7.

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