

BROWNIAN MOTION PENETRATING THE SIERPINSKI GASKET

by

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I. Introduction.

In the last few years there has been an increasing interest among probabilists for diffusions on fractals. Thanks to the efforts of Kusuoka [13], Goldstein [7], Barlow and Perkins [6], Kigami [11] and others, we have today a fairly good understanding of Brownian motion on the Sierpinski gasket, and this theory has been extended in various ways to cover more general classes of finitely ramified fractals by Lindstrøm [16], Kigami [12], Kusuoka [14], and - from a field theoretic point of view - Hattori, Hattori and Watanabe [8]. In a series of papers [2], [3], [4], [5], Barlow and Bass have studied Brownian motion on the simplest infinitely ramified fractal, the Sierpinski carpet. Since fractals can be considered as models for porous media and semiconductors (among other things), there is an extensive physics literature on diffusion on fractals; see Havlin and Ben-Avraham [9] for a survey.

In the present paper I shall restrict my attention to the Sierpinski gasket, but look at a kind of problem that has not been considered in the literature so far. The question is simply: can Brownian motion penetrate the Sierpinski gasket; i.e. is there a natural, continuous process which behaves like ordinary Brownian motion outside the gasket and like fractal Brownian motion inside it? The interesting part, of course, is what happens on the boundary where we have a delicate balance between excursions inside and outside the gasket; since the time scaling is drastically different in the two sets, it is not at all obvious how this balance can be achieved. In addition to its intrinsic appeal, the problem may also be of some interest to applications; if, e.g., we model porous rock by fractals, our processes will be useful in understanding the seismic properties of the rock.

As you can tell from the title of the paper, the answer to the question above is yes. More surprising, perhaps, is the number of solutions; given a pair of positive functions α, β , where α is defined and harmonic (w.r.t. fractal Brownian motion) inside the gasket and β is defined and harmonic outside the gasket, I shall construct a solution $z^{(\alpha, \beta)}$ of the problem. The functions α and β will describe the density of the process in equilibrium, and two solutions $z^{(\alpha, \beta)}$ and $z^{(\alpha', \beta')}$ will be equal only if there is a constant k such that $\alpha' = k\alpha$ and $\beta' = k\beta$.

In addition to looking like Brownian motion both outside and inside the gasket, the process $z^{(\alpha, \beta)}$ will in general have a highly nontrivial behaviour on the boundary between the two sets; unless α and β are both constant, the process on the boundary will look something like a Brownian motion run at infinite speed - although the average particle

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only spends time zero on the boundary, it still manages to get somewhere. As we shall see, this extremely singular behaviour on the boundary is necessary to keep the process in equilibrium.

To construct the processes $z^{(\alpha, \beta)}$, I shall use nonstandard analysis and random walks with infinitesimal increments, but readers unfamiliar with nonstandard analysis should not have any problems with the basic ideas of the paper as long as they are willing to think about infinitely large and infinitely small numbers in an intuitive way. A few words about notation and terminology may be useful: ${}^*\mathbb{R}$ is the set of nonstandard real numbers (basically \mathbb{R} with infinitesimal and infinite elements added), and “internal” is a technical condition on nonstandard sets and functions - in the present paper it will play a rôle analogous to “measurable” in basic probability theory; it’s not a condition you offer much thought, but your statements will be false if you omit it. Readers who want to know more about the formal background for nonstandard probability theory, may try [1] and [15].

Acknowledgement: I’m grateful to Martin Barlow who told me about the problem and encouraged me to work on it. I would also like to express my gratitude to the Taniguchi Foundation for generous financial support, and to Professors N. Ikeda, S. Watanabe and S. Kusuoka for an excellently organized conference.

II. Setting the stage.

The *Sierpinski gasket* is obtained by the following construction. Starting with the

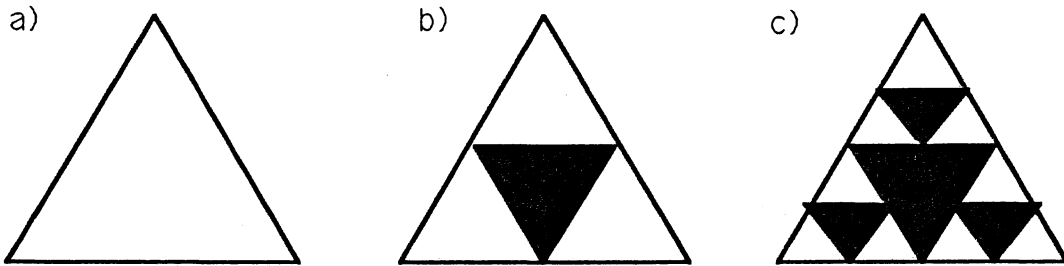


Figure II.1

equilateral triangle in Figure II.1.a), we first remove the black triangle in the middle as shown in b). This leaves us with three triangles similar two but smaller than the original one, and repeating the procedure with each one of these, we get the figure in c). Again we repeat the procedure with each of the nine triangles we now have etc. The set we get in the limit is the Sierpinski gasket.

In this paper we shall consider a Sierpinski gasket with sides of length one sitting in the middle of a larger triangle with sides of length four as shown in Figure II.2. The processes we shall construct will have behave like ordinary Brownian motion in the area between the two triangles (with normal reflection at the outer boundary), and like fractal Brownian motion inside the gasket. (The only reason for putting in the outer triangle is to make some of the more technical arguments a little more transparent.) We shall not

allow the process to enter the black triangles removed in the construction of the gasket.

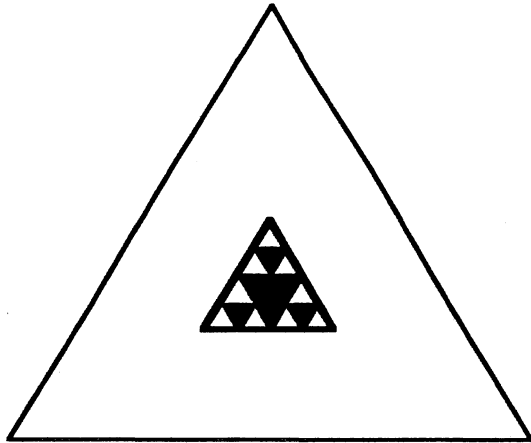


Figure 11.2

(An alternative version of the problem would be to construct a process which enters these areas and behaves like an ordinary Brownian motion inside them; this is actually a quite interesting question from the point of view of singular perturbation of the Laplacian as discussed in, e.g., chapter 6 of [1]. It can be solved by minor modifications of the methods in this paper).

As already mentioned, we shall obtain our processes from nonstandard random walks with infinitesimal increments. We begin by fixing an infinitely large, nonstandard integer N (if you don't like nonstandard analysis, just think of N as a big integer which will eventually go to infinity), and divide the sides of the outer triangle into intervals of length 2^{-N} . Connecting the division points by lines parallel to the sides of the triangle, we get a triangular lattice. Finally, we carry out the first N steps in the construction of the Sierpinski gasket (since the sides of the big triangle are four times those of the small triangle, the pieces we remove in the construction fit in nicely with the lattice). Figure 11.3 shows the situation for $N = 2$.

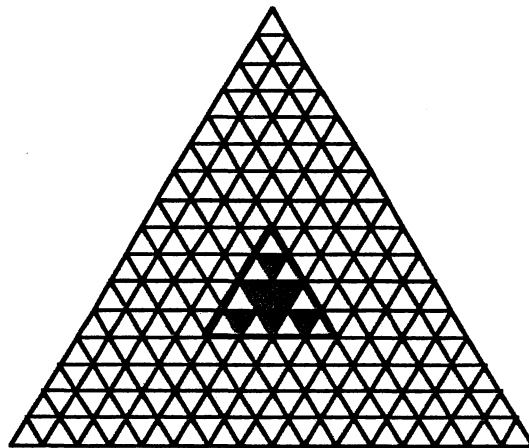


Figure 11.3

Our random walks will live on the set S_N of vertices that are left after these operations (since we only remove the interior of the black triangles, the sites on their boundaries are still there).

It is natural and convenient to write S_N as a disjoint union

$$S_N = S_N^i \cup S_N^e \cup S_N^b,$$

where the set S_N^i consists of the *interior sites* strictly inside the bold triangle in Figure II.3; the set S_N^e consists of the *exterior sites* strictly outside the same triangle; and the set S_N^b of *boundary sites* are the ones along the sides of this triangle. We shall also need the "closures"

$$\bar{S}_N^i = S_N^i \cup S_N^b$$

and

$$\bar{S}_N^e = S_N^e \cup S_N^b.$$

III. The interior process.

I shall construct the processes $z^{(\alpha,\beta)}$ in stages. Let us first take a look at what happens inside the Sierpinski gasket. If $\Delta t = 5^{-N}$ and

$$T = \{0, \Delta t, 2\Delta t, 3\Delta t, \dots\},$$

we shall study the nearest neighbor random walk

$$X_N : \Omega \times T \rightarrow \bar{S}_N^i$$

with the following transition probabilities: If x and y are neighboring sites in \bar{S}_N^i and X_N is at x at time t , then X_N will be at y at time $t + \Delta t$ with probability $1/4$ (if x is one of the three corners on the boundary, x has only two neighbors and the process will remain at x with probability $1/2$). It is well known that if we take the standard part of the nonstandard process X_N , we get Brownian motion on the Sierpinski gasket; this is just a nonstandard way of saying that the sequence $\{X_N\}_{N \in \mathbb{N}}$ converges to Brownian motion when N goes to infinity. Observe that X_N has time increments $\Delta t = 5^{-N}$ and space increments $\Delta x = 2^{-N}$, and hence we get

$$\Delta x = \Delta t^{\log 2 / \log 5}$$

instead of the usual

$$\Delta x = \Delta t^{\frac{1}{2}};$$

Brownian motions on fractals run much faster than ordinary Brownian motion to compensate for the fact that they live in labyrinths.

To be able to compute excursions from the boundary, we shall need to know a few basic facts about X_N .

III.1 Lemma. Let q_N be the probability that X_N started at A in Figure III.1 hits B before it hits the line segment CD . Then

$$(3.1) \quad q_N = \frac{3}{13 - 10q_{N-1}},$$

where q_{N-1} is the corresponding probability on S_{N-1} .

Proof: Start the process at A and let it run until it hits one of the points E, F, G or H . If it is stopped at either E or F , start it again and let it run until it hits either the point A or the line segment CJ .

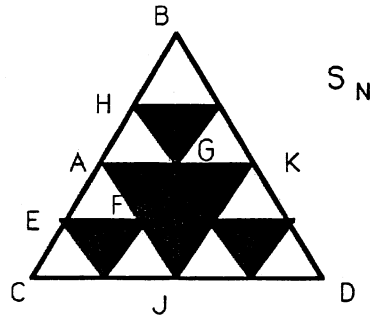


Figure III.1

This distributes a mass of $q_{N-1}/2$ at A and $(1 - q_{N-1})/2$ along CJ . If the process was first stopped at H or G , start it again and let it run until it hits A, B or K . A trivial calculation shows that this distributes a mass of $3/20$ at B and K and $1/5$ at A . Because of the symmetry of the problem we can consider A and K as the same state, and hence we end up with a mass of $(1 - q_{N-1})/2$ along CJ , a mass of $q_{N-1}/2 + 1/5 + 3/20 = \frac{q_{N-1}}{2} + 7/20$ at A , and a mass of $3/20$ at B . Repeating the experiment with the particles now at A will not change the ratio between the mass of B and the mass along CJ , and hence

$$\frac{q_N}{1 - q_N} = \frac{3/20}{(1 - q_{N-1})/2}.$$

Solving this equation for q_N , we get (3.1). □

The function

$$f(q) = \frac{3}{13 - 10q}$$

has a stable fixed point at $q = \frac{3}{10}$ and an unstable one at $q = 1$. A trivial calculation shows that $q_1 = 1/3$, and since f is a contraction on an interval containing $(\frac{3}{10}, \frac{1}{3})$, we get:

III.2 Lemma. The sequence $\{q_n\}$ converges to $3/10$ at a geometric rate. Moreover, there are positive constants $C, K \in \mathbb{R}$ such that

$$C\left(\frac{3}{10}\right)^N \leq \prod_{n=1}^N q_n \leq K\left(\frac{3}{10}\right)^N$$

for all N .

Proof: That q_n converges to $\frac{3}{10}$ at a geometric rate follows immediately from the fixed point theorem for contractions. Hence there are positive constants M and $r, r < 1$, such

that

$$\left|q_n - \frac{3}{10}\right| \leq Mr^n,$$

from which it follows that $q_n/(3/10)$ and $(3/10)/q_n$ are both less than $1 + M'r^n$ for some constant M' . Thus it suffices to show that the products

$$\prod_{n=1}^N (1 + M'r^n)$$

are bounded. But this is almost trivial since

$$\log \prod_{n=1}^N (1 + M'r^n) = \sum_{n=1}^N \log(1 + M'r^n) \leq \sum_{n=1}^N M'r^n \leq \frac{M'r}{1-r}.$$

□

In Figure III.2 the points $A_0, A_1, A_2, A_3, \dots$ along the side of S_N have been chosen such that $|B - A_n| = 2^{-n}$. Note that the triangle $A_n B C_n$ is a (scaled) copy of S_{N-n} .

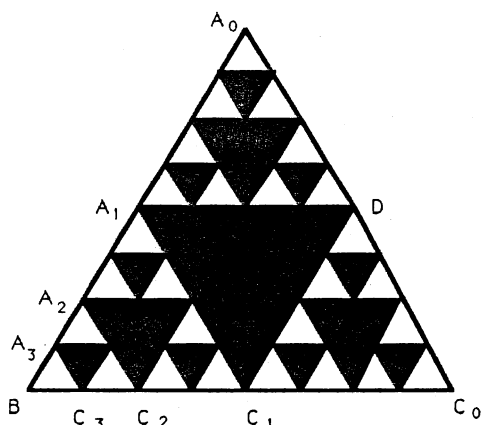


Figure III.2

III.3 Proposition. The probability p_n that X_n starting at A_n hits A_0 before it hits the line segment BC_0 , is

$$(3.2) \quad p_n = q_N q_{N-1} q_{N-2} \cdots q_{N-n+1}.$$

There are positive, real constants C and K such that

$$(3.3) \quad C\left(\frac{3}{10}\right)^n \leq p_n \leq K\left(\frac{3}{10}\right)^n.$$

Proof: Obviously, $p_0 = 1$ and since

$$p_{k+1} = q_{N-k} p_k,$$

(3.2) follows. The second formula (3.3) is an immediate consequence of (3.2) and Lemma III.2. □

The following simple corollary will not be used in the present paper, but it is of some independent interest.

III.4 Corollary: Assume that Figure III.2 shows the full Sierpinski gasket and not just S_N . Then the probability that a Brownian path starting at A_n will hit A_0 before it hits the line segment BC_0 is $(\frac{3}{10})^n$.

For the final results of this section we return once again to Figure III.1. Starting a particle at A and stopping it the first time it hits either B or the line segment CD , we now want to know the stopping distribution; what is the probability that the particle is stopped in B , in the interval CJ , and in the interval JD ? (If the particle is stopped in the common point J of the two intervals, we shall count it as belonging to CJ if it approaches J through the triangle ΔACJ , and to JD if it approaches J through ΔKJD .) The results will only be needed in one place in this paper, and may be skipped at the first reading.

III.5 Lemma. The hitting probabilities of the point B and the line segments CJ and CD are, respectively,

$$(3.4) \quad p_B = \frac{3}{13 - 10q_{N-1}}$$

$$(3.5) \quad p_{CJ} = \frac{20(1 - q_{N-1})(8 - 5q_{N-1})}{(13 - 10q_{N-1})(19 - 10q_{N-1})}$$

$$(3.6) \quad p_{JD} = \frac{30(1 - q_{N-1})}{(13 - 10q_{N-1})(19 - 10q_{N-1})}$$

Proof: We have already established (3.4), and hence we know that

$$(3.7) \quad p_{CJ} + p_{JD} = 1 - p_B = \frac{10(1 - q_{N-1})}{13 - 10q_{N-1}}.$$

To get a second equation, we copy the proof of Lemma III.1; starting the process at A , we first stop it when it hits one of the points E, F, G or H , then start it again and let it run until it hits either the segment CJ or one of the points A, K or B . According to the proof of Lemma III.1, the particle will now be somewhere on the segment CJ with probability $\frac{1 - q_{N-1}}{2}$, it will be back at A with probability $\frac{q_{N-1}}{2} + \frac{1}{5}$, and it will be in K with probability $3/20$. Since the situation is left-right symmetric, this means that

$$(3.8) \quad p_{CJ} = \frac{1 - q_{N-1}}{2} + \left(\frac{q_{N-1}}{2} + \frac{1}{5}\right)p_{CJ} + 3/20p_{JD}$$

Solving (3.7) and (3.8), we get (3.5) and (3.6). □

Since q_N tends to $\frac{3}{10}$ as N goes to infinity, it is convenient to rewrite (3.4)-(3.6) slightly. Introducing

$$r_N = \frac{3}{10} - q_N,$$

we get

$$(3.9) \quad p_B = \frac{3}{10(1+r_{N-1})}$$

$$(3.10) \quad p_{CJ} = \frac{91(1+\frac{10}{7}r_{N-1})(1+\frac{10}{13}r_{N-1})}{160(1+r_{N-1})(1+\frac{5}{8}r_{N-1})}$$

$$(3.11) \quad p_{JD} = \frac{21(1+\frac{10}{7}r_{N-1})}{160(1+r_{N-1})(1+\frac{5}{8}r_{N-1})}$$

Since r_N tends to zero at a geometric rate,

$$p_B \rightarrow \frac{3}{10}, p_{CJ} \rightarrow \frac{91}{160}, p_{JD} \rightarrow \frac{21}{160}$$

all at geometric rates. Again we get a corollary for Brownian motion which we shall not need in this paper, but which is of some independent interest.

III.6 Corollary: Assume that Figure III.1 shows the full Sierpinski gasket and not only S_N . Start a Brownian motion at A and let it run until it hits either the point B or the line segment CD . Then the particle is stopped at B with probability $3/10$, in the segment CJ with probability $\frac{91}{160}$, and in the segment JD with probability $\frac{21}{160}$.

For the very last result in this section we first take a look at Figure III.3.

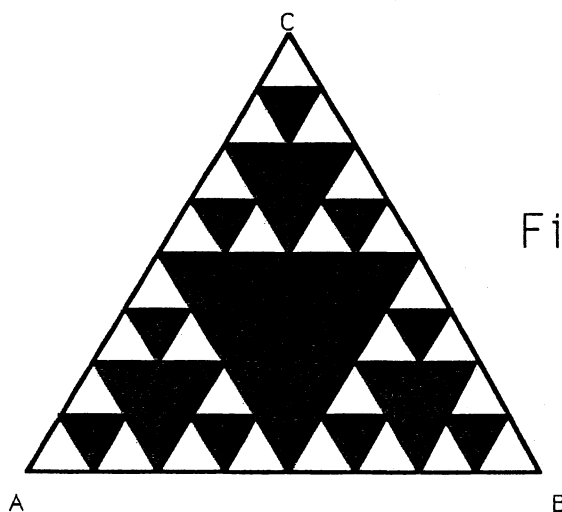


Figure III.3

Let a, b and c be three real numbers. Assume that $u : S_N \rightarrow \mathbb{R}$ is defined as follows: $u(A) = a, u(B) = b, u(C) = c$; u grows linearly on the line segment from A to B ; and u is harmonic outside $AB \cup \{C\}$. For each site $i \in AB$, let $N(i)$ be the set of neighbors in the interior of S_N .

III.7 Proposition. There is a real constant C (independent of a, b, c and N) such that

$$(3.12) \quad \left| \sum_{i \in AB} \sum_{j \in N(i)} [u(j) - u(i)]u(i) \right| \leq C \left(\frac{3}{5}\right)^N |(c-a)a + (c-b)b|$$

Remark: This formula is not as mysterious as it may seem at first glance; if we multiply it by $(\frac{5}{3})^N$, the left-hand side becomes basically the Dirichlet form of our process applied to the function u (since u is harmonic, there are no contributions from the interior). From this point of view the proposition just tells us that a harmonic function with linear growth along one of the edges, has finite Dirichlet integral. Since a fractal thinks that linear functions are very irregular (remember that the only C^1 -functions in the domain of the Laplacian are the constants), this is not at all obvious.

Proof of Proposition III.7: This is one long, terrible, but totally elementary calculation. For an integer $n < N$, let $AB^{(n)}$ be the sites on AB belonging to S_n , and for $i \in AB^{(n)}$, let $N^{(n)}(i)$ be its neighbors in the interior of S_n . We want to compare the two sums

$$(3.13) \quad \sum_{i \in AB^{(n)}} \sum_{j \in N^{(n)}(i)} [u(j) - u(i)]u(i)$$

$$(3.14) \quad \sum_{i \in AB^{(n+1)}} \sum_{j \in N^{(n+1)}(i)} [u(j) - u(i)]u(i)$$

where u is the function in the proposition.

Figure III.4 shows the situation; j is a neighbor of i_1 and i_2 in S_n , while in S_{n+1} j_1 is a neighbor of i_1 and i_3 , and j_2 is a neighbor i_3 and i_2 .

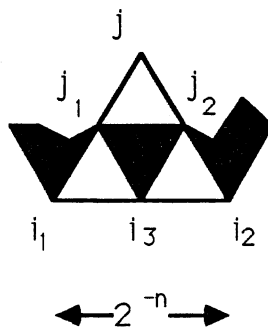


Figure III.4

Hence the terms

$$[u(j) - u(i_1)]u(i_1) \text{ and } [u(j) - u(i_2)]u(i_2)$$

in formula (3.13), will in (3.14) be replaced by the terms

$$\begin{aligned} & [u(j_1) - u(i_1)]u(i_1); \quad [u(j_1) - u(i_3)]u(i_3); \\ & [u(j_3) - u(i_3)]u(i_3); \quad [u(j_3) - u(i_2)]u(i_2). \end{aligned}$$

Since u is linear along the bottom line segment,

$$u(i_3) = \frac{u(i_1) + u(i_2)}{2}.$$

Using Lemma III.5 and the easy observation that the process started at j_1 will hit the intervals i_1i_3 and i_3i_2 in a uniform way, we can also compute $u(j_1)$:

$$u(j_1) = \frac{91}{160}(1 + A_n) \frac{3u(i_1) + u(i_2)}{4} + \frac{21}{160}(1 + B_n) \frac{u(i_1) + 3u(i_2)}{4} + \frac{3}{10}(1 + C_n)u(j),$$

where A_n, B_n, C_n converge to zero geometrically. Similarly,

$$u(j_2) = \frac{21}{160}(1 + B_n) \frac{3u(i_1) + u(i_2)}{4} + \frac{91}{160}(1 + A_n) \frac{u(i_1) + 3u(i_2)}{4} + \frac{3}{10}(1 + C_n)u(j).$$

We are now ready to compute the contribution to (3.14) from the points in Figure III.4:

$$\begin{aligned} & [u(j_1) - u(i_1)]u(i_1) + [u(j_1) - u(i_3)]u(i_3) + \\ & + [u(j_2) - u(i_3)]u(i_3) + [u(j_2) - u(i_2)]u(i_2) = \\ & = u(j_1) \frac{3u(i_1) + u(i_2)}{2} + u(j_2) \frac{u(i_1) + 3u(i_2)}{2} \\ & - \frac{3}{2}u(i_1)^2 - u(i_1)u(i_2) - \frac{3}{2}u(i_2)^2 = \\ & = \frac{819}{1280}(1 + A_n)u(i_1)^2 + \frac{546}{1280}(1 + A_n)u(i_1)u(i_2) + \frac{91}{1280}(1 + A_n)u(i_2)^2 + \\ & + \frac{63}{1280}(1 + B_n)u(i_1)^2 + \frac{210}{1280}(1 + B_n)u(i_1)u(i_2) + \frac{63}{1280}(1 + B_n)u(i_2)^2 + \\ & + \frac{9}{20}(1 + C_n)u(j)u(i_1) + \frac{3}{20}(1 + C_n)u(j)u(i_2) + \\ & + \frac{63}{1280}(1 + B_n)u(i_1)^2 + \frac{210}{1280}(1 + B_n)u(i_1)u(i_2) + \frac{63}{1280}(1 + B_n)u(i_2)^2 \\ & + \frac{91}{1280}(1 + A_n)u(i_1)^2 + \frac{546}{1280}(1 + A_n)u(i_1)u(i_2) + \frac{819}{1280}(1 + A_n)u(i_2)^2 \\ & + \frac{3}{20}(1 + C_n)u(j)u(i_1) + \frac{9}{20}(1 + C_n)u(j)u(i_2) \\ & - \frac{3}{2}u(i_1)^2 - u(i_1)u(i_2) - \frac{3}{2}u(i_2)^2 = \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{5}(1 + C_n)[(u(j) - u(i_1))u(i_1) + (u(j) - u(i_2))u(i_2)] - \\
&\quad - \frac{116}{1280}u(i_1)^2 + \frac{332}{1280}u(i_1)u(i_2) - \frac{116}{1280}u(i_2)^2 + \\
&\quad + \frac{910}{1280}A_nu(i_1)^2 + \frac{1092}{1280}A_nu(i_1)u(i_2) + \frac{910}{1280}A_nu(i_2)^2 \\
&\quad + \frac{126}{1280}B_nu(i_1)^2 + \frac{420}{1280}B_nu(i_1)u(i_2) + \frac{126}{1280}B_nu(i_2)^2 \\
&\quad + \frac{3}{5}C_nu(i_1)^2 + \frac{3}{5}C_nu(i_2)^2 \\
&= \frac{3}{5}(1 + C_n)[(u(j) - u(i_1))u(i_1) + (u(j) - u(i_2))u(i_2)] - \\
&\quad - \frac{116 + 546A_n + 210B_n}{1280}[u(i_1) - u(i_2)]^2
\end{aligned}$$

where the last step uses that $\frac{91}{160}A_n + \frac{21}{160}B_n + \frac{3}{10}C_n = 0$. Summing over all triangles of the form shown in Figure III.4, we get

$$\begin{aligned}
&\sum_{i \in AB^{(n+1)}} \sum_{j \in N^{(n+1)}} [u(j) - u(i)]u(i) \leq \\
&\leq \frac{3}{5}(1 + C_n) \sum_{i \in AB^{(n)}} \sum_{j \in N^{(n)}} [u(j) - u(i)]u(i) \\
&\quad - k^2 \cdot 2^{-n} \frac{116 + 546A_n + 210B_n}{1280},
\end{aligned}$$

where k is the slope of u along the bottom edge of the gasket. Since A_n, B_n and C_n tend to zero at a geometric rate, there must be a real constant C such that

$$\begin{aligned}
&| \sum_{i \in AB^{(n)}} \sum_{j \in N^{(n)}} [u(j) - u(i)]u(i) | \leq \\
&\leq C \left(\frac{3}{5}\right)^n | \sum_{i \in AB^{(0)}} \sum_{j \in N^{(0)}} [u(j) - u(i)]u(i) | \\
&= C \left(\frac{3}{5}\right)^n |(c - a)a + (c - b)b|,
\end{aligned}$$

and the proposition is proved. □

IV. The exterior process.

Let us now describe a process

$$Y_N : \Omega \times T \rightarrow \bar{S}_N^e$$

which approximates reflected Brownian motion. Since the time increment $\Delta t = 5^{-N}$ is smaller than usual, we have to be a little careful with the transition probabilities, but if we say that a particle which is at x at time t , will be at each one of x 's neighbors with probability

$$(4.1) \quad \frac{1}{3} \left(\frac{4}{5}\right)^N$$

at $t + \Delta t$, and stay at x with the remaining probability, it is easy to check that we get the right variance. Hence the standard part of Y_N is reflected Brownian motion in the area between the two triangles.

As with the interior process, we shall need to know a little about the escape probabilities from the boundary. Given a positive integer k , let Δ_k be the collection of all points $x \in S_N^e$ whose graph distance to the inner boundary S_N^b is k (i.e. the shortest path from x to S_N^b has exactly k steps.) Figure IV.1 shows Δ_3 . Note that the (euclidean) distance from a point in Δ_k to S_N^b is of order of magnitude $k \cdot 2^{-N}$.

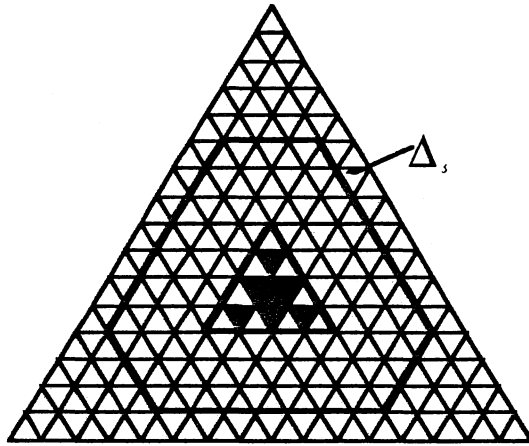


Figure IV.1

Define $u_k(x)$ to be the probability that Y_N starting at x will hit Δ_k before it hits S_N^b .

IV.1 Lemma. Assume that $x \in \Delta_l$ for $l \leq k$. Then

$$u_k(x) \geq \frac{l}{k}.$$

Proof: Define a function v by

$$v(x) = \frac{m}{k} \text{ if } x \in \Delta_m.$$

It is easy to check that v is subharmonic with respect to Y_N ; in fact, v is harmonic at $x \in \Delta_m$ unless x is a corner in Δ_m .

Since u_k is harmonic, $v - u_k$ is a subharmonic function which is zero on Δ_k , and hence $v - u_k \leq 0$ inside Δ_k .

□

V. The boundary process.

Fix a finite, internal function

$$\alpha : \bar{S}_N^i \rightarrow {}^*\mathbf{R}_+$$

which is harmonic in S_N^i (with respect to X_N), and a finite, internal function

$$\beta : \bar{S}_N^e \rightarrow {}^*\mathbf{R}_+$$

which is harmonic in S_N^e (with respect to Y_N). We want to construct a process Z on S_N which behaves like X_N at interior points, like Y_N at exterior points, and whose equilibrium measure is given by α and β . More precisely, since there are order of magnitude 3^N points in S_N^i and 4^N points in S_N^e , we shall let the equilibrium measure m have the form

$$(5.1) \quad m(x) = \begin{cases} \frac{\alpha(x)}{3^N} & \text{if } x \in S_N^i \\ \frac{\beta(x)}{4^N} & \text{if } x \in S_N^e \end{cases}$$

(what m looks like on the boundary S_N^b will be determined later). Note that since α and β are harmonic functions, there will never be any problem in maintaining the equilibrium at points in the interior of S_N^i and S_N^e .

To define the transition probabilities of our process on the boundary, let us first assume that $x \in S_N^b$ and $y \in S_N^i$ are neighbors. If we were just dealing with the interior process X_N and the distribution $\frac{\alpha(x)}{3^N}$, the mass passing from x to y would be

$$\frac{1}{4} \frac{\alpha(x)}{3^N}.$$

To keep the equilibrium, we want the same mass transfer in the present setting, and hence the transition probability p_{xy} must satisfy

$$p_{xy}m_x = \frac{1}{4} \frac{\alpha(x)}{3^N};$$

i.e.

$$(5.2) \quad p_{xy} = \frac{1}{4} \frac{\alpha(x)}{3^N m(x)} \text{ when } x \in S_N^b, y \in S_N^i.$$

Similarly, if $y \in S_N^e$ is an exterior neighbor of $x \in S_N^b$, the mass transfer from x to y is

$$\frac{1}{3} \left(\frac{4}{5}\right)^N \cdot \frac{\beta(x)}{4^N}$$

according to formula (4.1), and hence

$$p_{xy}m_x = \frac{1}{3} \left(\frac{4}{5}\right)^N \frac{\beta(x)}{4^N},$$

i.e.

$$(5.3) \quad p_{xy} = \frac{1}{3} \frac{\beta(x)}{5^N m(x)} \text{ when } x \in S_N^b, y \in S_N^e.$$

It still remains to describe the transition probabilities between two neighboring points x and y on the boundary S_N^b , and here we need to be quite careful if m is going to be an equilibrium measure. To see the problem, assume that we start our process Z with initial distribution m and run it one step from 0 to Δt . Since α and β are harmonic functions, we necessarily have

$$(5.4) \quad P\{Z(\Delta t) = x\} = m(x)$$

for all $x \in S_N^i \cup S_N^e$, but there is no reason why this formula should hold on the boundary; all we can say is that since the mass of all non-boundary sites are preserved, the *total* mass on the boundary must also be preserved. The question is whether we can use the transition probabilities along the boundary to shift the mass around in such a way that (5.4) also holds on the boundary. Of course, we have to do this in a reasonably controlled manner if the resulting process is going to make sense. The following elementary lemma will help us to get started:

V.1 Lemma. Let L be a positive integer and assume that $f : \mathbb{N} \rightarrow \mathbb{R}$ is periodic with period L (i.e. $f(k) = f(k + L)$ for all k). Then the inhomogeneous difference equation

$$(5.5) \quad x_{n+1} - 2x_n + x_{n-1} = f(n) \quad n \in \mathbb{N}$$

has a solution with period L if and only if $\sum_{i=1}^L f(i) = 0$. When such a solution exists, we can choose it such that

$$(5.6) \quad 0 \leq x_n \leq 4 \max_{m \leq L} \left| \sum_{k=1}^{m-1} \sum_{i=1}^k f(i) \right|$$

for all n .

Proof. The general solution of (5.5) is

$$x_n = \begin{cases} D & \text{if } n = 0 \\ C + D & \text{if } n = 1 \\ \sum_{k=1}^{n-1} \sum_{i=1}^k f(i) + Cn + D & \text{if } n > 1 \end{cases}$$

where C and D are arbitrary constants. Note that $\{x_n\}$ is periodic if and only if the equations $x_0 = x_L$ and $x_1 = x_{L+1}$ are both satisfied, i.e.

$$D = \sum_{k=1}^{L-1} \sum_{i=1}^k f(i) + CL + D$$

and

$$C + D = \sum_{k=1}^L \sum_{i=1}^k f(i) + C(L+1) + D.$$

From the first equation we get

$$C = \frac{1}{L} \sum_{k=1}^{L-1} \sum_{i=1}^k f(i)$$

and from the second

$$C = \frac{1}{L} \sum_{k=1}^L \sum_{i=1}^k f(i),$$

and these expressions are compatible if and only if $\sum_{i=1}^L f(i) = 0$.

To prove (5.6), first observe that for $n \leq L$

$$\begin{aligned} |x_n - D| &\leq \left| \sum_{k=1}^{n-1} \sum_{i=1}^k f(i) \right| + \left| \sum_{k=1}^{n-1} \sum_{i=1}^k f(i) \right| \frac{n}{L} \\ &\leq 2 \left| \sum_{k=1}^{n-1} \sum_{i=1}^k f(i) \right|. \end{aligned}$$

Hence if we choose $D = 2 \max_{m \leq L} \left| \sum_{k=1}^{m-1} \sum_{i=1}^k f(i) \right|$, formula (5.6) holds.

□

To apply the lemma to our problem, fix a site $x \in S_N^b$. At each moment, the total mass leaving x for a neighboring point not on the boundary, is

$$\sum_{y \in S_N^i} \frac{1}{4} \frac{\alpha(x)}{3^N} + \sum_{y \in S_N^e} \frac{1}{3} \frac{\beta(x)}{5^N}$$

where the summation is only over neighbors. Similarly, the total mass arriving at x from neighbors not on the boundary, is

$$\sum_{y \in S_N^i} \frac{1}{4} \frac{\alpha(y)}{3^N} + \sum_{y \in S_N^e} \frac{1}{3} \frac{\beta(y)}{5^N}.$$

Hence the total gain of mass at x due to interaction with neighbors not on the boundary, is

$$(5.7) \quad g(x) = \sum_{y \in S_N^i} \frac{1}{4} \frac{\alpha(y) - \alpha(x)}{3^N} + \sum_{y \in S_N^e} \frac{1}{3} \frac{\beta(y) - \beta(x)}{5^N}.$$

As we have already observed, the total gain of mass on the boundary is zero, i.e.

$$(5.8) \quad \sum_{x \in S_N^b} g(x) = 0.$$

Let us try to compensate for the gain of mass received from neighbors *not* on the boundary by giving away mass to neighbors *on* the boundary. Assume that site x gives away mass η_x to each of its neighbors x' and x'' on the boundary, and in return receives $\eta_{x'}$ and $\eta_{x''}$ from each of them. Then if the mass at x is going to remain the same, we need to have

$$(5.9) \quad \eta_{x''} - 2\eta_x + \eta_{x'} = -g(x).$$

This is exactly the kind of difference equation we solved in the lemma, and due to (5.8) we know that it has a periodic solution.

It should now be clear how to finish the construction of our process; we simply have to choose the equilibrium measure m and the transition probabilities between neighboring boundary sites such that at each instant a mass η_x passes from x to each one of its neighbors. For this to make sense we have to choose our solution η of (5.9) to be positive, and $m(x)$ has to be at least as large as $2\eta(x)$. We could be in trouble here; perhaps these conditions force us to assign noninfinitesimal (or - even worse - infinite) measure to the boundary S_N^b . Fortunately, this is not the case - at least not under the following, mild continuity conditions:

V.2 Condition. (a) β is Lipschitz continuous; i.e. there is a real constant C such that $|\beta(x) - \beta(y)| \leq C|x - y|$ for all $x, y \in \bar{S}_N^e$.

(b) There exist positive, real constants K and ϵ such that

$$(5.10) \quad |\alpha(x) - \alpha(y)| \leq K \frac{|x - y|^{\log(5/3)/\log 2}}{(\log |\frac{1}{x-y}|)^{1+\epsilon}}$$

for all boundary sites $x, y \in S_N^b$.

Remark: Barlow and Perkins [6, Theorem 5.22] have shown that functions in the domain of the Laplacian on the Sierpinski gasket typically are Hölder continuous with exponent $\log(\frac{5}{3})/\log 2$. I do not know if they satisfy the slightly stronger condition (5.10), and for that reason I have decided only to impose (5.10) on the boundary, although it is going to cost us some extra work. As we shall see, the logarithmic correction is needed in the proof of Proposition V.3.

Here is the result we are working towards:

V.3 Proposition. Assume that α and β satisfy condition V.2. Then (5.9) has a positive and periodic solution η such that

$$(5.11) \quad \max_{x \in S_N^b} \eta(x) \leq K \left(\frac{2}{5}\right)^N$$

for some $K \in \mathbb{R}$.

Remark: That $\eta(x)$ is of order of magnitude $(\frac{2}{5})^N$ means that we can choose $m(x)$ to be of the same order for $x \in S_N^b$. The total mass of the boundary S_N^b is then of order $2^N(\frac{2}{5})^N$, hence infinitesimal.

The first step on the way to Proposition V.3 will be to show that the condition (5.10) on α 's behavior on the boundary, can be used to obtain some information about α 's behavior near the boundary. Figure V.1 shows a piece of S_N 's boundary of length 2^{-n} . Let B be the set of sites on the line segment between x_L and x_R . We want to estimate $|\alpha(x) - \alpha(x_T)|$ where $x \in B$ and x_T is the top vertex of the triangle.

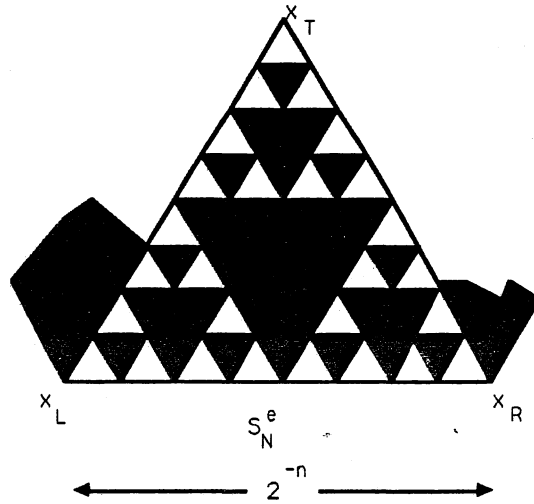


Figure V.1

V.3 Lemma. If α satisfies (5.10) there is a real constant C (independent of n) such that

$$(5.12) \quad |\alpha(x) - \alpha(x_T)| \leq Cn^{-(1+\epsilon)}\left(\frac{3}{5}\right)^n$$

for all $x \in B$.

Proof: Start the process in x_T and let it run until it hits S_N^b . If $y \in S_N^b$, let π_y be the probability that the process is stopped at y . Since α is harmonic, $\alpha(x_T) = \sum \alpha(y)\pi_y$, and hence

$$\alpha(x) - \alpha(x_T) = \sum_{y \in S_N^b} (\alpha(x) - \alpha(y))\pi_y.$$

Let us try to estimate π_y for a point y a given distance away from x . Figure V.2 shows a bigger piece of the fractal where the triangle $x_Lx_Rx_T$ in Figure V.1 now is the small triangle in the lower left corner.

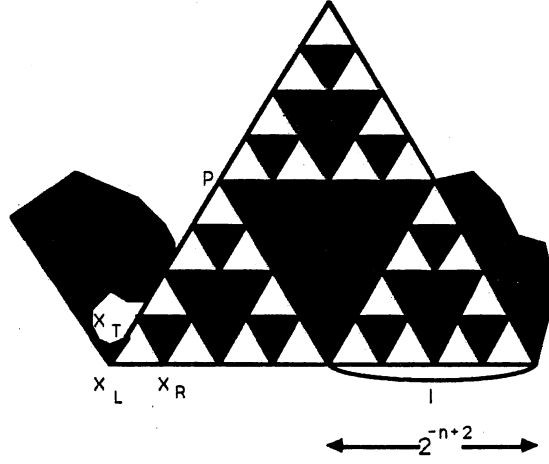


Figure V.2

Assume that y belongs to the interval I a distance no more than 2^{-n+3} away. A path stopping at y has to pass P before it hits the boundary, and according to Lemma III.2 the probability for this is bounded by a constant times $(\frac{3}{10})^2$. Similarly, the probability of stopping in an interval of length 2^{-n+k-1} no more than a distance 2^{-n+k} away from x , is bounded by the same constant times $(\frac{3}{10})^{k-1}$. (I'm neglecting the possibility that the path may stop in an interval on one of the other sides of the gasket, but this case can be treated in an analogous way). Hence

$$\begin{aligned}
& \sum_{y \in S_N^b} |\alpha(x) - \alpha(y)| \pi_y \leq \\
& \sum_{k=0}^{n-1} C_1 \frac{|2^{-n+k} \log(\frac{5}{3}) / \log 2}{|\log 2^{n-k}|^{1+\epsilon}} \left(\frac{3}{10}\right)^{k-1} \leq \\
& \leq n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^n \sum_{k=0}^{n-1} C_2 \frac{2^{-k}}{\left(1 - \frac{k}{n}\right)^{1+\epsilon}} \leq \\
& \leq n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^n \sum_{k=0}^{\infty} C_2 \frac{2^{-k}}{\left(1 - \frac{k}{k+1}\right)^{1+\epsilon}} \leq \\
& \leq n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^n \sum_{k=0}^{\infty} C_2 (k+1)^{1+\epsilon} 2^{-k} \leq C_3 n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^n
\end{aligned}$$

□

For the next lemma let us return for a moment to Figure V.1. Recall that the set of all sites on the boundary between x_L and x_R is called B . The set of all interior neighbors of B is called N (we only include those neighbors which belong to the triangle $x_L x_R x_T$). We want to estimate

$$\rho_n = \frac{\alpha(x_L) + \alpha(x_R)}{2} + \sum_{x \in B_0} \alpha(x) - \sum_{y \in N} \alpha(y),$$

where $B_0 = B \setminus \{x_L, x_R\}$ (we treat the endpoints x_L and x_R separately as they have only one neighbor each in N).

V.4 Lemma. Assume that α satisfies (5.10). There is a real number C (independent of n) such that

$$(5.13) \quad |\rho_n| \leq C n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^N$$

Proof: Pick a point $y \in N$, start the process in y and stop it when it hits either B or x_T (recall that x_T is the top vertex in Figure V.1). If $x \in B \cup \{x_T\}$, let π_{yx} be the probability that the process is stopped at x . Since α is harmonic

$$(5.14) \quad \alpha(y) = \sum_{x \in B} \alpha(x) \pi_{yx} + \alpha(x_T) \pi_{yx_T}.$$

The probability π_{yx_T} of hitting x_T first is easily seen to be independent of y , and we shall just denote it by π_T . According to Proposition III.3,

$$(5.15) \quad \pi_T = q_{N-n} \cdot q_{N-n-1} \cdots q_1 \leq C_1 \left(\frac{3}{10}\right)^{N-n}.$$

If we sum (5.14) over all $y \in N$, we get

$$(5.16) \quad \sum_{y \in N} \alpha(y) = \sum_{x \in B} \alpha(x) \sum_{y \in N} \pi_{yx} + 2^{N-n} \pi_T \alpha(x_T).$$

Using the symmetry of the situation, it is easy to check that $\sum_{y \in N} \pi_{yx}$ must be the same for all $x \in B_0$, and that

$$\sum_{y \in N} \pi_{yx_L} = \sum_{y \in N} \pi_{yx_R} = \frac{1}{2} \sum_{y \in N} \pi_{yx}$$

for $x \in B_0$. Since

$$\sum_{x \in B} \sum_{y \in N} \pi_{yx} + 2^{N-n} \pi_T = 2^{N-n},$$

we get

$$\sum_{y \in N} \pi_{yx} = 1 - \pi_T \text{ for } x \in B.$$

and

$$\sum_{y \in N} \pi_{yx_L} = \sum_{y \in N} \pi_{yx_R} = \frac{1 - \pi_T}{2}$$

Substituting this into (5.16), we get

$$\sum_{y \in N} \alpha(y) = (1 - \pi_T) \left(\sum_{x \in B_0} \alpha(x) + \frac{\alpha(x_L) + \alpha(x_R)}{2} \right) + 2^{N-n} \pi_T \alpha(x_T),$$

and hence

$$\rho_n = \pi_T \left(\sum_{x \in B_0} [\alpha(x) - \alpha(x_T)] + \frac{\alpha(x_L) - \alpha(x_T)}{2} + \frac{\alpha(x_R) - \alpha(x_T)}{2} \right)$$

According to (5.15), $\pi_T \leq C_1 \left(\frac{3}{10}\right)^{N-n}$, and according to the preceding lemma

$$|\alpha(x) - \alpha(x_T)| \leq C_2 n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^n$$

for all $x \in B$. Hence

$$|\rho_n| \leq C_1 \left(\frac{3}{10}\right)^{N-n} 2^{N-n} \cdot C_2 n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^n \leq C_3 n^{-(1+\epsilon)} \left(\frac{3}{5}\right)^N \quad \square$$

The last lemma we shall need is a slight variation on the previous one. The situation in Figure V.3 is exactly like the one in Figure V.1 except that the triangle depicted is now sitting in one of the corners of the gasket, and hence two of its edges now belong to the boundary S_N^b . Let B be the piece of the boundary between x_L and x_R , and let N be the sites in the triangle $x_L x_C x_R$ bordering on B .

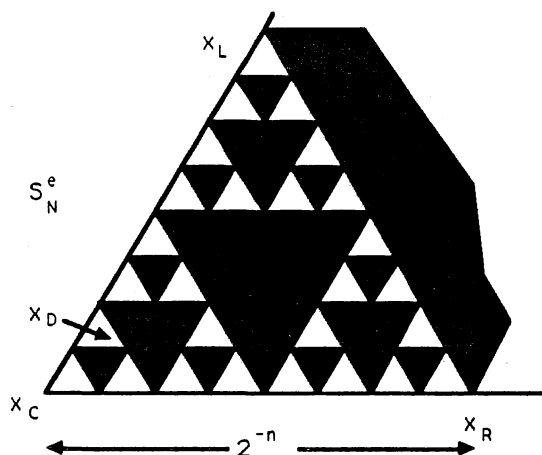


Figure V.3

Define weight functions $\mu : B \rightarrow \mathbb{N}$ and $\nu : N \rightarrow \mathbb{N}$ by

$$\mu(x) = \begin{cases} 1/2 & \text{if } x = x_L \text{ or } x = x_R \text{ or } x \text{ borders on } x_C \\ 0 & \text{if } x = x_C \\ 1 & \text{otherwise} \end{cases}$$

$$\nu(x) = \begin{cases} 2 & \text{if } x \text{ is the corner } x_D \\ 1 & \text{otherwise} \end{cases}$$

Note that

$$\sum_{x \in B} \mu(x) = \sum_{x \in N} \nu(x) = 2^{N-n+1} - 2$$

V.5 Lemma $\sum_{x \in B} \alpha(x) \mu(x) = \sum_{y \in N} \alpha(y) \nu(y)$.

Proof: Pick a point $y \in N$, start the process at y and let it run until it hits B . Let π_{yx} be the probability that the process is stopped at x . Since α is harmonic,

$$\alpha(y) = \sum_{x \in B} \alpha(x) \pi_{yx},$$

and summing over y , we get

$$(5.17) \quad \sum_{y \in N} \alpha(y) \nu(y) = \sum_{x \in B} \alpha(x) \sum_{y \in N} \pi_{yx} \nu(y)$$

Exploiting the geometric structure of the fractal it is not hard to see that the total hitting distribution $x \rightarrow \sum_{y \in N} \pi_{yx} \nu(y)$ must be proportional to μ , and since μ and ν have the same total mass, this means that

$$\mu(x) = \sum_{y \in N} \pi_{yx} \nu(y),$$

and hence the lemma follows from (5.17). □

We now have all the information we shall need:

Proof of Proposition V.3: Enumerate the sites on the boundary $x_1, x_2, x_3, \dots, x_{3 \cdot 2^N}$ counter-clockwise starting at the midpoint of one of the sides (see Figure V.4). According to Lemma V.1 and formula (5.7), we have to show that

$$\sum_{k=1}^{n-1} \sum_{i=1}^k \sum_{y \in N(x_i)} \left[\frac{1}{4} \frac{\alpha(y) - \alpha(x_i)}{3^N} + \frac{1}{3} \frac{\beta(y) - \beta(x_i)}{5^N} \right]$$

is bounded by a constant times $(\frac{2}{5})^N$. (In this formula $N(x_i)$ is the set of interior and exterior neighbors of x_i). The β -part is trivial; since β is Lipschitz-continuous

$$\begin{aligned} & \left| \sum_{k=1}^{n-1} \sum_{i=1}^k \sum_{y \in N(x_i)} \frac{\beta(y) - \beta(x_i)}{5^N} \right| \leq \\ & \leq \sum_{k=1}^{3 \cdot 2^N} \sum_{i=1}^{3 \cdot 2^N} 4 \cdot C \frac{2^{-N}}{5^N} \leq C' \left(\frac{2}{5}\right)^N. \end{aligned}$$

The α -part is much more subtle and we shall have to use our lemmas. We first note that in order to estimate the sum

$$\sum_{i=1}^k \sum_{y \in N(x_i)} (\alpha(y) - \alpha(x_i)),$$

it is convenient to break the summation over i into parts, each part corresponding to a triangle of the kind shown in figures V.1 and V.3. We want to choose each of these triangles as large as possible. Figure V.4 shows the idea; in order to sum from x_0 to x_k , we first sum over Σ_1 , then over Σ_2 , and so on.

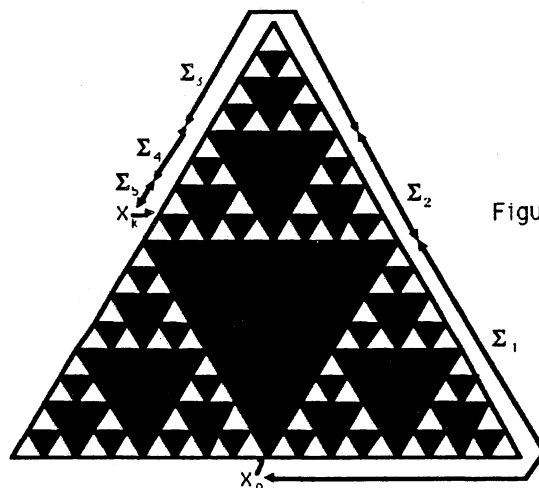


Figure V.4

The sum over each Σ_i can be estimated using lemmas V.4 and V.5; a “round the corner” summation like Σ_1 and Σ_3 on the figure will be free, while a “straight” summation like Σ_2 , Σ_4 and Σ_5 will cost us $Cn^{-(1+\epsilon)}(\frac{3}{5})^N$, where 2^{-n} is the length of the segment we are summing over. Since we shall only have to do that at most one “straight” summation of a given length, the total cost is bounded by

$$\sum_{n=1}^N Cn^{-(1+\epsilon)}(\frac{3}{5})^N \leq C'(\frac{3}{5})^N$$

since the series $\sum_{n=1}^{\infty} n^{-(1+\epsilon)}$ converges.

The rest is easy; we simply note that

$$\sum_{k=1}^{n-1} \sum_{i=1}^k \sum_{y \in N(x_i)} \frac{\alpha(y) - \alpha(x_i)}{3^N} \leq 3 \cdot 2^N \frac{C'(\frac{3}{5})^N}{3^N} = 3C'(\frac{2}{5})^N$$

and Proposition V.3 is proved. □

It may be useful to sum up our findings in a theorem. To make the notation a little more compact, let me write $m(x) = O((\frac{2}{5})^N)$ to say that there is a finite constant C (independent of N) such that $m(x) \leq C(\frac{2}{5})^N$ for all N .

V.6 Theorem: Let α and β be two finite, internal functions which are harmonic in S_N^i and S_N^e , respectively, and which satisfy Condition V.2. Then there is a nearest neighbor random walk Z on S_N whose invariant measure $m(x)$ and transition probabilities p_{xy} satisfy the following requirements:

- (a) $m(x) = \frac{\alpha(x)}{3^N}$ when $x \in S_N^i$
- (b) $m(x) = \frac{\beta(x)}{4^N}$ when $x \in S_N^e$
- (c) $m(x) = O(\frac{2}{5})^N$ when $x \in S_N^b$

In the following we always assume that x and y are neighbors:

- (d) $p_{xy} = \frac{1}{4}$ when $x \in S_N^i$
- (e) $p_{xy} = \frac{1}{3}(\frac{4}{5})^N$ when $x \in S_N^e$
- (f) $p_{xy} = O(\frac{5}{6})^N$ when $x \in S_N^b, y \in S_N^i$
- (g) $p_{xy} = O(\frac{1}{2})^N$ when $x \in S_N^b, y \in S_N^e$
- (h) $p_{xy} = O(1)$ when $x \in S_N^b, y \in S_N^b$

Proof: The only parts we haven't already checked are (f),(g) and (h). To prove (f), just observe that since we need to have

$$m(x)p_{xy} = \frac{1}{4} \frac{\alpha(x)}{3^N},$$

and $m(x) = O(\frac{2}{5})^N$ by (c), we must have $p_{xy} = O(\frac{5}{6})^N$. The same argument works for (g); since

$$m(x)p_{xy} = \frac{1}{3}(\frac{4}{5})^N \frac{\beta(x)}{4^N}$$

and $m(x) = O(\frac{2}{5})^N$, we get $p_{xy} = O(\frac{1}{2})^N$. Finally, to prove (h) just observe that

$$m(x)p_{xy} = \eta(x),$$

and that according to Proposition V.3, $\eta(x) = O(\frac{2}{5})^N$. Since $m(x)$ is also of order $(\frac{2}{5})^N$, we get $p_{xy} = O(1)$. □

As you will have observed, there are more than one process Z satisfying the theorem above; if I have one such process, I can always get another one by multiplying the values of m on S_N^b by a constant factor larger than one, and then readjusting the transition probabilities to maintain the equilibrium. These different nonstandard processes will probably be indistinguishable from a standard point of view (in standard terms this corresponds to slightly different random walks converging to the same limit process), and I shall just work with any one of them. Unless otherwise specified, I shall always assume that the process is started with the equilibrium distribution m , but normalized so that I have a probability measure.

VI. Continuity

So far our process Z (satisfying Theorem V.6) exists only as a nonstandard object, and to show that it induces a reasonable standard process, we must prove that it is continuous in the following nonstandard sense: A nonstandard process $Z : \Omega \times T \rightarrow \mathbb{R}^d$ is S -continuous if there is a set $\Omega_0 \subset \Omega$ of measure zero (w.r.t. to the natural Loeb measure on Ω) such that $Z(\omega, t) \approx Z(\omega, s)$ whenever $\omega \notin \Omega_0$ and s and t are two infinitely close, finite elements in T . Throughout this section I shall assume that Condition V.2 is satisfied.

We shall need a little more terminology from the nonstandard theory of processes. Assume that $X : \Omega \times T \rightarrow \mathbb{R}$ is an internal process adapted to an internal filtration $\{\mathcal{F}_t\}_{t \in T}$. I shall write $\Delta X(t)$ for the increment $X(t + \Delta t) - X(t)$, and I shall use the summation convention

$$\sum_{s=r}^t X(s) = X(r) + X(r + \Delta t) + \cdots + X(t - \Delta t);$$

note that $X(t)$ is not included in the sum. Define

$$[X](t) = \sum_{s=0}^t (\Delta X(s))^2$$

$$\langle X(t) \rangle = \sum_{s=0}^t E(\Delta X(s)^2 | \mathcal{F}_s)$$

Our main tool will be the following theorem (Hoover and Perkins [10, Theorem 8.5]):

VI.1 Theorem. Let X be a locally square S -integrable \mathcal{F}_t -martingale.

- (a) X is S -continuous iff and only if $[X]$ is.
- (b) If all X 's increments $\Delta X(\omega, t)$ are infinitesimal, then X is S -continuous if and only if $\langle X \rangle$ is.

(I have adapted - and weakened - the theorem for our purposes; see Hoover's and Perkins' paper for the full story).

Returning to our process Z , we first split it into three parts

$$Z^e(t) = \sum_{s=0}^t 1_{S_N^e}(Z(s)) \Delta Z(s)$$

$$Z^b(t) = \sum_{s=0}^t 1_{S_N^b}(Z(s)) \Delta Z(s)$$

$$Z^i(t) = \sum_{s=0}^t 1_{S_N^i}(Z(s)) \Delta Z(s).$$

To show that the exterior process Z^e is continuous is straightforward.

VI.2 Lemma. Z^e is S -continuous.

Proof: Apply Theorem VI.1(b) to each component, and check that nothing wrong happens at the boundary. □

The boundary process Z^b is more subtle and we shall need all our estimates from previous sections.

VI.3 Lemma. Z^b is S -continuous.

Proof: Let us first replace Z^b by a one-dimensional martingale M . If Pr is the projection onto the triangle determined by S_N^b , let

$$\Delta M(\omega, t) = \epsilon |Pr(\Delta Z^b(\omega, t))|$$

where ϵ is $+1$ if $\Delta Z^b(\omega, t)$ is a step in the anti-clockwise direction, and -1 otherwise. Clearly, Z^b must be continuous if M is.

To show that M is a square integrable martingale, it suffices to show that $E([M](t))$ is finite. According to Theorem V.6 there will at any time be $O(\frac{4}{5})^N$ particles on the boundary and with probability $O(1)$ they will each take a step of size 2^{-N} . Hence

$$E(\Delta M(t)^2) = O((2^{-N})^2) \cdot O(\frac{4}{5})^N = O(5^{-N}),$$

and since $5^{-N} = \Delta t$, this means that there is a finite C such that

$$E[M(t)] \leq Ct.$$

Hence M is square integrable (note that the calculations above would break down if our estimate for $m(x)$, $x \in S_N^b$, were any bigger than $O(\frac{2}{5})^N$; this is one of the reasons why we had to be so careful in the previous section).

S -integrability ensures us that M is reasonably well behaved (see Chapter IV of [1] for details), but it doesn't give us continuity. However, by Theorem VI.1 it is enough to show that $[M]$ is S -continuous. Assume that it is not, then it is easy to check that there must be a real, positive number ϵ and an internal set $\Omega' \subset \Omega$ with noninfinitesimal measure α such that for each $\omega \in \Omega'$ there exist finite $s, t \in T$, $s \approx t$, such that

$$(6.1) \quad [M](\omega, t) - [M](\omega, s) > \epsilon.$$

Since $|\Delta M| \leq 2^{-N}$, this means that Z must have made at least $\epsilon \cdot 4^N$ visits to S_N^b between s and t . Since s and t are infinitely close, M has not had time to make noninfinitesimal excursions into S_N^e or S_N^i in the mean time. So what we have to calculate is the probability that Z will make $\epsilon \cdot 4^N$ consecutive visits to S_N^b without starting a noninfinitesimal excursion.

As a preliminary step, let us assume that $Z(t) = x \in S_N^b$, and let A, B , and C be the vertices of the miniature Sierpinski gasket of size 2^{-n} ($n \in \mathbb{N}$) that x belongs to (see Figure VI.1).

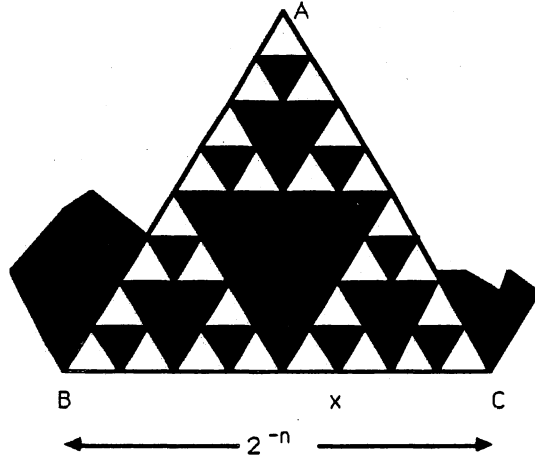


Figure VI.1

What is the probability that Z will immediately leave S_N^b and hit A before it hits S_N^b ? According to Theorem V.1, the probability of going from x to one of its interior neighbors is $O(\frac{5}{6})^N$, and once that neighbor is reached, the probability of hitting A before returning to BC is $O(\frac{3}{10})^{N-n}$ (Proposition III.3). Hence the total probability of starting an "excursion of size 2^{-n} " in the interior, is

$$O(\frac{5}{6})^N O(\frac{3}{10})^{N-n} = O[(\frac{10}{3})^n 4^{-N}].$$

We have bounds below as well as above, and hence there are constants C and K such that

$$C(\frac{10}{3})^n \cdot 4^{-N} \leq p(x, 2^{-n}) \leq K(\frac{10}{3})^n 4^{-N},$$

where $p(x, 2^{-n})$ is the probability that a particle at x will immediately start an interior excursion of size 2^{-n} . (A simpler, but similar calculation using Lemma IV.1 shows that the probability of starting an exterior excursion of size l is $O(l^{-1} 4^{-N})$; we shall not need this, but it is reassuring to see that the two probabilities are of the same order).

We are ready for the main argument. Let M be a (large) real number, and let $t_1, t_2, t_3, t_4, \dots, t_{M \cdot 4^N}$ be the $M \cdot 4^N$ first times Z visits S_N^b . Partition $\{t_1, t_2, t_3, \dots\}$ into intervals I_1, I_2, \dots of size $\frac{\epsilon}{2} \cdot 4^N$;

$$I_1 = \{t_1, t_2, \dots, t_{\frac{\epsilon}{2} \cdot 4^N}\}, I_2 = \{t_{\frac{\epsilon}{2} \cdot 4^N + 1}, \dots, t_{2 \cdot \frac{\epsilon}{2} \cdot 4^N}\}, \dots$$

(ϵ is as in formula (6.1); we can clearly choose it such that $\frac{\epsilon}{2} \cdot 4^N$ is an integer). Note that if there is a sequence of consecutive t_i 's of length $\epsilon \cdot 4^N$ where no excursion is started, then there must be an interval I_j where no excursion is started. The probability that no excursion of size 2^{-n} is started in a given interval I_j is clearly

$$(1 - O(\frac{10}{3})^n 4^{-N})^{\frac{\epsilon}{2} \cdot 4^N} \approx e^{-\frac{\epsilon}{2} O(\frac{10}{3})^n}.$$

Hence the probability that all the intervals I_1, I_2, \dots contain an excursion of size 2^{-n} is

$$(1 - e^{-\frac{\epsilon}{2} O(\frac{10}{3})^n})^{2M/\epsilon},$$

which we can get as close to 1 as we want by choosing n large enough.

This would close the argument if we only knew that the typical particle visits the boundary S_N^b at most $O(4^N)$ times in any finite time interval. Again we get the information we need from Theorem V.6; since $O(\frac{4}{5})^N$ of the particles are at S_N^b at any given time and there are $O(5^N)$ points in a finite time interval, only a negligible number of particles can make more than $O(4^N)$ visits to S_N^b in finite time. □

VI.4 Lemma. Z^i is S -continuous.

Proof: Since Brownian motion on the Sierpinski gasket is continuous (more precisely: since the nonstandard random walk inducing Brownian motion is S -continuous), Z^i can only fail to be S -continuous when it is infinitely close to the boundary. Indeed, it is easy to see that if Z^i is not S -continuous then there must be a set $\Omega_1 \subset \Omega$ of noninfinitesimal measure and an infinite integer k such that the following happens: For each $\omega \in \Omega_1$, there is a sequence

$$t_1 < t_2 \leq t_3 < t_4 \leq t_5 < \dots < t_{2n}$$

of elements in T such that $Z(t_m) \in S_N^b$ for all m ; $Z(t) \in S_N^i$ and $d(Z(t), S_N^b) < \frac{\sqrt{3}}{2}2^{-k}$ if $t_{2m-1} < t < t_{2m}$ for some m ; $t_{2n} - t_1$ is infinitesimal but

$$\left| \sum_{m=1}^n (Z(t_{2m}) - Z(t_{2m-1})) \right|$$

is not infinitesimal. Figure VI.2 shows what is going on; in each interval (t_{2m-1}, t_{2m}) the process makes an excursion into one of the miniature gaskets D_1, D_2, D_3, \dots . The excursions add up to a noninfinitesimal passage in infinitesimal time. Note that we can clearly assume that all the excursions are along the same edge of the gasket.

Since Z is not even a semimartingale, it's difficult to estimate the excursions of Z directly. The trick we shall use is basically to estimate $u(Z)$ instead, where u is a suitable harmonic function, but we have to be a little careful how we set things up.

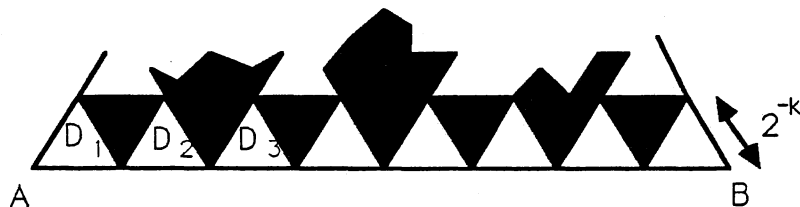


Figure VI.2

Let u be the linear function on the line segment AB which is zero in A and one in B . At the top vertex of one of the small gaskets D_i let u have the average of the two values at the other vertices (see Figure VI.3).

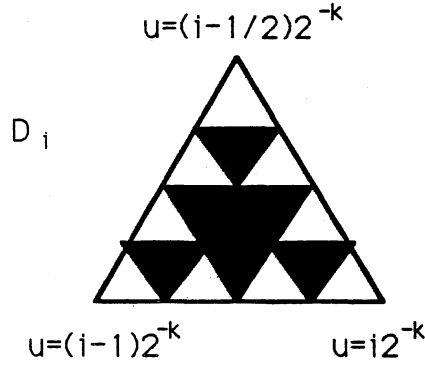


Figure VI.3

If D_i^0 is what remains of D_i when we remove the bottom edge and the top vertex, we now extend u to a harmonic function in D_i^0 . The process

$$U(t) = \sum_{s=0}^t 1_{\cup D_i^0}(Z_s)(u(Z(s+\Delta t)) - u(Z(s)))$$

is clearly a martingale since u is harmonic in $\cup D_i^0$, and by definition of u ,

$$\left| \sum_{m=1}^n (Z(t_{2m}) - Z(t_{2m-1})) \right| = \left| \sum_{m=1}^n (U(t_{2m}) - U(t_{2m-1})) \right| = |U(t_{2n}) - U(t_1)|.$$

Hence if we can show that u is S -continuous we shall have the contradiction we are working toward. We shall, in fact, prove much more; not surprisingly it turns out that U is almost constant.

To begin the calculations, note that

$$\begin{aligned} E([U](t)) &= \sum_{s=0}^t \sum_{x \in \cup D_i^0} \sum_y [u(y) - u(x)]^2 p_{xy} \frac{\alpha(x)}{3^N} \\ (6.2) \quad &\leq \|\alpha\|_\infty \sum_{s=0}^t \sum_{i=0}^{2^k-1} \sum_{x \in D_i^0} \sum_y [u(y) - u(x)]^2 p_{xy} 3^{-N} \\ &\leq \|\alpha\|_\infty \cdot t \cdot 5^N \sum_{i=0}^{2^k-1} \sum_{x \in D_i} \sum_y [u(y) - u(x)]^2 p_{xy} \frac{\alpha(x)}{3^N} \end{aligned}$$

The sum $5^N \sum_{x \in D_i} \sum_{y \in D_i} [u(y) - u(x)]^2 p_{xy} 3^{-N}$ is just the Beurling-Deny expression for the Dirichlet form of X_N restricted to D_i , and it can be rewritten as

$$\left(\frac{5}{3}\right)^N \sum_{x \in D_i} \sum_{y \in D_i} [u(y) - u(x)] u(x) p_{xy}.$$

Since u is harmonic in D_i^0 , all the contributions from elements $x \in D_i^0$ vanish. In fact, so does the contribution from the top vertex thanks to our choice of value for u . Hence the expression reduces to

$$\left(\frac{5}{3}\right)^N \sum_{x \in \underline{D}_i} \sum_y [u(y) - u(x)] u(x) p_{xy},$$

where \underline{D}_i is the bottom edge of D_i . According to Proposition III.7, this is less than

$$\begin{aligned} & \left(\frac{5}{3}\right)^N C \left(\frac{3}{5}\right)^{N-k} \left[\left(\left(i - \frac{1}{2}\right) 2^{-k} - (i-1) 2^{-k} \right) (i-1) 2^{-k} + \left(\left(i - \frac{1}{2}\right) 2^{-k} - i 2^{-k} \right) i 2^{-k} \right] \\ & = \hat{C} \left(\frac{5}{3}\right)^k 4^{-k}. \end{aligned}$$

Returning to (6.2), we see that

$$\begin{aligned} E([U](t)) & \leq \|\alpha\|_\infty t 2^k \hat{C} \left(\frac{5}{3}\right)^k \cdot 4^{-k} \\ & = \hat{C} \|\alpha\|_\infty t \left(\frac{5}{6}\right)^k \end{aligned}$$

which is infinitesimal. Hence $U(t)$ is infinitesimal almost everywhere, and the lemma follows. □

Combining the three lemmas above, we now have:

VI.5 Theorem. Z is S -continuous. □

Any nonstandard, S -continuous process induces a continuous, standard process called the standard part. If z is the standard part of Z , then z is clearly a continuous process which behaves like fractal Brownian motion inside the Sierpinski gasket and like ordinary Brownian motion outside. The equilibrium measure of z is given by the two harmonic functions α and β . More precisely, our results can be summed up as follows.

VI.6 Theorem. Let α and β be two real-valued functions defined on the outer boundary of the Sierpinski gasket. Assume that β is Lipschitz continuous and that there are constants $K, \epsilon \in \mathbb{R}_+$ such that

$$|\alpha(x) - \alpha(y)| \leq K \frac{|x - y|^{\log(5/3)/\log 2}}{(\log |\frac{1}{x-y}|)^{1+\epsilon}}$$

for all x, y on the boundary. Then there is a continuous process $z (= z^{(\alpha, \beta)})$ which behaves like fractal Brownian motion inside the gasket and like ordinary Brownian motion outside it, and whose equilibrium measure is $\bar{\alpha} d\mu + \bar{\beta} d\lambda$, where μ is the Hausdorff measure on the gasket, λ is the Lebesgue measure on the complement of the gasket, $\bar{\alpha}$ is the harmonic extension (w.r.t. to fractal Brownian motion) of α to the gasket, and $\bar{\beta}$ is the harmonic

extension of β outside the gasket. Particles starting outside the gasket will penetrate it and vice versa.

□

Remark: The last statement in the theorem is obviously true (since Z is a nonstandard Markov process), but quite clumsy. It would have been more satisfactory to say that z is a strong Markov process, but I haven't actually proved that, and the paper is already long enough!

□

Let me end with a few words about possible future developments. An alternative approach to the whole problem would be to use Dirichlet forms; this is probably difficult in the general case where α and β are not constant (because z is then not symmetric), but seems quite promising when α and β are constant - the problem is simply to make sense out of the formal expression

$$\alpha\xi_1(f, f) + \beta\xi_2(f, f)$$

where ξ_1 is the Dirichlet form of fractal Brownian motion on the Sierpinski gasket and ξ_2 is the Dirichlet form of reflected Brownian motion in the complement of the gasket. The main obstacle for this approach is probably that we today have very little positive information about the domain of ξ_1 . One way to get more information is by extending and systematizing the results from sections III and V of the present paper, and I hope to return to that in the near future.

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