On imaginary Verma modules over the affine Lie algebra $A_1^{(1)}$

by
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Abstract

The imaginary Verma modules (IVM) as an alternative to the Verma modules are introduced. The properties of such modules and their irreducible quotients are given including the criterion of irreducibility of the IVM, their composition structure and multiplicities of weights in the IVM and in the irreducible quotients.

1 Introduction

Let L be an affine Lie algebra over $\mathbf{C}$, H be a Cartan subalgebra of L [1]. L-module V is called weight module if $V = \bigoplus_{\lambda \in H^*} V_\lambda$, where

$$V_\lambda = \{v \in V | hv = \lambda(h)v \, \forall h \in H\}.$$

While these modules are natural and important in the applications still there is no complete classification of irreducible weight modules even in the case of finite-dimensional simple Lie algebras except in case of $sl(2)$ which is well-known. Among the weight L-modules the most famous are Verma modules, i.e. modules generated by vacuum or highest vector, and integrable one, i.e. such that the generators e_i, f_i of L act locally nilpotently on L. The integrable irreducible quotients of Verma modules are called standard modules. The genberator c of the centre of L acts on these modules except the trivial one as a non-zero scalar. All another irreducible integrable L-modules with finite-dimensional weight spaces are called loop modules and have a trivial action of the element c. In the same time there are irreducible integrable L-modules with all infinite dimensional weight subspaces and non-trivial action of c constructed by V. Chari and A. Pressley [2].

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In the present article we introduce a new family of weight modules which are called the imaginary Verma modules. They never have a vacuum vector and they are not an integrable. Moreover the irreducible quotients of such modules have the both finite and infinite dimensional weight subspaces together with any scalar action of c.

As a simplest case we consider the algebra $L = A_1^{(1)}$, using the following convenient realization. Let $L' = sl(2, \mathbf{C})$ with standard basis e, f, h, where

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f\,.$$

Consider the algebra of Laurent polynomials $\mathbf{C}[t, t^{-1}]$ and the loop algebra

$$\hat{L} = L' \bigotimes \mathbf{C}[t, t^{-1}]$$

with Lie bracket

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} \qquad \text{for any } x, y \in L'$$

and integer m, n. We will consider L' as the subalgebra of $\hat{L}$. Let $\tilde{L}$ denote the one dimensional central extension of $\hat{L}$:

$$\tilde{L} = \hat{L} \oplus \mathbf{C}z\,,$$

where $[x \otimes t^n, z] = 0$ for all $x \in L'$, $n \in \mathbf{Z}$ and

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + n\delta_{m+n,0}\langle x, y\rangle z\,,$$

$\langle\,,\rangle$ denotes the non-singular bilinear form on L'. Now the algebra $L = A_1^{(1)}$ will be the extension of $\tilde{L}$ by derivation d:

$$L = \tilde{L} \oplus \mathbf{C}d = sl(2, \mathbf{C}) \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}z \oplus \mathbf{C}d\,.$$

where $[z, d] = 0$, $[d, x \otimes t^n] = n\, x \otimes t^n$.

Denote by Δ the root system of the affine Lie algebra L, and let $\pi = \{\alpha, \beta\}$ be some basis of Δ, $\delta = \alpha + \beta$, $\Delta^{im} = \{k\delta | k \in \mathbf{Z} \setminus \{0\}\}$ the set of imaginary roots in Δ. Set

$$P = P_{\pi,\beta} = \{-\beta + k\delta | k \in \mathbf{Z}\} \bigcup \{k\delta | k \in \mathbf{N}\}.$$

Then $\Delta = P \cup -P$ and $P \cap -P = \emptyset$.

We have a root space decomposition of L:

$$L = H \oplus \sum_{\varphi \in \Delta} L_\varphi$$

with Cartan subalgebra $H=\mathbf{C}h\oplus\mathbf{C}z\oplus\mathbf{C}d$ and root subspaces $L_\varphi=\{X\in L|[\hat{h},X]=\varphi(\tilde{h})X$ for all $\tilde{h}\in H\}$. In our case $\dim L_\varphi=1$ for all $\varphi\in\Delta$. Choose the following basis X_φ in each subspace L_φ, $\varphi\in\Delta$:

$$\begin{aligned} X_{\beta+k\delta} &= f\otimes t^{k+1}, \quad X_{-\beta+k\delta}=e\otimes t^{k-1}, \quad k\in\mathbf{Z}, \\ X_{n\delta} &= h\otimes t^n, \qquad n\in\mathbf{Z}\setminus\{0\}. \end{aligned}$$

Then for any integer k,m we obtain

$$[X_{k\delta},X_{m\delta}]=\begin{cases} 0, & m+k\neq 0 \\ 2kz, & m+k=0, \end{cases}$$

$$[X_{k\delta},X_{\beta+m\delta}]=-2X_{\beta+(m+k)\delta}\,, [X_{k\delta},X_{-\beta+m\delta}]=2X_{\beta+(m+k)\delta}\,,$$

$$[X_{\beta+k\delta},X_{-\beta+m\delta}]=\begin{cases} -X_{(m+k)\delta}\,, & m+k\neq 0 \\ -h+(k+1)z\,, & m+k=0, \end{cases}$$

2 The imaginary Verma modules

Consider the following subalgebras in L:

$$L_+=\sum_{\varphi\in P}L_\varphi\,, \qquad L_-=\sum_{\varphi\in -P}L_\varphi\,.$$

Then we have decomposition L into direct sum

$$L=L_+\oplus H\oplus L_-\,.$$

Let V be a weight L-module. We will denote by $P(V)$ the set of all $\mu\in H^*$ such that $V_\mu\neq 0$.

Definition 1. Let $\lambda\in P(V)$. A non-zero element $v\in V_\lambda$ is called an imaginary vacuum vector of weight λ if $L_+v=0$.

Now we can construct the universal L-modules generated by imaginary vacuum vector analogically to Verma modules.

Denote by $\mathsf{U}(L)$ the universal enveloping algebra of L. Let $\lambda\in H^*$. Define on a complex field $\mathbf{C}$ the structure of an one-dimensional $H\oplus L_+$-module with action of $H\oplus L_+$

$$(h+x)1=\lambda(h)\cdot 1\,.$$

Consider the induced L-module

$$\widetilde{M}(\lambda)=\mathsf{U}(L)\bigotimes_{\mathsf{U}(H\oplus L_+)}\mathbf{C}$$

associated with π, β, λ.

Obviously, the module $\tilde{M}(\lambda)$ is weight module and $\tilde{M}(\lambda)_\lambda=1\otimes\mathbf{C}\simeq\mathbf{C}$.

Proposition 1.

1. $\tilde{M}(\lambda)$ is free $U(L_-)$-module
2. $\tilde{M}(\lambda)$ has the unique maximal L-submodule.
3. $P(\tilde{M}(\lambda)) = \{\lambda + k\beta + m\delta | k, m \in Z, k > 0\} \cup \{\lambda - n\delta | n \geq 0\}$.
4. $\dim \tilde{M}(\lambda)_{\lambda + k\beta + m\delta} = \infty, \ k > 0$.

$$\dim \widetilde{M}(\lambda)_{\lambda - n\beta} = \begin{cases} 1, & n = 0 \\ P(n), & n > 0, \end{cases}$$

where $P(n)$ is the number of partitions of n.

The proof of the proposition follows from the definition of $\tilde{M}(\lambda)$.

The module $\tilde{M}(\lambda)$ will be called the imaginary Verma module (IVM).

Denote by $\tilde{N}$ the unique maximal submodule in $\tilde{M}(\lambda)$ and by $\tilde{L}(\lambda)$ the irreducible quotient $\tilde{M}(\lambda)/\tilde{N}$. Certainly, the module $\tilde{L}(\lambda)$ is weight module with imaginary vacuum vector.

Suppose we have some weight L-module V generated by imaginary vacuum vector v of weight λ. Then there is a unique epimorphism $f : \tilde{M}(\lambda) \rightarrow V$ such that $f(1 \otimes 1) = v$. It follows from the universal properties of tensor product. Using this fact we can easily obtain

Proposition 2. Let V be an irreducible weight L-module with imaginary vacuum vector of weight λ. Then

$$V \simeq \tilde{L}(\lambda).$$

Now we establish the criterion of irreducibility of module $\tilde{M}(\lambda)$.

Theorem 1. The module $\tilde{M}(\lambda)$ is irreducible if and only if $\lambda(z) \neq 0$.

For proving this theorem we will need some previous calculations.

Lemma 1.

1. $\tilde{M}(\lambda)_{\lambda - n\delta} = \tilde{L}(\lambda)_{\lambda - n\delta}$ for all $n > 0$ if and only if $\lambda(z) \neq 0$.
2. $\tilde{L}(\lambda)_{\lambda - n\delta} = 0$ for all $n > 0$ if and only if $\lambda(z) = 0$.

Proof. Fixed the integer $n>0$ and consider the weight subspace $\tilde{M}(\lambda)_{\lambda-n\delta}$. The basis of this vector space consists of the elements $X^{l_1}_{-k_1\delta}\dots X^{l_m}_{-k_m\delta}\otimes 1$, where $\sum_{i=1}^m l_ik_i = n$, $k_1<\dots<k_m$.

Assume that there exists a non-zero element $u\in N\cap\tilde{M}(\lambda)_{\lambda-n\delta}$. Then

$$u = \sum_{\substack{k_1<\dots<k_m\\ k_1+\dots+k_m=n}} a_{k_1,\dots,k_m,l_1,\dots,l_m}X^{l_1}_{-k_1\delta}\dots X^{l_m}_{-k_m\delta}\bigotimes 1$$

with complex coefficients $a_{k_1,\dots,k_m,l_1,\dots,l_m}$. Choose the maximal number $\bar{k}_m$ among all k_m and maximal $\bar{l}_m$ among all l_m with the same $\bar{k}_m$. Consider in u only monomials

$$X^{l_1}_{-k_1\delta}\dots X^{\bar{l}_m}_{-\bar{k}_m\delta}\bigotimes 1$$

and repeat the same procedure as above only among k_{m-1} and l_{m-1}. Then consider only monomials

$$X^{l_1}_{-k_1\delta}\dots X^{l_{m-2}}_{-k_{m-2}\delta}X^{\bar{l}_{m-1}}_{-\bar{k}_{m-1}\delta}X^{\bar{l}_m}_{-\bar{k}_m\delta}\bigotimes 1\,.$$

Continue we obtain the unique monomial

$$X^{\bar{l}_1}_{-\bar{k}_1\delta}X^{\bar{l}_2}_{-\bar{k}_2\delta}\dots X^{\bar{l}_m}_{-\bar{k}_m\delta}\bigotimes 1\,.$$

Let $\omega = X^{\bar{l}_1}_{\bar{k}_1\delta}\dots X^{\bar{l}_m}_{\bar{k}_m\delta}u\in\tilde{M}(\lambda)_\lambda$. Then

$$\omega = a_{\bar{k}_1,\dots,\bar{k}_m,\bar{l}_1,\dots,\bar{l}_m}(2\lambda(z))^{\bar{l}_1+\dots+\bar{l}_m}\prod_{i=1}^m \bar{k}_i^{\bar{l}_i}\cdot\bar{l}_i!$$

But, $\tilde{M}(\lambda)_\lambda\cap N = 0$ and $\omega\in\tilde{M}(\lambda)_\lambda\cap N$. Therefore, $\omega = 0$ and $\lambda(z) = 0$. It proves that if $\lambda(z)\neq 0$ then $\tilde{M}(\lambda)_{\lambda-n\delta} = \tilde{L}(\lambda)_{\lambda-n\delta}$ for all $n>0$, and thus if $\tilde{L}(\lambda)_{\lambda-n\delta} = 0$ for all $n>0$ then $\lambda(z)=0$. Now, assume that $\lambda(z)=0$ and consider an arbitrary element

$$u = X^{l_1}_{-k_1\delta}\dots X^{l_m}_{-k_m\delta}\bigotimes 1\,.$$

Let $Y\in \mathsf{U}(L)$ and $Yu\in\tilde{M}(\lambda)_\lambda$. But, Y is the linear combination of the elements $X^{s_1}_{t_1\delta}\dots X^{s_k}_{t_k\delta}$ and each of them acts trivially on u. Therefore, $Yu = 0$ and $u\in N$. We obtain that $\tilde{M}(\lambda)_{\lambda-n\delta}\subset N$ for all $n>0$ and therefore $\tilde{L}(\lambda)_{\lambda-n\delta} = 0$ for all $n>0$. It completes the proof of lemma.

Lemma 2. Let $n\geq 1$ and $\tilde{M}(\lambda)_{\lambda+n\beta} = \tilde{L}(\lambda)_{\lambda+n\beta}$. Then $\tilde{M}(\lambda)_{\lambda+n\beta+k\delta} = \tilde{L}(\lambda)_{\lambda+n\beta+k\delta}$ for all integer k.

Proof. Let $k > 0$. Suppose that there exists a non-zero element $u \in \tilde{M}(\lambda)_{\lambda+n\beta+k\delta} \cap N$. Then $\omega = X_{-k\delta}u \neq 0$ by Poincare-Birkhgof-Witt theorem and $\omega \in \tilde{M}(\lambda)_{\lambda+n\beta} \cap N$. Therefore, we have a contrudiction. Now assume that $k < 0$ and $u \in \tilde{M}(\lambda)_{\lambda+n\beta+k\delta} \cap N$ again. Then u is the finite linear combination of elements

$$X_{\beta+k_1\delta} \dots X_{\beta+k_n\delta} X_{-l_1\delta} \dots X_{-l_s\delta} \bigotimes 1 ,$$

with coefficients $a_{k_1,\dots,k_n l_1,\dots,l_s} \in \mathbf{C}$, where $k_1 + \dots + k_n - l_1 \dots - l_s = k$ and $k_1 \geq \dots \geq k_n$.

Consider the element $X_{-k\delta}u \in \tilde{M}(\lambda)_{\lambda+n\beta} \cap N$. Show that $X_{-k\delta}u \neq 0$. Indeed,

$$\begin{aligned} X_{-k\delta}u &= \sum a_{k_1,\dots,k_n,l_1,\dots,l_s} \Big[X_{\beta+k_1\delta} \dots X_{\beta+k_n\delta} X_{-k\delta} X_{-l_1\delta} \dots X_{-l_s\delta} \bigotimes 1 \\ &\quad + \sum_{i=1}^{n} X_{\beta+k_1\delta} \dots X_{\beta+(k_i-k)\delta} \dots X_{\beta+k_n\delta} X_{-l_1\delta} \dots X_{-l_s\delta} \bigotimes 1 \Big] \end{aligned} \tag{1}$$

Let $\bar{k}_1$ be a maximal among all k_1 in u. Then we have in (1) the elements of kind

$$X_{\beta+(\bar{k}_1-k)\delta} X_{\beta+k_2\delta} \dots X_{\beta+k_n\delta} X_{-l_1\delta} \dots X_{-l_s\delta} \bigotimes 1$$

and, therefore, $X_{-k\delta}u \neq 0$. We get a contridiction again. Lemma is proved.

Lemma 3. For any integer $m \geq 1$ and any nonzero element $u \in \tilde{M}(\lambda)_{\lambda+m\beta}$ there exists integer $n > 0$ such that $X_{-\beta-n\delta}u \neq 0$.

Proof. 1) Suppose that $u = X_\beta^m \otimes 1$. Then for any $n > 0$

$$X_{-\beta-n\delta}u = X_{-\beta-n\delta} X_\beta^m \bigotimes 1 \neq 0 .$$

2) Let $u \neq X_\beta^m \otimes 1$. Then u is a finite linear combination of the elements

$$X_{\beta+k_1^i\delta} \dots X_{\beta+k_m^i\delta} X_{-l_1^i\delta} \dots X_{-l_s^i\delta} \bigotimes 1 ,$$

$i = \overline{1, N}$ for some N and

$$\sum_{j=1}^{m} k_j^i = \sum_{j=1}^{s} l_j^i, \qquad k_1^i \geq \dots \geq k_m^i, \qquad k_j^i \in \mathbf{Z}, \quad l_j^i > 0$$

for all i. Denote

$$n(i) = 2 \sum_{j=1}^{m} k_j^i , \qquad n = \max_{1 \leq i \leq N} n(i) .$$

Show that $X_{-\beta-n\delta}u \neq 0$. Consider the number i such that $\sum_{j=1}^{m-1} k_j^i > 0$. Such number exists because otherwise $k_1^i = \ldots = k_m^i = 0 = l_1^i = \ldots = l_s^i$ for all i and $u = X_\beta^m \otimes 1$. Then

$$X_{-\beta-n\delta}(X_{\beta+k_1^i\delta} \ldots X_{\beta+k_m^i\delta} X_{-l_1^i\delta} \ldots X_{l_s^i\delta} \bigotimes 1) =$$
$$= X_{\beta+k_1^i\delta} \ldots X_{\beta+k_{m-1}^i\delta} X_{(k_m^i-n)\delta} X_{-l_1^i\delta} \ldots X_{-l_s^i\delta} \bigotimes 1 + \ldots \qquad (2)$$

We have $k_m^i - n < 0$ and if $k_j^i > k_m^i$ then of course $k_j^i - n > k_m^i - n$. Moreover,

$$k_m^i - n + l_j^i \leq k_m^i - \sum_{j=1}^{m} k_j^i - \sum_{r=1}^{s} l_r^i + l_j^i = - \sum_{j=1}^{m-1} k_j^i - \sum_{\substack{r=1 \\ r \neq j}}^{s} l_r^i < 0$$

and thus $k_m^i - n < -l_j^i$ for any $j = \overline{1, s}$. It proves that $k_m^i - n$ is the minimal coefficient in (2). Using this fact it is easy to complete the proof of lemma.

Proof of the theorem. Let $\tilde{M}(\lambda)$ be the irreducible. Then $\tilde{M}(\lambda)_\mu = \tilde{L}(\lambda)_\mu$ for all $\mu \in H^*$. Therefore, $\lambda(z) \neq 0$ by Lemma 1. Let $\lambda(z) \neq 0$. Then

$$\tilde{M}(\lambda)_{\lambda-n\delta} = \tilde{L}(\lambda)_{\lambda-n\delta}$$

for all $n \geq 0$. It implies that

$$\tilde{M}(\lambda)_{\lambda+\beta} = \tilde{L}(\lambda)_{\lambda+\beta}$$

by Lemma 3 and

$$\tilde{M}(\lambda)_{\lambda+\beta+k\delta} = \tilde{L}(\lambda)_{\lambda+\beta+k\delta}$$

for all integer k by Lemma 2.

We can continue by induction. Therefore, $\tilde{M}(\lambda)_\mu = \tilde{L}(\lambda)_\mu$ for all $\mu \in H^*$ and $\tilde{M}(\lambda) = \tilde{L}(\lambda)$, i.e. $\tilde{M}(\lambda)$ is irreducible. The theorem is proved.

3 The irreducible quotients of IVM

If $\lambda(z) \neq 0$ then $\tilde{M}(\lambda) = \tilde{L}(\lambda)$ and the structure of the irreducible module $\tilde{L}(\lambda)$ is given by Proposition 1. Further, we will assume that $\lambda(z) = 0$. We will denote such functional by λ_0.

Consider in L the vector subspace

$$L^+ = \sum_{\varphi \in P \cup \Delta^{im}} L_\varphi$$

and commutative subalgebra

$$L^- = \sum_{\varphi \in -P \setminus \Delta^{im}} L_\varphi .$$

Then we have the following decomposition of L as the vector space

$$L = L^+ \bigoplus H \bigoplus L^- .$$

Definition 2. Let V be a weight L-module, $\lambda \in P(V)$. A nonzero element $v \in V_\lambda$ is called a special imaginary vacuum vector (s.i.v.v.) of weight λ if $L^+ v = 0$.

Let $\lambda_0 \in H^*$, $\lambda_0(z) = 0$. Define on $\mathbf{C}$ the structure of $H \oplus L^+$-module, where

$$(\tilde{h} + X)1 = \lambda_0(\tilde{h}) \cdot 1$$

for all $\tilde{h} \in H$, $X \in L^+$.

Consider the induced L-module

$$\hat{M}(\lambda_0) = U(L) \bigotimes_{U(H \oplus L^+)} \mathbf{C}.$$

The definition implies the following properties of $\hat{M}(\lambda_0)$.

Proposition 2.

1. $\hat{M}(\lambda_0)$ is weight L-module generated by s.i.v.v. $1 \otimes 1$.
2. $\hat{M}(\lambda_0)$ has a unique maximal submodule $\hat{N}$.
3. There exists an epimorphism $f : \tilde{M}(\lambda_0) \to \hat{M}(\lambda_0)$.
4. Let V be an irreducible weight L-module with s.i.v.v. of weight λ_0. Then $V \simeq \hat{M}(\lambda_0) / \hat{N} \simeq \tilde{L}(\lambda_0)$.
5. $P(\hat{M}(\lambda_0)) = \{\lambda_0 + n\beta + k\delta | k_1 n \in Z, n > 0\} \cup \{\lambda_0\}$, $\dim \hat{M}(\lambda_0)_{\lambda_0 + n\beta + k\delta} = \infty$, $n > 1$, $\dim \hat{M}(\lambda_0)_{\lambda_0 + \beta + k\delta} = \dim \hat{M}(\lambda_0)_{\lambda_0} = 1$.

The main result of this chapter is the following.

Theorem 2. (i) $\hat{M}(\lambda_0)$ is irreducible if and only if $\lambda_0(h) \neq 0$.
(ii) If $\lambda_0(h) = 0$ then $\tilde{L}(\lambda_0)$ is trivial one-dimensional module.

Lemma 4. Let $n \geq 1$ and $\hat{M}(\lambda_0)_{\lambda_0 + n\beta} = \tilde{L}(\lambda_0)_{\lambda_0 + n\beta}$. Then

$$\hat{M}(\lambda_0)_{\lambda_0 + n\beta + k\delta} = \tilde{L}(\lambda_0)_{\lambda_0 + n\beta + k\delta}$$

for all integer k.

Proof. Fixed $k \in Z$. Suppose that there is a nonzero element $u \in \hat{M}(\lambda_0)_{\lambda_0 + n\beta + k\delta} \cap \hat{N}$. Then u is the finite linear combination of the elements

$$X^{s_m}_{\beta + k_1 \delta} \cdots X^{s_m}_{\beta + k_m \delta} \bigotimes 1,$$

where $\sum_{i=1}^m s_i k_i = k$, $\sum_{i=1}^m s_i = n$, $k_1 < \ldots < k_m$.

One can check that $\omega = X_{-k\delta} u \neq 0$. Therefore we have a nonzero element in $\hat{M}(\lambda_0)_{\lambda_0 + n\beta} \cap \hat{N}$ and get a contrudition. Lemma is proved.

Lemma 5. Let $\lambda_0(h) \neq 0$. Then for any integer $n \geq 1$ and any nonzero element $u \in \hat{M}(\lambda_0)_{\lambda_0+n\beta}$ there exists an integer p such that $X_{-\beta+p\delta}u \neq 0$.

Proof. Let

$$u = \sum a_{k_1,\ldots,k_m,s_1,\ldots,s_m} X^{s_1}_{\beta+k_1\delta} \ldots X^{s_m}_{\beta+k_m\delta} \bigotimes 1 ,$$

where $k_i \in Z$, $s_i \in \mathrm{N}$, $\sum_{i=1}^m s_i = n$, $\sum_{i=1}^m s_i k_i = 0$, $k_1 < \ldots < k_m$.

Consider in u a summand

$$u_{\min} = X^{\bar{s}_1}_{\beta+\bar{k}_1\delta} \ldots X^{\bar{s}_M}_{\beta+\bar{k}_M\delta} \bigotimes 1 ,$$

where for any i $\bar{k}_i$ is minimal among all k_i and $\bar{s}_i$ is maximal among all s_i in the elements

$$X^{\bar{s}_1}_{\beta+\bar{k}_1\delta} \ldots X^{\bar{s}_{i-1}}_{\beta+\bar{k}_{i-1}\delta} X^{s_i}_{\beta+k_i\delta} \ldots \bigotimes 1 .$$

1. Suppose that $M=1$. Then $u = a_{0,n} X^n_\beta \otimes 1$ and $X_{-\beta+p\delta}u \neq 0$ for any $p>0$ if $n>1$ and $X_{-\beta}u \neq 0$ if $n=1$.

2. Suppose that $M>1$ and $\bar{s}_M + \bar{s}_{M-1} > 2$. For $p \neq -\bar{k}_i$, $i = \overline{1,M}$ we have

$$X_{-\beta+p\delta}u_{\min} = -2\sum_{i=1}^{M} \bar{s}_i \Big[\sum_{j=1}^{M-i} s_{i+j} X^{\bar{s}_1}_{\beta+\bar{k}_1\delta} \ldots X^{\bar{s}_i}_{\beta+\bar{k}_i\delta} \ldots$$
$$\ldots X_{\beta+(\bar{k}_i+\bar{k}_{i+j}+p)\delta} X^{\bar{s}_{i+j}-1}_{\beta+\bar{k}_{i+j}\delta} \ldots X^{\bar{s}_M}_{\beta+\bar{k}_M\delta} \otimes 1 + \frac{1}{2}\big(s_i - 1\big) X^{\bar{s}_1}_{\beta+\bar{k}_1\delta} \ldots$$
$$\ldots X^{\bar{s}_i-2}_{\beta+\bar{k}_i\delta} X_{\beta+(2\bar{k}_i+p)\delta} \ldots X^{\bar{s}_M}_{\beta+\bar{k}_M\delta} \otimes 1\Big] . \qquad (3)$$

It is easy to see that $X_{-\beta+p\delta}u \neq 0$.

3. Let $M>1$ and $\bar{s}_M = \bar{s}_{M-1} = 1$. Then

$$u_{\min} = X^{\bar{s}_1}_{\beta+\bar{k}_1\delta} \ldots X^{\bar{s}_{M-2}}_{\beta+\bar{k}_{M-2}\delta} X_{\beta+\bar{k}_{M-1}\delta} X_{\beta+\bar{k}_M\delta} \bigotimes 1$$

and in (3) we have a following element

$$X^{\bar{s}_1}_{\beta+\bar{k}_1\delta} \ldots X^{\bar{s}_{M-2}}_{\beta+\bar{k}_{M-2}} X_{\beta+(\bar{k}_{M-1}+\bar{k}_M+p)\delta} \bigotimes 1 .$$

Denote

$$A = a_{\bar{k}_1,\ldots,\bar{k}_M,\bar{s}_1,\ldots,\bar{s}_{M-2},1,1} ,$$

$$A_{k_{M-1},k_M} = a_{\bar{k}_1,\ldots,\bar{k}_{M-2},k_{M-1},k_M,\bar{s}_1,\ldots,\bar{s}_{M-2},1,1} ,$$

$$A_{k_{M-1}} = a_{\bar{k}_1,\dots,\bar{k}_{M-2},k_{M-1},\bar{s}_1,\dots,\bar{s}_{M-2},2}\,.$$

Suppose that $X_{-\beta+p\delta}u = 0$ for some integer $p \neq -\bar{k}_i$, $i=\overline{1,M}$. Then we have an equality

$$A + A_{k_{M-1}} \sum A_{k_{M-1},k_M} = 0 \tag{4}$$

Set $p = -\bar{k}_M$ and assume that $X_{-\beta+p\delta}u = 0$. Then we obtain the following equality

$$-2A + \lambda_o(h)A - 2\sum A_{k_{M-1},k_M} - 2A_{k_{M-1}} = 0 \tag{5}$$

Comparing (4) and (5) we get

$$\lambda_0(h)A = 0$$

and thus $A = 0$. Contrudiction. Therefore, there exists an integer p such that $X_{-\beta+p\delta}u \neq 0$. Lemma is proved.

Corollary 1. If $\lambda_0(h) \neq 0$ then $\hat{M}(\lambda_0)$ is the irreducible.

It follows immediately from Lemmas 4 and 5.

Lemma 6. Let $\lambda_0(h) = 0$ Then $\hat{N} = \sum_{\mu\neq\lambda_0} \hat{M}(\lambda_0)_\mu$.

Proof. Fixed $n \in Z$ and consider the element $\omega = X_{\beta+n\delta} \otimes 1$. Let $Y \in U(L)$ and $Y\omega \in \hat{M}(\lambda_0)_{\lambda_0}$. It is easy to check that in this case $Y\omega = 0$ because of $\lambda_0(h) = 0$. It means that

$$\sum_{n\in Z} \hat{M}(\lambda_0)_{\lambda_0+\beta+n\delta} \subset \hat{N}$$

and therefore $\hat{N} = \sum_{\mu\neq\lambda_0} \hat{M}(\lambda_0)$. Lemma is proved.

Proof of the Theorem 2. On one hand if $\lambda_0(h) \neq 0$ then $\hat{M}(\lambda_0)$ is irreducible by Corollary 1. On the other hand if $\lambda_0(h) = 0$ then $\hat{N} \neq 0$ by Lemma 3 and thus $\hat{M}(\lambda_0)$ is reducible. It completes the proof of (i). Point (ii) follows directly from Lemma 3.

4 The imaginary Verma modules $\tilde{M}(\lambda_0)$

We will assume that $\lambda_0(h) \neq 0$.

Denote by $\tilde{N}$ the maximal submodule in $\tilde{M}(\lambda_0)$. Then $\tilde{M}(\lambda_0)/\tilde{N} = \tilde{L}(\lambda_0) \simeq \hat{M}(\lambda_0)$.

Lemma 7. The module $\tilde{N}$ is generated by subspace

$$\sum_{n=1}^{\infty} \tilde{M}(\lambda_0)_{\lambda_0 - n\delta} .$$

The proof is obvious.

Denote by $T(\lambda_0)$ the subspace of all imaginary vacuum vectors of $\tilde{M}(\lambda_0)$.

Lemma 8. $\sum_{n=0}^{\infty} \tilde{M}(\lambda_0)_{\lambda_0-n\delta} \subset T(\lambda_0)$.

Proof. If $u \in \tilde{M}(\lambda_0)_{\lambda_0-n\delta}$ then $X_{k\delta}u = 0$ for all $k > 0$. Therefore, $u \in T(\lambda_0)$ and $\tilde{M}(\lambda_0)_{\lambda_0-n\delta} \subset T(\lambda_0)$ for all $n \geq 0$. Lemma is proved.

Consider an arbitrary proper submodule N in $\tilde{M}(\lambda_0)$ and denote $[N] = N \cap \sum_{n=1}^{\infty} \tilde{M}(\lambda_0)_{\lambda_0-n\delta}$.

Lemma 9. $[N] \neq 0$ and N is generated by $[N]$ as L-module.

Proof. If $u \in N$ then $0 \neq Yu \in \sum_{n=1}^{\infty} \tilde{M}(\lambda_0)_{\lambda_0-n\delta}$ for some $Y \in U(L)$ by Lemma 3. Let N' be a submodule of N generated by $[N]$ and suppose that $N' \neq N$. Then there exists a weight element $u \in N \setminus N'$. Let $\bar{u}$ be an image of u in a factormodule N/N'. Then $\bar{u}$ is the finite linear combination of the elements

$$X_{\beta+n_1^i\delta} \dots X_{\beta+n_{s_i}^i\delta} X_{-k_1^i\delta} \dots X_{-k_{t_i}^i\delta} \bigotimes 1$$

by Lemma 7, where $k_1^i + \dots + k_{t_i}^i \leq k_1^{i+1} + \dots + k_{t_{i+1}}^{i+1}$ for all i.

Denote $a_i = X_{-k_1^i\delta} \dots X_{-k_{t_i}^i\delta} \otimes 1$. Consider the submodule $\tilde{N}'$ of $\tilde{N}$ generated by $[N]$ and all elements $X_{-k_1\delta} \dots X_{-k_n\delta} \otimes 1$ such that

$$\sum_{j=1}^{n} k_j \geq k_1^1 + \dots + k_{t_1}^1$$

except a_1. Then

$$N \big/ N \cap \tilde{N}' \subset \tilde{N} \big/ \tilde{N}'$$

and u has a nonzero image $\boldsymbol{u}$ in $N/N \cap \tilde{N}'$. Let $\bar{a}_1$ be an image of a_1 in $\tilde{N}/\tilde{N}'$. Then a_1 is the s.i.v.v. by Lemma 8. The module generated by $\bar{a}_1$ is irreducible by Theorem 2 and contains $\boldsymbol{u}$. Therefore there exists $Y \in U(L)$ such that

$$0 \neq Y\boldsymbol{u} = \bar{a}_1 \in \sum_{k=1}^{\infty} (\tilde{N}/\tilde{N}')_{\lambda_0-k\delta} .$$

But, $Y\boldsymbol{u} \in N/N \cap \tilde{N}'$ and thus

$$\sum_{k=1}^{\infty}(N/N \cap \tilde{N}')_{\lambda_0-k\delta} \neq 0\,.$$

We have a contradiction. Therefore $N'=N$. Lemma is proved.

Next theorem describes all imaginary vacuum vectors of $\tilde{M}(\lambda_0)$.

Theorem 3. $T(\lambda_0) = \sum_{n=0}^{\infty} \tilde{M}(\lambda_0)_{\lambda_0-n\delta}$.

Proof. Denote $T = \sum_{n=0}^{\infty} \tilde{M}(\lambda_0)_{\lambda_0-n\delta}$. Then $T \subset T(\lambda_0)$ by Lemma 8. Let $u \in T(\lambda_0)$ and $u \notin T$. Consider the submodule N generated by u. Then $N \cap T = 0$. But, it's impossible by Lemma 9. Therefore, $T = T(\lambda_0)$. The theorem is proved.

Corollary 2. 1) Let $\lambda_0, \lambda_0' \in H^*$, $\lambda_0(z) = \lambda_0'(z) = 0$. Then

$$Hom\big(\tilde{M}(\lambda_0'), \tilde{M}(\lambda_0)\big) \neq 0$$

if and only if $\lambda_0' = \lambda_0 - n\delta$ for some integer $n \geq 0$.

2) $\dim Hom(\tilde{M}(\lambda_0 - n\delta), \tilde{M}(\lambda_0)) = P(n)$.

Consider the composition structure of $\tilde{M}(\lambda_0)$.

Lemma 10. Every irreducible factor of $\tilde{M}(\lambda_0)$ has the imaginary vacuum vector of the weight $\lambda_0 - n\delta$ for some integer $n \geq 0$.

Proof. Let N_1, N_2 be the submodules of $\tilde{M}(\lambda_0)$, N_2 is the proper submodule of N_1 and the factor N_1/N_2 is irreducible. Then $[N_1] \neq [N_2]$ and $(N_1/N_2)_{\lambda_0-n\delta} \neq 0$ for some integer $n \geq 0$ by Lemma 9. Therefore, there exists an imaginary vacuum vector in N_1/N_2 by Theorem 3.

Theorem 4. (i) $\tilde{M}(\lambda_0)$ has an infinite composition series.

(ii) The irreducible modules $\tilde{L}(\lambda_0 - n\delta)$, $n \geq 0$ completely exhaust all irreducible factors of the composition series.

Proof. The point (ii) follows immediately from Lemma 10. Denote by N_i, $i \geq 0$ L-submodule of $\tilde{M}(\lambda_0)$ generated by the subspace $\sum_{k=i}^{\infty} \tilde{M}(\lambda_0)_{\lambda_0-k\delta}$.

Here $N_0 = \tilde{M}(\lambda_0)$, $N_1 = \tilde{N}$. Then

$$N_i\big/N_{i+1} \simeq P(i)\tilde{L}(\lambda_0 - i\delta)$$

and we can easyly complete the construction of the series. The Theorem is proved.

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