ON THE UNIQUENESS OF THE TRACE ON SOME SIMPLE C*-ALGEBRAS.

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1.Introduction.

By a trace on a C^*-algebra, we will always mean a tracial state. In other words, we will only consider finite traces. Many C^*-algebras, especially simple C^*-algebras, are equipped with a natural trace and it is often interesting to know that this trace is unique. Within the theory of von Neumann algebras, this is the characteristic feature of finite factors, but it also plays a decisive role in the classification of some families of C^*-algebras (see for example [21] and [26]). Further, in view of the emerging comparison theory for simple C^*-algebras which is surveyed in [7], it is be clear that one should first study the fundamental questions of this theory in their easiest setting, which is the class of simple C^*-algebras with a unique trace. We feel therefore it is of some interest to produce new examples belonging to this class.

A lot of simple C^*-algebras arise as crossed products, eventually with some twisting allowed. Our first aim in this note is to give a sufficient condition on the (twisted) action of a discrete group acting on a simple unital C^*-algebra $\mathcal{A}$ with a unique trace in aim that the resulting reduced (twisted) crossed product inherits the same properties. Loosely, this condition says that the action is outer when extended to the von Neumann algebra generated by $\mathcal{A}$ in the GNS-representation associated to the trace. As pointed out by Longo in [17], the outerness of the action at the C^*-level is not sufficient in general.

The rest of this note is devoted to several applications of this result. Recall from [3] that a discrete group G is said to be C^*-simple if its reduced group C^*-algebra $C_r^*(G)$ is simple, and to have a unique trace if the canonical trace on $C_r^*(G)$ is unique. Suppose that a discrete group G contains a normal C^*-simple subgroup H with trivial centralizer. In [3], we showed that G is then C^*-simple too. However, we left the following problem open: If we suppose further that H has a unique trace, does G have a unique trace too? Our first application will be to show that the answer is yes. As an example of this situation, we may start with a C^*-simple group H whith a unique trace, and let G be the automorphism group or the holomorph of H. Hence, this generalizes the result of [19], where Nitica and Török consider the automorphism group of a non-amenable free product of groups.

Recall further from [3] that a group G is said to be an ultraweak Powers group if G contains a normal weak Powers group with trivial centralizer. Weak Powers groups are C^*-simple and have a unique trace ([5]), so we see that ultraweak Powers groups possess the same properties. Suppose now that K is an ultraweak Powers group (or is obtained by repeated

extensions of ultraweak Powers groups) and let

$$1 \to H \to G \to K \to 1$$

denote an exact sequence where H is a C^*-simple group with a unique trace. Then we show that G is C^*-simple with a unique trace, which generalizes [6; Theorem]. Our result is obtained as a consequence of a more general statement about reduced twisted crossed products by G.

Before our work in [3], all C^*-simple groups known so far were equipped with a unique trace, merely as a by-product of the proof of simplicity, which in all cases but one was related to the Dixmier property of the algebras. The only exception to this was the case of the matrix groups considered in [13], where the uniqueness of the trace is pointed out in [22] (We are indebted to Alain Valette for drawing our attention to [22]). Our results in the present note imply that all known C^*-simple goups until now do have a unique trace. A general proof of this fact is still lacking.

We next turn to crossed products associated with characters on groups, which are considered by Yin in [26]. There, Yin asks whether the crossed product of $C_r^*(G)$ by the automorphism induced by a character of infinite order has a unique trace when G is a Powers group. In the note added in proof, he says that the answer is yes, thanks to [12]. However, it is unclear to us how [12] may be used, at least directly. Anyway, we will prove a more general result which answers the question positively: Let G denote a C^*-simple group with a unique trace. Then the crossed product $C_r^*(C_r^*(G), \Gamma, \alpha)$ is simple with a unique trace whenever Γ is any subgroup of the character group of G and $\alpha : \Gamma \to Aut(C_r^*(G))$ is the induced action.

As our last application, we consider crossed products of irrational rotation algebras by subgroups of toral automorphisms. For each irrational $\theta \in (0,1)$, let $\mathcal{A}_\theta$ denote the corresponding rotation algebra ([21], [23]), which is known to be simple with a unique trace. For each $A \in SL(2,\mathbf{Z})$, Watatani has defined in [27] the toral automorphism α_A of $\mathcal{A}_\theta$ induced by A (see also [9]). Let G be any subgroup of $SL(2,\mathbf{Z})$ and $\alpha : G \to Aut(\mathcal{A}_\theta)$ the induced mapping which is known to be an action ([27]). Then we show that $C_r^*(\mathcal{A}_\theta, G, \alpha)$ is simple with a unique trace. The case when $G \simeq \mathbf{Z}_2$ is generated by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ has recently been studied in [8], while the case when $G \simeq \mathbf{Z}$ is generated by a matrix A of infinite order may be seen to be equivalent to the study of the twisted group C^*-algebra $C_r^*(\mathbf{Z}^2 \mathrm{x}_A \mathbf{Z}, \sigma_\theta)$ (for a suitable choice of multiplier σ_θ) which simplicity is determined in [20;§3].

Our notation will be as in [2] and [3]. *All C^*-algebras are supposed to be unital and all groups are considered as discrete.*

2. The main result.

Let A be a simple C^*-algebra with a unique trace φ and let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ denote the GNS-triple associated to φ. Since φ is necessarily faithful, $\mathcal{A}$ is $*$-isomorphic to $\pi_\varphi(\mathcal{A})$ and we will identify $\mathcal{A}$ with $\pi_\varphi(\mathcal{A})$.

Now, let $\alpha \in Aut(\mathcal{A})$. By uniqueness of φ, φ is α-invariant. Therefore, α is implemented on $\mathcal{H}_\varphi$ by the unitary operator U_α determined by $U_\alpha(a\xi_\varphi) = \alpha(a)\xi_\varphi$, $a \in \mathcal{A}$. Let us denote by $\tilde{\alpha}$ the extension of α to $\mathcal{A}''$ which is defined by $\tilde{\alpha} = ad(U_\alpha)$ on $\mathcal{A}''$.

Definition: We say that α *is* $\varphi - outer$ if $\tilde{\alpha}$ is outer. Further, if (α, u) denotes a cocycle crossed action of a group G on $\mathcal{A}$ ([3]), we say that (α, u) *is* $\varphi - outer$ if each α_g is φ-outer, $g \neq e$.

Obviously, φ-outerness is a stronger condition than usual outerness. Our interest in introducing this notion lies in the following:

Theorem 1: Suppose $\mathcal{A}$ is a simple C^*-algebra with a unique trace φ and (α, u) is a φ-outer cocycle crossed action of a group G on $\mathcal{A}$. Then the reduced twisted crossed product $\mathcal{B} = C_r^*(\mathcal{A}, G, \alpha, u)$ is simple with a unique trace τ given by $\tau = \varphi_{\circ E}$, where E denotes the canonical conditional expectation from $\mathcal{B}$ onto $\mathcal{A}$.

Proof: Since (α, u) is outer, the simplicity of $\mathcal{B}$ follows from the extended version of Kishimoto's theorem given in [3; Theorem 3.2]. Further, since φ is α-invariant, $\tau = \varphi_{\circ E}$ defines a faithful trace on $\mathcal{B}$.

Let now ζ be a trace on $\mathcal{B}$. We will show that $\zeta = \tau$, hence that τ is unique, by checking the equality on each of the generators of $\mathcal{B}$.

We start by representing $\mathcal{A}$ via the GNS-representation associated to φ and, as before, we identify $\mathcal{A}$ with $\pi_\varphi(\mathcal{A})$. We will write $\mathcal{H}$ for $\mathcal{H}_\varphi$ and $\xi_\circ$ for ξ_φ. We will also identify $\mathcal{A}$ with its canonical copy in $\mathcal{B}$. Hence, by definition, $\mathcal{B}$ is generated on $l^2(G, \mathcal{H})$ by $\{a, \lambda_u(g); a \in \mathcal{A}, g \in G\}$ where λ_u is the u-projective left regular representation of G on $l^2(G, \mathcal{H})$ (cf. [3]).

Since (α, u) clearly extends to a cocycle crossed action $(\tilde{\alpha}, u)$ of G on $\mathcal{A}''$, we may also form the regular extension $\mathcal{A}'' \mathrm{x}_{(\tilde{\alpha}, u)} G$ which, when identifying $\mathcal{A}''$ with its canonical copy in $\mathcal{A}'' \mathrm{x}_{(\tilde{\alpha}, u)} G$, is the von Neumann algebra generated by $\{x, \lambda_u(g);\ x \in \mathcal{A}'', g \in G\}$ on $l^2(G, \mathcal{H})$ (cf.[2]), so that we have $\mathcal{B}'' = \mathcal{A}'' \mathrm{x}_{(\tilde{\alpha}, u)} G$.

For each $x \in \mathcal{A}''$, set $\widetilde{\varphi}(x) = (x\xi_0, \xi_0)$. By [25; Prop.3.19], $\widetilde{\varphi}$ is a faithful normal trace on $\mathcal{A}''$ which extends φ. Further, by uniqueness of φ, $\mathcal{A}''$ is a factor ([17; lemma 1]), and $\widetilde{\varphi}$ is therefore unique. Since $(\widetilde{\alpha}, u)$ is outer by assumption, we also have that $\mathcal{A}''\mathrm{x}_{(\widetilde{\alpha},u)}G$ is a factor too (the proof goes along the same line as the one for usual crossed products and may be deduced from [11; Theorem 2]). By the uniqueness of $\widetilde{\varphi}$, $\widetilde{\varphi}$ is $\widetilde{\alpha}$-invariant and $\widetilde{\tau} = \widetilde{\varphi} \circ \widetilde{E}$ defines therefore the (unique) faithful normal trace on the finite factor $\mathcal{B}'' = \mathcal{A}''\mathrm{x}_{(\widetilde{\alpha},u)}G$, where $\widetilde{E}$ denotes the canonical conditional expectation from $\mathcal{B}''$ onto $\mathcal{A}''$ (see [10;Theorem 6] or [3; Proof of theorem 2.2]) which is such that $\widetilde{E}(\lambda_u(g)) = 0$ for all $g \in G$, $g \neq e$. Since E is, by definition, the restriction of $\widetilde{E}$ to $\mathcal{B}$, and $\widetilde{\varphi} = \varphi$ on $\mathcal{A}$, we see that $\widetilde{\tau} = \tau$ on $\mathcal{B}$.

Let now $(\pi_\zeta, \mathcal{H}_\zeta, \xi_\zeta)$ denote the GNS-triple associated to $(\mathcal{B}, \zeta)$, and observe that $\overline{\zeta}(\cdot) = (\cdot\xi_\zeta, \xi_\zeta)$ defines a faithful normal trace on $\pi_\zeta(\mathcal{B})''$ (again by [25; Prop.3.19]). Especially, $\tilde{\zeta} = \overline{\zeta}|\pi_\zeta(\mathcal{A})''$ is a faithful normal trace on $\pi_\zeta(\mathcal{A})''$. Further, since $\zeta|\mathcal{A} = \varphi$ by uniqueness of φ, we have for all $a \in \mathcal{A}$:

$$\widetilde{\varphi}(a) = \varphi(a) = \zeta(a) = (\pi_\zeta(a)\xi_\zeta, \xi_\zeta) = \widetilde{\zeta}(\pi_\zeta(a)).$$

By simplicity of $\mathcal{A}$, $\pi_\zeta|\mathcal{A}$ is $*$-isomorphism form $\mathcal{A}$ onto $\pi_\zeta(\mathcal{A})$, and from what we just have seen, it follows (cf. [14; 7.2.7]) that $\pi_\zeta|\mathcal{A}$ extends to a $*$-isomorphism from $\mathcal{A}''$ onto $\pi_\zeta(\mathcal{A})''$, which we denote by θ.

For $g, h \in G$, we set:

$$\omega(g) = \pi_\zeta(\lambda_u(g)) \in \mathcal{U}(\pi_\zeta(\mathcal{B})) \subseteq \mathcal{U}(\pi_\zeta(\mathcal{B})''),$$

$$\beta_g = \theta\widetilde{\alpha}_g\theta^{-1} \in Aut(\pi_\zeta(\mathcal{A})''),$$

$$v(g,h) = \pi_\zeta(u(g,h)) \in \mathcal{U}(\pi_\zeta(\mathcal{A}) \subseteq \mathcal{U}(\pi_\zeta(\mathcal{A})'').$$

Since $\omega(g)\pi_\zeta(a)\omega(g)^* = \pi_\zeta(\lambda_u(g)a\lambda_u(g)^*) = \pi_\zeta(\alpha_g(a)) = \beta_g(\pi_\zeta(a))$ for all $g \in G, a \in \mathcal{A}$, it follows by ultraweak continuity that

$$\beta_g = ad(\omega(g)) \text{ on } \pi_\zeta(\mathcal{A})'', \quad g \in G.$$

Since $\lambda_u(g)\lambda_u(h) = ad(u(g,h))\lambda_u(g,h)$ for $g, h \in G$, we clearly have

$$\omega(g)\omega(h) = ad(v(g,h))\omega(g,h), \quad g, h \in G.$$

Further, by assumption, each $\widetilde{\alpha}_g(g \neq e)$ is outer on $\mathcal{A}''$. Since $\mathcal{A}''$ is a factor, Kallman's theorem ([15]) implies that each $\widetilde{\alpha}_g(g \neq e)$ is freely acting on $\mathcal{A}''$, from which it follows that each $\beta_g(g \neq e)$ is freely acting on $\pi_\zeta(\mathcal{A})''$.

As we have seen before, $\pi_\zeta(\mathcal{B})''$ has a faithful normal trace $\overline{\zeta}$ and there exists therefore a faithful normal conditional expectation P from $\pi_\zeta(\mathcal{B})''$ onto $\pi_\zeta(\mathcal{A})''$ (cf. [24; Corollary 10.6] or [25; Proposition 2.36]). Since $\beta_g(x)\omega(g) = \omega(g)x$ for all $g \in G$, $x \in \pi_\zeta(\mathcal{A})''$, we have $\beta_g(x)P(\omega(g)) = P(\omega(g))x$, so that $P(\omega(g)) = 0$ if $g \neq e$, by freeness of β. At last, since $\{\pi_\zeta(\mathcal{A}),\ \omega(G)\}$ clearly generates $\pi_\zeta(\mathcal{B})$ as a C^*-algebra, we see that $\{\theta(\mathcal{A}''),\ \omega(G)\} = \{\pi_\zeta(\mathcal{A})'',\ \omega(G)\}$ generates $\pi_\zeta(\mathcal{B})''$ as a von Neumann algebra. We are now in position to invoke the analog version of [24; Prop.22.2], cf. [10; Theorem 7], which characterizes regular extensions of countably decomposable von Neumann algebras by discrete groups. It says that there exist a $*$-isomorphism Φ from $\pi_\zeta(\mathcal{B})''$ onto $\mathcal{A}''\mathrm{x}_{(\tilde{\alpha},u)}G = \mathcal{B}''$ such that

$$\Phi(\theta(x)) = x \quad , x \in \mathcal{A}'',$$
$$\Phi(\omega(g)) = \lambda_u(g), \quad g \in G \text{ (and } \Phi(v(g,h)) = u(g,h), \quad g,h \in G).$$

As $\mathcal{B}''$ is a finite factor, $\pi_\zeta(\mathcal{B})''$ is then a finite factor too, and its trace $\overline{\zeta}$ is therefore unique. Hence, $\overline{\zeta} = \widetilde{\tau} \circ \Phi$, which implies that, for each $a \in \mathcal{A}$, $g \in G$,

$$\zeta(a) = \overline{\zeta}(\pi_\zeta(a)) = \widetilde{\tau}(\Phi(\pi_\zeta(a)) = \widetilde{\tau}(a) = \tau(a),$$

and

$$\zeta(\lambda_u(g)) = \overline{\zeta}(\pi_\zeta(\lambda_u(g)) = \overline{\zeta}(\omega(g)) = \widetilde{\tau}(\Phi(\omega(g))) = \widetilde{\tau}(\lambda_u(g)) = \tau(\lambda_u(g)).$$

By norm-continuity, we obtain $\zeta = \tau$ as required. □

2. Some applications.

To ease our exposition, we first introduce som terminology.

Let H be a group and let σ be an automorphism of H. We will say that σ *acts freely on* H if the set $\{\sigma(h)ph^{-1};\ h \in H\}$ is infinite for all $p \in H$. This definition is motivated by [15; Theorem 2.2], where Kallman shows that the $*$-automorphism of the group von Neumann algebra $L(H)$ of H induced by σ is freely acting if and only if the above condition is satisfied.

Suppose now H is a normal subgroup of a group G. We will say that G/H *acts freely on* H if $ad(g)$ acts freely on H for all $g \in G \setminus H$ (where $ad(g)$ denotes the automorphism of H implemented by g).

The reader will easily check that the following conditions are equivalent:

i) G/H acts freely on H

ii) the set $\{hgh^{-1}, h \in H\}$ is infinite for all $g \in G \setminus H$.

Condition ii) appears in [10; Lemma 10], where it is shown to be equivalent to the fact that the relative commutant of $L(H)$ in $L(G)$ is contained in $L(H)$. Further, it is clear that if G is a semi-direct product of H by K under a homomorphism $\sigma : K \to Aut(H)$, then $G/H \simeq K$ acts freely on H if and only if each σ_k acts freely on H $(k \in K, k \neq e)$.

We will be mainly interested in the following example:

- Suppose H is an ICC-group. Then G/H acts freely on H if and only if H has trivial centralizer in G. This fact may be extracted from [3;§3] or proved directly form ii) above.

Let us also mention here another example to which we will refer at the end of this paper:

- Let $A \in SL(2, \mathbf{Z})$ and consider A as an automorphism of $\mathbf{Z}^2$. Then, trivially, A acts freely on $\mathbf{Z}^2$ whenever $A \neq I$.

After these preliminaries, we now give an analog version of [3; Theorem 3.5] (which only deals with the simplicity of the involved algebras). Let us at once remark that in [3; Theorem 3.5] one may replace the assumption that H is ICC and with trivial centralizer in G by the assumption that G/H acts freely on H.

Theorem 2: Suppose (α, u) denotes a cocycle crossed action of a group G on a C^*-algebra $\mathcal{A}$ which possess a faithful α-invariant trace φ. Suppose further H is a normal subgroup of G such that G/H acts freely on H and $C_r^*(\mathcal{A}, H, \alpha, u)$ is simple with a unique trace σ. Then $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple with a unique trace.

Proof: The proof is nearly related to the proof of [3; Theorem 3.5], but some preparations are required first. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ denote the GNS-triple of $\mathcal{A}$ associated to φ. Again, we will write $\mathcal{H}$ for $\mathcal{H}_\varphi, \xi_0$ for ξ_φ and identify $\mathcal{A}$ with $\pi_\varphi(\mathcal{A})$ (by faithfulness of φ). As before, the α-invariance of φ enables us to extend (α, u) to a cocycle crossed action $(\widetilde{\alpha}, u)$ of G on $\mathcal{A}''$. As in the proof of theorem 1, $C_r^*(\mathcal{A}, G, \alpha, u)$ and $\mathcal{A}'' \mathrm{x}_{(\tilde{\alpha}, u)} G$ are then defined on $l^2(G, \mathcal{H})$. Further, by restriction, $C_r^*(\mathcal{A}, H, \alpha, u)$ and $\mathcal{A}'' \mathrm{x}_{(\tilde{\alpha}, u)} H$ are well-defined as acting on $l^2(H, \mathcal{H})$, and we have $C_r^*(\mathcal{A}, H, \alpha, u)'' = \mathcal{A}'' \mathrm{x}_{(\tilde{\alpha}, u)} H$. Let now $(\pi_\sigma, \mathcal{H}_\sigma, \xi_\sigma)$ denote the GNS-triple of $C_r^*(\mathcal{A}, H, \alpha, u)$ associated to σ. By the assumed simplicity of $C_r^*(\mathcal{A}, H, \alpha, u)$, σ is faithful. Further, by the uniqueness of σ, we have $\sigma = \varphi_{\circ E}$ on $C_r^*(\mathcal{A}, H, \alpha, u)$, where E denotes the canonical conditional expectation from $C_r^*(\mathcal{A}, H, \alpha, u)$ onto $\mathcal{A}$. If we define $\eta_0 \in l^2(H, \mathcal{H})$ by

$$\eta_0(h) = \begin{cases} \xi_0 & ,h = e \\ 0 & ,h \neq e,\ h \in H \end{cases}$$

then η_0 is easily seen to be a cyclic unit vector for $C_r^*(\mathcal{A}, H, \alpha, u)$ in $l^2(H, \mathcal{H})$ and it follows

from $\sigma = \varphi_{\circ E}$ that

$$\sigma(y) = (y\eta_0, \eta_0)\ ,\ y \in C_r^*(\mathcal{A}, H, \alpha, u).$$

This means that we may identify $\mathcal{H}_\sigma$ with $l^2(H, \mathcal{H}), \xi_\sigma$ with η_0, $\pi_\sigma(C_r^*(\mathcal{A}, H, \alpha, u))$ with $C_r^*(\mathcal{A}, H, \alpha, u)$ and $\mathcal{A}''\mathrm{x}_{(\tilde{\alpha}, u)}H$ with $\pi_\sigma(C_r^*(\mathcal{A}; H, \alpha, u))''$.

By the decomposition theorems [3; Theorem 2.1] and [2; Theorem 1], we have:

$$C_r^*(\mathcal{A}, G, \alpha, u) \simeq C_r^*(C_r^*(\mathcal{A}, H, \alpha, u), K, \beta, v)$$

and

$$\mathcal{A}''\mathrm{x}_{(\widetilde{\alpha}, u)}G \simeq (\mathcal{A}''\mathrm{x}_{(\widetilde{\alpha}, u)}H)\mathrm{x}_{(\widetilde{\beta}, v)}K,$$

where K denotes the factor group G/H and $\beta, \widetilde{\beta}$ and v are defined as follows:
First, recall that, for each $g \in G$, there exists $\gamma_g \in Aut(C_r^*(\mathcal{A}, H, \alpha, u))$
(resp. $\widetilde{\gamma}_g \in \mathcal{A}''x_{(\tilde{\alpha}, u)}G$) such that

$$\gamma_g(a) = \alpha_g(a), \quad a \in \mathcal{A}, \text{ and}$$
$$\gamma_g(\lambda_u(h)) = u(g, h)u(ghg^{-1}, g)^*\lambda_u(ghg^{-1}), \quad h \in H$$

$$(\mathrm{resp.}\widetilde{\gamma}_g(x) = \widetilde{\alpha}_g(x),\ x \in \mathcal{A}'', \text{ and}$$
$$\widetilde{\gamma}_g(\lambda_u(h)) = \gamma_g(\lambda_u(h)),\ h \in H).$$

It is clear that $\widetilde{\gamma}_g = \gamma_g$ on $C_r^*(\mathcal{A}, H, \alpha, u)$, $g \in G$.
For a chosen section $n : K \to G$ for the canonical homomorphism from G onto K with $n(e) = e$, $\beta(resp \cdot \widetilde{\beta})$ and v are defined by

$$\beta_k = \gamma_{n(k)} \quad (resp.\ \widetilde{\beta}_k = \widetilde{\gamma}_{n(k)}),\ k \in K \text{ and}$$
$$v(k, l) = u(n(k), n(l))u(m(k, l), n(kl))^*\lambda_u(m(k, l)),$$
$$\text{where } m(k, l) = n(k)n(l)n(kl)^{-1}, \quad k, l \in K.$$

If we can show that each $\widetilde{\gamma}_g, g \in G \setminus H$ is outer, it will follow from the above identifications that (β, v) is σ-outer on $C_r^*(\mathcal{A}, H, \alpha, u)$, and Theorem 1 will therefore imply that $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple with a unique trace as desired.

Now, our assumption that G/H acts freely on H implies precisely that each $\widetilde{\gamma}_g, g \in G \setminus H$, is outer. The proof of this assertion is essentially the same as the one of [3; Lemma 3.3], where the outerness of γ_g is shown. We mention here the required changes: Let $\widetilde{\varphi}$ denote the normal trace on $\mathcal{A}''$ associated to ξ_0. Then $\widetilde{\varphi}$ is faithful (again by [25; Prop.3.19]) and

$\widetilde{\varphi}$ is $\widetilde{\alpha}$-invariant (by the α-invariance of φ and the ultraweak continuity of $\widetilde{\alpha}$ and $\widetilde{\varphi}$). In the proof of [3; Lemma 3.3], replace then $\mathcal{A}$ by $\mathcal{A}''$, (α, u) by $(\widetilde{\alpha}, u), \sigma$ by $\mathrm{ad}(g), \varphi$ by $\widetilde{\varphi}$, $\mathcal{B}$ by $\mathcal{A}''\mathrm{x}_{(\widetilde{\alpha},u)}H$, γ_g by $\widetilde{\gamma}_g$, E by the canonical conditional expectation $\widetilde{E}$ from $\mathcal{A}''\mathrm{x}_{(\widetilde{\alpha},u)}H$ onto $\mathcal{A}''$ and define the Fourier coefficients of operators in $\mathcal{A}''\mathrm{x}_{(\widetilde{\alpha},u)}H$ with respect to $\widetilde{E}$ (as in the proof of [3; Theorem 2.2]). Then the proof goes through verbatimly.

□

Corollary 3: Let H denote a normal subgroup of a group G and ω a two-cocycle of G with values in the circle group T (i.e. ω is a multiplier of G). Suppose G/H acts freely on H and $C_r^*(H,\omega)$ is simple with a unique trace. Then $C_r^*(G,\omega)$ is simple with a unique trace.

Proof: Set $\mathcal{A}=\mathbf{C}$, $\alpha = id$ and $u = \omega$ in Theorem 2.□

As promised in the introduction, we may now answer positively the problem raised in [3].

Corollary 4: Let H denote a normal subgroup of a group G with trivial centralizer in G and suppose H is C^*-simple with a unique trace. Then G is C^*-simple with a unique trace.

Proof: Set $\omega = 1$ in Corollary 3 and recall that a C^*-simple group is necessarily ICC, so that G/H acts freely on H. □

This corollary generalizes Akemann and Lee's result [1; Theorem 3], where H is supposed to be a non-abelian free group. If H is a C^*-simple group with a unique trace, we may let G be the automorphism group of H or the holomorph of H, cf. [3; Corollaries 3.7 and 3.8]. Hence, Corollary 3 also generalizes the recent result of Nitica and Török [18]. At last, it shows that an ultraweak Powers group is C^*-simple *and* has a unique trace. Moreover, we have:

Corollary 5: Suppose (α, u) is a cocycle crossed action of an ultraweak Powers group G on a simple C^*-algebra $\mathcal{A}$ with a unique trace. Then $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple with a unique trace.

Proof: The group G contains a normal weak Powers group H with trivial centralizer. Then H is ICC and $C_r^*(\mathcal{A}, H, \alpha, u)$ is simple with a unique trace by [3; Theorems 4.1 and 4.2]. Hence, $C_r^*(\mathcal{A}, G, \alpha, u)$ is simple with a unique trace by Theorem 2. □

By an inductive argument based on the decomposition theorem, the conclusion of Corollary

5 still holds if we only assume that G may be obtained by successive extensions of ultraweak Powers groups. The same is true for the next corollary:

Corollary 6: Suppose $1 \to H \to \Gamma \to G \to 1$ is an exact sequence of groups, where H is C^*-simple with a unique trace and G is an ultraweak Powers group. Then Γ is C^*-simple with a unique trace.

Proof: Since $C_r^*(\Gamma)$ decomposes as $C_r^*(C_r^*(H), G, \alpha, u)$ for some (α, u), this follows from Corollary 5. □

This generalizes Boca and Nitica's [6; Theorem].

We next give an application of theorem 1 to crossed products associated with characters of groups.
Let $\widehat{G}$ denote the character group of a group G. Recall that a character on G is defined to be a group homomophism from G to the circle group T, and that $\widehat{G}$ is the abelian group consisting of all characters of $\widehat{G}$ under pointwise multiplication.
For each $\gamma \in \widehat{G}$, we denote by α_γ (resp. $\widetilde{\alpha}_\gamma$) the *-automorphism of $C_r^*(G)$ (resp. $C_r^*(G)''$) induced by γ (cf.[4],[26]). Then $\widetilde{\alpha}_\gamma = \alpha_\gamma$ on $C_r^*(G)$. If τ denotes the canonical trace on $C_r^*(G)$, it is well known that we may identify $\mathcal{H}_\tau$ with $l^2(G)$ and π_τ with the identity representation. Further, if G is ICC, then the induced action $\widetilde{\alpha} : \widehat{G} \to Aut(C_r^*(G)'')$ is outer ([4; Theorem 5.4]), which implies that $\alpha : \widehat{G} \to Aut(C_r^*(G))$ is also outer (see also [26; Prop. 2.13]).

Theorem 7: Let K denote a subgroup of $\widehat{G}$. Then the crossed product $C_r^*(C_r^*(G), K, \alpha)$ is simple (resp. simple with a unique trace) whenever G is C^*-simple (resp. C^*-simple with a unique trace).

Proof: A C^*-simple group being ICC, the first part follows from Kishimoto's theorem [16; Theorem 3.1] and the fact that α is outer, while the respective part follows from theorem 1 and the fact that $\widetilde{\alpha}$ is outer. □

In [26; Conjecture 1, p.117], Yin conjectured that $C_r^*(C_r^*(G), K, \alpha)$ has a unique trace whenever G is a Powers group and $K = \{\gamma^n, n \in \mathbf{Z}\}$ for som $\gamma \in \widehat{G}$ of infinite order. Theorem 7 provides a more general answer to this conjecture (see our comment in the introduction).

Finally, we consider cross products of irrational rotation algebras by groups of toral automorphisms. Let θ be a fixed irrational number in $(0, 1)$ and let $\mathcal{A}_\theta$ denote the corresponding

rotation algebra, which is known to be simple and have a unique trace φ (cf. [21], [23]). It is the (universal) C^*-algebra generated by two unitaries U and V satisfying $UV = e^{2\pi i\theta}VU$.

For each $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbf{Z})$, let α_A denote the toral automorphism of $\mathcal{A}_\theta$ induced by A ([9], [27]), which is determined by

$$\alpha_A(U) = e^{-\pi i\theta ac}\, U^a V^c \quad \text{and}$$
$$\alpha_A(V) = e^{-\pi i\theta bd}\, U^b V^d.$$

Let also $\alpha : SL(2,\mathbf{Z}) \to Aut(\mathcal{A}_\theta)$ denote the induced action. Then we have:

Theorem 8: Let G be any subgroup of $SL(2,\mathbf{Z})$. Then the reduced crossed product $C_r^*(\mathcal{A}_\theta, G, \alpha)$ is simple with a unique trace.

Proof: The result may be obtained without too much difficulty as a direct consequence of Theorem 1. We will sketch a slightly different approach, based on the well known fact that $\mathcal{A}_\theta$ may be realized as $C_r^*(\mathbf{Z}^2,\sigma)$, where $\sigma : \mathbf{Z}^2 \times \mathbf{Z}^2 \to T$ is defined by

$$\sigma(x,y) = \exp(\pi i x^t \left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right) y).$$

Then α_A is determined by $\alpha_A(\lambda_\sigma(x)) = \lambda_\sigma(Ax), (x \in \mathbf{Z}^2)$.

Now, let Γ denote the semidirect product of $\mathbf{Z}^2$ by G under the natural action of G on $\mathbf{Z}^2$, and let $\omega : \Gamma \times \Gamma \to T$ be defined by

$$\omega((x,A),(y,B)) = \exp(\pi i x^t \left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right) Ay).$$

Then one verifies easily that ω is a multiplier of Γ, which clearly coincides with σ on $\Gamma_1 = \mathbf{Z}^2 \mathrm{x} \{I\}$. As remarked at the beginning of this section, each $A \in SL(2,\mathbf{Z})$, $A \neq I$, acts freely on $\mathbf{Z}^2$, and it follows therefore that Γ/Γ_1 acts freely on Γ_1. Since $\mathcal{A}_\theta \simeq C_r^*(\Gamma_1,\omega)$ is simple with a unique trace, the same is true for $C_r^*(\Gamma,\omega)$ by Corollary 3. A straightforward application of the decomposition theorem [3; Theorem 2.1] gives that

$$C_r^*(\Gamma,\omega) \simeq C_r^*(C_r^*(\Gamma_1,\omega), G, \beta, v)$$

where β coincides with α on $C_r^*(\Gamma_1,\omega) \simeq \mathcal{A}_\theta$ and $v \equiv 1$. Thus, $C_r^*(\Gamma,\omega) \simeq C_r^*(\mathcal{A}_\theta, G, \alpha)$ and the result follows. □

The case $G = \{I, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$ has recently been studied in [8]. We also refer the reader to [19] and [20] for some related results, especially when G is generated by a matrix of infinite order.

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