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TWO-DIMENSIONAL HOMOGENEOUS SPACES

by

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International Sophus Lie Center

HOMOGENEOUS SPACES
B. KOMRAKOV SEMINAR

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CONTENTS

Foreword

This is a preliminary version of an expository text intended for the Summer School *Lie group analysis of differential equations* (Nordfjoreid, Norway, 1993). We feel that the theory of two-dimensional homogeneous spaces, or, if one prefers, in local terms the theory of finite-dimensional subalgebras of the Lie algebra of all vector fields on the plane, and to a greater extent their applications, have not really become part of mainstream mathematics, although they are absolutely basic, in particular to the theory of differential equations, and were introduced over a century ago by Sophus Lie (see the recent discussion on this topic in [7]). We are in the possession of the classification of all two-dimensional homogeneous spaces, obtained by purely algebraic methods (via the description of so-called effective pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2). We do not present the proof of this classification here, partly to save space and time and, partly because of the didactic character of this text. To conclude the Foreword, we list a few references that are relevent to the present exposition.

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CHAPTER I

NAIVE APPROACH

§1. Smooth functions

1.1. Smooth functions on the line. Let us recall some basic definitions of differential calculus.

Definition 1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *differentiable at the point* $a \in \mathbb{R}$ if there exists a finite limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Then this limit is called the *derivative of the function f at the point a* and is denoted by $f'(a)$ or $\frac{df}{dx}(a)$.

For any function f differentiable at the point a, the following condition holds:

$$f(x) = f(a) + f'(a)(x - a) + o(x - a).$$

Conversely, suppose that there exist $A,\ B \in \mathbb{R}$ such that

$$f(x) = A + B(x - a) + o(x - a).$$

Then it is easy to show that f is differentiable at a and $A = f(a)$, $B = f'(a)$. So, we see that all functions differentiable at a point $a \in \mathbb{R}$ are exactly those functions which can be approximated by linear mappings up to infinitesimals of the first order.

A function $f : \mathbb{R} \to \mathbb{R}$ is called *differentiable* if it is differentiable at each point $a \in \mathbb{R}$. To every differentiable function f we assign the function $f' : \mathbb{R} \to \mathbb{R}$ that takes any point $a \in \mathbb{R}$ to $f'(a) \in \mathbb{R}$.

Definition 2. A function $f : \mathbb{R} \to \mathbb{R}$ is called *continuously differentiable* if it is differentiable and f' is continuous.

By $C^1(\mathbb{R})$ denote the set of all continuously differentiable functions.

Exercise. Show that the set $C^1(\mathbb{R})$ is closed under addition and multiplication of functions.

Below we shall construct by induction the chain of embedded classes of functions:

$$C^1(\mathbb{R}) \supset C^2(\mathbb{R}) \supset \dots \supset C^k(\mathbb{R}) \supset \dots$$

Definition 3. A function $f \in C^k(\mathbb{R})$ is called $(k+1)$ *times continuously differentiable* if $f' \in C^k(\mathbb{R})$ (i.e. f' is k times continuously diffirentiable). In this case the function

$f^{(k+1)} = (f')^{(k)}$ is called the $(k+1)$*-th derivative* of f. By $C^{k+1}(\mathbb{R})$ denote the set of all $(k+1)$ times continuously differentiable functions.

Exercise.
1) Show that $C^k(\mathbb{R})$ is closed under addition and multiplication of functions.
2) Show that the functions $f(x) = x^k \cdot |x|$ belong to $C^k(\mathbb{R})$ but do not belong to $C^{k+1}(\mathbb{R})$.

Definition 4. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *smooth* if $f \in C^k(\mathbb{R})$ for each $k \in \mathbb{N}$.

By $C^\infty(\mathbb{R})$ denote the set of all smooth functions:

$$C^\infty(\mathbb{R}) = \bigcap_{k=1}^{\infty} C^k(\mathbb{R}).$$

Examples.
1. The simplest examples of smooth functions are constant mappings, linear functions, and polynomials.
2. Since $e^x \in C^1(\mathbb{R})$ and $(e^x)' = e^x$, we see that e^x is a smooth function.
3. Let f and g be smooth functions. Then the functions $f \cdot g$, $f+g$, $f \circ g$ are smooth. Moreover, if $f(x) \neq 0$ for all $x \in \mathbb{R}$, then the function $\frac{1}{f(x)}$ is also smooth. This gives us some more examples of smooth functions, for instance

$$\frac{1}{(e^{-\frac{1}{2}x^2} + 2x^2e^{e^{2x}})} \in C^\infty(\mathbb{R}).$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then the functions $f^{(k)}(x)$ and $F(x) = \int\limits_0^x f(t)dt$ are also smooth. For example,

$$F(x) = \int\limits_0^x e^{-t^2} dt \in C^\infty(\mathbb{R}).$$

Exercise. Show that the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases} \tag{1}$$

is smooth.

Let U be an open subset of $\mathbb{R}$ (for instance, an open interval). Similarly, we can introduce the concept of a function differentiable on U and define the classes $C^k(U)$, $C^\infty(U)$. For example, $f(x) = \sqrt{x}$ belongs to $C^\infty(U)$, where $U = (0, +\infty)$.

Exercise. Show that the function $f(x) = x\sqrt[3]{x}$ belongs to $C^\infty(0,+\infty)$, but does not belong to $C^\infty(\mathbb{R})$.

Suppose $f \in C^\infty(\mathbb{R})$ and $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Then it is possible to show that f is a bijection of $\mathbb{R}$ onto $f(\mathbb{R})$. Moreover, $U = f(\mathbb{R})$ is an open subset of $\mathbb{R}$ and the inverse function $f^{-1} : U \to \mathbb{R}$ also belongs to $C^\infty(\mathbb{R})$. For instance, the function $\ln x$, which can be uniquely determined from the equation $\ln(e^x) = x$, is smooth on the interval $(0, +\infty)$.

In the sequel we shall make use of the following fact:

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R} \in C^\infty(\mathbb{R})$ and $f(a) = 0$ for some point $a \in \mathbb{R}$. Then there exists a smooth mapping $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = (x - a)g(x)$ for all $x \in \mathbb{R}$.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and $a \in \mathbb{R}$. Consider the following power series:

$$T_f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots,$$

which is called the *Taylor series* of f at the point a. This series is not necessarily convergent. But even if it does converge, the sum $T_f(x)$ is not necessarily equal to $f(x)$.

Definition 5. A smooth function f is called *analytic* at a point $a \in \mathbb{R}$ if the Taylor series of f at a converges in some neighborhood of a and its sum is equal to $f(x)$.

Example. Consider the function f given by (1) and put $a = 0$. It can be verified that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$. Therefore, the Taylor series of f at the point $a = 0$ converges to the zero function. But f is a nonzero function on any neighborhood of $a = 0$. Thus, f is not an analytic function at $a = 0$.

By $C^\omega(\mathbb{R})$ denote the set of all *analytic functions* on $\mathbb{R}$, i.e. functions that are analytic at each point $a \in \mathbb{R}$. As a matter of fact, all smooth functions considered in the previous examples are analytic.

Theorem 1 implies that the following result is true:

Theorem 1'. *Let $f \in C^\omega(\mathbb{R})$ be a nonzero function and $f(a) = 0$ for some point $a \in \mathbb{R}$. Then the function f can be uniquely represented as*

$$f(x) = (x-a)^n h(x),$$

where $h \in C^\omega(\mathbb{R})$ and $h(a) \neq 0$.

Remark. Generally speaking, theorem 1' is no longer valid for smooth functions. For example, so is the case when f has the form (1) and $a = 0$.

1.2. Smooth functions on the plane. By the plane we shall mean the set of all pairs of reals:

$$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

We can consider $\mathbb{R}^2$ as a real vector space. This means that pairs of numbers can be called vectors. These vectors can naturally be added and multiplied by real numbers:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$

$$\lambda \cdot (x_1, x_2) = (\lambda x_1, \lambda x_2).$$

We shall denote the plane by V^2 if we want to emphasize that we consider it as a vector space.

Fix a point $a \in \mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function on the plane. To every vector $v \in V^2$ we assign the function $g_v : \mathbb{R} \to \mathbb{R}$ by the formula

$$g_v(t) = f(a + tv).$$

We say that the function f *is differentiable at the point a along the vector $v \in V^2$* if g_v is differentiable at $t = 0$. Then $g_v'(0)$ is called the *derivative of f at the point a along the vector v* and is denoted by $f_v'(a)$. From this definition it follows that

$$f_v'(a) = \lim_{t \to 0} \frac{f(a + tv) - f(a)}{t}.$$

The derivatives along the vectors $(1, 0)$ and $(0, 1)$ are called *the first and the second partial derivatives* of f at a and are denoted by $D_1 f(a)$ and $D_2 f(a)$ respectively. (Sometimes we shall use another notation: $\frac{\partial f}{\partial x_1}(a)$ and $\frac{\partial f}{\partial x_2}(a)$.) If the partial derivatives of f exist at each point $a \in \mathbb{R}^2$, then we can define the following functions:

$$D_i f : \mathbb{R}^2 \to \mathbb{R}, \ a \mapsto D_i f(a), \ i = 1, 2.$$

For example, for $f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$ we have

$$D_1 f = 2x_1 + x_2, \quad D_2 f = x_1 + 2x_2.$$

Exercise. Find the functions $D_1 f$ and $D_2 f$ for $f : \mathbb{R}^2 \to \mathbb{R}$, where

$$f(x_1, x_2) = \begin{cases} x_1 x_2 \dfrac{x_1^2 - x_2^2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq 0, \\ 0 & \text{if } (x_1, x_2) = 0. \end{cases}$$

Definition. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *continuously differentiable* if the functions $D_1 f$ and $D_2 f$ are defined at each point of the plane and are continuous.

By $C^1(\mathbb{R}^2)$ we denote the set of all continuously differentiable functions on the plane. As well as for functions on the line, we shall construct by induction the chain of embedded classes of functions:

$$C^1(\mathbb{R}^2) \supset C^2(\mathbb{R}^2) \supset \cdots \supset C^\infty(\mathbb{R}^2).$$

Definition. A function $f \in C^k(\mathbb{R}^2)$ is called $(k+1)$ *times continuously differentiable* if the functions $D_1 f$ and $D_2 f$ are k times continuously differentiable. The set of all $(k+1)$ times continuously differentiable functions is denoted by $C^{k+1}(\mathbb{R}^2)$.

We say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is *smooth* if it is k times continuously differentiable for all $k \in \mathbb{N}$. The set of all smooth functions on $\mathbb{R}^2$ is denoted by $C^\infty(\mathbb{R}^2)$:

$$C^\infty(\mathbb{R}^2) = \bigcap_{k=1}^{\infty} C^k(\mathbb{R}).$$

Examples.

1) As in the case of $\mathbb{R}$, the simplest examples of smooth functions on $\mathbb{R}^2$ are constant mappings, linear functions, and polynomials in two variables x_1 and x_2.

2) The set $C^\infty(\mathbb{R}^2)$ is closed under addition and multiplication of functions, i.e. $C^\infty(\mathbb{R}^2)$ is a commutative algebra. Besides, if $f \in C^\infty(\mathbb{R}^2)$ and $f(a) \neq 0$ for all $a \in \mathbb{R}^2$, then the function $1/f$ is also smooth.

3) Suppose $f, f_1, f_2 \in C^\infty(\mathbb{R}^2)$ and $g \in C^\infty(\mathbb{R})$. Then the functions

$$(x_1, x_2) \mapsto g(f(x_1, x_2))$$

and

$$(x_1, x_2) \mapsto f(f_1(x_1, x_2), f_2(x_1, x_2))$$

are also smooth. For example,

$$\frac{x_1^2 + e^{2x_1 - x_2^3}}{(x_1^2 + x_2^2 + 1)e^{-x_1 x_2}} \in C^\infty(\mathbb{R}^2).$$

We shall now formulate some important results omitting the proofs.

Theorem 2. *Suppose $f_1, f_2 \in C^1(\mathbb{R})$ and $g \in C^1(\mathbb{R}^2)$. Then the function $h(x) = g(f_1(x), f_2(x))$ belongs to $C^1(\mathbb{R})$ and*

$$h'(a) = \frac{\partial g}{\partial x_1}(f_1(a), f_2(a)) \cdot f_1'(a) + \frac{\partial g}{\partial x_2}(f_1(a), f_2(a)) \cdot f_2'(a).$$

Corollary. *Let $f \in C^1(\mathbb{R}^2)$, $v = (v_1, v_2) \in V^2$. Then*

$$f_v'(a) = v_1 \frac{\partial f}{\partial x_1}(a) + v_2 \frac{\partial f}{\partial x_2}(a)$$

for all $a \in \mathbb{R}^2$.

Thus, the knowledge of the partial derivatives of a function at each point of $\mathbb{R}^2$ allows to find its derivative along any vector $v \in V^2$.

Theorem 3. *Suppose $f \in C^2(\mathbb{R}^2)$. Then*

$$D_1 D_2 f = D_2 D_1 f.$$

In the sequel, for the sake of convenience, we shall write

$$D_{i_1 i_2 \dots i_n} f \quad \text{or} \quad \frac{\partial^n f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_n}}$$

instead of $D_{i_1} D_{i_2} \dots D_{i_n} f$. Besides, if $f \in C^n(\mathbb{R}^2)$, then we can permute the indeces $i_1, i_2, \dots, i_n$.

For functions on the plane, as well as for functions on $\mathbb{R}$, the following fact is true:

Theorem 4. *Let $f \in C^\infty(\mathbb{R}^2)$ and $f(a) = 0$ for a certain point $a = (a_1, a_2) \in \mathbb{R}^2$. Then there exist functions $g_1, g_2 \in C^\infty(\mathbb{R}^2)$ such that*

$$f(x_1, x_2) = (x_1 - a_1) g_1(x_1, x_2) + (x_2 - a_2) g_2(x_1, x_2).$$

Suppose U is an open subset on the plane. Then, in the similar way, we can introduce the concept of directional derivative of a function $f : U \to \mathbb{R}$ at the point $a \in U$. We can also define the classes of functions $C^k(U)$ and $C^\infty(U)$. For example, the function $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ is smooth on the set $U = \mathbb{R}^2 \backslash \{0\}$ but is not smooth on $\mathbb{R}^2$.

§2. Diffeomorphisms of the plane

2.1. The group of diffeomorphisms of $\mathbb{R}^2$. A one-to-one mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called a *transformation* of $\mathbb{R}^2$. The set of all transformations of $\mathbb{R}^2$ forms a group with respect to composition of mappings and is denoted by $\mathrm{Aut}(\mathbb{R}^2)$ or $\mathrm{Bij}(\mathbb{R}^2)$.

Transformations of the vector space V^2 are those transformations of $\mathbb{R}^2$ that preserve addition of vectors and multiplication of vectors by scalars. They are called *linear* and have the form:

$$(x_1, x_2) \mapsto (a_{11} x_1 + a_{12} x_2, a_{21} x_1 + a_{22} x_2), \tag{1}$$

where $a_{ij} \in \mathbb{R}$, $a_{11} a_{22} - a_{12} a_{21} \neq 0$.

The set of all transformations of the vector space V^2 (= the set of all linear transformations of the plane) forms a subgroup of $\mathrm{Bij}(\mathbb{R}^2)$ and is denoted by $\mathrm{Aut}(V^2)$ or $\mathrm{GL}(\mathbb{R}^2)$. Thus, considering the plane as a vector space, we assume that every admissible transformation has form (1). Similarly, we can consider the plane as an *affine space* corresponding to the vector space V^2.

Any transformation of the affine plane has the form

$$(x_1, x_2) \mapsto (a_{11} x_1 + a_{12} x_2 + b_1, a_{21} + x_1 a_{22} + b_2),$$

where $a_{ij}, b_i \in \mathbb{R}$, $a_{11}a_{22} - a_{12}a_{21} \neq 0$. These transformations are called *affine transformations*. Each affine transformation can be uniquely written as $t_v \circ \varphi$, where φ is a linear transformation and

$$t_v : u \mapsto u + v \ (u \in \mathbb{R}^2)$$

is a parallel translation by the vector v. The set of all transformations of the affine space A^2 (= the set of all affine transformations of the plane) forms a group. This group is denoted by $\mathrm{Aut}(A^2)$ or $\mathrm{Aff}(\mathbb{R}^2)$.

Note that the set of admissible transformations performs a significant part in the study of the plane as a set, a vector space, and an affine space. Our aim is to study the plane as a smooth manifold. We shall not give any rigorous definition of a smooth manifold, but we shall describe the transformation group of the plane considered as a smooth manifold.

Every transformation of $\mathbb{R}^2$ has the form:

$$\varphi : (x_1, x_2) \mapsto (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2)),$$

where φ_1, φ_2 are certain mappings of $\mathbb{R}^2$ into $\mathbb{R}$.

Definition 1. The mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$ is called a *diffeomorphism* or *smooth transformation* if the following conditions hold:

1° $\varphi \in \mathrm{Bij}(\mathbb{R}^2)$;
2° $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^2)$;
3° $(\varphi^{-1})_1, (\varphi^{-1})_2 \in C^\infty(\mathbb{R}^2)$.

Examples.

1) Any linear and even affine transformation is a diffeomorphism.
2) Define the transformation by the rule

$$(x_1, x_2) \mapsto (x_1 + 1, e^{x_1} x_2).$$

It is a diffeomorphism of the plane $\mathbb{R}^2$. It can be easily shown that the inverse of this transformation has the form

$$(x_1, x_2) \mapsto (x_1 - 1, e^{-x_1+1} x_2).$$

3) Suppose $f \in C^\infty(\mathbb{R}^2)$; then the transformation

$$(x_1, x_2) \mapsto (x_1, x_2 + f(x_1))$$

is a diffeomorphism of the plane and is called a *shift*.

Exercise. Describe the inverse of the transformation given in example 3).

4) Consider the following mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$:

$$(x_1, x_2) \mapsto (x_1, x_1 x_2);$$

$$(x_1, x_2) \mapsto (x_1 + x_2, \sqrt[3]{x_1 - x_2});$$

$$(x_1, x_2) \mapsto (x_1^5, x_2).$$

Since these mappings do not satisfy conditions $1°, 2°, 3°$ of definition 1, respectively, we see that they are not diffeomorphisms.

Exercise. Show that linear and affine transformations could also be defined as mappings $\mathbb{R}^2 \to \mathbb{R}^2$ satisfying conditions 1°–3° of definition 1 if we replaced the set $C^\infty(\mathbb{R}^2)$ in this definition by the following classes of functions:

$$\{(x_1, x_2) \mapsto a_1x_1 + a_2x_2 | a_1, a_2 \in \mathbb{R}\},$$

$$\{(x_1, x_2) \mapsto a_1x_1 + a_2x_2 + b | a_1, a_2, b \in \mathbb{R}\}$$

respectively. Prove that in this case condition 3° is redundant.

Further, let M^2 denote the plane considered as a two-dimensional manifold. The set of all diffeomorphisms of the plane M^2 forms a group, which is denoted by $\mathrm{Aut}(M^2)$ or $\mathrm{Diff}(\mathbb{R}^2)$.

2.2. Local diffeomorphisms. Now we shall give some variations of definition 1.

Let U and V be two open domains on the plane.

Definition 2. The mapping $\varphi : U \mapsto V, (x_1, x_2) \mapsto (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$ is called a *diffeomorphism* of U onto V if the following conditions hold:

1° φ is a one–to–one mapping;

2° $\varphi_1, \varphi_2 \in C^\infty(U)$;

3° $(\varphi^{-1})_1, (\varphi^{-1})_2 \in C^\infty(V)$.

Examples.

1. Let $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a certain diffeomorphism of the plane and U some open domain in $\mathbb{R}^2$. Then $\varphi(U)$ is an open domain in $\mathbb{R}^2$ and $\varphi|_U : U \to \varphi(U)$ is a diffeomorphism of U onto $\varphi(U)$.

2. The mapping

$$\varphi : (x_1, x_2) \mapsto (x_1 \cos x_2, x_1 \sin x_2)$$

is a diffeomorphism of the domain

$$U = \{(x_1, x_2) \mid x_1 > 0,\ 0 < x_2 < 2\pi\}$$

onto

$$V = \mathbb{R}^2 \backslash \{(x, 0) | x \geq 0\}.$$

Exercise. Describe the largest domain on the plane such that the restriction of the following mapping to it is a diffeomorphism:

a) $(x_1, x_2) \mapsto (x_1 + x_2, (x_1 - x_2)^3)$;

b) $(x_1, x_2) \mapsto (x_1, x_1x_2)$.

The local form of definition 1 is

Definition 3. A mapping $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is said to be a *local diffeomorphism* at the point $a \in \mathbb{R}^2$ if there exist two neighborhoods U and V of the points a and $\varphi(a)$, respectively, such that $\varphi|_U$ is a diffeomorphism of U onto V.

Note. Let W be a certain domain on the plane such that $a \in W$. Then we can extend definition 3, assuming that φ is defined only on W.

Every diffeomorphism $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a local diffeomorphism at each point of the plane. Moreover, U can be chosen arbitrarily. In general, the converse is not true. To prove this we consider the mapping given by $(x_1, x_2) \mapsto (x_1^2, x_2)$. It is a local diffeomorphism if $x_1 \neq 0$, but is not even a one-to-one mapping of the plane.

Examples.

1. Let the mapping φ be given by

$$(x_1, x_2) \mapsto (\cos x_1, x_2^3).$$

Then φ is a local diffeomorphism at a point (x_1, x_2) such that $x_1 \neq \pi n$, $x_2 \neq 0$.

2. Define the mapping φ by the rule

$$(x_1, x_2) \mapsto (x_1 x_2, x_1^2 x_2^2 + 1).$$

Then φ is a local diffeomorphism at no point of the plane.

The mapping $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}^2, (x_1, x_2) \mapsto (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$ is smooth if $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^2)$. For example, diffeomorphisms are smooth one-to-one mappings of the plane that have smooth inverses. Let $a \in \mathbb{R}^2$. There is a simple method to determine whether φ is a local diffeomorphism at the point a or not. Consider the matrix

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1}(a) & \frac{\partial \varphi_1}{\partial x_2}(a) \\ \frac{\partial \varphi_2}{\partial x_1}(a) & \frac{\partial \varphi_2}{\partial x_2}(a) \end{pmatrix},$$

which is called the *Jacobi matrix of the mapping φ at the point a.*

Theorem 1. *The smooth mapping $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a local diffeomorphism at $a \in \mathbb{R}^2$ if and only if the Jacoby matrix of φ at a is non-singular.*

§3. Vector fields on the plane

Now we shall introduce the concept of a vector field on the plane. Since the concept is extremely important, we shall give several different definitions and set the correspondence between them.

3.1. Naive definition. To every point on the plane we assign a vector such that its coordinates are smooth functions of coordinates on the plane. For example, suppose that a liquid flows on the plane. Then to every point of $\mathbb{R}^2$ we can assign the velocity vector of the liquid at this point. More rigorously,

Definition 1. A *vector field on the plane* is a smooth mapping $\mathbf{v} : \mathbb{R}^2 \to V^2$ that takes every point $a \in \mathbb{R}$ to a vector $\mathbf{v}_a = \mathbf{v}(a)$.

Let us recall that a function $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth if

$$\mathbf{v}(x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2)),$$

where $v_1, v_2 \in C^\infty(\mathbb{R}^2)$. By $\mathcal{D}(\mathbb{R}^2)$ denote the set of all vector fields on the plane. The set $\mathcal{D}(\mathbb{R}^2)$ can be supplied with the operations of addition and multiplication by constants:

$$(\mathbf{v}_1 + \mathbf{v}_2)(a) = \mathbf{v}_1(a) + \mathbf{v}_2(a), \text{ where } \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{D}(\mathbb{R}^2);$$

$$(\lambda \mathbf{v})(a) = \lambda \cdot \mathbf{v}(a), \text{ where } \lambda \in \mathbb{R}, \ \mathbf{v} \in \mathcal{D}(\mathbb{R}^2).$$

Thus, $\mathcal{D}(\mathbb{R}^2)$ is a vector space. It is also possible to multiply vector fields by smooth functions:

$$(f\mathbf{v})(a) = f(a) \cdot \mathbf{v}(a), \text{ where } f \in C^\infty(\mathbb{R}^2), \ \mathbf{v} \in \mathcal{D}(\mathbb{R}^2).$$

It is easy to verify that $f\mathbf{v}$ is indeed a vector field.

By $\frac{\partial}{\partial x_1}$ (respectively, $\frac{\partial}{\partial x_2}$) denote the constant vector field

$$(x_1, x_2) \mapsto (1, 0)$$

(respectively, $(x_1, x_2) \mapsto (0, 1)$). This strange notation will be clear from other interpretations of vector fields.

Exercise. Show that any vector field $\mathbf{v}$ can be written uniquely in the form:

$$\mathbf{v} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}, \text{ where } v_1, v_2 \in C^\infty(\mathbb{R}^2).$$

Let $\mathbf{v}$ be a vector field and $a \in \mathbb{R}^2$ some point on the plane. The vector $\mathbf{v}_a \in V^2$ is called a *tangent vector to the plane at the point a*. By $T_a\mathbb{R}^2$ denote the set of all tangent vectors to the plane at a point $a \in \mathbb{R}^2$:

$$T_a\mathbb{R}^2 = \{\mathbf{v}_a \mid \mathbf{v} \in \mathcal{D}(\mathbb{R}^2)\}.$$

Exercise. For each vector $v \in V$, find a vector field $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$ such that $\mathbf{v}_a = \mathbf{v}$.

This exercise shows that $T_a\mathbb{R}^2$ is a vector space V "attached" to the plane at the point a.

3.2. Algebraic point of view. Let $f \in C^\infty(\mathbb{R}^2)$ be a smooth function and $\mathbf{v}$ a vector field. Fix a point $a \in \mathbb{R}^2$. Let us consider the derivative of f along the vector $\mathbf{v}_a$:

$$f'_{\mathbf{v}_a}(a) = \lim_{t \to 0} \frac{f(a + t\mathbf{v}_a) - f(a)}{t}.$$

As we can see, it is a number. Let us assign to every point $a \in \mathbb{R}^2$ the derivative of f along the vector $\mathbf{v}_a$:

$$a \mapsto f'_{\mathbf{v}_a}.$$

Thus, we obtain a new function of $\mathbb{R}^2$ to $\mathbb{R}$, which is denoted by $\mathbf{v}(f)$. It shows the rate of change of the function f along the vector field $\mathbf{v}$.

Exercise. Show that
1) $(\mathbf{v}_1 + \mathbf{v}_2)(f) = \mathbf{v}_1(f) + \mathbf{v}_2(f)$;
2) $(g\mathbf{v})(f) = g \cdot \mathbf{v}(f)$,
where $\mathbf{v}_1, \mathbf{v}_2, v \in \mathcal{D}(\mathbb{R}^2)$ $f, g \in C^\infty(\mathbb{R}^2)$.

Examples.
1) Let $\mathbf{v} = \frac{\partial}{\partial x_1}$. Then for any point $a = (x_1, x_2)$ we have:

$$f'_{\mathbf{v}_a}(a) = \lim_{t \to 0} \frac{f(x_1 + t, x_2) - f(x_1, x_2)}{t} = \frac{\partial f}{\partial x_1}(a).$$

Thus, $\mathbf{v}(f) = \frac{\partial f}{\partial x_1}$. This gives the explanation of the notation of section 3.1.

2) Suppose $\mathbf{v} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}$. Then from the previous exercise it immediately follows that $\mathbf{v}(f) = v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\partial f}{\partial x_2}$ for all $f \in C^\infty(\mathbb{R}^2)$. In particular, this shows that $\mathbf{v}(f)$ also belongs to $C^\infty(\mathbb{R}^2)$.

So, each vector field $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$ defines the mapping:

$$C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2), \quad f \mapsto \mathbf{v}(f).$$

In the following we shall denote this mapping in the same way as the vector field itself.

Problem. Prove that
1° $\mathbf{v}$ is a linear mapping over $\mathbb{R}$;
2° for any $f, g \in C^\infty(\mathbb{R}^2)$ we have:

$$\mathbf{v}(f \cdot g) = v(f) \cdot g + f \cdot \mathbf{v}(g).$$

We see that a mapping $\mathbf{v}$ is a generalization of the concept of differentiation of a function. So, it gives us some reasons to consider the concept of a vector field from the other point of view. A mapping of $C^\infty(\mathbb{R}^2)$ to $C^\infty(\mathbb{R}^2)$ is called a *derivation of the algebra of functions* $C^\infty(\mathbb{R}^2)$ if it satisfies conditions 1° and 2° above.

Definition 2. A derivation of the algebra of functions $C^\infty(\mathbb{R}^2)$ is called a *vector field.*

The following theorem establishes the relationship between Definitions 1 and 2.

Theorem 1. *Any derivation d of the algebra of functions $C^\infty(\mathbb{R}^2)$ has the form:*

$$d : f \mapsto v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\partial f}{\partial x_2}$$

for certain $v_1, v_2 \in C^\infty(\mathbb{R}^2)$.

Proof. Put $v_1 = d(x_1)$ and $v_2 = d(x_2)$. Note that

$$d(1) = d(1 \cdot 1) = d(1) \cdot 1 + 1 \cdot d(1) = 2d(1).$$

It follows that $d(1) = 0$ and therefore $d(c) = c \cdot d(1) = 0$ for all $c \in \mathbb{R}$. Fix an arbitrary point $a = (a_1, a_2)$ of $\mathbb{R}^2$. From Theorem 4, §1, it follows that any function f can be written as

$$f(x_1, x_2) = f(a) + (x_1 - a_1) g_1(x_1, x_2) + (x_2 - a_2) g_2(x_1, x_2),$$

where $g_1, g_2 \in C^\infty(\mathbb{R}^2)$. Finding the partial derivatives of the left- and right-hand sides of the equality at the point a, we obtain:

$$g_1(a) = \frac{\partial f}{\partial x_1}(a) \text{ and } g_2(a) = \frac{\partial f}{\partial x_2}(a).$$

Now, using the properties of derivations, we have:

$$d(f) = d((x_1 - a_1) \cdot g_1) + d((x_2 - a_2) \cdot g_2) = (x_1 - a_1) d(g_1) + v_1 g_1 + (x_2 - a_2) d(g_2) + v_2 g_2.$$

Hence,

$$d(f)(a) = v_1(a) g_1(a) + v_2(a) g_2(a) = v_1(a) \frac{\partial f}{\partial x_1}(a) + v_2(a) \frac{\partial f}{\partial x_2}(a).$$

Since the last equality holds for each point $a \in \mathbb{R}^2$, we have

$$d(f) = v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\partial f}{\partial x_2}.$$

Exercise. Show that every tangent vector to the plane at a point a can be identified with a linear mapping $p : C^\infty(\mathbb{R}^2) \to \mathbb{R}$ such that

$$p(f \cdot g) = p(f) \cdot g(a) + f(a) \cdot p(g).$$

3.3 Geometric point of view. Consider another interpretation of a tangent vector to the plane at a fixed point. A *smooth curve* is a smooth mapping $s : I \to \mathbb{R}^2$, where I is some open interval of $\mathbb{R}$. The condition of smoothness means that $s(t) = (s_1(t), s_2(t))$, where $s_1, s_2 \in C^\infty(I)$. At each point $t_0 \in I$, it is possible to find the tangent vector to the curve s:

$$s'(t_0) = (s_1'(t_0), s_2'(t_0)) = \lim_{t \to 0} \frac{s(t_0 + t) - s(t_0)}{t}.$$

Now, let a be a fixed point on the plane. Consider the curves $s : I \to \mathbb{R}^2$ passing through the point a. Without loss of generality we can assume that $0 \in I$ and $s(0) = a$.

Exercise. For each vector $v \in V^2$ find a smooth curve $s : \mathbb{R} \to \mathbb{R}^2$ such that $s(0) = a$ and $s'(0) = v$.

But there is the possibility that different curves have the same tangent vector. For example, the curves $t \mapsto (t, 0)$ and $t \mapsto (t, t^2)$ have the same tangent vector at the point $t = 0$.

Definition 3. Let s_1, s_2 be smooth curves passing through a point $a \in \mathbb{R}$. We shall say that they are *equivalent at the point* a if their derivatives coincide at a. Equivalence classes of curves passing through a are called *vectors tangent to the plane at the point* a.

In addition, consider one way of constructing vector fields on the plane. Let $\{s_\alpha : I_\alpha \to \mathbb{R}^2\}$ be a set of curves on the plane such that their images $s_\alpha(I_\alpha)$ cover the plane without intersections. For instance, it can be parallel lines $\{s_\alpha :\ \mathbb{R} \to \mathbb{R}^2,\ t \to (\alpha, t)\}$. Then to every point $a \in \mathbb{R}^2$ we can assign the tangent vector to the curve passing through a. In this way (if certain conditions of smoothness hold) we obtain a vector field on the plane.

Example. It is easy to verify that the set of curves

$$\{s_r :\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (r\cos t, r\sin t),\ r \geqslant 0$$

satisfies the required condition. Let us find the corresponding vector field. Suppose $x = (x_1, x_2)$ is some point of the plane and $(x_1, x_2) = (r\cos t_0, r\sin t_0)$ for certain $r \geqslant 0,\ t_0 \in \mathbb{R}$. The tangent vector to the curve $t \mapsto (r\cos t, r\sin t)$ at $t = t_0$ is equal to $(-r\sin t_0, r\cos t_0) = (-x_2, x_1)$. Hence the corresponding vector field has the form:

$$-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}.$$

Exercise. Show that the following sets satisfy the required condition and find the corresponding vector fields:

a) $s_\alpha :\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (e^t\cos(\alpha + t), e^t\sin(\alpha + t))$, where $\alpha \in [0, 2\pi]$, and $s^0 :\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (0, 0)$;

b) $s_\alpha: \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (e^t \cos\alpha, e^t \sin\alpha)$, $\alpha \in [0, 2\pi]$,
and $s^0: \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (0,0)$.

Let U be an arbitrary open subset of the plane. Then all the definitions given above can be reformulated if we replace $\mathbb{R}^2$ (but not V^2) by U.

1) In accordance with Definition 1, a *vector field on* U is a smooth function $\mathbf{v}: U \to V^2$.

2) In accordance with Definition 2, a *vector field on* U is a derivation of the algebra of functions $C^\infty(U)$.

3) In accordance with Definition 3, a *tangent vector to* U *at a point* $a \in U$ is the class of equivalent curves $s: I \to U$ passing through a.

Examples.

1) If $\mathbf{v}: \mathbb{R}^2 \to V^2$ is an arbitrary vector field on the plane, then its restriction $\mathbf{v}|_U$ to U is a vector field on U.

2) Let $U = \mathbb{R}^2 \backslash \{0\}$. Then

$$\mathbf{v}: U \to V^2, \ (x_1, x_2) \mapsto \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}; \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right)$$

is a vector field on U, which cannot be represented as a restriction of a certain vector field on the plane to U.

3.4. Lie algebras of vector fields. We can consider vector fields as mappings of $C^\infty(\mathbb{R}^2)$ into $C^\infty(\mathbb{R}^2)$. In such a situation one natural operation appears—the composition of vector fields.

If $\mathbf{v}, \mathbf{w} \in \mathcal{D}(\mathbb{R}^2)$, then

$$(\mathbf{v} \circ \mathbf{w})(f) = \mathbf{v}(\mathbf{w}(f)) \text{ for } f \in C^\infty(\mathbb{R}^2).$$

Exercise. Let $\mathbf{v} = \frac{\partial}{\partial x_1}$. Prove that the mapping $\mathbf{v} \circ \mathbf{v}: C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ is not a vector field, i.e. $\mathbf{v} \circ \mathbf{v}$ is not a derivation of the algebra $C^\infty(\mathbb{R}^2)$.

However, if we make a slight improvement and instead of composition of two vector fields consider their commutator

$$[\mathbf{v}, \mathbf{w}] = \mathbf{v} \circ \mathbf{w} - \mathbf{w} \circ \mathbf{v},$$

then we shall get again a vector field.

Proposition. *Let* $\mathbf{v}, \mathbf{w} \in \mathcal{D}(\mathbb{R}^2)$. *Then* $[\mathbf{v}, \mathbf{w}]$ *also belongs to* $\mathcal{D}(\mathbb{R}^2)$.

Proof. Since the operators $\mathbf{v}$ and $\mathbf{w}$ are linear, we see that the operator $[\mathbf{v}, \mathbf{w}]$ is also linear. Suppose $f, g \in C^\infty(\mathbb{R}^2)$. Then

$$\begin{aligned}
[\mathbf{v}, \mathbf{w}](f \cdot g) = \mathbf{v}(\mathbf{w}(fg)) - \mathbf{w}(\mathbf{v}(fg)) = \mathbf{v}(\mathbf{w}(f)g + f\mathbf{w}(g)) - \mathbf{w}(\mathbf{v}(f)g + f\mathbf{v}(g)) = \\
(\mathbf{v} \circ \mathbf{w})(f) \cdot g + \mathbf{w}(f) \cdot \mathbf{v}(g) + \mathbf{v}(f) \cdot \mathbf{w}(g) + f \cdot (\mathbf{v} \circ \mathbf{w})(g) - \\
(\mathbf{w} \circ \mathbf{v})(F) \cdot g - \mathbf{v}(f) \cdot \mathbf{w}(g) - \mathbf{w}(f) \cdot \mathbf{v}(g) - f \cdot (\mathbf{w} \circ \mathbf{v})(g) = \\
[\mathbf{v}, \mathbf{w}](f) \cdot g + f \cdot [\mathbf{v}, \mathbf{w}](g).
\end{aligned}$$

Hence, $[\mathbf{v}, \mathbf{w}]$ is also a derivation of the algebra $C^\infty(\mathbb{R}^2)$, i.e. $[\mathbf{v}, \mathbf{w}] \in \mathcal{D}(\mathbb{R}^2)$.

Let $\mathbf{v} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}$ and $\mathbf{w} = w_1 \frac{\partial}{\partial x_1} + w_2 \frac{\partial}{\partial x_2}$. Let us find the explicit expression for $[\mathbf{v}, \mathbf{w}]$. In order to do this, we have to know how the vector field $[\mathbf{v}, \mathbf{w}]$ acts on an arbitrary function f. We have

$$(\mathbf{v} \circ \mathbf{w})(f) = \mathbf{v}(w_1 \frac{\partial f}{\partial x_1} + w_2 \frac{\partial f}{\partial x_2}) = \mathbf{v}(w_1) \frac{\partial f}{\partial x_1} + \mathbf{v}(w_2) \frac{\partial f}{\partial x_2} + \\ \left(v_1 w_1 \frac{\partial^2 f}{\partial x_1^2} + (v_1 w_2 + v_2 w_1) \frac{\partial^2 f}{\partial x_1 \partial x_2} + v_2 w_2 \frac{\partial^2 f}{\partial x_2^2} \right).$$

In a similar way:

$$(\mathbf{w} \circ \mathbf{v})(f) = \mathbf{w}(v_1) \frac{\partial f}{\partial x_1} + \mathbf{w}(v_2) \frac{\partial f}{\partial x_2} + \\ \left(v_1 w_1 \frac{\partial^2 f}{\partial x_1^2} + (v_1 w_2 + v_2 w_1) \frac{\partial^2 f}{\partial x_1 \partial x_2} + v_2 w_2 \frac{\partial^2 f}{\partial x_2^2} \right).$$

Now we see that, in the expression $(\mathbf{v} \circ \mathbf{w})(f) - (\mathbf{w} \circ \mathbf{v})(f)$, the terms that contain second partial derivatives of the function f cancel. So,

$$[\mathbf{v}, \mathbf{w}](f) = (\mathbf{v}(w_1) - \mathbf{w}(v_1)) \frac{\partial f}{\partial x_1} + (\mathbf{v}(w_2) - \mathbf{w}(v_2)) \frac{\partial f}{\partial x_2}.$$

This means that

$$[\mathbf{v}, \mathbf{w}] = (\mathbf{v}(w_1) - \mathbf{w}(v_1)) \frac{\partial}{\partial x_1} + (\mathbf{v}(w_2) - \mathbf{w}(v_2)) \frac{\partial}{\partial x_2} = \sum_{i=1}^{2} (\mathbf{v}(w_i) - \mathbf{w}(v_i)) \frac{\partial}{\partial x_i}.$$

Example. Let $\mathbf{v} = \frac{\partial}{\partial x_1}$, $\mathbf{w} = f(x_1) \frac{\partial}{\partial x_2}$. Then

$$[\mathbf{v}, \mathbf{w}] = f'(x_1) \frac{\partial}{\partial x_1}.$$

Exercise. Suppose $\mathbf{v} = \frac{\partial}{\partial x_1}$. Find all vector fields $\mathbf{w}$ such that

$$[\mathbf{v}, \mathbf{w}] = 0.$$

Exercise. Check that commutation of vector fields has the following properties:

1° it is bilinear (over $\mathbb{R}$);
2° it is skew-symmetric: $[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}]$;
3° the Jacoby identity holds:

$$\big[\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_3]\big] + \big[\mathbf{v}_2, [\mathbf{v}_3, \mathbf{v}_1]\big] + \big[\mathbf{v}_3, [\mathbf{v}_1, \mathbf{v}_2]\big] = 0.$$

Definition. A *Lie algebra* is a vector space $\mathfrak{g}$ supplied with a binary operation

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y]$$

such that the conditions 1°–3° hold.

Thus, we see that $\mathcal{D}(\mathbb{R}^2)$ is an infinite-dimensional Lie algebra. Note that commutation is a bilinear operation over $\mathbb{R}$, but not over $C^\infty(\mathbb{R}^2)$.

Exercise. Show that

$$[\mathbf{v}, f \cdot \mathbf{w}] = \mathbf{v}(f) \cdot w + f \cdot [\mathbf{v}, \mathbf{w}]$$

for all $\mathbf{v}, \mathbf{w} \in \mathcal{D}(\mathbb{R}^2)$ and $f \in C^\infty(\mathbb{R}^2)$. Find the value of the expression $[f\mathbf{v}, g\mathbf{w}]$, where $f, g \in C^\infty(\mathbb{R}^2)$.

Definition. A *subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace of the vector space $\mathfrak{g}$ closed under commutation.

In the sequel we shall be especially interested in finite-dimensional subalgebras of the Lie algebra $\mathcal{D}(\mathbb{R}^2)$.

Examples.

1) The vector space

$$\left\{ \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

is a two-dimensional subalgebra of the Lie algebra $\mathcal{D}(\mathbb{R}^2)$.

2) The space

$$\left\{ \alpha \frac{\partial}{\partial x_1} + f(x_1, x_2) \frac{\partial}{\partial x_2} \;\middle|\; \alpha \in \mathbb{R},\ f \in C^\infty(\mathbb{R}^2) \right\}$$

forms an infinite-dimensional subalgebra.

Exercise. Show that the following spaces are subalgebras of the Lie algebra $\mathcal{D}(\mathbb{R}^2)$:

a) $\left\{ (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2) \frac{\partial}{\partial x_1} \;\middle|\; \alpha_i \in \mathbb{R} \right\}$;

b) $\left\{ f(x_1) \frac{\partial}{\partial x_1} \;\middle|\; f \in C^\infty(\mathbb{R}^2) \right\}$;

c) $\left\{ (\alpha_{11} x_1 + \alpha_{12} x_2) \frac{\partial}{\partial x_1} + (\alpha_{21} x_1 + \alpha_{22} x_2) \frac{\partial}{\partial x_2} \;\middle|\; \alpha_{ij} \in \mathbb{R}^2 \right\}$;

d) $\left\{ \alpha \frac{\partial}{\partial x_1} + (\beta_0 + \beta_1 x_1 + \dots + \beta_n x_1^n) \frac{\partial}{\partial x_2} \;\middle|\; \alpha, \beta_0, \dots \beta_n \in \mathbb{R} \right\}$;

e)* $\left\{ \frac{\partial f}{\partial x_2} \cdot \frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1} \cdot \frac{\partial}{\partial x_2} \;\middle|\; f \in C^\infty(\mathbb{R}^2) \right\}$.

Which of them are finite-dimensional?

It is possible to consider the Lie algebra $\mathcal{D}(U)$ and its subalgebras for an arbitrary open subset U on the plane. All the constructions are analogous to those of the case of $\mathbb{R}^2$.

§4. Action of diffeomorphisms on functions and vector fields

4.1. Differentials. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a certain smooth mapping. This means that

$$\varphi(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2)),$$

where $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^2)$. Let $\mathbf{v}_a \in T_a\mathbb{R}^2$ be the tangent vector to the plane at some point a. Recall that $\mathbf{v}_a$ can be considered as an equivalence class of straight lines passing through the point a. Using this definition of a tangent vector, we shall define the differential of the mapping φ.

Theorem 1. *Let curves $t \mapsto s^i(t), i = 1, 2$ be equivalent at the point $t = 0$. Then the curves $t \mapsto (\varphi \circ s^i)(t), i = 1, 2,$ are equivalent at $t = 0$.*

Proof. Suppose $(s^1)'(0) = (s^2)'(0) = v = (v_1, v_2) \in T_a\mathbb{R}^2$. Then

$$(\varphi \circ s^i)(t) = (\varphi_1(s_1^i(t), s_2^i(t)), \varphi_2(s_1^i(t), s_2^i(t)).$$

From theorem 2, §1, it follows that

$$(\varphi \circ s^i)'(0) = \left(\frac{\partial \varphi_1}{\partial x_1}(a)v_1 + \frac{\partial \varphi_1}{\partial x_2}(a)v_2, \frac{\partial \varphi_2}{\partial x_1}(a)v_1 + \frac{\partial \varphi_2}{\partial x_2}(a)v_2\right) \tag{1}$$

for $i = 1, 2$. Thus, the mapping φ takes equivalence classes of curves passing through the point a to equivalence classes of curves passing through the point $b = \varphi(a)$.

Identifying tangent vectors $\mathbf{v}_a$ with the equivalence classes of curves, we obtain the mapping

$$d_a\varphi : T_a\mathbb{R}^2 \to T_b\mathbb{R}^2.$$

This mapping is called the *differential of φ at the point a*. From formula (1) it follows that the mapping $d_a\varphi$ takes a tangent vector $(v_1, v_2) \in T_a\mathbb{R}^2$ to

$$\left(\frac{\partial \varphi_1}{\partial x_1}(a)v_1 + \frac{\partial \varphi_1}{\partial x_2}(a)v_2, \frac{\partial \varphi_2}{\partial x_1}(a)v_1 + \frac{\partial \varphi_2}{\partial x_2}(a)v_2\right) \in T_b\mathbb{R}^2.$$

In other words, φ is a linear mapping of tangent spaces and, in the standard basis, its matrix has the form:

$$J(a) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1}(a) & \frac{\partial \varphi_1}{\partial x_2}(a) \\ \frac{\partial \varphi_2}{\partial x_1}(a) & \frac{\partial \varphi_2}{\partial x_2}(a) \end{pmatrix}.$$

Note that it is the Jakobi matrix of the mapping φ at the point a.

Examples.

1) Let φ be the identity mapping of the plane. Obviously, $d_a\varphi$ is the identity mapping of the tangent space $T_a\mathbb{R}^2$ for each $a \in \mathbb{R}^2$. 2) Let φ be the parallel translation by a vector $v = (v_1, v_2)$. We have

$$\varphi : (x_1, x_2) \mapsto (x_1 + v_1, x_2 + v_2).$$

It can be easily shown that at every point $a \in \mathbb{R}^2$, the Jakobi matrix of φ is the identity matrix.

3) Let φ be a linear transformation of the plane:

$$\varphi : (x_1, x_2) \mapsto (a_{11}x_1 + a_{21}x_2, a_{21}x_1 + a_{22}x_2).$$

Then

$$J(a) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

for each $a \in \mathbb{R}^2$. This means that the Jakobi matrix of φ is scalar and equal to the matrix of φ (cp. $(ax)' = a,\ \forall x \in \mathbb{R}$).

Exercise. Show that if the Jakobi matrix of a smooth mapping φ is constant, then φ is an affine mapping.

Further, let φ and ψ be two smooth mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ and a some point of the plane. From the definition of the differential it follows that

$$d_a(\psi \circ \varphi) = d_{\varphi(a)}\psi \circ d_a\varphi.$$

In particular, suppose φ is a diffeomorphism of the plane and $\psi = \varphi^{-1}$; then the last expression has the form

$$d_a(\mathrm{Id}_{\mathbb{R}^2}) = d_{\varphi(a)}(\varphi^{-1}) \circ d_a\varphi.$$

Since the differential of the identity of the plane at a point a is the identity of the tangent space $T_a\mathbb{R}^2$, we have

$$d_{\varphi(a)}(\varphi^{-1}) = (d_a\varphi)^{-1}.$$

Thus, the differential of a diffeomorphism at every point is a non-singular linear mapping.

4.2. Action of diffeomorphisms on vector fields. Suppose $\mathbf{v}$ is a vector field on the plane and φ is some diffeomorphism. We can consider the vector field $\varphi.\mathbf{v}$ given by

$$(\varphi.\mathbf{v})_{\varphi(a)} = d_a\varphi(\mathbf{v}_a). \tag{2}$$

Since $d_a\varphi$ is a mapping of $T_a\mathbb{R}^2$ into $T_\varphi(a)\mathbb{R}^2$, we see that $\varphi.\mathbf{v}$ is well-defined.

Example. Suppose $\varphi(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2))$ and $\mathbf{v} = \frac{\partial}{\partial x_i}, i = 1, 2$. Then

$$(\varphi.\mathbf{v})_{\varphi(a)} = d_a\varphi(1, 0) = \left(\frac{\partial \varphi_1}{\partial x_i}(a); \frac{\partial \varphi_2}{\partial x_i}(a)\right).$$

Substituting a for $\varphi^{-1}(a)$ in the expression above, we obtain

$$(\varphi.\mathbf{v})_a = \left(\frac{\partial \varphi_1}{\partial x_i}(\varphi^{-1}(a)), \frac{\partial \varphi_2}{\partial x_i}(\varphi^{-1}(a))\right)$$

or

$$\varphi.\mathbf{v} = \left(\frac{\partial\varphi_1}{\partial x_i} \circ \varphi^{-1}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial\varphi_2}{\partial x_i} \circ \varphi^{-1}\right)\frac{\partial}{\partial x_2}.$$

For example, suppose $\varphi(x_1, x_2) = (x_1 e^{x_2}, x_2)$. Then $\varphi^{-1}(x_1, x_2) = (x_1 e^{-x_2}, x_2)$ and

$$\varphi.\left(\frac{\partial}{\partial x_1}\right) = e^{x_2}\frac{\partial}{\partial x_1}, \quad \varphi.\left(\frac{\partial}{\partial x_2}\right) = x_1\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.$$

Suppose $\varphi(x_1, x_2) = (x_1 + f(x_2), x_2)$, where $f \in C^\infty(\mathbb{R}^2)$. Then $\varphi^{-1}(x_1, x_2) = (x_1 - f(x_2), x_2)$ and

$$\varphi.\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_1}; \quad \varphi.\left(\frac{\partial}{\partial x_2}\right) = f'(x_2)\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.$$

Exercise. Let φ be the parallel translation by a vector $a = (a_1, a_2)$ and

$$\mathbf{v} = v_1(x_1, x_2)\frac{\partial}{\partial x_1} + v_2(x_1, x_2)\frac{\partial}{\partial x_2}.$$

Find the field $\varphi.\mathbf{v}$.

Exercise. Show that the following relations are true:
1° $\varphi.(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2) = \lambda_1\varphi.\mathbf{v}_1 + \lambda_2\varphi.\mathbf{v}_2$;
2° $(\varphi_1 \circ \varphi_2).\mathbf{v} = \varphi_1.(\varphi_2.\mathbf{v})$;
3° $\varphi.(f\mathbf{v}) = (f \circ \varphi^{-1})\varphi.\mathbf{v}$,
where $\lambda_1, \lambda_2 \in \mathbb{R}$, $\varphi, \varphi_1, \varphi_2 \in \mathrm{Diff}(\mathbb{R}^2)$, $f \in C^\infty(\mathbb{R}^2)$, $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{D}(\mathbb{R})$.

In particular, from the previous example and 3° it follows that for

$$\varphi(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2)) \in \mathrm{Diff}(\mathbb{R}^2)$$

and

$$\mathbf{v} = v_1(x_1, x_2)\frac{\partial}{\partial x_1} + v_2(x_1, x_2)\frac{\partial}{\partial x_2} \in \mathcal{D}(\mathbb{R}^2)$$

the vector field $\varphi.\mathbf{v}$ has the form:

$$\varphi.\mathbf{v} = \sum_{i=1}^{2}\sum_{j=1}^{2}\left(v_j\frac{\partial\varphi_i}{\partial x_j}\right) \circ \varphi^{-1}\frac{\partial}{\partial x_i}.$$

4.3. Action of diffeomorphisms on functions. We can also define the action of diffeomorphisms on smooth functions on the plane:

$$\varphi.f \overset{def}{=} f \circ \varphi^{-1}.$$

Examples.

1) Let $\varphi = \mathrm{Id}_{\mathbb{R}^2}$. It is evident that $\varphi.f = f$ for all $f \in C^\infty(\mathbb{R}^2)$.

2) Suppose φ is the parallel translation by a vector $v = (v_1, v_2)$. Then

$$(\varphi.f)(x_1, x_2) = f(x_1 - v_1, x_2 - v_2).$$

Exercise. Check that the action of diffeomorphisms on functions has the following properties:

1° it is linear over $\mathbb{R}$:

$$\varphi.(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \cdot \varphi.f_1 + \lambda_2 \cdot \varphi.f_2;$$

2° $\varphi.(f_1 f_2) = (\varphi.f_1)(\varphi.f_2)$;

3° $(\varphi_1 \circ \varphi_2).f = \varphi_1.(\varphi_2.f)$.

What form would property 3° take if we defined the action as follows: $\varphi.f = f \circ \varphi$?

Let us describe a relationship between the actions of diffeomorphisms on functions and on vector fields.

Theorem 2. *For all $\varphi \in \mathrm{Diff}(\mathbb{R}^2)$, $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$, $f \in C^\infty(\mathbb{R}^2)$, we have*

1° $\varphi.(f\mathbf{v}) = (\varphi.f)(\varphi.\mathbf{v})$;

2° $\varphi.(\mathbf{v}(f)) = (\varphi.\mathbf{v})(\varphi.f)$.

Proof.

1°. Let

$$\mathbf{v} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}, \quad \varphi(x_1, x_2) = (\varphi_1(x_1, x_2), \varphi_2(x_1, x_2)),$$

where $v_1, v_2, \varphi_1, \varphi_2 \in C^\infty(\mathbb{R}^2)$. Then

$$\varphi.(f\mathbf{v}) = \sum_{i=1}^{2} \sum_{j=1}^{2} \left(f v_j \frac{\partial \varphi_i}{\partial x_j} \right) \circ \varphi^{-1} \frac{\partial}{\partial x_i} =$$

$$(f \circ \varphi^{-1}) \cdot \sum_{i=1}^{2} \sum_{j=1}^{2} \left(v_j \frac{\partial \varphi_i}{\partial x_j} \right) \circ \varphi^{-1} \frac{\partial}{\partial x_i} = (\varphi.f)(\varphi.\mathbf{v}).$$

2°. The proof is quite analogous to that of 1° and involves only direct calculation.

Exercise. Do this calculation.

Now we introduce one of the most important properties of the action of diffeomorphisms on vector fields:

Theorem 3.

$$\varphi.[\mathbf{v},\mathbf{w}] = [\varphi.\mathbf{v},\varphi.\mathbf{w}] \tag{3}$$

for all $\varphi \in \mathrm{Diff}(\mathbb{R}^2)$, $\mathbf{v},\mathbf{w} \in \mathcal{D}(\mathbb{R}^2)$.

Proof. It is clear that (3) is true when $\mathbf{v} = \mathbf{w}$. Let

$$\mathbf{v} = \frac{\partial}{\partial x_1}, \quad \mathbf{w} = \frac{\partial}{\partial x_2}, \quad \text{and } \varphi(x_1,x_2) = (\varphi_1(x_1,x_2), \varphi_2(x_1,x_2)).$$

Then $\varphi.[\mathbf{v},\mathbf{w}] = \varphi.0 = 0$ and

$$\begin{aligned}
&[\varphi.\mathbf{v},\varphi.\mathbf{w}] = \\
&\quad\left[\varphi.\left(\frac{\partial\varphi_1}{\partial x_1}\right)\frac{\partial}{\partial x_1} + \varphi.\left(\frac{\partial\varphi_2}{\partial x_1}\right)\frac{\partial}{\partial x_2}, \varphi.\left(\frac{\partial\varphi_1}{\partial x_2}\right)\frac{\partial}{\partial x_1} + \varphi.\left(\frac{\partial\varphi_2}{\partial x_2}\right)\frac{\partial}{\partial x_2}\right] = \\
&\qquad\left((\varphi.\mathbf{v})\left(\varphi.\frac{\partial\varphi_1}{\partial x_2}\right) - (\varphi.\mathbf{w})\left(\varphi.\frac{\partial\varphi_1}{\partial x_1}\right)\right)\frac{\partial}{\partial x_1} + \\
&\qquad\left((\varphi.\mathbf{v})\left(\varphi\frac{\partial\varphi_2}{\partial x_2}\right) - (\varphi.\mathbf{w})\left(\varphi.\frac{\partial\varphi_2}{\partial x_1}\right)\right)\frac{\partial}{\partial x_2} = \\
&\quad\varphi.\left(\mathbf{v}\left(\frac{\partial\varphi_1}{\partial x_2}\right) - \mathbf{w}\left(\frac{\partial\varphi_1}{\partial x_1}\right)\right)\frac{\partial}{\partial x_1} + \varphi.\left(\mathbf{v}\left(\frac{\partial\varphi_2}{\partial x_2}\right) - \mathbf{w}\left(\frac{\partial\varphi_2}{\partial x_1}\right)\right)\frac{\partial}{\partial x_2} = 0.
\end{aligned}$$

Thus, equality (3) holds for $\mathbf{v} = \frac{\partial}{\partial x_i}$, $\mathbf{w} = \frac{\partial}{\partial x_j}$, $i,j = 1,2$. But if equality (3) is true for some vector fields $\mathbf{v},\mathbf{w} \in \mathcal{D}(\mathbb{R}^2)$ and f is a smooth function on the plane, then it is true for the vector fields $\mathbf{v}, f\mathbf{w}$:

$$\begin{aligned}
\varphi([\mathbf{v}, f\mathbf{w}]) = \varphi(\mathbf{v}(f)\mathbf{w} + f\cdot[\mathbf{v},\mathbf{w}]) = \varphi.(\mathbf{v}(f))(\varphi.\mathbf{w}) + (\varphi.f)\varphi.[\mathbf{v},\mathbf{w}] = \\
(\varphi.\mathbf{v})(\varphi.f)\varphi.\mathbf{w} + (\varphi.f)[\varphi.\mathbf{v},\varphi.\mathbf{w}] = [\varphi.\mathbf{v},\varphi.(f\mathbf{w})].
\end{aligned}$$

This proves the theorem.

Definition. Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be Lie algebras. A *homomorphism of the Lie algebras* $\mathfrak{g}_1$, $\mathfrak{g}_2$ is a linear mapping $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$\varphi([x,y]) = [\varphi(x),\varphi(y)] \text{ for all } x,y \in \mathfrak{g}_1.$$

If φ is an isomorphism of the vector spaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$, then φ is called an *isomorphism of the algebras* $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Finally, if $\mathfrak{g}_1 = \mathfrak{g}_2$, then homomorphisms and isomorphisms are called *endomorphisms* and *automorphisms* of the Lie algebra $\mathfrak{g}_1$ respectively.

4.4. Equivalence of vector fields. So, for any $\varphi \in \mathrm{Diff}(\mathbb{R}^2)$ the mapping $\mathbf{v} \mapsto \varphi.\mathbf{v}$ is an automorphism of the Lie algebra $\mathcal{D}(\mathbb{R}^2)$.

Definition. Let $\mathbf{v}_1, \mathbf{v}_2$ be two vector fields ($\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras of vector fields) on the plane. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ ($\mathfrak{g}_1$ and $\mathfrak{g}_2$) are said to be *equivalent* if there exists a diffeomorphism φ of the plane such that

$$\varphi.\mathbf{v}_1 = \mathbf{v}_2 \quad (\varphi.\mathfrak{g}_1 = \mathfrak{g}_2).$$

Examples.

1) From the example of item 4.2 it follows that all vector fields of the form $\frac{\partial}{\partial x_1} + f(x_1)\frac{\partial}{\partial x_2}$ are equivalent.

2) The vector fields $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ are equivalent. For example, the diffeomorphism $\varphi : (x_1, x_2) \mapsto (x_2, x_1)$ takes one vector field into the other.

3) The vector fields $\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ and $x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ are not equivalent, because the first vector field is not equal to zero at any point $a \in \mathbb{R}^2$, whereas the second one is equal to zero at the point $(0, 0)$.

4) The Lie algebras $\mathfrak{g}_1 = \{a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} | a_1, a_2 \in \mathbb{R}\}$ and $\mathfrak{g}_2 = \{a_1\frac{\partial}{\partial x_1} + (a_1x_2 + a_2e^{x_1})\frac{\partial}{\partial x_2} | a_1, a_2 \in \mathbb{R}\}$ are equivalent. The desired diffeomorphism has the form:

$$\varphi\colon (x_1, x_2) \mapsto (x_1, e^{x_1}x_2).$$

Note that the restriction of the mapping $\mathbf{v} \mapsto \varphi.\mathbf{v}$ to $\mathfrak{g}_1$ is an isomorphism of the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Thus, if the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are equivalent, then they are necessarily isomorphic. As we shall see later, the converse statement, generally speaking, is not true.

Let us now describe the local analogues of the definitions given above. Let U and V be open subsets of the plane and let $\mathcal{D}(U)$ and $\mathcal{D}(V)$ be the corresponding Lie algebras of vector fields. Then every diffeomorphism $\varphi\colon U \to V$ generates two mappings: $\mathcal{D}(U) \to \mathcal{D}(V)$ and $C^\infty(U) \to C^\infty(V)$. All the properties of these mappings still hold in this case.

Example. Let

$$U = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1, 0 < x_2 < 2\pi\},$$

$$V = \mathbb{R}^2 \backslash \{(x_1, 0) \mid x_1 \geqslant 0\}$$

and let

$$\varphi\colon (x_1, x_2) \mapsto (x_1 \cos x_2, x_1 \sin x_2)$$

be a diffeomorphism of U onto V. Under the action of φ the vector fields $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ are taken to the fields

$$\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_1} + \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\frac{\partial}{\partial x_2}$$

and

$$-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$$

respectively.

Now, let $U \subset V$. To every vector field $\mathbf{v}$ on V we assign its restriction $\mathbf{v}|_U$ to the subset U. Thus, we obtain the mapping $\mathcal{D}(U) \to \mathcal{D}(V)$. It is easy to verify that it is a homomorphism of Lie algebras. Suppose $\mathfrak{g}$ is a Lie algebra of vector fields on V (i.e. $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathcal{D}(V)$). Then by $\mathfrak{g}|_U$ we denote its image by this diffeomorphism.

Let a be an arbitrary point on the plane and φ some local diffeomorphism of the plane at the point a such that $\varphi(a) = a$. Then it generates a diffeomorphism $U \to V$ for certain neighborhoods U and V of a and therefore an isomorphism $\mathcal{D}(U) \to \mathcal{D}(V)$ of Lie algebras.

Definition. Let $\mathbf{v}_1, \mathbf{v}_2$ be two vector fields ($\mathfrak{g}_1, \mathfrak{g}_2$ two Lie algebras of vector fields) on the plane. Then $\mathbf{v}_1, \mathbf{v}_2$ (respectively, $\mathfrak{g}_1, \mathfrak{g}_2$) are called *locally equivalent at the point* $a \in \mathbb{R}^2$ if the following conditions hold:

(1) there exists a local diffeomorphism φ of the plane at a such that $\varphi(a) = a$;

(2) there exist neighborhoods U, $V = \varphi(U)$ of the point a such that the diffeomorphism $\varphi|_U : U \to V$ takes the vector field $\mathbf{v}_1|_U$ into the vector field $\mathbf{v}_2|_V$ (i.e. generates an isomorphism of the Lie algebras $\mathfrak{g}_1|_U$ and $\mathfrak{g}_2|_V$).

Remark. All objects in this definition (the vector fields, the diffeomorphism φ, etc.) can be defined only on some neighborhood of the point a.

Examples.

1) Since the local diffeomorphism

$$(x_1, x_2) \mapsto \left(\frac{x_1}{x_1 + x_2 + 1}, x_2\right)$$

takes the vector field $\frac{\partial}{\partial x_2}$ into the field

$$\frac{x_1^2 - x_1}{x_2 + 1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},$$

we see that these fields are locally equivalent at the point 0.

2) Let $f(x_1)\frac{\partial}{\partial x_1}$ be a vector field such that $f(0) \neq 0$. Then it is locally equivalent to the vector field $\frac{\partial}{\partial x_1}$ at the point 0 . Indeed, let $(x_1, x_2) \mapsto (\varphi(x_1), x_2)$ be a local diffeomorphism of the plane at the point 0. This is equivalent to the following fact: $\varphi'(0) \neq 0$. Hence, it takes the vector field $\frac{\partial}{\partial x_1}$ into the field $\varphi' \circ \varphi^{-1}(x_1)\frac{\partial}{\partial x_1}$; therefore the following condition is true for the function φ:

$$\varphi'(x) = f(\varphi(x)), \ \varphi(0) = 0.$$

This ordinary differential equation is uniquely solvable in some neighborhood of 0 and, in addition, $\varphi'(0) = f(\varphi(0)) = f(0) \neq 0$. So, the mapping $(x_1, x_2) \mapsto (\varphi(x_1), x_2)$ is really a local diffeomorphism of the plane and takes the vector field $\frac{\partial}{\partial x_1}$ into the vector field $f(x_1)\frac{\partial}{\partial x_1}$.

In a similar way one can prove the following theorem:

Theorem 4. *Let* $\mathbf{v}$ *be a vector field on the plane and* $\mathbf{v}_0 \neq 0$. *Then* $\mathbf{v}$ *is locally equivalent to the vector field* $\frac{\partial}{\partial x_1}$ *at the point* 0.

§5. One-parameter transformation groups

5.1. Groups of transformations.

Definition 1. A set G of diffeomorphisms of $\mathbb{R}^2$ is called a *transformation group of the plane* if the following conditions hold:

(1) $\mathrm{Id} \in G$;
(2) for all $\varphi_1, \varphi_2 \in G, \quad \varphi_1 \circ \varphi_2 \in G$;
(3) for all $\varphi \in G, \quad \varphi^{-1} \in G$.

In other words, a transformation group of the plane is a subgroup of the group $\mathrm{Diff}(\mathbb{R}^2)$.

Examples.

1) Obviously the set that consists of the identity mapping is a trivial example of a transformation group of the plane.

2) The set of all parallel translations, the sets of all linear and affine transformations are transformation groups of the plane.

3) The set of all Euclidean transformations is a transformation group of the plane.

4) *The symmetry group of a regular polygon* is the set of all Euclidean transformations of the plane that take the polygon into itself. Then it is a finite transformation group of the plane.

5) The set of all diffeomorphisms

$$(x_1, x_2) \mapsto (x_1, x_2 + f(x_1)), \quad f \in C^\infty(\mathbb{R}),$$

is a transformation group of the plane.

Exercise. Consider the following sets:

a) the set of Euclidean transformations of the plane that preserve orientation;
b) the set of Euclidean transformations of the plane that change orientation;
c) $\{(u_1, x_2) \mapsto (u_1, e^{\lambda u_1} x_2) | \lambda \in \mathbb{R}\}$;
d) $\{(u_1, u_2) \mapsto (u_1 + a, u_2 + e^{u_1 + a}) | a \in \mathbb{R}\}$;
f) the set of all affine transformations of the plane that preserve area.

Which of them are transformation groups of the plane?

5.2. One-parameter transformation groups. We shall say that a family of diffeomorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$ smoothly depends on parameter t if the mapping $t \mapsto \varphi_t(a)$ is a smooth curve on the plane for each $a \in \mathbb{R}^2$.

Definition 2. Let $\{\varphi_t\}$ be a family of diffeomorphisms that smoothly depends on t. Then $\{\varphi_t\}$ is called a *one-parameter transformation group of the plane* if the mapping $t \mapsto \varphi_t$ is a homomorphism of the group $\mathbb{R}$ into $Diff(\mathbb{R}^2)$.

The last condition means that

(1) $\varphi_0 = \mathrm{Id}$;
(2) $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$;
(3) $\varphi_{-t} = (\varphi_t)^{-1}$.

Examples.

1) The trivial one-parameter transformation group is $\varphi_t = \mathrm{Id}_{\mathbb{R}^2}$ for each $t \in \mathbb{R}$.

2) The group of parallel translations along a vector $v = (v_1, v_2) \in V^2$:

$$\varphi_t : (x_1, x_2) \mapsto (x_1 + v_1 t,\ x_2 + v_2 t).$$

3) The group of rotations around the origin:

$$\varphi_t : (x_1, x_2) \mapsto (x_1 \cos t - x_2 \sin t,\ x_1 \sin t + x_2 \cos t).$$

4) The group of shifts:

$$\varphi_t : (x_1, x_2) \mapsto (x_1, x_2 + t f(x_1)),$$

where f is some fixed smooth function on the line.

Exercise. Invent a one-parameter transformation group different from the groups mentioned above.

Let us find a connection between vector fields $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$ and one-parameter transformation groups $\{\varphi_t\}$ of the plane. This can be done in the following way. Put

$$\mathbf{v}_a = \lim_{t \mapsto 0} \frac{\varphi_t(a) - a}{t} = \lim_{t \mapsto 0} \frac{\varphi_t(a) - \varphi_0(a)}{t}$$

for each $a \in \mathbb{R}^2$.

Note that $\mathbf{v}_a$ is the tangent vector to the curve $t \mapsto \varphi_t(a)$ at $t = 0$. This is in agreement with the interpretation of tangent vectors at a point as equivalence classes of curves passing through this point.

The vector field $\mathbf{v}$ is called the *infinitesimal generator of the one-parameter transformation group* $\{\varphi_t\}$.

Examples.

1) The infinitesimal generator of the trivial one-parameter transformation group is the zero vector field.

2) Let us find the infinitesimal generator of the group of rotations around the origin. We get

$$\mathbf{v}_{(x_1,x_2)} = \frac{d}{dt}(x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t)|_{t=0} = (-x_2, x_1).$$

Thus $\mathbf{v} = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$.

Exercise. Find the infinitesimal generators of the one-parameter transformation groups from examples 2) and 4).

Let us show that a one–parameter transformation group is uniquely determined by its infinitesimal generator.

Theorem 1. *Let $\{\varphi_t\}$ be a one–parameter transformation group of the plane, $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$ its infinitesimal generator.*

1°. *The differential equation*

$$s'_a(t) = \mathbf{v}_{s_a(t)}, \quad s_a(0) = a, \tag{1}$$

where $s_a : \mathbb{R} \mapsto \mathbb{R}^2$ is a smooth curve on the plane, is uniquely solvable for each $a \in \mathbb{R}^2$. The solution is defined at each point $t \in \mathbb{R}$ and $s_a(t) = \varphi_t(a)$.

2°. *Suppose $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$ is a vector field on the plane such that the differential equation (1) is globally solvable for each $a \in \mathbb{R}^2$. Then the set of mappings $\{\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2\}$ determined by the equation $\varphi_t(a) = s_a(t)$ forms a one–parameter transformation group of the plane.*

Proof.

1°. Suppose $s_a(t) = \varphi_t(a)$. Then

$$s'_a(t) = \lim_{\varepsilon \mapsto 0} \frac{s_a(t+\varepsilon) - s_a(t)}{\varepsilon} = \lim_{\varepsilon \mapsto 0} \frac{\varphi_{t+\varepsilon}(a) - \varphi_t(a)}{\varepsilon} =$$
$$\lim_{\varepsilon \mapsto 0} \frac{\varphi_\varepsilon(\varphi_t(a)) - \varphi_t(a)}{\varepsilon} = \mathbf{v}_{\varphi_t(a)} = \mathbf{v}_{s_a(t)}.$$

Thus the curve $s_a(t)$ is a solution of differential equation (1). This differential equation is uniquely solvable. Hence, the curve $s_a(t)$ is the unique solution and it is determined at each point of the line.

2°. Consider the diffeomorphisms φ_t, $t \in \mathbb{R}$, determined by the equation $\varphi_t(a) = s_a(t)$. Let us prove that $\{\varphi_t\}$ is a one-parameter transformation group. Clearly $\varphi_0 = \mathrm{Id}_{\mathbb{R}^2}$. We claim that

$$\varphi_{t_1+t_2}(a) = \varphi_{t_1}(\varphi_{t_2}(a)), \ \forall t_1, t_2 \in \mathbb{R}, a \in \mathbb{R}^2. \tag{2}$$

Indeed, the curve $s(t) = s_a(t + t_2)$ is a solution of the differential equation

$$s'(t) = \mathbf{v}_{s(t)}, \quad s(0) = \varphi_{t_2}(a)$$

But this equation has exactly one solution, which is equal to $s_{\varphi_{t_2}(a)}(t)$. Hence we get

$$s_a(t_1 + t_2) = s_{\varphi_{t_2}}(a)(t_1).$$

This completes the proof.

Example. Consider the vector field $\mathbf{v} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$. In this case the differential equation (1) has the form:

$$\begin{cases} s'_1(t) = s_1(t), & s_1(0) = a_1; \\ s'_2(t) = s_2(t), & s_2(0) = a_2, \end{cases}$$

where $s(t) = (s_1(t), s_2(t))$. Hence we get

$$s_1(t) = a_1e^t, s_2(t) = a_2e^t.$$

The corresponding one-parameter transformation group has the form:

$$\varphi_t : (x_1, x_2) \mapsto (e^t x_1, e^t x_2)$$

In other words, φ_t is the homothety with ratio e^t.

Now we assume that $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$ is a vector field such that equation (1) does not have any global solutions.

For example, put $\mathbf{v} = x_1^2\frac{\partial}{\partial x_1}$. The corresponding differential equation has the form:

$$\begin{cases} s_1' = s_1^2, & s_1(0) = a_1 \\ s_2' = 0, & s_2(0) = a_2. \end{cases}$$

Solving this equation we obtain

$$s_1(t) = \frac{a_1}{1 - a_1t}, \; s_2(t) = a_2.$$

We see that the solution is defined only in a certain neighborhood of the point 0, for example, on the interval $\left(-\frac{1}{|a_1|}; \frac{1}{|a_1|}\right)$. Moreover, we cannot define the diffeomorphism φ_t whenever $t \neq 0$. However, for each $t \in \mathbb{R}$ it is possible to find a domain on the plane such that the diffeomorphism φ_t is defined. In our case φ_t can be defined on the open subset $V_t = \{(x_1, x_2) | tx_1 \neq 1;\; x_2 \in \mathbb{R}\}$.

Now we introduce the following concept.

Definition 3. Let $\{\varphi_t\}$ be a family of diffeomorphisms that smoothly depends on t and suppose that the following conditions hold:

(1) each diffeomorphism φ_t is defined on some open domain $V_t \subset \mathbb{R}^2$ and $V_0 = \mathbb{R}^2$;

(2) the set $\{(t, a) \in \mathbb{R} \times \mathbb{R}^2 | a \in V_t\}$ is open;

(3) $\varphi_0 = \mathrm{Id}_{\mathbb{R}^2} : \; \varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ whenever both sides of the equality make sense.

Then $\{\varphi_t\}$ is called a *local one-parameter transformation group of the plane.*

Examples.

1) Any one-parameter transformation group is a local one-parameter transformation group, where for V_t one can take the whole plane for all $t \in \mathbb{R}^2$.

2) The family of diffeomorphisms

$$\varphi_t : \; (x_1, x_2) \mapsto \left(\frac{x_1}{1 - x_1t}, x_2\right)$$

is also a local one-parameter transformation group. Moreover, for V_t we can take

$$V_t = \begin{cases} \mathbb{R}^2, & t = 0; \\ (-\infty; \frac{1}{t}), & t > 0; \\ (\frac{1}{t}; \infty), & t < 0. \end{cases}$$

Indeed,

$$\varphi_{t_1}(\varphi_{t_2}(x_1,x_2)) = \varphi_{t_1}\left(\frac{x_1}{1-x_1t_2}; x_2\right) =$$
$$\left(\frac{\frac{x_1}{1-x_1t_2}}{1-\frac{x_1}{1-x_1t_2}\cdot t_1};\ x_2\right) = \left(\frac{x_1}{1-x_1(t_1t_2)};\ x_2\right)$$

for all $t_1, t_2 \in \mathbb{R}^2$, $(x_1, x_2) \in \mathbb{R}^2$ such that these expressions make sense.

As before, to every local one-parameter transformation group $\{\varphi_t\}$ of the plane we can assign a certain vector field. Indeed, for each point $a \in \mathbb{R}^2$ the curve $s_a(t) = \varphi_t(a)$, according the definition, is defined in some neighborhood of 0. If we assign to every point a the tangent vector to the curve s_a at the point $t = 0$, we shall get the vector field $\mathbf{v}$:

$$\mathbf{v}_a = s_a'(0) \text{ for all } a \in \mathbb{R}^2.$$

This vector field is said to be the *infinitesimal generator* of the local one-parameter transformation group $\{\varphi_t\}$. Moreover, from the proof ot theorem 1 it follows that the following statement is true:

Theorem 2. *There is a one-to-one correspondence between vector fields on the plane and local one-parameter transformation groups of the plane.*

Let us now show how local one-parameter transformation groups appear naturally. Let S^2 be the two-dimensional sphere given by the equation $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$. Let us introduce a parametrization on the sphere by means of two parameters x_1 and x_2. Since the sphere is not homeomorphic to the plane, we can do it only in some neighborhood U on the sphere. For U we take the set $S^2 \backslash \{0,0,1\}$ and project it on the plane $\{x_3 = 0\}$ by means of stereographic projection. In this case a point (x_1, x_2, x_3) of the sphere is transformed into the point $\left(\frac{x_1}{1-x_3}; \frac{x_2}{1-x_3}\right)$ of the plane. The inverse mapping is given by

$$(y_1, y_2) \mapsto \left(\frac{2y_1}{y_1^2+y_2^2+1}; \frac{2y_2}{y_1^2+y_2^2+1}; \frac{y_1^2+y_2^2-1}{y_1^2+y_2^2+1}\right).$$

Consider the one-parameter group of rotations of $\mathbb{R}^3$ with respect to $0x_2$-axis:

$$\varphi_t:\ (x_1, x_2, x_3) \mapsto (x_1\cos t - x_3\sin t, x_2, x_1\sin t + x_3\cos t).$$

Since the sphere is stable under these rotations, we see that they induce a one-parameter transformation group of the sphere. Consider its action on points of U in the coordinates introduced above:

$$\varphi_t:\ (y_1, y_2) \mapsto \left(\frac{2y_1\cos t - (y_1^2+y_2^2-1)\sin t}{y_1^2+y_2^2+1-2y_1\sin t-(y_1^2+y_2^2-1)\cos t};\right.$$
$$\left.\frac{2y_2}{y_1^2+y_2^2+1-2y_1\sin t-(y_1^2+y_2^2-1)\cos t}\right).$$

Consequently, diffeomorphisms φ_t are defined not for all points of the plane, i.e. $\{\varphi_t\}$ is a local one-parameter transformation group.

Exercise.

1) Find all points of $\mathbb{R}^2$ where φ_t is not defined.

2) Show that the infinitesimal generator of the given-above local one-parameter transformation group of the sphere has the form:

$$\mathbf{v} = \frac{2}{(y_1^2 + y_2^2 + 1)^2} \cdot \left((y_1^2 - y_2^2 + 1)\frac{\partial}{\partial y_1} + 2y_1 y_2 \frac{\partial}{\partial y_2} \right)$$

Thus, we see that *the global one-parameter transformation group of the sphere becomes local if we consider its action on some parametrized neighborhood, which we identify with the plane.*

§6. Introduction to Lie transformation groups

6.1. Lie transformation groups. In the sequel we shall be interested only in those transformation groups of the plane that possess an additional topological structure (not arbitrary transformation groups of the plane).

A *parametrization* of a transformation group G is a homeomorphism of some open domain in $\mathbb{R}^r$ onto a neighborhood of the identity mapping in G:

$$p\colon \mathcal{O} \to G,$$

where $\mathcal{O} \subset \mathbb{R}^r$, $p(\mathcal{O})$ is a neighborhood of identity in G.

In the sequel we shall assume that $\mathcal{O} \ni 0$ and $p(0) = \mathrm{Id}_{\mathbb{R}^2}$.

Definition. A group G of transformations of the plane is called an *r-parameter transformation group* (or an *r-dimensional Lie transformation group*) if there exists a parametrization $p\colon \mathcal{O} \to G$, where $\mathcal{O} \subset \mathbb{R}^r$, such that the following condition holds:

if $s\colon \mathbb{R} \to \mathcal{O}$ is a smooth curve in $\mathcal{O}$, then the family of diffeomorphisms $\varphi_t = p(s(t))$ of the plane smoothly depends on parameter t.

Examples.

1) Any one-parameter transformation group is a 1-parameter transformation group in the sense of the definition given above.

2) The group of parallel translations on the plane is a 2-parameter transformation group, where for $\mathcal{O}$ one can take the whole plane:

$$p\colon (x_1, x_2) \mapsto T_{(x_1,x_2)}.$$

Here $T_{(x_1,x_2)}$ is a parallel translation by a vector $(x_1, x_2) \in V^2$. Note that in this case $p(\mathcal{O}) = G$.

3) The group of all linear transformations of the plane is a 4-parameter transformation group; the parametrization, for instance, has the form: $p(x_1, x_2, x_3, x_4)$ is a linear transformation with matrix

$$\begin{pmatrix} 1 + x_1 & x_2 \\ x_3 & 1 + x_4 \end{pmatrix}$$

Exercise. Find the largest number $r \in \mathbb{R}$ such that the open ball

$$B_r = \{(x_1, x_2, x_3, x_4) | x_1^2 + x_2^2 + x_3^2 + x_4^2 < r^2\}$$

can be chosen for $\mathcal{O}$.

4) Similarly, it is possible to show that the group of all affine transformations of the plane is a 6-parameter transformation group.

5) The group of all Euclidean transformations of the plane is a 3-parameter transformation group. The parametrization, for example, has the form:

$$p(\varphi, x_1, x_2) = T_{(x_1, x_2)} \circ R_\varphi,$$

where R_φ is a rotation by the angle φ around the origin of coordinates. For $\mathcal{O}$ we can take the neighborhood

$$\{(\varphi, x_1, x_2) \in \mathbb{R}^3 \mid -\pi < \varphi < \pi\}.$$

6) The group of all transformations of the plane is not an r-parameter transformation group whenever $r \in \mathbb{N}$. It is really so, because there does not exist a neighborhood of the identity mapping that can be parametrized by a finite number of parameters.

Exercise. Show that the group of all linear transformations of the plane with determinant 1 is a 3-parameter transformation group.

To every r-parameter transformation group of the line it is possible to assign some r-dimensional Lie algebra of vector fields on the plane. Let $s: \mathbb{R} \to \mathcal{O}$ be a smooth curve in a neighbourhood $\mathcal{O}$, where $s(0) = 0$, and $\{\varphi_t = p(s(t))\}$ a corresponding family of diffeomorphisms. Then to the curve S we can assign the vector field $\mathbf{v}^s$ on the plane:

$$\mathbf{v}_a^s = \lim_{t \to 0} \frac{\varphi_t(a) - a}{t}.$$

The following theorem is a fundamental result of theory of Lie groups and is given without proof.

Theorem 1.

1) The set $\mathfrak{g} = \{\mathbf{v}^s \mid s: \mathbb{R} \to \mathcal{O},\ s(0) = 0\}$ *is an* r*-dimensional Lie algebra on the plane and does not depend on parametrization* p*.*

2) The vector field $\mathbf{v}^s$ *depends only on the vector* $v = s'(0)$*. The map* $\mathbb{R}^r \ni v \mapsto \mathbf{v}^s \in \mathfrak{g}$ *is an isomorphism of vector spaces.*

3) For each $\mathbf{v} \in \mathfrak{g}$*, differential equation* (1) *is globally solvable. Diffeomorphisms from corresponding one-parameter subgroups lie in* G *and generate the connected component of the identity of the group* G*.*

The Lie algebra $\mathfrak{g}$ is called *the Lie algebra of vector fields* of the Lie transformation group G.

This theorem shows that the Lie algebra $\mathfrak{g}$ almost completely determines the Lie transformation group G.

Examples.

1) Assume that G is the group of parallel translations on the plane and $s: t \mapsto (a_1 t, a_2 t)$ is a smooth curve in a neighborhood $\mathcal{O}$. Then the family of diffeomorphisms corresponding to s has the form:

$$\varphi_t\colon (x_1, x_2) \mapsto (x_1 + a_1 t, x_2 + a_2 t).$$

Therefore, the corresponding vector field has the form:

$$a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2}.$$

From item 2 of theorem 1 it follows that the Lie algebra $\mathfrak{g}$ does not contain any other vector fields except those given above. Thus, $\mathfrak{g} = \{\, a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} \mid a_1, a_2 \in \mathbb{R} \,\}$ and the map

$$\mathbb{R}^2 \ni (a_1, a_2) \mapsto a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} \in \mathfrak{g}$$

is really an isomorphism of vector spaces.

2) Let G be the group of all linear transformations of the plane. Then to a curve of the form $t \mapsto (a_1 t, a_2 t, a_3 t, a_4 t) \in \mathcal{O}$ there corresponds the family of diffeomorphisms

$$\varphi_t\colon (x_1, x_2) \mapsto \big((1 + a_1 t)x_1 + a_3 t x_2,\ a_2 t x_1 + (1 + a_4 t) x_2\big).$$

The corresponding vector field at a point (x_1, x_2) has the form:

$$\frac{d}{dt}\big((1 + a_1 t)x_1 + a_3 t x_2, a_2 t x_1 + (1 + a_4 t)x_2\big)\big|_{t=0} = (a_1 x_1 + a_3 x_2, a_2 x_1 + a_4 x_2).$$

Hence,

$$\mathfrak{g} = \{(a_1 x_1 + a_3 x_2)\frac{\partial}{\partial x_1} + (a_2 x_1 + a_4 x_2)\frac{\partial}{\partial x_2} \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}.$$

Exercise. Find the Lie algebras of vector fields for the following Lie groups of transformations of the plane:

a) a one-parameter transformation group with infinitesimal generator $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$;

b) the group of affine transformations of the plane;

c) the group of Euclidean transformations of the plane.

6.2. Local Lie transformation groups. Let us now consider local analogues of Lie transformation group.

Let G be a Lie transformation group. Now we assume that diffeomorphisms from G are defined not on the entire plane but on some open domain except for the identity map, defined everywhere. We also assume that for every point $a \in \mathbb{R}^2$ there exists an open neighborhood of the identity of G such that it contains only those diffeomorphisms which are defined at the point a. Transformation groups of this type are called *local Lie transformation groups.*

Example. To every non-singular matrix from $\mathrm{Mat}_{3\times 3}(\mathbb{R})$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

assign the following transformation of the plane:

$$\varphi_A\colon (x_1, x_2) \mapsto \left(\frac{a_{11}x_1 + a_{12}x_2 + a_{13}}{a_{31}x_1 + a_{32}x_2 + a_{33}}, \frac{a_{21}x_1 + a_{22}x_2 + a_{23}}{a_{31}x_1 + a_{32}x_2 + a_{33}} \right).$$

Note that it is defined only at those points (x_1, x_2) that satisfy the condition

$$a_{31}x_1 + a_{32}x_2 + a_{33} \neq 0.$$

Transformations of this type are called *linear fractional* or *projective.*

Exercise. Check that

$$1^\circ. \quad \varphi_A \varphi_B = \varphi_{AB};$$
$$2^\circ. \quad \varphi_{\lambda A} = \varphi_A,$$

for all non-singular matrices A, B and $\lambda \in \mathbb{R}^*$.

In particular, we have $(\varphi_A)^{-1} = \varphi_{A^{-1}}$, i.e. $\varphi_A \circ \varphi_{A^{-1}}(a) = \varphi_{A^{-1}} \circ \varphi_A(a) = a$ for all points $a \in \mathbb{R}^2$ for which these equalities make sense.

Since $\varphi_{\lambda A} = \varphi_A$, in the sequel we shall consider only matrices with determinant 1.

Exercise.

1) Show that for any two different matrices A, B with determinant 1 the transformations φ_A and φ_B are different.

2) Find all matrices A such that the transformation φ_A is defined everywhere.

It is possible to show that the set of all projective transformations of the plane is an 8-parameter Lie transformation group.

To every local transformation group of the plane there also corresponds a certain Lie algebra of vector fields on the plane. We can construct it in just the same way.

6.3. Transitive transformation groups of the plane.

Definition. A transformation group of the plane G is called *transitive* if for any points $a, b \in \mathbb{R}^2$ there exists a diffeomorphism $\varphi \in G$ such that $\varphi(a) = b$.

Examples.

1) The groups of parallel translations and affine transformations are obviously transitive.

2) The group of linear transformations of the plane is not transitive, since the point $(0, 0)$ is stable under any linear transformation of the plane.

Exercise. Which of the followings transformation groups are transitive:

a) the group of Euclidean transformations of the plane;

b) the group of shifts:

$$G = \{(x_1, x_2) \mapsto (x_1, f(x_1) + x_2) \mid f \in C^\infty(\mathbb{R}^2)\}?$$

Now suppose G is a Lie transformation group of the plane and $\mathfrak{g}$ is the corresponding Lie algebra of vector fields. There is a simple criterion for the group G to be transitive.

Theorem 2. *A Lie transformation group of the plane G is transitive if and only if for each point $a \in \mathbb{R}^2$ the space $\mathfrak{g}_a = \{ \mathbf{v}_a \mid \mathbf{v} \in \mathfrak{g} \}$ coincides with $T_a\mathbb{R}^2$.*

Exercise.

1) Check that for the group of parallel translations the condition of theorem 2 is satisfied.

2) Let G be the group of linear transformations of the plane and $\mathfrak{g}$ the corresponding Lie algebra of vector fields. Find a point $a \in \mathbb{R}^2$ such that $\mathfrak{g}_a \neq T_a\mathbb{R}^2$. Theorem 2 is a basis for a local analogue of the concept of transitivity.

Definition. A Lie algebra $\mathfrak{g}$ of vector fields on the plane is said to be *transitive at a point* $a \in \mathbb{R}^2$ if $\mathfrak{g}_a = T_a\mathbb{R}^2$.

Thus, a Lie transformation group G of the plane is transitive if the corresponding Lie algebra of vector fields is transitive at every point of the plane.

Exercise. Suppose that a Lie algebra $\mathfrak{g}$ of vector fields on the plane is transitive at a point $a \in \mathbb{R}^2$. Show that there exists a neighborhood U of a such that $\mathfrak{g}$ is transitive at each point $b \in U$.

§7. Local classification

7.1. One-dimensional case. Our nearest aim is to describe all finite-dimensional transitive Lie algebras of vector fields on the plane up to local equivalence.

However, now we consider the same problem on the line. It is obvious that all definitions given above can be formulated for any space $\mathbb{R}^n$, where $n \in \mathbb{N}$.

Each vector field on the line has the form $f\frac{\partial}{\partial x}$ for some $f \in C^\infty(\mathbb{R}^2)$ and the operation of commutation is given by

$$\left[f\frac{\partial}{\partial x}, g\frac{\partial}{\partial x} \right] = (fg' - g'f)\frac{\partial}{\partial x}.$$

We shall employ the following notation: for linearly independent vectors $e_1, \ldots, e_k$ of some vector space V by $\langle e_1, \ldots, e_k \rangle$ we shall denote the subspace of V spanned by these vectors.

Theorem 1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra of vector field on the line and suppose that $\mathfrak{g}$ is transitive at $x = 0$. Then $\mathfrak{g}$ is locally equivalent to one and only one of the following Lie algebras:*

1°.

$$\langle \frac{\partial}{\partial x} \rangle;$$

2°.

$$\langle \frac{\partial}{\partial x}; \ x\frac{\partial}{\partial x} \rangle;$$

3°.

$$\langle \frac{\partial}{\partial x}; \ x\frac{\partial}{\partial x}; \ x^2\frac{\partial}{\partial x} \rangle.$$

Proof. Since $\mathfrak{g}$ is transitive at 0, there exists a vector field $\mathbf{v} \in \mathfrak{g}$ such that $\mathbf{v}_0 \neq 0$. Then, as it was shown in section 3, it is locally equivalent to the vector field $\frac{\partial}{\partial x}$. Now consider the set of functions

$$\mathcal{F} = \left\{ f \in C^\infty(\mathbb{R}) \mid f\frac{\partial}{\partial x} \in \mathfrak{g} \right\}.$$

It is easy to show that $\mathcal{F}$ is a finite-dimensional vector space. Besides, since $\frac{\partial}{\partial x} \in \mathfrak{g}$ and $\left[\frac{\partial}{\partial x}, f\frac{\partial}{\partial x}\right] = f'\frac{\partial}{\partial x}$, we see that it is closed under differentiation of functions. Hence, $\mathcal{F}$ is the set of all solutions of some homogeneous linear differential equation with constant coefficients (see appendix B). In particular, it follows that all functions from $\mathcal{F}$ are analytic in a certain neighborhood of the point $a = 0$. To complete the proof of the theorem we need the following lemmas:

Lemma 1. *Let* $\mathbf{v}, \mathbf{w}$ *be nonzero vector fields on the line. If* $[\mathbf{v}, \mathbf{w}] = 0$, *then there exists* $\lambda \in \mathbb{R}^*$ *such that* $\mathbf{v} = \lambda\mathbf{w}$.

Proof of lemma 1. Indeed, let $\mathbf{v} = f(x)\frac{\partial}{\partial x}$, $\mathbf{w} = g(x)\frac{\partial}{\partial x}$. Then the equality $[\mathbf{v}, \mathbf{w}] = 0$ is equivalent to the equality

$$fg' - f'g = 0.$$

Consequently,

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = 0$$

for all $x \in \mathbb{R}$ such that $g(x) \neq 0$. Similarly, $\left(\frac{g}{f}\right)' = 0$ for all points x of the line such that $f(x) \neq 0$. Hence, the zero sets of the functions f and g coincide and $f = \lambda g$ for a certain $\lambda \in \mathbb{R}$.

In the sequel we shall say that a vector field $\mathbf{v} \in \mathcal{D}(\mathbb{R})$ *has zero of order* n at a point $a \in \mathbb{R}$ if $\mathbf{v} = (x - a)^n \cdot f(x)\frac{\partial}{\partial x}$, where $f(a) \neq 0$. If $\mathbf{v}_a \neq 0$, then we shall say that the vector field $\mathbf{v}$ has zero of order 0 at the point a.

Lemma 2. *Suppose that vector fields* $\mathbf{v}, \mathbf{w} \in \mathcal{D}(\mathbb{R})$ *have zeros of order* n *and* m, *respectively, at the point* 0. *Then the vector field* $[\mathbf{v}, \mathbf{w}]$ *has zero of order not less than* $n + m - 1$ *at* 0.

Proof of lemma 2. Indeed, if $\mathbf{v} = x^n f(x)\frac{\partial}{\partial x}$, $\mathbf{w} = x^m g(x)\frac{\partial}{\partial x}$, then we have

$$\begin{aligned}[\mathbf{v}, \mathbf{w}] = \big(x^n f(x) \cdot (x^m g'(x) + m x^{m-1} g(x)) - \\ (x^n f'(x) + n x^{n-1} f(x)) \cdot x^m g(x)\big) \frac{\partial}{\partial x} = \\ x^{n+m-1} \left(x f(x) g'(x) + (m - n) f(x) g(x) - x f'(x) g(x)\right) \frac{\partial}{\partial x};\end{aligned}$$

i.e. the vector field $[\mathbf{v}, \mathbf{w}]$ has the zero of order greater or equal to $n + m - 1$ at the point 0.

Lemma 3. *Dimension of the Lie algebra $\mathfrak{g}$ is not greater than 3.*

Proof of lemma 3. Choose a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ of the Lie algebra $\mathfrak{g}$ so that $\mathbf{v}_1 = \frac{\partial}{\partial x}$ and every next vector field has zero of order greater then that of the previous one at the point 0. We can always do it by means of linear transformations of an arbitrary basis. Now, let us assume that $n \geqslant 4$ and p, q are orders of zeros of the vector fields $\mathbf{v}_{n-1}$ and $\mathbf{v}_n$ respectively (since all functions from $\mathcal{F}$ are analytic, we see that p and q are finite). Then $n-2 \leqslant p < q$. Consequently, the vector field $[\mathbf{v}_{n-1}, \mathbf{v}_n]$ has zero of order not less than $p+q-1 \geqslant q+n-3 \geqslant q+1$ at the point 0. Since all nonzero vector fields from $\mathfrak{g}$ have zero of order not greater than q at the origin and $[\mathbf{v}_{n-1}, \mathbf{v}_n]$ is a nonzero vector field from $\mathfrak{g}$, we come to a contradiction. This proves the lemma.

Consider now the following cases:

1) $\dim \mathfrak{g} = 1$. Since $\frac{\partial}{\partial x} \in \mathfrak{g}$, we see that $\mathfrak{g}$ has form 1°.

2) $\dim \mathfrak{g} = 2$. In such a situation $\mathcal{F}$ is the solution set some second-order differential equation and $1 \in \mathcal{F}$. Hence, this equation has the form:

$$f'' + \lambda f' = 0.$$

If $\lambda = 0$, then $\mathfrak{g}$ has form 2°. Suppose $\lambda \neq 0$. Then $\mathfrak{g} = \langle \frac{\partial}{\partial x};\ e^{-\lambda x}\frac{\partial}{\partial x} \rangle$. It is easy to show that the local diffeomorphism $x \mapsto 1 - e^{x/\lambda}$ takes this subalgebra into the subalgebra $\langle \frac{\partial}{\partial x},\ x\frac{\partial}{\partial x} \rangle$.

3) $\dim \mathfrak{g} = 3$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of the Lie algebra $\mathfrak{g}$ chosen as in the proof of lemma 3. In this case it is not difficult to show that at the point 0 the vector fields $\mathbf{v}_2$ and $\mathbf{v}_3$ have zero of order 1 and 2 respectively. Without loss of generality we can assume that

$$\mathbf{v}_1 = \frac{\partial}{\partial x}; \quad \mathbf{v}_2 = (x + o(x))\frac{\partial}{\partial x}; \quad \mathbf{v}_3 = (x^2 + o(x^2))\frac{\partial}{\partial x}.$$

Then we get

$$\begin{aligned}
[\mathbf{v}_1, \mathbf{v}_2] &= \mathbf{v}_1 + \alpha\mathbf{v}_2 + \beta\mathbf{v}_3;\\
[\mathbf{v}_1, \mathbf{v}_3] &= 2\mathbf{v}_2 + \gamma\mathbf{v}_3;\\
[\mathbf{v}_2, \mathbf{v}_3] &= \mathbf{v}_3
\end{aligned}$$

for certain numbers $\alpha, \beta, \gamma \in \mathbb{R}$. Passing to the new basis

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 + \left(\beta - \frac{\alpha\gamma}{2}\right)\mathbf{v}_3,\\
\mathbf{w}_2 &= 2\mathbf{v}_2 + \frac{\gamma}{2}\mathbf{v}_3,\\
\mathbf{w}_3 &= \mathbf{v}_3,
\end{aligned}$$

we obtain

$$[\mathbf{w}_1, \mathbf{w}_2] = \mathbf{w}_1 + \delta\mathbf{w}_2, \quad [\mathbf{w}_1, \mathbf{w}_3] = 2\mathbf{w}_2, \quad [\mathbf{w}_2, \mathbf{w}_3] = \mathbf{w}_3, \tag{1}$$

where $\delta = \alpha + \gamma$.

From the Jacoby identity

$$\big[[\mathbf{w}_1, \mathbf{w}_2], \mathbf{w}_3\big] + \big[[\mathbf{w}_2, \mathbf{w}_3], \mathbf{w}_1\big] + \big[[\mathbf{w}_3, \mathbf{w}_1], \mathbf{w}_2\big] = 0$$

it follows that $\delta = 0$.

Since the vector field $\mathbf{w}_1$ is nonzero at the point 0, we see that there exists a local diffeomorphism that takes $\mathbf{w}_1$ into the vector field $\frac{\partial}{\partial x}$. Suppose as well that the vector fields $\mathbf{w}_2$ and $\mathbf{w}_3$ pass into $f(x)\frac{\partial}{\partial x}$ and $g(x)\frac{\partial}{\partial x}$, respectively, under this diffeomorphism. Then from equalities (1) it follows that

$$f' = 1, \quad g' = 2f, \quad fg' - f'g = g.$$

Hence, $f(x) = x,\ g(x) = x^2$, and we have $\mathfrak{g} = \langle \frac{\partial}{\partial x},\ x\frac{\partial}{\partial x},\ x^2\frac{\partial}{\partial x}\rangle$. This completes the proof of the theorem.

7.2. Two–dimensional case. The analogous theorem for finite-dimensional Lie algebras on the plane is more complicated and has the following form:

Theorem 2. *Any finite-dimensional transitive Lie algebra of vector fields on the plane is locally equivalent to one of the following Lie algebras:*

1.1. $\langle \frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2}\rangle$.

2.1. $\langle \frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1} + \lambda x_2\frac{\partial}{\partial x_2}\rangle, \quad |\lambda| \leqslant 1$.

2.2. $\langle \frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ (\lambda x_1 - x_2)\frac{\partial}{\partial x_1} + (x_1 + \lambda x_2)\frac{\partial}{\partial x_2}\rangle, \quad \lambda \geqslant 0$.

3.1. $\langle \frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2}\rangle$.

3.2. $\langle \frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2};\ -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\rangle$.

4.1. $\langle \frac{\partial}{\partial x_1};\ \varphi_1(x_1)\frac{\partial}{\partial x_2};\dots;\varphi_n(x_1)\frac{\partial}{\partial x_2}\rangle$, *where functions* $\varphi_1,\dots,\varphi_n$ *form a basis of solutions of some homogeneous linear differential equation with constant coefficients.*

5.1. $\langle \frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2};\ \varphi_1(x_1)\frac{\partial}{\partial x_2};\dots;\varphi_n(x_1)\frac{\partial}{\partial x_2}\rangle$, *where functions* $\varphi_1,\dots,\varphi_n$ *are as in 4.1.*

6.1. $\langle \frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1} + \lambda x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-1}\frac{\partial}{\partial x_2}\rangle, \quad \lambda \neq n-1$.

7.1. $\langle \frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1} + (n-1)\frac{\partial}{\partial x_2} + \frac{x_1^{n-1}}{(n-1)!};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-2}\frac{\partial}{\partial x_2}\rangle$.

8.1. $\langle \frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-1}\frac{\partial}{\partial x_2}\rangle$.

9.1. $\langle \frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1} + (n-1)x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-1}\frac{\partial}{\partial x_2}\rangle$.

10.1. $\langle \frac{\partial}{\partial x_1};\ 2x_1\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\rangle$.

11.1. $\langle x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1};\ (1-x_1^2-x_2^2)\frac{\partial}{\partial x_1} - 2x_1x_2\frac{\partial}{\partial x_2};\ 2x_1x_2\frac{\partial}{\partial x_1} + (1+x_1^2+x_2^2)\frac{\partial}{\partial x_2}\rangle$.

11.2. $\langle x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1};\ (1-x_1^2+x_2^2)\frac{\partial}{\partial x_1} - 2x_1x_2\frac{\partial}{\partial x_2};\ -2x_1x_2\frac{\partial}{\partial x_1} + (1+x_1^2-x_2^2)\frac{\partial}{\partial x_2}\rangle$.

11.3. $\langle x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1};\ (1+x_1^2-x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2};\ 2x_1x_2\frac{\partial}{\partial x_1} + (1-x_1^2+x_2^2)\frac{\partial}{\partial x_2}\rangle$.

12.1. $\langle \frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1};\ x_1^2\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_2};\ x_2^2\frac{\partial}{\partial x_2}\rangle$.

12.2. $\langle \frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2};\ -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2};$
$(x_1^2 - x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2};\ -2x_1x_2\frac{\partial}{\partial x_1} + (x_1^2 - x_2^2)\frac{\partial}{\partial x_2}\rangle$.

13.1. $\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_2};$
$x_1\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right);\ x_2\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)\rangle.$

14.1. $\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_1}\rangle.$

15.1. $\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_1}\rangle.$

16.1. $\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1};\ x_1^2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}\rangle.$

17.1. $\langle\frac{\partial}{\partial x_1};\ 2x_1\frac{\partial}{\partial x_1}+nx_2\frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1}+nx_1x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^n\frac{\partial}{\partial x_2}\rangle.$

18.1. $\langle\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1}+nx_1x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^n\frac{\partial}{\partial x_2}\rangle.$

§8. Global realizations

Let us show how the Lie algebras of vector fields that have been listed in theorem 2 can be realized as Lie algebras of local Lie transformation groups of the plane. Moreover, we shall show that every local Lie transformation group like this is a global Lie transformation group of a certain two-dimensional surface with the action written in coordinates of some domain on this surface. (This way, for example, in §6 we obtain a local one-parameter transformation group of the sphere.)

8.1. One-dimensional case. Let us first consider at first global realizations of the transformation groups of the line.

1°. Let $\mathfrak{g}=\langle\frac{\partial}{\partial x}\rangle$. Then it is easy to see that the one-parameter transformation group of the line corresponding to the vector field $a\frac{\partial}{\partial x}$, $a\in\mathbb{R}$. has the form:

$$\varphi_t\colon x\mapsto x+at.$$

Besides, for all $t\in\mathbb{R}$ diffeomorphisms φ_t are completely defined on the line. So, the corresponding Lie transformation group of the line is the group of parallel translations:

$$G=\{x\mapsto x+a\mid a\in\mathbb{R}\}.$$

2°. Let $\mathfrak{g}=\langle\frac{\partial}{\partial x},\ x\frac{\partial}{\partial x}\rangle$. Then the one-parameter transformation group of the line corresponding to the vector field $\mathbf{v}=(a+bx)\frac{\partial}{\partial x}$ has the form:

$$\varphi_t\colon x\mapsto e^{bt}x+a\cdot\frac{e^{bt}-1}{b}\quad\text{when } b\neq 0 \text{ and}\tag{1}$$

$$\varphi_t\colon x\mapsto x+at\quad\text{when } b=0.$$

Diffeomorphisms φ_t for all $t\in\mathbb{R}$ are also defined on the whole line. All transformations of type (1) generate the group of affine transformations on the line:

$$G=\{x\mapsto ax+b\mid a\in\mathbb{R}_+^*,\ b\in\mathbb{R}\}.$$

3°. Now let $\mathfrak{g}=\langle\frac{\partial}{\partial x};\ x\frac{\partial}{\partial x};\ x^2\frac{\partial}{\partial x}\rangle$. In this case the one-parameter transformation group of the line corresponding to the vector field $(a+bx+cx^2)\frac{\partial}{\partial x}$, where $c\neq 0$, can be determined only locally.

Consider the group of linear fractional transformations of the line:

$$G = \left\{ x \mapsto \frac{ax+b}{cx+d} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \right\}.$$

Since transformation of the form $x \mapsto \frac{ax+b}{cx+d}$, where $c \neq 0$, is not defined for the point $x = -\frac{d}{c}$, we see that G is a local Lie transformation group. Let us describe its Lie algebra of vector fields on the line. To do it, we shall draw smooth curves in G passing trough the identity and then find the vector fields on the line corresponding to these curves. For example, suppose

$$s\colon t \mapsto \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

is one of these curves. Then the corresponding family of diffeomorphisms of the line that depends smoothly on parameter t has the form:

$$\varphi_t\colon x \mapsto e^{2t}x.$$

The corresponding vector field has the form:

$$\mathbf{v}_x = \lim_{t\to 0} \frac{\varphi_t(x) - x}{t} = 2x, \;\; x \in \mathbb{R},$$

i.e.

$$\mathbf{v} = 2x\frac{\partial}{\partial x}.$$

Exercise. Check that the curves

$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } t \mapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

determine the vector fields $\frac{\partial}{\partial x}$ and $-x^2\frac{\partial}{\partial x}$ respectively. Since the group G is a 3-parameter transformation group, we see that the corresponding Lie algebra $\mathfrak{g}$ of vector fields is three-dimensional. We already know that $\frac{\partial}{\partial x}$; $2x\frac{\partial}{\partial x}$; $-x^2\frac{\partial}{\partial x}$ belong to $\mathfrak{g}$. Therefore,

$$\mathfrak{g} = \langle \frac{\partial}{\partial x};\; x\frac{\partial}{\partial x};\; x^2\frac{\partial}{\partial x} \rangle.$$

Let us now describe the global realization of this transformation group.

By $\mathbb{R}\mathrm{P}^n$ denote the set of all straight lines in $\mathbb{R}^{n+1}$ passing through the origin of coordinates. This set is called the *n-dimensional projective space*.(If $n = 1$ or 2, the set $\mathbb{R}\mathrm{P}^n$ is also called the *projective line* and *projective plane* respectively.) It can also be described as the quotient set of $\mathbb{R}^{n+1}/\{0\}$ by the following equivalence relation:

$$(x_1, x_2, \ldots, x_{n+1}) \sim (\lambda x_1, \lambda x_2, \ldots, \lambda x_{n+1}), \quad \lambda \in \mathbb{R}^*.$$

The equivalence class determined by a point $(x_1, x_2, \dots, x_{n+1})$ is denoted by $[x_1 : x_2 : \dots : x_{n+1}]$. The set of elements $[x_1 : x_2 : \dots : x_{n+1}] \in \mathbb{R}\mathrm{P}^n$ such that $x_{n+1} \neq 0$ is called the *affine chart* and can be identified with $\mathbb{R}^n$ by means of the following correspondence:

$$\mathbb{R}\mathrm{P}^n \ni [x_1 : x_2 : \dots : x_{n+1}] \mapsto \left(\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}} \right) \in \mathbb{R}^n.$$

Now let $n = 1$. Consider the following transformations of the projective line $\mathbb{R}\mathrm{P}^1$:

$$[x_1 : x_2] \mapsto [(ax_1 + bx_2) : (cx_1 + dx_2)].$$

They are well-defined and form a transformation group G. Let us identify the affine chart

$$U = \{[x_1, x_2] \in \mathbb{R}\mathrm{P}^1 | x_2 \neq 0\}$$

with the line $\mathbb{R}$ by means of the following correspondence:

$$t \mapsto [t : 1], \quad [x_1 : x_2] \mapsto \frac{x_1}{x_2}.$$

If we now consider the action of transformations from G on the domain U in the new coordinates, we obtain:

$$G = \left\{ t \mapsto \frac{at + b}{ct + d} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \right\}.$$

As we see, this is exactly the group of linear fractional transformations of the line. However, diffeomorphisms of G are still defined not on the whole line.

8.2. Examples for dimension 2. Now for each Lie algebra $\mathfrak{g}$ of vector fields on the plane from theorem 2 we describe a two-dimensional surface S, a Lie group G of its transformations, and a parametrization $\pi: \mathbb{R}^2 \to S$ of some open domain $U = \pi(\mathbb{R}^2)$ on S such that the restriction of the action of G to U determines the Lie algebra $\mathfrak{g}$. Let us first consider the most important examples.

1) Let $S = \mathbb{R}^2$ and let G be the group of affine transformations of the plane. Then the corresponding Lie algebra of vector fields is exactly algebra 15.1 of theorem 3.

2) Let

$$f^{(n)} + a_{n-1} f^{(n-1)} + \dots + a_1 f' + a_0 f = 0 \tag{2}$$

be some homogeneous linear differential equation and $\mathcal{F}$ the space of its solutions. Consider the set G of transformations of the plane such that

$$(x_1, x_2) \mapsto (x_1 + a, bx_2 + f(x_1 + a)),$$

where $a \in \mathbb{R},\ b \in \mathbb{R}^*_+,\ f \in \mathcal{F}$.

Lemma 1. *The set G is a transitive $(n+2)$-parameter transformation group of the plane.*

Proof. Denote by (a, b, f) the transformation of the plane given by

$$(x_1, x_2) \mapsto (x_1 + a, bx_2 + f(x_1 + a)).$$

It is not difficult to show that

$$(a, b, f) = (0, 1, f) \circ (a, b, 0).$$

Let us show that G is a group. Indeed, $\mathrm{Id}_{\mathbb{R}^2} = (0, 1, 0)$. Suppose

$$(a_1, b_1, f_1), (a_2, b_2, f_2) \in G.$$

Then

$$(a_1, b_1, f_1) \circ (a_2, b_2, f_2) = (0, 1, f_1) \circ (a_1, b_1, 0) \circ (0, 1, f_2) \circ (a_2, b_2, 0).$$

Let us calculate $(a_1, b_1, 0) \circ (0, 1, f_2)$. We have:

$$(x_1, x_2) \overset{(0,1,f_2)}{\longmapsto} (x_1, x_2 + f_2(x_1)) \overset{(a_1,b_1,0)}{\longmapsto} (x_1 + a_1, b_1x_2 + b_1f_2(x_1)) =$$

$$= (x_1 + a_1, b_1x_2 + b_1 (L_{a_1} f_2)(x_1 + a_1)),$$

where $L_{a_1} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is a linear operator of the space $C^\infty(\mathbb{R})$ such that $(L_{a_1} f)(x) = f(x - a_1)$. But the space $\mathcal{F}$ is invariant under the action of L_a whenever $a \in \mathbb{R}$. Hence, $L_{a_1} f_2 \in \mathcal{F}$ and

$$(a_1, b_1, 0) \circ (0, 1, f_2) = (a_1, b_1, b_1 L_{a_1} f_2).$$

It means that

$$(a_1, b_1, f_1) \circ (a_2, b_2, f_2) = (0, 1, f_1) \circ (a_1, b_1, b_1 L_{a_1} f_2) \circ (a_2, b_2, 0) =$$

$$= (0, 1, f_1) \circ (0, 1, b_1 L_{a_1} f_2) \circ (a_1, b_1, 0) \circ (a_2, b_2, 0) =$$

$$= (0, 1, f_1 + b_1 L_{a_2} f_2) \circ (a_1 + a_2, b_1b_2, 0) = (a_1 + a_2, b_1b_2, f_1 + b_1 L_{a_2} f_2).$$

It is easy to verify that

$$(a, b, f)^{-1} = (-a, \frac{1}{b}, -\frac{1}{b} L_{-a} f).$$

Consequently, G is a group of transformations of the plane. Let us show that it is transitive. Let f be an arbitrary nonzero function from $\mathcal{F}$. Since $L_a f \in \mathcal{F}$ for all

$a \in \mathbb{R}$, we can assume that $f(0) \neq 0$. Then any point (x_1, x_2) of the plane can be taken into the point $(0, 0)$ by means of the following transformations:

$$\left(-x_1, 1, -\frac{x_2}{f(0)} f\right) : (x_1, x_2) \longmapsto (0, 0).$$

Further, let $\omega_1, \omega_2, \ldots, \omega_n$ be a basis of the space $\mathcal{F}$. Then the mapping

$$\pi \colon \mathbb{R}^{n+2} \to G, \qquad (x, y, z_1, z_2, \ldots, z_n) \mapsto (x, e^y, z_1\omega_1 + \cdots + z_n\omega_n)$$

is a parametrization of the group. It shows that G is an $(n+2)$-parameter transformation group.

Exercise. Prove that the corresponding Lie algebra of vector fields has form 4.1 from theorem 2.

3) The example given below is some modification of the previous one. So, let $\mathcal{F} = \mathbb{R}_{n-1}[x]$ be the set of all polynomials of degree not greater than $n-1$. It can be considered as the set of all solutions of the differential equation $f^{(n)} = 0$. Note that $\mathcal{F}$ is stable under the following transformations of the space $C^\infty(\mathbb{R})$:

$$L_{(a,b)} \colon f \mapsto L_{(a,b)} f,$$

where $L_{(a,b)} f(x) = f(ax + b)$ for all $a \in \mathbb{R}^*$, $b \in \mathbb{R}$. This enables us to construct the following transformation group of the plane:

$$G = \{\ (x_1, x_2) \mapsto (ax_1 + b, cx_2 + f(ax_1 + b)) \mid a, c \in \mathbb{R}^*_+,\ b \in \mathbb{R},\ f \in \mathcal{F}\}.$$

The proof of the following lemma is similar to that of lemma 1.

Lemma 2. *The group G is a transitive $(n+3)$-parameter transformation group of the plane.*

The corresponding Lie algebra of vector fields is 5.1 from theorem 2.

4) As above let $\mathcal{F} = \mathbb{R}_n[x]$. Consider the set G of all local diffeomorphisms of the plane such that

$$(x_1, x_2) \longmapsto \left(\frac{ax_1 + bx_2}{cx_1 + dx_2}, \frac{ex_2}{(cx_1 + dx_2)^n} + f\left(\frac{ax_1 + bx_2}{cx_1 + dx_2}\right)\right).$$

Lemma 3. *The set G is an $(n+5)$-parameter local transformation group of the plane.*

The proof of the lemma is analogous to that of lemma 1 if we take into account that $\mathcal{F}$ is invariant under the following transformations

$$f \mapsto \tilde{f}, \text{ where } \tilde{f}(t) = (ct + d)^n \cdot f\left(\frac{at + b}{ct + d}\right).$$

In other words, if

$$f(t) = p_n t^n + \dots + p_1 t + p_0,$$

then

$$\tilde{f}(t) = p_n(at+b)^n + \dots + p_1(at+b)\cdot(ct+b)^{n-1} + p_0(ct+b)^n.$$

Direct calculation shows that the corresponding Lie algebra of vector fields has form 18.1 of theorem 2.

Let us describe the global realization of this transformation group. For S consider the quotient set of $(\mathbb{R}^2\backslash\{0\})\times\mathbb{R}$ by the following equivalence relation:

$$(y_1, y_2, z) \sim (\lambda y_1, \lambda y_2, \lambda^n z), \quad \lambda \in \mathbb{R}^*.$$

Denote by $\mathbb{R}^n[y_1, y_2]$ the set of all homogeneous polynomials of degree n in variables y_1, y_2:

$$\mathbb{R}^n[y_1, y_2] = \{a_0 y_1^n + a_1 y_1^{n-1} y_2 + \dots + a_{n-1} y_1 y_2^{n-1} + a_n y_2^n | a_i \in \mathbb{R},\ 0 \leqslant i \leqslant n\}.$$

Consider the following set of transformations of S:

$$G = \{(y_1, y_2, z) \longmapsto (ay_1 + by_2, cy_1 + dy_2, ez + f(ay_1 + by_2, cy_1 + dy_2)) | \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),\ e \in \mathbb{R}_+^*,\ f \in \mathbb{R}^n[y_1, y_2]\Big\}.$$

Exercise. Check that these transformations are well-defined and form a group.

By U denote the following domain in S:

$$U = \{(y_1, y_2, z) \in S | y_2 \neq 0\}.$$

Let us identify U with the plane by means of the mapping

$$\pi : (y_1, y_2, z) \longmapsto \left(\frac{y_1}{y_2}, \frac{z}{y_2^n}\right).$$

Then the inverse mapping has the form:

$$\pi^{-1} : (x_1, x_2) \longmapsto (x_1, 1, x_2).$$

It can be easily seen that transformations from G, written in coordinates (x_1, x_2) of U, have the desired form.

5) Let us construct the global realization of Lie algebra 13.1 of theorem 3. It is analogous to the global realization of linear fractional transformations of the line. Namely, let $S = \mathbb{R}P^2$ and let G be the following set of transformations of the projective plane:

$$G = \{[y_1 : y_2 : y_3] \longmapsto [(a_{11}y_1 + a_{12}y_2 + a_{13}y_3) : (a_{21}y_1 + a_{22}y_2 + a_{23}y_3) : \\ (a_{31}y_1 + a_{32}y_2 + a_{33}y_3)] \mid (a_{ij})_{1\leqslant i,j\leqslant 3} \in SL(3, \mathbb{R})\}.$$

For a domain $U \subset \mathbb{R}P^2$ take the affine chart

$$U = \{[y_1 : y_2 : y_3] \in \mathbb{R}P^2 | y_3 \neq 0\}.$$

Let us parametrize U by the mapping

$$\pi : \mathbb{R}^2 \to U, \quad (x_1, x_2) \mapsto [x_1 : x_2 : 1].$$

Then

$$\pi^{-1}\colon U \to \mathbb{R}^2, \quad [y_1 : y_2 : y_3] \mapsto \left(\frac{y_1}{y_3}; \frac{y_2}{y_3}\right)$$

and transformations from the group G written in the coordinates (x_1, x_2) of the plane are exactly linear fractional transformations of the plane. The corresponding Lie algebra of vector fields is the algebra 13.1 of theorem 3.

6) Let $S = \mathbb{R}^2 \backslash \{0\}$ and let G be the group of linear transformations of the plane. Consider the following parametrization of S:

$$\pi\colon (x_1, x_2) \mapsto (x_1 e^{x_2},\ e^{x_2}).$$

The inverse mapping is given by

$$\pi^{-1}\colon (y_1, y_2) \mapsto \left(\frac{y_1}{y_2};\ \ln y_2\right)$$

and defined on the domain $U = \{(y_1, y_2) \mid y_2 > 0\}$. It is easy to show that the action of G restricted to U and written in the coordinates (x_1, x_2) has the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (x_1, x_2) \mapsto \left(\frac{ax_1 + b}{cx_1 + d},\ x_2 + \ln(cx_2 + d)\right).$$

It is a local 4-parameter transformation group of the plane such that its Lie algebra is exactly the Lie algebra 16.1 from theorem 3.

CHAPTER II

ABSTRACT APPROACH

§1. Basic constructions

1.1. Action of groups. The central concept of theory of homogeneous spaces is an action of a group on a set.

Definition 1. Let $\overline{G}$ be an arbitrary group. An *action of the group $\overline{G}$ on a set M* is a homomorphism

$$\alpha : \overline{G} \to \operatorname{Aut}(M).$$

Then any element $g \in \overline{G}$ can be regarded as the transformation $\alpha(g)$ of the set M. In the sequel, for the sake of convenience we shall write $g.m$ instead of $\alpha(g)(m)$ $\big(g \in \overline{G}, m \in M\big)$. It is immediate from the definition that the following conditions hold:

$$(g_1g_2).m = g_1.(g_2.m) \qquad \forall\, g_1, g_2 \in \overline{G}, m \in M$$

$$e.m = m \qquad \forall\, m \in M.$$

Taking this into account, definition 1 can be reformulated as follows:

Definition 1'. Let $\overline{G}$ be an arbitrary group. An *action of $\overline{G}$ on a set M* is a mapping $\alpha : \overline{G} \times M \to M,\quad (g, m) \mapsto g.m$ such that

$$(g_1g_2).m = g_1.(g_2.m) \qquad \forall\, g_1, g_2 \in \overline{G}, m \in M$$

$$e.m = m \qquad \forall\, m \in M.$$

Examples.

1) For any group $\overline{G}$ and any set M, put $g.m = m$ for all $g \in \overline{G}, m \in M$. This action is called trivial.

2) The permutation group S_n of degree n acts naturally on a set of n elements.

3) The group of Euclidean (affine) transformations acts on the plane.

4) The group of rotations acts on the sphere.

5) Let $M = C^\infty(\mathbb{R})$ be the set of all smooth functions on $\mathbb{R}$ and $\overline{G}$ the group of translations on the line. Note that $\overline{G}$ can be identified with the set of real numbers. Then put

$$(a.f)(x) = f(x - a) \quad \text{for } a \in \overline{G}, f(x) \in M,$$

which gives an action of $\overline{G}$ on M.

Let us introduce some concepts connected with an action of a group on a set.

Definition 2. Suppose that a group $\overline{G}$ acts on a set M and m is an arbitrary element of M. The *stabilizer of the element* m is the set

$$\overline{G}_m = \{g \in \overline{G} | g.m = m\}.$$

It is easy to show that $\overline{G}_m$ is a subgroup of $\overline{G}$.

The set M can be supplied with an equivalence relation in the following way: $m_1 \sim m_2$ whenever there exists a $g \in \overline{G}$ such that $g.m_1 = m_2$.

Exercise. Make sure that the relation introduced above is truly an equivalence relation.

Definition 3. Equivalence classes with respect to the introduced equivalence relation are called *orbits of the action of* $\overline{G}$ on M.

Exercise. Suppose $m_1, m_2 \in M$ are two elements lying in the same orbit. Show that the subgroups $\overline{G}_{m_1}$ and $\overline{G}_{m_2}$ are conjugate.

Definition 4. We say that a group $\overline{G}$ acts *transitively* on a set M if the action of $\overline{G}$ on M has only one orbit, which is equal to M.

This is equivalent to the fact that for any two elements $m_1, m_2 \in M$ there exists a $g \in \overline{G}$ such that $g.m_1 = m_2$. If the element g is unique, then the action is called *simply transitive*. For example, the action of a group $\overline{G}$ on itself by means of left shifts ($g.h = gh$ for $g, h \in \overline{G}$) is simply transitive.

Exercise. Show that any simply transitive action can be reduced to the action like this.

Consider one very important example of a transitive action. Suppose $\overline{G}$ is an arbitrary group and G is its subgroup. Let $M = \overline{G}/G$ be the set of left cosets $\{gG | g \in \overline{G}\}$ relative to G in $\overline{G}$. Consider the action of $\overline{G}$ on M defined by

$$g.(hG) = ghG.$$

Exercise. Check that the action is well-defined and transitive. Find the stabilizer of the point eG.

The following lemma shows that all transitive actions can be reduced to this example.

Lemma. *Suppose that a group* $\overline{G}$ *acts transitively on a set* M, $m \in M$, *and* $G = \overline{G}_m$ *is a stabilizer of the point* m. *Then the mapping* $\pi : \overline{G}/G \to M$, $gG \mapsto g.m$ *establishes the natural isomorphism of the sets* $\overline{G}/G$ *and* M, *and the isomorphism* π *is in agreement with the action of* $\overline{G}$, *i.e.*

$$g.\pi(x) = \pi(g.x) \qquad \text{for all } g \in \overline{G}, x \in \overline{G}/G.$$

The proof is trivial and is left as an exercise.

Thus, the study of transitive actions can essentially be reduced to the study of groups and their subgroups.

The concept of a homogeneous space is based on the consideration of smooth objects in the previous definitions. In doing so groups turn into Lie groups, while sets into smooth manifolds.

Definition 5. Let $\overline{G}$ be a Lie group that acts on a smooth manifold M so that the mapping $\overline{G} \times M \to M$, $(g, m) \mapsto g.m$ is a smooth mapping of manifolds. Then the pair $(\overline{G}, M)$ is called a *homogeneous space.*

All given earlier definitions can be extended to the case of homogeneous spaces. Moreover, the following facts are true.

Proposition.
1) Let $(\overline{G}, M)$ be a homogeneous space and $m \in M$. Then $G = \overline{G}_m$ is a closed subgroup of the Lie group $\overline{G}$.
2) Let $\overline{G}$ be an arbitrary Lie group and G its closed subgroup. Then the set $M = \overline{G}/G$ of left cosets can be uniquely supplied with a structure of a smooth manifold so that the natural action of $\overline{G}$ on M is smooth.

Therefore, as in general case, the study of homogeneous spaces reduces to the study of pairs $(\overline{G}, G)$, where $\overline{G}$ is a Lie group and G is a closed subgroup of $\overline{G}$.

1.2. Examples of homogeneous spaces. Examples 1), 2), and 4) of actions of groups on sets are in fact examples of homogeneous spaces.

1) The groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ act transitively on the set $\mathbb{R}^n\backslash\{0\}$, which can be naturally supplied with a structure of a smooth manifold.

2) The action of the groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ on $\mathbb{R}^n$ generates their action on the set of straight lines passing through the origin of coordinates. This way we obtain the action of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$ on the projective space $\mathbb{R}\mathrm{P}^{n-1}$. Let us write it out in an explicit form in terms of homogeneous coordinates of the manifold $\mathbb{R}\mathrm{P}^{n-1}$. Let $[x_0 : x_1 : \cdots : x_{n-1}]$ be homogeneous coordinates of a point $X \in \mathbb{R}\mathrm{P}^{n-1}$ and let A be an element of $\mathrm{GL}(n,\mathbb{R})$. Then the homogeneous coordinates of the point $A.X$ have the form $[y_0 : y_1 : \cdots : y_{n-1}]$, where the column

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$$

can be obtained as a product of the matrix A by the column

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

on the right. In particular, this immediately implies that the action is smooth.

3) Let $(\overline{G}, M)$ be a homogeneous space and $\overline{H}$ a closed Lie subgroup of $\overline{G}$. If $\overline{H}$ acts transitively on M, we obtain the new homogeneous space $(\overline{H}, M)$. In this case $(\overline{H}, M)$ is called a *restriction* of the homogeneous space $(\overline{G}, M)$. It may turn out that the action of $\overline{H}$ on M is not transitive but has an open orbit $\widetilde{M}$. In such a situation

the structure of a smooth manifold can be induced to $\widetilde{M}$ and the pair $(\overline{H}, \widetilde{M})$ can be turned into a homogeneous space. In this case $(\overline{H}, \widetilde{M})$ is said to be a *local restriction* of the homogeneous space $(\overline{G}, M)$.

For example, let $\overline{G} = \mathrm{Aff}(n)$ be the group of affine transformations of the affine space $\mathbb{A}^n$ and let x be an arbitrary point of $\mathbb{A}^n$. Let $\overline{H}$ be the stabilizer of the point x. Then $\overline{H}$ is a closed Lie subgroup of $\overline{G}$ and it can be identified with $\mathrm{GL}(n, \mathbb{R})$. The action of $\overline{H}$ on $\mathbb{A}^n$ is not transitive. However, $\overline{H}$ acts transitively on $\mathbb{A}^n \backslash \{x\}$. So we obtain the new homogeneous space $(\mathrm{GL}(n, \mathbb{R}), \mathbb{A}^n \backslash \{x\})$, which coincides essentially with that from example 1).

Exercise. Describe the stabilizer of some point for each of the homogeneous spaces mentioned above.

We shall say that two homogeneous spaces $(\overline{G}_1, M_1)$ and $(\overline{G}_2, M_2)$ are *equivalent* if there exists a pair of mappings (f, φ) such that

$$\begin{aligned} &f : \overline{G}_1 \to \overline{G}_2 \text{ is an isomorphism of Lie groups,} \\ &\varphi : M_1 \to M_2 \text{ is a diffeomorphism of manifolds,} \end{aligned}$$

and $\varphi(g.m) = f(g).\varphi(m)$ for all $g \in \overline{G}_1, m \in M_1$.

By the *dimension of a homogeneous space* $(\overline{G}, M)$ we shall mean the dimension of the manifold M.

In the sequel, we shall mainly be interested in describing and studying small-dimensional homogeneous spaces ($\dim M \leqslant 4$), especially two-dimensional ones. (One-dimensional homogeneous spaces are rather simple, whereas those of dimensions 3 and 4 are much more unwieldy.)

Preparatory to this, we shall impose some more restrictions on homogeneous spaces to be considered.

Definition 6. A homogeneous space $(\overline{G}, M)$ is called *effective* if the identity element $e \in \overline{G}$ is the only element of the group $\overline{G}$ which leaves all points of the manifold M stable. The subgroup $K = \cap_{x \in M} \overline{G}_x$ is called the *kernel of ineffectiveness* of $(\overline{G}, M)$. In other words, K is the set of all elements of $\overline{G}$ that leave all points of M stable.

Note that a homogeneous space $(\overline{G}, M)$ is effective if and only if its kernel of ineffectiveness is trivial. Suppose $\alpha : \overline{G} \to \mathrm{Diff}(M)$ is the homomorphism of groups that determines the action of $\overline{G}$ on M; then K is exactly the kernel of α. In particular, this implies that K is a normal subgroup of the Lie group $\overline{G}$. Since K is an intersection of closed Lie subgroups ($K = \cap_{x \in M} \overline{G}_x$), then K is also a closed Lie subgroup of $\overline{G}$. This allows to consider the factor group $\overline{H} = \overline{G}/K$, which can be naturally supplied with a structure of a Lie group. Moreover, it is possible to define an action of the Lie group $\overline{H}$ on M:

$$(xK).m = x.m \quad \text{for } x \in \overline{G}, m \in M.$$

It is easy to show that the action is well-defined, which turns $(\overline{H}, M)$ into a homogeneous space.

Exercise. Check that $(\overline{H}, M)$ is an effective homogeneous space.

In the study of homogeneous spaces (for instance, while various different invariants) it is important to consider not a group $\overline{G}$ itself but its image in $\mathrm{Diff}(M)$. Therefore, we shall further be interested only in effective homogeneous spaces.

Another restriction that we impose on the homogeneous spaces to be studied is that $\overline{G}$ is a connected Lie group. It can always be achieved by considering the action of the connected component of the identity element instead of the whole Lie group's action.

So, our immediate task is to describe small-dimensional effective homogeneous spaces $(\overline{G}, M)$ such that $\overline{G}$ is a connected Lie group.

Since there is a one-to-one correspondence between the set of homogeneous spaces and the set of pairs $(\overline{G}, G)$, where $\overline{G}$ is a Lie group and G its connected Lie subgroup, we can extend the previous definitions to the case of pairs $(\overline{G}, G)$.

Theorem.

1. The homogeneous spaces defined by pairs $(\overline{G}_1, G_1)$ and $(\overline{G}_2, G_2)$ are equivalent if and only if there exists an isomorphism of Lie groups $f : \overline{G}_1 \to \overline{G}_2$ such that $f(G_1) = G_2$.

2. The kernel of ineffectiveness of the homogeneous space defined by a pair $(\overline{G}, G)$ is exactly the largest normal Lie subgroup of $\overline{G}$ that belongs to G.

3. The dimension of the homogeneous space defined by a pair $(\overline{G}, G)$ is equal to the codimension of the subgroup G in the Lie group $\overline{G}$.

Proof.

1. Indeed, suppose $f : \overline{G}_1 \to \overline{G}_2$ is an isomorphism of Lie groups such that $f(G_1) = G_2$. Then consider the smooth mapping

$$\varphi : \overline{G}_1/G_1 \to \overline{G}_2/G_2$$

defined by $\varphi(xG_1) = f(x)G_2$ for $x \in \overline{G}_1$. It is easy to check that φ is well-defined and is a diffeomorphism of manifolds, and that the pair (f, φ) establishes equivalence of the homogeneous spaces $(\overline{G}_1, \overline{G}_1/G_1)$ and $(\overline{G}_2, \overline{G}_2/G_2)$.

Conversely, suppose that a pair of mappings $(\tilde{f}, \varphi)$ establishes equivalence of the homogeneous spaces $(\overline{G}_1, \overline{G}_1/G_1)$ and $(\overline{G}_2, \overline{G}_2/G_2)$. There exists a $g \in \overline{G}_2$ such that $\varphi(eG_1) = gG_2$. Put

$$f(x) = g^{-1}\tilde{f}(x)g \qquad \text{for all } x \in \overline{G}_1.$$

It is easy to verify that $f : \overline{G}_1 \to \overline{G}_2$ is an isomorphism of Lie groups and $f(G_1) = G_2$.

2. Suppose K is the kernel of ineffectiveness of the homogeneous space $(\overline{G}, \overline{G}/G)$. Then K is a normal Lie subgroup of the Lie group $\overline{G}$. Show that $K \subset G$. Indeed, if $x \in K$, then

$$xG = x.(eG) = eG.$$

Therefore $x \in G$.

Let K' be an arbitrary normal Lie subgroup of $\overline{G}$ such that $K' \subset G$. Then for $x \in K'$ and $g \in \overline{G}$, since $g^{-1}xg \in K' \subset G$, we have

$$x.(gG) = xgG = g(g^{-1}xg)G = gG.$$

This means that $K' \subset K$ and that K is really the largest normal Lie subgroup of $\overline{G}$ lying in G.

The converse can be proved in a similar way.

3. The proof is trivial.

Definition 7.

1. We say that pairs $(\overline{G}_1, G_1)$ and $(\overline{G}_2, G_2)$ are *equivalent* if there exists an isomorphism of Lie groups $f : \overline{G}_1 \to \overline{G}_2$ such that $f(G_1) = G_2$.
2. We say that a pair $(\overline{G}, G)$ is *effective* if G contains no nontrivial normal Lie subgroups of $\overline{G}$.
3. By the *codimension of a pair* $(\overline{G}, G)$ we shall mean the codimension of the Lie subgroup G in the Lie group $\overline{G}$.

Thus, our problem reduces to finding (up to equivalence) all effective pairs $(\overline{G}, G)$ of small codimension.

Example. Let $\overline{G} = \mathrm{SL}(2, \mathbb{R})$. Suppose $\widetilde{G}$ is the set of all upper triangular matrices of the Lie group $\mathrm{SL}(2, \mathbb{R})$. This set is normally denoted by $\mathrm{ST}(2, \mathbb{R})$. Since $\widetilde{G}$ contains the center of $\overline{G}$, which is equal to $\{\pm E\}$, we see that the pair $(\overline{G}, G)$ is not effective. Instead of the subgroup $\widetilde{G}$, consider its connected component of the identity element

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix} \,\middle|\, x \in \mathbb{R}_+^*, y \in \mathbb{R} \right\}.$$

The pair $(\overline{G}, G)$ is effective. Let us describe the homogeneous space $(\overline{G}, \overline{G}/G)$. Any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

can be uniquely written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1/x \end{pmatrix}, x \in \mathbb{R}_+^*, y \in \mathbb{R}, \varphi \in [0, 2\pi). \tag{1}$$

Here x and φ are uniquely determined from the condition $xe^{i\varphi} = a + ic$, and

$$y = b\cos\varphi + d\sin\varphi.$$

Therefore, each left coset $gG, g \in \overline{G}$, is uniquely determined by the angle φ in (1), which is the same for all elements of the coset. Thus, the factor space $M = \overline{G}/G$ can be identified with the circle S^1. If S^1 is regarded as the set of those vectors of the Euclidean vector space $\mathbb{R}^2$ whose norm is equal to 1:

$$S^1 = \{v \in \mathbb{R}^2, |v| = 1\},$$

then the identification has the form

$$\varphi \mapsto \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$$

Now describe the action of $\overline{G}$ on M. Let

$$v = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

be an arbitrary element of M and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an element of $\overline{G}$. The left coset corresponding to $g.v$ has the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\overline{G} = \begin{pmatrix} a\cos\varphi + b\sin\varphi & -a\sin\varphi + b\cos\varphi \\ c\cos\varphi + d\sin\varphi & -c\sin\varphi + d\cos\varphi \end{pmatrix} G.$$

Therefore $g.v = \begin{pmatrix} \cos\varphi' \\ \sin\varphi' \end{pmatrix}$, where $\varphi' = \arg((a\cos\varphi + b\sin\varphi) + i(c\cos\varphi + d\sin\varphi))$. This can also be written as

$$g.v = \frac{g(v)}{|g(v)|},$$

where $g(v)$ is the image of v by the linear transformation of $\mathbb{R}^2$ with matrix g.

Exercise. Show that the homogeneous space defined by the pair $(\overline{G}, \widetilde{G})$ is equivalent to $(\mathrm{SL}(2,\mathbb{R}), \mathbb{R}P^1)$.

1.3. Linearization. The study of Lie groups can in many respects be reduced to the study of their Lie algebras. Similarly, the study of pairs $(\overline{G}, G)$ (and therefore, the study of homogeneous spaces) can be reduced to the study of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}$ is a Lie algebra and $\mathfrak{g}$ its subalgebra.

In the sequel, we shall employ the following three equivalent definitions for the Lie algebra $\mathfrak{g}$ of a Lie group G.

Definition 8. The *Lie algebra* $\mathfrak{g}$ *of a Lie group* G is
a) the tangent space to the identity element of G;
b) the set of left-invariant vector fields on G;
c) the set of one-parameter subgroups of G.

All three definitions are in agreement:
a) $\Leftrightarrow$ b)

To each vector $v \in T_eG$ assign the left-invariant vector field $V \in D(G)$ defined by

$$V_g = dL_g(v),$$

where $g \in G$, $L_g : G \to G$, $x \mapsto gx$ is the diffeomorphism of G corresponding to the element g, and $dL_g : T_eG \to T_gG$ is the differential of L_g at the point $x = e$.

Conversely, if V is a left-invariant vector field on G, then V is uniquely determined by the vector $V_e \in T_eG$.

a) $\Leftrightarrow$ c)

Let v be an arbitrary element of the tangent space T_eG. Define a curve $g(t)$ in the Lie group G as a solution of the differential equation

$$\dot{g}(t) = dL_{g(t)}(v)$$

with the initial condition $g(0) = e$. It is easy to show that $g(t)$ is defined for all $t \in \mathbb{R}$ and that $g(t_1 + t_2) = g(t_1)g(t_2)$. In other words, $g(t)$ is a one-parameter subgroup of G.

Conversely, each one-parameter subgroup $g(t)$ of G is uniquely determined by the tangent vector to $g(t)$ at the point $g(0)$. Since $g(0) = e$, this vector belongs to T_eG.

If $\mathfrak{g}$ is regarded as T_eG, then $\mathfrak{g}$ can be supplied with the structure of a vector space of dimension equal to that of G. Furthermore, if V_1 and V_2 are two left-invariant vector fields on G, then their Lie bracket $[V_1, V_2]$ is also a left-invariant vector field on G. Thus, we have the skew-symmetric bilinear form on $\mathfrak{g}$:

$$(x, y) \mapsto [x, y] \qquad \text{for } x, y \in \mathfrak{g},$$

satisfying the Jacobi identity:

$$\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0 \qquad \forall\, x, y, z \in \mathfrak{g}.$$

This turns $\mathfrak{g}$ into a finite-dimensional real Lie algebra.

Examples.

1) The Lie algebra of the Lie group $\mathrm{GL}(n, \mathbb{R})$ can be identified with the set of all n-by-n matrices. It is denoted by $\mathfrak{gl}(n, \mathbb{R})$. The bracket operation has the form:

$$[A, B] = AB - BA \qquad \text{for } A, B \in \mathfrak{gl}(n, \mathbb{R}).$$

Similarly, the Lie algebra of the Lie group $\mathrm{SL}(n, \mathbb{R})$ is the set of all n-by-n matrices with zero trace. It is denoted by $\mathfrak{sl}(n, \mathbb{R})$ and has the same bracket operation.

2) Let $G = Aff(n)$. Fixing a point in $\mathbb{A}^n$, we can identify G with the group $\mathrm{GL}(n, \mathbb{R}) \leftthreetimes \mathbb{R}^n$. The group operation has the form:

$$(A_1, v_1)(A_2, v_2) = (A_1A_2, v_1 + A_1v_2),$$

where $A_1, A_2 \in \mathrm{GL}(n, \mathbb{R})$, $v_1, v_2 \in \mathbb{R}^n$.

The Lie algebra of the Lie group G can be identified with the set $\mathfrak{gl}(n,\mathbb{R}) \rightthreetimes \mathbb{R}^n$. The bracket operation has the form:

$$[(A_1, v_1), (A_2, v_2)] = ([A_1, A_2], A_1 v_2 - A_2 v_1).$$

Let $\bar{\mathfrak{g}}$ be a Lie algebra and $\mathfrak{g}$ a subspace of $\bar{\mathfrak{g}}$ closed under the bracket operation. Then $\mathfrak{g}$ is called a *subalgebra* of $\bar{\mathfrak{g}}$. In particular, $\mathfrak{g}$ is a Lie algebra itself. If $[\bar{\mathfrak{g}}, \mathfrak{g}] \subset \mathfrak{g}$, then $\mathfrak{g}$ is called an *ideal* in $\bar{\mathfrak{g}}$. For example, the subalgebra $\mathfrak{sl}(n,\mathbb{R})$ is an ideal in the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$.

Exercise. Let $\mathfrak{t}(n,\mathbb{R})$ denote the set of all upper triangular square matrices of order n. Show that $\mathfrak{t}(n,\mathbb{R})$ is a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$. Prove that $\mathfrak{t}(n,\mathbb{R})$ is not an ideal whenever $n \geqslant 2$.

Let $\overline{G}$ be a Lie group and $\bar{\mathfrak{g}}$ its Lie algebra. Then to any Lie subgroup G of $\overline{G}$ we can assign a subalgebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$. If $\bar{\mathfrak{g}}$ is regarded as $T_e\overline{G}$, then $\mathfrak{g}$ is exactly the subspace T_eG of the vector space $T_e\overline{G}$. Therefore, to any pair of Lie groups $(\overline{G}, G)$ we can assign the pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$. Moreover, if G is a normal Lie subgroup of $\overline{G}$, then $\mathfrak{g}$ is an ideal in $\bar{\mathfrak{g}}$.

Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras. A *homomorphism* of $\mathfrak{g}_1$ into $\mathfrak{g}_2$ is a linear mapping $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$f([x,y]) = [f(x), f(y)]$$

for all $x, y \in \mathfrak{g}_1$.

Suppose $f : G_1 \to G_2$ is an arbitrary homomorphism of Lie groups. Consider its differential $df : T_eG_1 \to T_eG_2$ at the identity element of G_1. If we identify the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of the Lie groups G_1 and G_2 with the tangent spaces T_eG_1 and T_eG_2, respectively, then the mapping $df : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of Lie algebras. If f is an isomorphism of Lie groups, then df is obviously an isomorphism of Lie algebras.

According to all the preceding, we approach the problem of classifying pairs of Lie groups $(\overline{G}, G)$ in the following way: first, to classify pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras, and then for each of the obtained pairs to find all corresponding pairs of Lie groups.

In terms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, definition 7 corresponds to the following one:

Definition 9.

1) Two pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are said to be *equivalent* if there exists an isomorphism of Lie algebras $f : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ such that $f(\mathfrak{g}_1) = \mathfrak{g}_2$.

2) A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *effective* if $\mathfrak{g}$ contains no nontrivial ideals of $\bar{\mathfrak{g}}$.

3) The *codimension of a pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the codimension of the subspace $\mathfrak{g}$ in the vector space $\bar{\mathfrak{g}}$.

The correspondence between definitions 7 and 9 can be established by the following

Theorem.

1) The equivalence of pairs $(\overline{G}_1, G_1)$ and $(\bar{G}_2, G_2)$ implies the equivalence of the corresponding pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$.

2) If a pair $(\bar{G}, G)$ is effective, then the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is also effective.
3) The codimension of a pair $(\overline{G}, G)$ is equal to that of the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Proof. It is immediate from the outlined correspondence between Lie groups and Lie algebras.

So, we shall first turn to the description of effective pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of small codimension.

§2. One-dimensional homogeneous spaces

2.1. Local description. We now turn to the description of one-dimensional homogeneous spaces. First we describe effective pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 1.

Theorem. *Any effective pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 1 is equivalent to one and only one of the following pairs:*

1. $\bar{\mathfrak{g}} = \mathbb{R}e_1, \quad \mathfrak{g} = \{0\}$
2. $\bar{\mathfrak{g}} = \mathbb{R}e_1 \oplus \mathbb{R}e_2,$ *where* $[e_1, e_2] = e_2, \quad \mathfrak{g} = \mathbb{R}e_1$
3. $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}), \quad \mathfrak{g} = \mathfrak{st}(2, \mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$

Proof. We shall make use of Morozov's theorem and some results of semisimple Lie algebras theory.

Theorem [Morozov]. *Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair such that $\mathfrak{g}$ is a maximal subalgebra.*
(i) If $\bar{\mathfrak{g}}$ is not semisimple, then there exists a commutative ideal $\mathfrak{a}$ such that $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$ and $\mathfrak{g}$ acts faithfully on $\mathfrak{a}$ (i.e. the set $\{x \in \mathfrak{g} | [x, \mathfrak{a}] = \{0\}\}$ is zero).
(ii) If $\bar{\mathfrak{g}}$ is semisimple but not simple, then there exists a simple Lie algebra $\tilde{\mathfrak{g}}$ such that $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ and $\mathfrak{g} = \{(x, x) | x \in \tilde{\mathfrak{g}}\}$.

In our case $\mathfrak{g}$ is a subalgebra of codimension 1. Therefore, $\mathfrak{g}$ is maximal. Consider the following cases:

(i) The Lie algebra $\bar{\mathfrak{g}}$ is not semisimple. Then $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is a one-dimensional ideal, and the set $\{x \in \mathfrak{g} | [x, \mathfrak{a}] = \{0\}\}$ equals $\{0\}$. It easily follows that $\dim \mathfrak{g} \leqslant 1$. If $\dim \mathfrak{g} = 0$, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form 1. If $\dim \mathfrak{g} = 1$, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair 2.

(ii) The Lie algebra $\bar{\mathfrak{g}}$ is semisimple but not simple. Then there exists a simple Lie algebra $\tilde{\mathfrak{g}}$ such that $\bar{\mathfrak{g}} = \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ and $\mathfrak{g} = \{(x, x) | x \in \tilde{\mathfrak{g}}\}$. It follows that $\dim \bar{\mathfrak{g}} - \dim \mathfrak{g} = \dim \tilde{\mathfrak{g}}$. However, there exist no simple Lie algebras of dimension less than 3.

(iii) The Lie algebra $\bar{\mathfrak{g}}$ is simple. It is known from semisimple Lie algebras theory that in this case rank of $\bar{\mathfrak{g}}$ equals 1. There exist two real simple Lie algebras of rank 1: $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{su}(2)$. However, $\mathfrak{su}(2)$ contains no subalgebras of codimension 1. Therefore $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R})$. There exists a unique (up to the group $\mathrm{Aut}(\bar{\mathfrak{g}})$) two-dimensional subalgebra in $\mathfrak{sl}(2, \mathbb{R})$. So, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair 3.

2.2. Globalization. Now we proceed to fulfillment of the second part of our plan. For each pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ we shall describe all pairs of Lie groups $(\overline{G}, G)$ and the corresponding homogeneous spaces $(\overline{G}, \overline{G}/G)$.

The process of globalization was described by Mostow in 1950 (G.D.Mostow "The extensibility of local Lie groups of transformations and groups on surfaces", Ann. of Math, v.52, No 3.) Let us cite the basic results of this work.

Let $(\overline{G}, G)$ be a pair of Lie groups, $\bar{\mathfrak{g}}$ the Lie algebra of $\overline{G}$, and $\mathfrak{g}$ the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. In this case we say that the pair $(\overline{G}, G)$ is *associated* with the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Theorem 1. *Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair of codimension $\leqslant 4$. Then there exists a unique effective pair $(\overline{G}, G)$ associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that G is a connected subgroup of $\overline{G}$ and the manifold $\overline{G}/G$ is simply connected.*

The pair $(\overline{G}, G)$ mentioned in theorem 1 can be constructed in the following way. For the Lie algebra $\bar{\mathfrak{g}}$ there exists a unique simply connected Lie group $\overline{H}$ such that $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{H}$. The group $\overline{H}$ contains a uniquely determined connected Lie subgroup H (which is not necessarily closed) corresponding to the subalgebra $\mathfrak{g}$. In particular, if the codimension of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is not greater than 4, then it is immediate from theorem 1 that the subgroup H is closed. Since $\overline{H}$ is a simply connected Lie group, we see that the quotient manifold $\overline{H}/H$ is simply connected. However the pair $(\overline{H}, H)$ is not necessarily effective.

Lemma. *The kernel of ineffectiveness of the pair $(\overline{H}, H)$ is discrete and equals $\mathcal{Z}(\overline{H}) \cap H$.*

Proof. Let K be the kernel of ineffectiveness of the pair $(\overline{H}, H)$. Then K is a normal closed Lie subgroup of the Lie group $\overline{H}$ and $K \subset H$. Therefore, K is discrete; otherwise the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ would not be effective. On the other hand, any normal discrete subgroup belongs to the center $\mathcal{Z}(\overline{H})$. Thus $K \subset \mathcal{Z}(\overline{H}) \cap H$. Conversely, $\mathcal{Z}(\overline{H}) \cap H$ is a normal Lie subgroup of $\overline{H}$ lying in H. Therefore $K \supset \mathcal{Z}(\overline{H}) \cap H$. It follows that $K = \mathcal{Z}(\overline{H}) \cap H$.

The lemma implies that the pair $(\overline{H}/K,\ H/K)$ satisfies the conditions of theorem 1.

Examples.

1. Let $(\bar{\mathfrak{g}}, \mathfrak{g}) = (\mathbb{R}, \{0\})$. The simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$ is also $\mathbb{R}$; the connected subgroup corresponding to the subalgebra $\mathfrak{g}$ is zero. The pair of Lie groups $(\mathbb{R}, \{0\})$ is effective. So, we have $(\overline{G}, G) = (\mathbb{R}, \{0\})$. The corresponding homogeneous space is the group of translations on the line.

2. Let $\bar{\mathfrak{g}} = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, where $[e_1, e_2] = e_2$, and $\mathfrak{g} = \mathbb{R}e_1$. Then the simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$ is the set $\overline{G} = \mathbb{R}_+^* \times \mathbb{R}$ supplied with the following group operation:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 + x_1 y_2).$$

The subgroup corresponding to the subalgebra $\mathfrak{g}$ is $\mathbb{R}_+^* \times \{0\}$.

Exercise. Check that $\mathcal{Z}(\overline{G}) = \{e\}$.

Therefore, the pair $(\overline{G}, G)$ is effective. Consider the homogeneous space $(\overline{G}, \overline{G}/G)$. Note that each element $(x, y) \in \overline{G}$ can be uniquely written as

$$(x, y) = (1, y) \cdot (x, 0).$$

It follows that we can take the number $p = y$ for a coordinate on the coset $(x, y)G$. Then the action of $\overline{G}$ on $M = \overline{G}/G$ can be written as follows:

$$(x, y).p = (x, y).((1, p)G) = (x, y + px)G = y + px.$$

Thus, the homogeneous space $(\overline{G}, \overline{G}/G)$ is exactly the group of affine transformations of $\mathbb{R}$ preserving the orientation (since $x > 0$).

The following result obtained by Mostow gives the description of all effective pairs $(\overline{G}, G)$ associated with a given pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Theorem 2. *Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair of codimension $\leqslant 4$ and $(\overline{G}, G)$ the effective pair associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that $\overline{G}$ is connected and $\overline{G}/G$ is simply connected. Now suppose $\mathcal{Z}$ is the center of $\overline{G}$ and $N(G)$ is the normalizer of G in $\overline{G}$. Then any effective pair associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:*

$$(\overline{G}/(S \cap \mathcal{Z}), G/(S \cap \mathcal{Z})),$$

where S is a Lie subgroup of the Lie group $N(G)$ such that $S \supset G$ and the Lie group S/G is discrete.

Example. Let us describe all pairs associated with the pair $(\mathbb{R}, \{0\})$. Recall that in this case $\overline{G} = \mathbb{R}$ and $G = \{0\}$. It is easy to check that $\mathcal{Z} = \overline{G}$ and $N(G) = \overline{G}$. Thus, all discrete subgroups of the group of real numbers are to be found. All of them have the form $S_a = \{an | n \in \mathbb{Z}\}$. Since the mapping $x \mapsto \alpha x$, where $x \in \mathbb{R}$ and $\alpha \neq 0$, is an automorphism of $\overline{G}$ preserving G, we see that (up to equivalence) S is equal to either $\mathbb{Z}$ or $\{0\}$. In the latter case we obtain no new homogeneous spaces. On the contrary, if $S = \mathbb{Z}$, we obtain the new pair $(\mathbb{R}/\mathbb{Z}, \{0\})$. The corresponding homogeneous space is the group of rotations of the circle.

Exercise. Turn back to example 2 and show that $S = G$ and that no new homogeneous spaces can be obtained.

2.3. The simply connected covering group $\widetilde{\mathrm{SL}(2, \mathbb{R})}$. Let us describe all homogeneous spaces corresponding to the pair

$$(\mathfrak{sl}(2, \mathbb{R}), \mathfrak{st}(2, \mathbb{R})).$$

We have already considered two of them. One of the associated pairs is

$$\overline{G} = \mathrm{SL}(2, \mathbb{R}), \qquad G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a > 0 \right\}.$$

The corresponding homogeneous space can be described as follows: the group $\mathrm{SL}(2,\mathbb{R})$ acts on the circle $S^1 = \{v \in \mathbb{R}^2 \,|\, |v| = 1\}$ so that

$$g.v = \frac{g(v)}{|g(v)|}, \qquad v \in S^1,\ g \in \mathrm{SL}(2,\mathbb{R}).$$

In the second case the group $\mathrm{SL}(2,\mathbb{R})$ acts on the projective space $\mathbb{R}\mathrm{P}^1$, which is also homeomorphic to the circle. Since the element $-E$ acts trivially on $\mathbb{R}\mathrm{P}^1$, we see that the action is not effective. However, we can consider the group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm E\}$, which acts effectively on $\mathbb{R}\mathrm{P}^1$.

As we can see, in both cases the manifold M is not simply connected. This follows from the fact that the Lie group $\mathrm{SL}(2,\mathbb{R})$ is not simply connected and is diffeomorphic to $S^1 \times \mathbb{R}^2$. By $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ denote the simply connected Lie group such that its Lie algebra equals $\mathfrak{sl}(2,\mathbb{R})$. It is called the simply connected covering group of $\mathrm{SL}(2,\mathbb{R})$. So, in order to describe the homogeneous spaces to be found, we have first to describe the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Recall that every element of the group $\mathrm{SL}(2,\mathbb{R})$ can be uniquely written as

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} x & y \\ o & x^{-1} \end{pmatrix},$$

where $\varphi \in [0, 2\pi)$, $x \in \mathbb{R}^*_+$, $y \in \mathbb{R}$. This establishes a diffeomorphism of the Lie groups $\mathrm{SL}(2,\mathbb{R})$ and $S^1 \times \mathbb{R}^*_+ \times \mathbb{R} \approx S^1 \times \mathbb{R}^2$. Therefore, the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is diffeomorphic to $\mathbb{R}^3$. For a covering mapping it is convenient to consider the mapping

$$\pi : \mathbb{R}^*_+ \times \mathbb{R}^2 \to \mathrm{SL}(2,\mathbb{R}),$$

$$\pi : (x, y, z) \mapsto \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix} \cdot \begin{pmatrix} \sqrt{x} & \frac{y}{\sqrt{x}} \\ 0 & \frac{1}{\sqrt{x}} \end{pmatrix}.$$

Now it remains to introduce a group operation on $\mathbb{R}^*_+ \times \mathbb{R}^2$ so that π would be a surjection of Lie groups.

Let us just give the result, omitting detailed calculation.

Theorem. *The operation on $\mathbb{R}^*_+ \times \mathbb{R}^2$ given by*

$$\begin{aligned}(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) &= \\ &= (X(x_1, y_1, z_2)x_2, Y(x_1, y_1, z_2) + X(x_1, y_1, z_2)y_2, z_1 + Z(x_1, y_1, z_2)),\end{aligned}$$

where

$$X(x, y, z) = \frac{(x\cos z + y\sin z)^2 + \sin^2 z}{x},$$

$$Y(x, y, z) = \frac{(x\cos z + y\sin z)(-x\sin z + y\cos z) + \sin z\cos z}{x},$$

$$Z(x, y, z) = \int_0^z \frac{x\,dt}{(x\cos t + y\sin t)^2 + \sin^2 t},$$

turns $\mathbb{R}_+^* \times \mathbb{R}^2$ *into a Lie group such that* π *is a surjective homomorphism of Lie groups.*

Exercise. Check that
a) $Z(x, y, \pi k) = \pi k$, whenever $k \in \mathbb{Z}$;
b) for $\pi k < z < \pi(k+1)$ the following condition holds:

$$Z(x, y, z) = \pi k + \cot^{-1}(x \cot z + y).$$

Exercise. Check that

$$\begin{aligned} X(x,y,0) &= x, & X(1,0,z) &= 1, \\ Y(x,y,0) &= y, & Y(1,0,z) &= 0, \\ Z(x,y,0) &= 0, & Z(1,0,z) &= z. \end{aligned}$$

Let us find the center of the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Suppose that an element (x, y, z) belongs to the center. Then for any $s \in \mathbb{R}$ we have

$$(x, y, z) \cdot (1, 0, s) = (1, 0, s) \cdot (x, y, z).$$

It follows that

$$\begin{cases} X(x,y,s) = x, \\ Y(x,y,s) = y, \\ Z(x,y,s) = s \end{cases}$$

for all $s \in \mathbb{R}$.

The first of the equations may be rearranged to give

$$(x^2 - y^2 - 1)\cos 2s + 2xy \sin 2s = x^2 - y^2 - 1$$

for all $s \in \mathbb{R}$.

This takes place only if

$$x^2 - y^2 - 1 = 2xy = 0.$$

Since $x \in \mathbb{R}_+^*$, we have $y = 0$ and $x = 1$. Then two other equalities of the system above are also satisfied. So, every element of the center has the form $(1, 0, z)$. If $(1, 0, z)$ belongs to the center, then the following conditions hold:

$$\begin{cases} X(x,y,z) = x, \\ Y(x,y,z) = y, \\ Z(x,y,z) = z \end{cases}$$

for all $x \in \mathbb{R}_+^*$, $y \in \mathbb{R}$.

The first equation can be rearranged to give

$$x^2(\cos 2z - 1) + 2xy \sin 2z - y^2(\cos 2z - 1) = \cos 2z - 1$$

for all $x \in \mathbb{R}_+^*$, $y \in \mathbb{R}$.

This takes place only if

$$\sin 2z = \cos 2z - 1 = 0.$$

It follows that $z = \pi k$, $k \in \mathbb{Z}$.

Conversely, it is easy to show that for $k \in \mathbb{Z}$, the element $(1, 0, \pi k)$ belongs to the center.

So, we have proved the following

Theorem. *The center of the Lie group* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *is infinite and has the form*

$$\{(1, 0, \pi k |\, k \in \mathbb{Z}\}.$$

The connected subgroup of $\mathrm{SL}(2,\mathbb{R})$ corresponding to the subalgebra $\mathfrak{st}(2,\mathbb{R})$ of $\mathfrak{sl}(2,\mathbb{R})$ has the form

$$H = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \middle| \, x \in \mathbb{R}_+^*, \, y \in \mathbb{R} \right\}.$$

Then the inverse image of H by π has the same Lie algebra, and the connected subgroup of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ corresponding to the subalgebra $\mathfrak{st}(2,\mathbb{R})$ is the connected component of the identity of $\pi^{-1}(H)$. It is easy to check that

$$\pi^{-1}(H) = \{(x, y, 2\pi k) |\, x \in \mathbb{R}_+^*, \, y \in \mathbb{R}, \, k \in \mathbb{Z}\}.$$

Then $G = \{(x, y, 0) |\, x \in \mathbb{R}_+^*, \, y \in \mathbb{R}\}$.

Exercise. Show that $N(H) = \pi(N(G))$.

Note that

$$N(H) = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \middle| \, x \in \mathbb{R}_+^*, \, y \in \mathbb{R} \right\}$$

and

$$\pi^{-1}(N(H)) = \{(x, y, \pi k) |\, x \in \mathbb{R}_+^*, \, y \in \mathbb{R}, \, k \in \mathbb{Z}\}.$$

Direct calculation shows that $\pi^{-1}(N(H)) \subset N(G)$. Therefore $N(G) = \pi^{-1}(N(H))$.

Let S be a Lie subgroup of the normalizer $N(G)$ such that $S \underset{\neq}{\supset} G$. Then S has the form:

$$S_n = \{\, (x, y, \pi n k) \mid x \in \mathbb{R}_+^*, \, y \in \mathbb{R}, \, k \in \mathbb{Z} \,\} \qquad (n \in \mathbb{N}).$$

For any $n \in \mathbb{N}$, the Lie group S_n/G is discrete and the intersection of S_n and the center $\mathcal{Z}$ of $\overline{G}$ has the form

$$Z_n = S_n \cap \mathcal{Z} = \{(1, 0, \pi n k) |\, k \in \mathbb{Z}\}.$$

So, any effective pair associated with the pair $(\mathfrak{sl}(2,\mathbb{R}), \mathfrak{st}(2,\mathbb{R}))$ is equivalent to one and only one of the following pairs:

a) $(\overline{G}, G)$,

b) $(\overline{G}/Z_n, S_n/Z_n)$, $n \in \mathbb{N}$.

§3. Two-dimensional homogeneous spaces. Local classification

3.1. Preliminaries. We now turn to the description of effective pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2. Preparatory to formulating the basic theorem, we shall introduce some auxiliary constructions.

1. Let V be an arbitrary finite-dimensional vector space and G a Lie subgroup of the Lie group $\mathrm{GL}(V)$. The set $G \times V$ can be turned into a Lie group by putting

$$(\varphi_1, v_1) \cdot (\varphi_2, v_2) = (\varphi_1\varphi_2, v_1 + \varphi_1(v_2))$$

for all $(\varphi_1, v_1), (\varphi_2, v_2) \in G \times V$.

Exercise. Check that the operation above actually turns $G \times V$ into a Lie group.

We shall denote the Lie group constructed this way by $G \leftthreetimes V$. For example, the group $\mathrm{Aff}(n)$ of affine transformations can be identified with $\mathrm{GL}(n, \mathbb{R}) \leftthreetimes \mathbb{R}^n$, and the group of transformations of the plane preserving the orientation with the Lie group

$$\mathrm{SO}(2) \leftthreetimes \mathbb{R}^2, \quad \text{where } \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \middle| \, \alpha \in \mathbb{R} \right\}.$$

Now find the Lie algebra of the Lie group $G \leftthreetimes V$. Let $\mathfrak{g}$ be the subalgebra of $\mathfrak{gl}(V)$ corresponding to the subgroup G. It can be defined as

$$\mathfrak{g} = \{x \in \mathfrak{gl}(V) \,|\, \exp(tx) \in G \ \forall t \in \mathbb{R}\}.$$

Then the Lie algebra to be found is the vector space $\mathfrak{g} \times V$ with the following bracket operation:

$$[(\varphi_1, v_1), (\varphi_2, v_2)] = ([\varphi_1, \varphi_2], \varphi_1(v_2) - \varphi_2(v_1)).$$

For example, the subalgebra corresponding to the Lie subgroup $\mathrm{SO}(2)$ has the form

$$\mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\};$$

then the Lie algebra of $\mathrm{SO}(2) \leftthreetimes \mathbb{R}^2$ is $\mathfrak{so}(2) \leftthreetimes \mathbb{R}^2$.

Remark. If V is regarded as a commutative Lie group (Lie algebra), then our construction is a particular case of a semidirect product of Lie groups (Lie algebras).

Exercise. Suppose H is the Lie group of similitude transformations of the plane. Find the Lie algebra of H.

2. Recall that an elementary Frobenius matrix is a matrix of the form:

$$\begin{pmatrix} 0 & . & . & . & . & 0 & -a_0 \\ 1 & 0 & . & . & . & 0 & -a_1 \\ 0 & 1 & . & . & . & 0 & -a_2 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & -a_{n-2} \\ 0 & 0 & . & . & . & 1 & -a_{n-1} \end{pmatrix}.$$

Its characteristic and minimal polynomials coincide. They are equal to

$$(-1)^n(x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0).$$

Let $p(x)$ be an arbitrary polynomial of nonzero degree. By $F(p)$ denote the elementary Frobenius matrix such that its characteristic polynomial equals (up to a constant factor) the polynomial p. The size of $F(p)$ is equal exactly to the degree of p. For example,

$$F(x^n) = \begin{pmatrix} 0 & . & . & . & 0 & 0 \\ 1 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & 1 & 0 \end{pmatrix} \quad (n\text{-by-}n \text{ matrix}).$$

Exercise. Show that $F(p)$ is a nilpotent matrix if and only if $p = ax^n$, $a \neq 0$.

Problem. Suppose $p \in \mathbb{R}[x]$ is an arbitrary polynomial of nonzero degree. Prove that matrices

(1) $F(p(x))$ and $F(p(x+\lambda)) + \lambda E$, $\lambda \in \mathbb{R}$;
(2) $F(p(x))$ and $\lambda F(p(x/\lambda))$, $\lambda \in \mathbb{R}^*$

are conjugate.

3. By $\mathbb{R}^n[x,y]$ denote the vector space of homogeneous polynomials in x, y of the n-th degree. Then $\dim \mathbb{R}^n[x,y] = n+1$. Note that differential operators of the form $(ax+cy)\frac{\partial}{\partial x} + (bx+dy)\frac{\partial}{\partial y}$ leave the space $\mathbb{R}^n[x,y]$ invariant and can be regarded as linear operators on $\mathbb{R}^n[x,y]$. Moreover, the set of these operators is a vector space and is closed under the bracket operation. In other words, it is a subalgebra of the Lie algebra $\mathfrak{gl}(\mathbb{R}^n[x,y])$.

Exercise. Show that the mapping

$$\pi_n : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ax+cy)\frac{\partial}{\partial x} + (bx+dy)\frac{\partial}{\partial y}$$

establishes an isomorphism of the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ and the subalgebra outlined above.

For a subalgebra $\mathfrak{a} \in \mathfrak{gl}(2,\mathbb{R})$, by $\mathfrak{a} \leftthreetimes_{\pi_n} \mathbb{R}^n[x,y]$ denote the Lie algebra $\pi_n(\mathfrak{a}) \leftthreetimes \mathbb{R}^n[x,y]$. If $n = 2$, the Lie algebra $\pi_n(\mathfrak{sl}(2,\mathbb{R}))$, in terms of the basis $\{x^2, xy, y^2\}$, has the form:

$$\left\{ \begin{pmatrix} 2x & y & 0 \\ 2z & 0 & 2y \\ 0 & z & -2x \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}.$$

Further we shall use the following notation:

(1) for the subalgebras of $\mathfrak{gl}(2,\mathbb{R})$

$$\mathfrak{t}(2,\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| \; x, y, z \in \mathbb{R} \right\},$$

$$\mathfrak{st}(2,\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\} = \mathfrak{t}(2,\mathbb{R}) \cap \mathfrak{sl}(2,\mathbb{R});$$

(2) for the Lie algebra of orthogonal matrices

$$\mathfrak{so}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) | A +^t A = 0\};$$

(3) for the n-by-n matrices

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \dots & \vdots & \\ 0 & & & \dots & 0 & 1 \\ 0 & 0 & & \dots & & 0 \end{pmatrix} \quad \text{and} \quad S_n = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & -2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -n \end{pmatrix}.$$

3.2. Classification of pairs. Now we are able to formulate the central theorem.

Classification theorem. *Any effective pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of codimension 2 is equivalent to one of the following pairs:*

1.1

$$\bar{\mathfrak{g}} = \mathbb{R}^2, \quad \mathfrak{g} = \{0\}.$$

2.1(λ)

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & \lambda x \end{pmatrix} \middle| x \in \mathbb{R} \right\}, |\lambda| \leqslant 1.$$

2.2(λ)

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} \lambda x & -x \\ x & \lambda x \end{pmatrix} \middle| x \in \mathbb{R} \right\}, \lambda \geqslant 0.$$

3.1

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}.$$

3.2

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}.$$

4.1(p)

$$\bar{\mathfrak{g}} = \{xF(p) | x \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$

$$\mathfrak{g} = \{0\} \times \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix} \middle| x_i \in \mathbb{R} \right\}, \text{ where } p \in \mathbb{R}[x], n = \deg p \geqslant 1;$$

in addition, if $p = \alpha x^n$, *then* $n \geqslant 2$.

5.1(p)

$$\bar{\mathfrak{g}} = \{xE_n + yF(p) | x, y \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$

$$\mathfrak{g} = \{xE_n | x \in \mathbb{R}\} \times \left\{ \left. \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix} \right| x_i \in \mathbb{R} \right\}, \text{ where } p \in \mathbb{R}[x], n = \deg p \geqslant 2.$$

6.1(n, λ)

$$\bar{\mathfrak{g}} = \{x(\lambda E_n + S_n) + yN_n | x, y \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$

$$\mathfrak{g} = \{x(\lambda E_n + S_n) | x \in \mathbb{R}\} \leftthreetimes \left\{ \left. \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2, \lambda \in \mathbb{R}, \lambda \neq n.$$

7.1(n)

$$\bar{\mathfrak{g}} = \left\{ \left. \left(x_n(nE_n + S_n) + yN_n, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \right| x_i \in \mathbb{R}, y \in \mathbb{R} \right\} \subset 9.1(n),$$

$$\mathfrak{g} = \left\{ \left. \left(x_n(nE_n + S_n), \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right) \right| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2.$$

8.1(n)

$$\bar{\mathfrak{g}} = \{xE_n + yS_n + zN_n | x, y, z \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$

$$\mathfrak{g} = \{xE_n + yS_n | x, y \in \mathbb{R}\} \times \left\{ \left. \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2.$$

9.1(n)

$$\bar{\mathfrak{g}} = \{x(nE_n + S_n) + yN_n | x, y \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$

$$\mathfrak{g} = \{x(nE_n + S_n) | x \in \mathbb{R}\} \times \left\{ \left. \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2.$$

10.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}.$$

11.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}.$$

11.2

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} = \mathfrak{so}(2).$$

11.3

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} \middle| \, x,y,z \in \mathbb{R} \right\}, \quad \mathfrak{g} = \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}.$$

12.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}), \quad \mathfrak{g} = \mathfrak{st}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R}).$$

12.2

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}, \quad \mathfrak{g} = \mathfrak{st}(2,\mathbb{C})_{\mathbb{R}}.$$

13.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(3,\mathbb{R}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| \, A \in \mathfrak{gl}(2,\mathbb{R}), B \in \mathrm{Mat}_{1\times 2}(\mathbb{R}) \right\}.$$

14.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \times \{0\}.$$

15.1

$$\bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{gl}(2,\mathbb{R}) \times \{0\}.$$

16.1

$$\bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \middle| \, x,y \in \mathbb{R} \right\}.$$

17.1(n)

$$\begin{aligned} &\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \mathbb{R}^n[x,y], \\ &\mathfrak{g} = \mathfrak{st}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \{a_0x^n + \dots + a_{n-1}x^{n-1}y | a_i \in \mathbb{R}\}, \ \textit{where } n \geqslant 0. \end{aligned}$$

18.1(n)

$$\begin{aligned} &\bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \mathbb{R}^n[x,y], \\ &\mathfrak{g} = \mathfrak{t}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \{a_0x^n + \dots + a_{n-1}x^{n-1}y | a_i \in \mathbb{R}\}, \ \textit{where } n \geqslant 1. \end{aligned}$$

When $n = 0$ $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times (\mathbb{R}_+^* \leftthreetimes \mathbb{R})$; $\mathfrak{g} = \mathfrak{st}(2,\mathbb{R}) \times (\mathbb{R}_+^* \times \{0\}$.

Pairs from different items are not equivalent to each other. Two pairs from 4.1 corresponding to polynomials p_1 and p_2 are equivalent if and only if there exist $\alpha, \lambda \in \mathbb{R}^*$ such that $p_1 = \alpha p_2(\lambda x)$.

Two pairs from 5.1 corresponding to polynomials p_1 and p_2 are equivalent if and only if there exist $\alpha, \lambda \in \mathbb{R}^*$ and $\mu \in \mathbb{R}$ such that $p_1 = \alpha p_2(\lambda x + \mu)$.

Any two pairs from any other item are not equivalent.

Remark 1. The Lie algebras $\bar{\mathfrak{g}}$ from items 1.1–9.1 are solvable; $\bar{\mathfrak{g}}$ from items 10.1–13.1 are semisimple; $\bar{\mathfrak{g}}$ from items 14.1–18.1 have the nontrivial radical.

Remark 2. If we omitted the restrictions imposed on parameters in the theorem, the following pairs would be equivalent:

$$\begin{aligned} 4.1(x) &\cong 1.1; & 6.1(1,\lambda) &\cong 2.1(\lambda-1); \\ 5.1(x-\lambda) &\cong 2.1(0); & 9.1(1) &\cong 2.1(0); \\ 6.1(n,n) &\cong 9.1(n); & 8.1(1) &\cong 3.1. \end{aligned}$$

§4. Two-dimensional homogeneous spaces. Global classification

4.1 The process of globalization. Let us modify the basic constructions described by Mostow. We shall first generalize the concept of an effective homogeneous space.

Definition. A homogeneous space $(\overline{G}, M)$ is called *locally effective* if its kernel of ineffectiveness is discrete.

Let $(\overline{G}, G)$ be a pair corresponding to a locally effective homogeneous space. This is equivalent to the fact that any normal Lie subgroup of $\overline{G}$ lying in G is discrete. In this case we say that the pair $(\overline{G}, G)$ is *locally effective.* Nevertheless, the corresponding pair of Lie algebras is effective. Moreover, the following is true:

Proposition 1. *Let $(\overline{G}, G)$ be a pair of Lie groups and $(\bar{\mathfrak{g}}, \mathfrak{g})$ the corresponding pair of Lie algebras. The pair $(\overline{G}, G)$ is locally effective if and only if the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is effective.*

Exercise. Prove the proposition above.

The proposition shows that the concept of a locally effective pair is somewhat more natural than that of an effective pair. To each locally effective pair $(\overline{G}, G)$ we assign the effective pair $(\overline{G}/K, G/K)$, where K is the kernel of ineffectiveness of $(\overline{G}, G)$.

Definition. We shall say that two locally equivalent pairs are *similar* if the corresponding effective pairs are equivalent.

For each effective pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ we shall find (up to similarity) all locally effective pairs associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$. We shall then be able to describe (up to equivalence) all effective pairs associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$ by passing from locally effective pairs just obtained to the corresponding effective pairs. In doing so we shall use the following result:

Proposition 2. *Let $(\overline{G}, G)$ be a locally effective pair and K its kernel of ineffectiveness. Then $K = G \cap \mathcal{Z}(\overline{G})$, where $\mathcal{Z}(\overline{G})$ is the center of $\overline{G}$.*

Proof. Since any subgroup of the center is normal, we see that $G \cap \mathcal{Z}(\overline{G}) \subset K$. Conversely, since K is a normal discrete subgroup, we see that the subgroup K is central (why?). Therefore $K \subset G \cap \mathcal{Z}(G)$, which concludes the proof.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair of codimension 2. Recall that there exists a unique pair $(\overline{G}, G)$ associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the subgroup G is connected and the manifold $\overline{G}/G$ is simply connected. The following result follows immediately from the results of Mostow.

Proposition 3. *Every locally effective pair associated with $(\bar{\mathfrak{g}}, \mathfrak{g})$ is similar to a pair of the form $(\overline{G}, S)$, where S is a Lie subgroup of $N(G)$ such that $S \supset G$ and the Lie group S/G is discrete.*

Note that there is a one-to-one correspondence between the set of subgroups S described in proposition 3 and the set of all discrete subgroups of the Lie group $N(G)/G$.

Theorem. *Let $(\overline{G}, M)$ be the homogeneous space corresponding to the pair $(\overline{G}, G)$.*

a) The mapping $g \mapsto gG$ gives a one-to-one correspondence between elements of the group $N(G)/G$ and points $x \in M$ such that $\overline{G}_x = G$.

b) Put $C = \{x \in M | \overline{G}_x = G\}$. Then the group operation on C given by

$$n_1 G \cdot n_2 G = n_1 n_2 G \qquad \text{for } n_1, n_2 \in N(G)$$

turns C into a Lie group isomorphic to $N(G)/G$.

c) The equality

$$nG.(gG) = (gn)G,$$

where $n \in N(G)$ and $g \in \overline{G}$, gives a right action of C on M.

d) The action of C on M commutes with the action of $\overline{G}$ on M, i.e.

$$c.(g.x) = g.(c.x)$$

for all $g \in \overline{G}$, $c \in C$, $x \in M$.

Proof.

a) Let $x = gG$. Then $\overline{G}_x = gGg^{-1}$. Therefore, $\overline{G}_x = G$ if and only if $g \in N(G)$.

b) It is sufficient to check that the operation is well-defined. Indeed, for $h_1, h_2 \in G$ we have

$$n_1 h_1 G \cdot n_2 h_2 G = n_1 h_1 n_2 h_2 G = n_1 h_1 n_2 G = n_1 n_2 (n_2^{-1} h_1 n_2) G = n_1 n_2 G.$$

The statements of items c) and d) can be proved in a similar way.

Remark. The action of C on M can be rewritten as

$$c.(gG) = nG.gG = gnG = g.(nG) = g.c,$$

where $c = nG \in C$, $n \in N(G)$, and $g \in \overline{G}$.

Now let D be an arbitrary discrete subgroup of C. Then D determines a regular equivalence relation on M: $x_1 \sim x_2$ if there exists a $d \in D$ such that $d.x_1 = x_2$.

Let M/D denote the quotient manifold of M by the relation just introduced. Item d) of the theorem above implies that this relation is invariant under the action of the Lie group $\overline{G}$. This allows to define an action of $\overline{G}$ on the manifold M/D. This way we obtain the new homogeneous space $(\overline{G}, M/D)$. It is easy to check that the corresponding pair of Lie groups has the form $(\overline{G}, S)$, where $S/G = D$.

We shall say that discrete subgroups D_1 and D_2 of the Lie group C are equivalent if there exists an automorphism (π, τ) of $(\overline{G}, M)$ such that $\tau(D_1) = D_2$.

Further, for each effective pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ from the classification theorem, we shall describe the corresponding pair $(\overline{G}, G)$, the homogeneous space $(\overline{G}, M)$, where $M = \overline{G}/G$, the set C, the action of C on M, discrete subgroups D of C (up to equivalence), and the manifolds M/D.

4.2. Examples.

1. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair 1.1 from the classification theorem. Then the corresponding pair $(\overline{G}, G)$ has the form $(\mathbb{R}^2, \{0\})$, where $\mathbb{R}^2$ is considered as an abelian additive group. The homogeneous space is the plane with the simply transitive action of the group of translations. In other words, $M = \mathbb{R}^2 = \{(p,q) | p, q \in \mathbb{R}\}$, $\overline{G} = \{(x,y) | x, y \in \mathbb{R}\}$, and the action of $\overline{G}$ on M can be written as

$$(x,y).(p,q) = (x+p, y+q)$$

for $(x,y) \in \overline{G}$, $(p,q) \in M$.

In this case the stabilizer of any point of the plane equals G. Therefore, the subgroup C coincides with M and the group operation on C has the form:

$$(p_1, q_1).(p_2, q_2) = (p_1 + p_2, q_1 + q_2).$$

Every discrete subgroup of C is equivalent to one and only one of the following subgroups:

a) $\{(0,0)\}$; b) $\{(n,0) | n \in \mathbb{Z}\}$; c) $\{(n,m) | n, m \in \mathbb{Z}\}$.

Factorization of M by the corresponding equivalence relation gives a transitive action of the group $\overline{G}$ on the plane $\mathbb{R}^2$, cylinder $S^1 \times \mathbb{R}$, and torus $S^1 \times S^1$ respectively.

2. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair 2.1 from the classification theorem. Then the corresponding pair $(\overline{G}, G)$ has the form

$$\overline{G} = A \leftthreetimes \mathbb{R}^2 \ , \ \overline{G} = A \times \{0\} \ , \ \text{where}$$

$$A = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^\lambda \end{pmatrix} \middle| \ x \in \mathbb{R}_+^* \right\}.$$

The manifold M can be identified with the plane $M = \mathbb{R}^2 = \{(p,q) | \ p, q \in \mathbb{R}\}$. (In the sequel, by the plane we shall always mean the manifold $\mathbb{R}^2$ with the standard coordinates (p,q)). The action of $\overline{G}$ on M can be written as

$$(x, (y,z)).(p,q) = (xp + y, x^\lambda q + z),$$

where $x \in \mathbb{R}_+^*$, $(y,z) \in \mathbb{R}^2$.

Now find all points $(p,q) \in M$ such that the stabilizer of (p,q) equals G. This takes place if and only if the following condition holds:

$$(p.q) = (xp, x^\lambda q) \text{ for all } x \in \mathbb{R}_+^* .$$

Therefore, if $\lambda \neq 0$, then $C = \{(0,0)\}$ and we obtain no new homogeneous spaces. If $\lambda = 0$, then $C = \{(0,a)|\ a \in \mathbb{R}\}$. We have

$$(0,a).(p,q) = (0,a).((1,p,q)G) = (1,p,q).(0,a) = (p,q+a).$$

So, the action of the group C on M has the form:

$$(0,a).(p,q) = (p,q+a).$$

All nonzero discrete subgroups of C are equivalent to the subgroup $\{(0,n)|\ n \in \mathbb{Z}\}$. The corresponding quotient manifold is the cylinder $\mathbb{R} \times S^1$. Identifying the circle S^1 with the factor group $\mathbb{R}/\mathbb{Z}$ ($x \mapsto e^{2\pi i x}$), we can write the action of $\overline{G}$ on $\mathbb{R} \times S^1$ as

$$(x,(y,z)).(p,q) = (xp+y, (q+z) \bmod 1).$$

3. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair 10.1 from the classification theorem:

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \ , \ \mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right| \ x \in \mathbb{R} \right\}.$$

Then the corresponding pair $(\overline{G}, G)$ has the form

$$\overline{G} = \widetilde{SL(2,\mathbb{R})} \ , \ G = \{(1,y,0)|y \in \mathbb{R}\}.$$

The manifold M can be identified with the infinite half-plane

$$\mathbb{R}_+^* \times \mathbb{R} = \{(p,q)|\ p \in \mathbb{R}_+^*,\ y \in \mathbb{R}\}.$$

The action of $\overline{G}$ on M has the form:

$$(x,y,z).(p,q) = (pX(x,y,q) \ , \ Z(x,y,q)+z).$$

Now find the group C. A point (p,q) belongs to C if

$$(p,q) = (1,y,0).(p,q) = (pX(1,y,q) \ , \ Z(1,y,q))$$

for all $y \in \mathbb{R}$. Therefore, for all $y \in \mathbb{R}$ the following condition holds:

$$q = \int_0^q \frac{dt}{(\cos t + y \sin t)^2 + \sin^2 t}.$$

This is true for $q = \pi k$, $k \in \mathbb{Z}$. If $\pi k < q < \pi(k+1)$, then the last condition is equivalent to the following one:

$$q - \pi k = \cot^{-1}(\cot q + y) \text{ for all } y \in \mathbb{R},$$

which is impossible. Hence $q = \pi k$, $k \in \mathbb{Z}$. Then $pX(1, y, q) = p$ and therefore

$$C = \{(a, \pi k) |\ a \in \mathbb{R}_+^* \ , \ k \in \mathbb{Z}\}.$$

The action of C on M has the form:

$$(a, \pi k).(p, q) = (a, \pi k).(p, 0, q)G = (p, 0, q).(a, \pi k) = (ap, q + \pi k).$$

Every nonzero discrete subgroup of C is equivalent to one and only one of the following subgroups:
a) $\{(1, \pi kn) |\ k \in \mathbb{Z}\}$, $n \in \mathbb{N}$;
b) $\{(\alpha^k, \pi kn) |\ k \in \mathbb{Z}\}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^*$;
c) $\{(\alpha^k, \pi ln) |\ k, l \in \mathbb{Z}\}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^*$.
The corresponding quotient manifolds are
a), b) the cylinder $\mathbb{R} \times S^1$;
c) the torus $S^1 \times S^1$.

4. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair 11.1 from the classification theorem:

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}) \ , \ \mathfrak{g} = \left\{ \left(\begin{smallmatrix} x & 0 \\ 0 & -x \end{smallmatrix} \right) \middle|\ x \in \mathbb{R} \right\}.$$

If we put $\overline{G} = \widetilde{SL(2, \mathbb{R})}$ with the group operation described earlier, it would hardly be possible to find the action of $\overline{G}$ on M in an explicit form (try to do it !). We shall do it in a different way. Let $\overline{G}$ be the set $\widetilde{SL(2, \mathbb{R})}$ supplied with the following group operation:

$$g_1 * g_2 = g_2 g_1 \qquad \text{for } g_1, g_2 \in \widetilde{SL(2, \mathbb{R})}.$$

We denote this group by $\widetilde{SL(2, \mathbb{R})}^*$. In coordinates, the group operation has the form:

$$\begin{aligned}
&(x_1, y_1, z_1) * (x_2, y_2, z_2) = \\
&\qquad = (X(x_2, y_2, z_1)x_1 \ , \ Y(x_2, y_2, z_1) + X(x_2, y_2, z_1)y_1 \ , \ z_2 + Z(x_2, y_2, z_1)).
\end{aligned}$$

(Compare with that on $\widetilde{SL(2, \mathbb{R})}$). The Lie group $\widetilde{SL(2, \mathbb{R})}^*$ is isomorphic to $\widetilde{SL(2, \mathbb{R})}$. The isomorphism can be established by the mapping $g \mapsto g^{-1}$, $g \in \widetilde{SL(2, \mathbb{R})}$. The subgroup G has the form:

$$G = \{(x, 0, 0) |\ x \in \mathbb{R}_+^*\}.$$

It is easy to check that the manifold $M = \overline{G}/G$ can be identified with the plane and the action of $\overline{G}$ on M can be written as

$$(x, y, z).(p, q) = (Y(x, y, q) + X(x, y, q)p \ , \ z + Z(x, y, q)).$$

Now find the set C. A point (p, q) belongs to C if

$$(p, q) = (x, 0, 0).(p, q) = (Y(x, 0, q) + X(x, 0, q)p \ , \ Z(x, 0, q))$$

for all $x \in \mathbb{R}_+^*$. Therefore,

$$q = \int_0^q \frac{x dt}{x^2 \cos^2 t + \sin^2 t} \qquad \text{for all } x \in \mathbb{R}_+^*.$$

Note that the condition holds if $q = \pi k \ , \ k \in \mathbb{Z}$. If $\pi k < q < \pi(k+1)$, this is equivalent to the following condition

$$q - \pi k = \cot^{-1}(x \cot q) \text{ for all } x \in \mathbb{R}_+^*.$$

This takes place if and only if $\cot q = 0$ or $q = \pi k + \pi/2$. Therefore $q = \pi k/2 \ , \ k \in \mathbb{Z}$. It immediately follows that $p = 0$ and

$$C = \{(0, \pi k/2) |\ k \in \mathbb{Z}\}.$$

The action of the group C has the form:

$$(0, \pi k/2).(p, q) = (0, \pi k/2).((1, p, q)G) = (1, p, q).(0, \pi k/2) = ((-1)^k p, q + \pi k/2).$$

Every nonzero discrete subgroup of C is equivalent to one and only one of the following subgroups:
a) $\{(0, \pi n k) |\ k \in \mathbb{Z}\},\ n \in \mathbb{N}$;
b) $\{(0, (2n-1)\pi k/2) |\ k \in \mathbb{Z}\},\ n \in \mathbb{N}$.
The corresponding quotient manifolds are the cylinder and Möbius strip respectively.
5. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair 4.1 given by the polynomial

$$p = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

For a smooth function $f \in C^\infty(\mathbb{R})$, put

$$E(f) = f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f.$$

Then $E(f) = 0$ is a linear ordinary differential equation with constant coefficients. It is know from theory of differential equations that solutions of our equation form an n-dimensional vector space. (See appendix B.) Let $\mathcal{F}_p$ denote the vector space of solutions. In addition, if

$$p(x) = \prod_{i=1}^{s}(x-\alpha_i)^{k_i} \cdot \prod_{j=1}^{t}((x-\lambda_j)^2+\mu_j^2)^{l_j},$$

then the functions

$$x^n e^{\alpha_i x}, \qquad 0 \leqslant n \leqslant k_i - 1,\ 1 \leqslant i \leqslant s,$$
$$x^n e^{\lambda_j x}\cos\mu_j x,\ x^n e^{\lambda_j x}\sin\mu_j x, \qquad 1 \leqslant n \leqslant l_{j-1},\ 1 \leqslant i \leqslant t$$

form a basis of $\mathcal{F}_p$.

Problem. Prove that there exists a solution $\omega(x)$ of the equation $E(f)=0$ such that the functions

$$\omega(x),\ \omega'(x),\dots,\omega^{n-1}(x)$$

form a basis of the space of solutions.

Let us describe the pair $(\overline{G}, G)$ corresponding to $(\bar{\mathfrak{g}}, \mathfrak{g})$. First, consider the action of the group $\mathbb{R}$ on the set $C^\infty(\mathbb{R})$ defined by

$$(x.f)(a) = f(a-x)$$

for $x \in \mathbb{R},\ a \in \mathbb{R},\ f \in C^\infty(\mathbb{R})$.

Problem.* Prove that the action is linear, i.e.

$$x.(f+g) = x.f + x.g \qquad \text{and} \qquad x.(\lambda f) = \lambda(x.f).$$

Prove that for any finite–dimensional subspace $V \subset C^\infty(\mathbb{R})$ invariant under this action there exists a polynomial $p \in \mathbb{R}[x]$ such V equals $\mathcal{F}_p$.

Now put $\overline{G} = \{\,(x,f) \mid x \in \mathbb{R},\ f \in \mathcal{F}_p\,\}$, where the group operation has the form:

$$(x_1, f_1)\cdot(x_2,f_2) = (x_1+x_2,\ f_1 + x_1.f_2),$$

and $G = \{(0,f) \mid f \in \mathcal{F}_p,\ f(0)=0\}$. It can be shown that $(\overline{G}, G)$ is the desired pair. The corresponding homogeneous space is the plane with the following action of the Lie group $\overline{G}$:

$$(x,f).(p,q) = (x+p,\ q+f(x+p)).$$

Exercise. Check that $\overline{G}_{(0,0)} = G$.

Let us now describe the set C. By $\widetilde{\mathcal{F}_p}$ denote the set $\{f \in \mathcal{F}_p \mid f(0)=0\}$. A point (p,q) belongs to C if

$$(p,q) = (0,f).(p,q) = (p, q+f(p))$$

for all $f \in \widetilde{\mathcal{F}_p}$. This takes place if $f(p)=0$ for all functions $f \in \widetilde{\mathcal{F}_p}$. It is obvious that $C \supset \{(0,a) \mid a \in \mathbb{R}\}$.

Lemma. *Let $C \neq \{(0,a) \mid a \in \mathbb{R}\}$. Then the polynomial $p(x)$ is equivalent to one of the following polynomials:*

$$a) \quad p(x) = \prod_{i=1}^{k}((x-\lambda)^2 + b_i^2), \qquad \lambda \in \mathbb{R},\ b_i \in 2\mathbb{Z};$$

$$b) \quad p(x) = \prod_{i=1}^{k}((x-\lambda)^2 + b_i^2), \qquad \lambda \in \mathbb{R},\ b_i \in 2\mathbb{Z}+1;$$

$$c) \quad p(x) = (x-\lambda)\prod_{i=1}^{k}((x-\lambda)^2 + b_i^2), \quad \lambda \in \mathbb{R},\ b_i \in 2\mathbb{Z};$$

Proof. Since $C \neq \{(0,a) \mid a \in \mathbb{R}\}$, we see that functions from $\widetilde{\mathcal{F}_p}$ have nonzero common roots. Let α be the nonzero common root with the least absolute value. Since the space $\widetilde{\mathcal{F}_p}$ is finite-dimensional and consists of smooth functions, we see that this root always exists. Two pairs 4.1 defined by polynomials $p(x)$ and $\lambda p(\mu x)$, $\lambda, \mu \in \mathbb{R}^*$, are equivalent. Therefore, it can be assumed that $\alpha = \pi$. Let L be the linear operator on $\mathcal{F}$ given by $Lf(x) = f(x+\pi)$. It is easy to show that the subspace $\widetilde{\mathcal{F}_p}$ is invariant with respect to L. Therefore, $\widetilde{\mathcal{F}_p}$ is invariant with respect to the operator L^k $(k \in \mathbb{Z})$ and the points πk, $k \in \mathbb{Z}$, are also common roots of all functions from $\widetilde{\mathcal{F}_p}$.

Consider the forms on $\mathcal{F}_p$ given by $f \mapsto f(\pi k)$, $k \in \mathbb{Z}$. Note that they are proportional. If the functions $\omega_1, ..., \omega_n$ form a basis of $\widetilde{\mathcal{F}_p}$, then in the dual basis, these forms have the coordinates

$$(\omega_1(\pi k), \omega_2(\pi k), ..., \omega_n(\pi k)).$$

Therefore, all basis functions are eigenvectors of L belonging to the same eigenvalue. This proves the lemma. Note that the eigenvalue equals $e^{\lambda\pi}$ in cases a), c) and $-e^{\lambda\pi}$ in case b).

We say that cases a), b), and c) are *special* and the other cases are *nonspecial.*

Now find the action of C on M. Not that for any point $(p,q) \in M$, an element $(p,f) \in \overline{G}$ such that $f(p) = q$ takes the point $(0,0)$ into (p,q). In nonspecial cases we have

$$(0,a).(p,q) = (0,a).(p,f)G = (p,f).(0,a) = (p, a + f(p)) = (p, q+a),$$

where $(0,a) \in C$.

All nonzero discrete subgroups of C are equivalent to the subgroup $\{(0,n) \mid n \in \mathbb{Z}\}$. The corresponding quotient manifold is the cylinder.

Consider the special cases. We have

$$(\pi k, a).(p,q) = (p,f).(\pi k, a) = (p + \pi k, a + f(p+\pi k)) = (p + \pi k, a + L^k f(p)).$$

Therefore, the action of C on M has the following form:

$$a), c)\ (\pi k, a).(p, q) = (p + \pi k, e^{\lambda \pi k} q + a),$$

$$b)\ (\pi k, a).(p, q) = (p + \pi k, (-1)^k e^{\lambda \pi k} q + a).$$

Let us now describe discrete subgroups D of the group C and the corresponding manifolds M/D.

a), c) It is easy to check that the group operation on C has the form:

$$(\pi k_1, x_1) \cdot (\pi k_2, x_2) = (\pi(k_1 + k_2), x_1 + e^{\lambda \pi k_1} x_2).$$

If $\lambda \neq 0$, then every nonzero discrete subgroup of C is equivalent to one and only one of the following subgroups:

$$\{(\pi n k, 0) \mid k \in \mathbb{Z}\},\ n \in \mathbb{N};$$

$$\{(0, k) \mid k \in \mathbb{Z}\}.$$

The corresponding quotient manifolds are cylinders. If $\lambda = 0$, we have one more subgroup in addition to the subgroups just mentioned:

$$\{(2\pi n k, m) \mid k, m \in \mathbb{Z}\},\ n \in \mathbb{N}.$$

The corresponding quotient manifold is the torus.

b) The group operation on C has the form:

$$(\pi k_1, x_1) \cdot (\pi k_2, x_2) = (\pi(k_1 + k_2), x_1 + (-1)^{k_1} e^{\lambda \pi k_1} x_2).$$

If $\lambda \neq 0$, then all nonzero discrete subgroups viewed up to equivalence have the form:

$$\{(2\pi n k, 0) \mid k \in \mathbb{Z}\},\ n \in \mathbb{N};$$

$$\{(0, k) \mid k \in \mathbb{Z}\};$$

$$\{((2n-1)\pi k, 0) \mid k \in \mathbb{Z}\},\ n \in \mathbb{N}.$$

In the first two cases the factorization gives the cylinder and in the last case the Möbius strip. If $\lambda = 0$, we obtain two more discrete subgroups:

$$\{(2\pi n k, m) \mid k, m \in \mathbb{Z}\},\ n \in \mathbb{N};$$

$$\{((2n-1)\pi k, m) \mid k, m \in \mathbb{Z}\},\ n \in \mathbb{N}.$$

The corresponding quotient manifolds are the torus and the Klein bottle respectively.

6. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair 17.1 from the classification theorem. Let us now find the corresponding pair of Lie groups $(\overline{G}, G)$. Just as $\mathfrak{gl}(2, \mathbb{R})$ can be realized as a subalgebra

of $\mathfrak{gl}(\mathbb{R}^n[x,y])$, the Lie group $\mathrm{GL}(2,\mathbb{R})$ can be realized as a group of automorphisms of the space $\mathbb{R}^n[x,y]$. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \text{ and } f(x,y) \in \mathbb{R}^n[x,y].$$

Then put

$$(g.f)(x,y) = f(ax+cy, bx+dy).$$

Exercise. Check that

a) $g.f \in \mathbb{R}_n[x,y]$ for all $g \in \mathrm{GL}(2,\mathbb{R}), f \in \mathbb{R}^n[x,y]$;

b) $(g_1g_2).f = g_1(g_2f)$ for all $g_1, g_2 \in \mathrm{GL}(2,\mathbb{R}), f \in \mathbb{R}^n[x,y]$.

Now let $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ be the universal covering group of the group $\mathrm{SL}(2,\mathbb{R})$ and $\pi : \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{SL}(2,\mathbb{R})$ the covering homomorphism. Then the action of the Lie group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{R}^n[x,y]$ can be defined by

$$g.f = \pi(g).f$$

for $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$, $f \in \mathbb{R}^n[x,y]$. The Lie group $\overline{G}$ has the form:

$$\overline{G} = \widetilde{\mathrm{SL}(2,\mathbb{R})} \leftthreetimes \mathbb{R}^n[x,y].$$

The group operation on $\overline{G}$ can be written as

$$(g_1, f_1) \cdot (g_2, f_2) = (g_1g_2, f_1 + g_1.f_2).$$

Then the subgroup G has the form:

$$G = \{(x,y,0) | x \in \mathbb{R}_+^*, y \in \mathbb{R}\} \leftthreetimes \{f \in \mathbb{R}^n[x,y] | f(0,1) = 0\}.$$

The corresponding homogeneous space is the plane with the following action of $\overline{G}$:

$$(x,y,z,f).(p,q) = (\alpha, X^{-n/2}q + f(-\sin\alpha, \cos\alpha)),$$

where $\alpha = z + Z(x,y,p)$, $X = X(x,y,p)$.

A necessary and sufficient condition for a point (p,q) to belong to the set C is that for all $(x,y,0,f) \in G$ the following condition holds:

$$(p,q) = (Z(x,y,p), X^{-n/2}(x,y,p)q + f(-\sin Z(x,y,p), \cos Z(x,y,p))).$$

It follows that

$$C = \{(\pi k, a) | k \in \mathbb{Z}, a \in \mathbb{R}\} \text{ if } n = 0$$

and

$$C = \{(\pi k, 0) | k \in \mathbb{Z}\} \text{ if } n > 0.$$

If $n = 0$, then all nonzero discrete subgroups viewed up to equivalence have the form:

$$\{(\pi nk, 0) | k \in \mathbb{Z}\}, \ n \in \mathbb{N};$$
$$\{(0, k) | k \in \mathbb{Z}\};$$
$$\{(\pi nk, l) | k, l \in \mathbb{Z}\}, \ n \in \mathbb{N}.$$

In the first two cases the factorization gives the cylinder and in the last case the torus.

Consider the case $n > 0$. The action of the Lie group C on M is given by

$$(\pi k, 0).(p, q) = (p + \pi k, (-1)^{kn} q).$$

Each nonzero discrete subgroup is equivalent to the following one:

$$\{(\pi mk, 0) | k \in \mathbb{Z}\}, \ m \in \mathbb{N}.$$

The corresponding quotient manifold is the cylinder whenever mn is even or the Möbius strip whenever mn is odd.

The complete list fo all two-dimensional homogeneous spaces is given in appendix C.

5. Structure of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$

5.1. Isotropically-faithful and reductive pairs. Let us give some definitions and then find out which of the pairs mentioned in the classification theorem satisfy their conditions.

Let $\mathfrak{g}$ be a Lie algebra and V a vector space. Recall that an arbitrary homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a representation of $\mathfrak{g}$ on V. If ρ is injective, then the representation is said to be faithful.

Let ρ be a representation of $\mathfrak{g}$ on V. Then each element $x \in \mathfrak{g}$ can be regarded as an endomorphism of V denoted by x_V. In this case the vector space V is called a $\mathfrak{g}$-*module* and we write $x.v = x_V(v)$ instead of $\rho(x)(v)$, $x \in \mathfrak{g}$, $v \in V$. A $\mathfrak{g}$-module V is called *faithful* if the set $\{x \in \mathfrak{g} | x.v = 0 \ \forall \ v \in V\}$ is zero. There is a one-to-one correspondence between the set of all $\mathfrak{g}$-modules and the set of all representations of $\mathfrak{g}$. In the sequel we shall use both of these terms.

To each pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ we can assign a representation of the Lie algebra $\mathfrak{g}$ on the vector space of dimension $\dim \bar{\mathfrak{g}} - \dim \mathfrak{g}$.

Definition. The *isotropic representation* of a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the representation of the Lie algebra $\mathfrak{g}$ on the vector space $(\bar{\mathfrak{g}}, \mathfrak{g})/\mathfrak{g}$ given by

$$\rho(x)(\bar{x} + \mathfrak{g}) = [x, \bar{x}] + \mathfrak{g}$$

for all $x \in \mathfrak{g}$, $\bar{x} \in \bar{\mathfrak{g}}$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *isotropically-faithful* if its isotropic representation is faithful.

Examples.

1) Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair 2.1(λ) from the classification theorem:

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \ \ \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & \lambda x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}.$$

The factor space $\bar{\mathfrak{g}}/\mathfrak{g}$ can be identified with $\mathbb{R}^2$. Then the action of an element

$$\left(\begin{pmatrix} x & 0 \\ 0 & \lambda x \end{pmatrix}, 0 \right) \in \mathfrak{g}$$

on $\bar{\mathfrak{g}}/\mathfrak{g}$ is given exactly by the matrix

$$\begin{pmatrix} x & 0 \\ 0 & \lambda x \end{pmatrix}.$$

Therefore, for any $\lambda \in \mathbb{R}$ the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is isotropically-faithful.

2) Consider the pair 4.1(p) from the classification theorem. Then

$$\bar{\mathfrak{g}} = \{xF(p) | x \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n, \ \ \mathfrak{g} = \{0\} \times \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix} \middle| \, x_i \in \mathbb{R} \right\},$$

where $n = \deg p \geqslant 1$ and if $p = x^n$ then $n \geqslant 2$. Put

$$u_1 = F(p) + 0 \text{ and } u_2 = 0 + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then the cosets $u_1 + \mathfrak{g}$ and $u_2 + \mathfrak{g}$ form a basis of the space $\bar{\mathfrak{g}}/\mathfrak{g}$. Let

$$e = 0 + \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix}$$

be an arbitrary element of $\mathfrak{g}$. Then

$$\rho(e)(u_1 + \mathfrak{g}) = [e, u_1] + \mathfrak{g} = \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} + \mathfrak{g} = x_{n-1} u_2 + \mathfrak{g},$$

$$\rho(e)(u_2 + \mathfrak{g}) = [e, u_2] + \mathfrak{g} = 0 + \mathfrak{g}.$$

Therefore, the isotropic representation of $(\bar{\mathfrak{g}}, \mathfrak{g})$, in terms of our basis, has the form:

$$\rho(e) = \begin{pmatrix} 0 & x_{n-1} \\ 0 & 0 \end{pmatrix}.$$

The class of isotropically-faithful pairs turns out to be rather important. In particular, if a homogeneous space has an invariant affine connection, then the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is isotropically-faithful.

Exercise. Show that if a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is isotropically-faithful, then it is effective.

Theorem. *Any isotropically-faithful pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2 is equivalent to one and only one of the following pairs from the classification theorem:*

$$\begin{aligned}
&1.1,\ 2.1,\ 2.2,\ 3.1,\ 3.2,\\
&4.1(p),\ \textit{where } p = x-1,\ (x-1)(x-\lambda)\ (|\lambda| \leqslant 1),\\
&\qquad x^2,\ (x-1)^2,\ (x-\lambda)^2+1\ (\lambda \geqslant 0);\\
&5.1(p),\ \textit{where } p = x^2,\ x(x-1),\ x^2+1;\\
&6.1(2,\lambda),\ 7.1(2),\ 7.1(3),\ 8.1(2),\ 9.1(2),\\
&10.1,\ 11.1,\ 11.2,\ 11.3,\ 14.1,\ 15.1,\ 16.1.
\end{aligned}$$

Proof. It is a matter of direct verification to prove that the pairs mentioned above are the only isotropically-faithful pairs among those mentioned in the classification theorem. A pair $4.1(p)$ is isotropically-faithful if and only if $\deg p \leqslant 2$. It is easy to check that the polynomial p viewed up to transformations of the form

$$p(x) \mapsto \lambda p(\mu x),\ \lambda, \mu \in \mathbb{R}_+^*,$$

is equivalent to one and only one of the polynomials mentioned in the theorem. Similarly, a pair $5.1(p)$ is isotropically-faithful if and only if $\deg p = 2$. In this case, p can be reduced to one and only one of the polynomials mentioned in the theorem by transformations of the form

$$p(x) \mapsto \lambda p(\mu x + \alpha),\ \lambda, \mu \in \mathbb{R}_+^*,\ \alpha \in \mathbb{R}.$$

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an arbitrary pair of codimension 2 and $\rho : \mathfrak{g} \to \mathfrak{gl}(\bar{\mathfrak{g}}/\mathfrak{g})$ its isotropic representation. Consider the matrix realizations of the subalgebra $\rho(\mathfrak{g})$ in different bases of the space $\bar{\mathfrak{g}}/\mathfrak{g}$. So, to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ we can assign the class of subalgebras of $\mathfrak{gl}(2, \mathbb{R})$ conjugate to each other.

Definition. Two pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are called *isotropically-equivalent* if the corresponding classes of conjugate subalgebras of $\mathfrak{gl}(2, \mathbb{R})$ coincide.

Problem. Prove that every class of conjugate subalgebras of the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ can be assigned to a certain isotropically-faithful pair.

Let us pick out one representative from each class of conjugate subalgebras of $\mathfrak{gl}(2, \mathbb{R})$. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(2, \mathbb{R})$ up to conjugation.

Proposition. *Any subalgebra of the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ is conjugate to one and only one of the following subalgebras:*

$$
\begin{array}{llll}
I & \{0\}; & VII & \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}; \\
II(\lambda) & \left\{ \begin{pmatrix} x & 0 \\ 0 & \lambda x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}, \ |\lambda| \leqslant 1; & VIII(\lambda) & \left\{ \begin{pmatrix} \lambda x & y \\ 0 & x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}; \\
III(\lambda) & \left\{ \begin{pmatrix} \lambda x & -x \\ x & \lambda x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}, \ \lambda \geqslant 0; & IX & \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\};
\end{array}
$$

$$
\begin{array}{llll}
IV & \left\{ \begin{pmatrix} x & x \\ 0 & x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}; & X & \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}; \\
V & \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}; & XI & \mathfrak{sl}(2,\mathbb{R}); \\
VI & \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}; & XII & \mathfrak{gl}(2,\mathbb{R}).
\end{array}
$$

Proof. See "Subalgebras of $\mathfrak{gl}(n,P)$", ISLC Abstracts, Lie-Lobachevsky Colloquium, Tartu, Estonia, 1992.

Representatives from classes of conjugate subalgebras corresponding to all effective pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2 are listed in appendix C.

Consider an arbitrary $\mathfrak{g}$-module V. A subspace $W \subset V$ is called *invariant* if $\mathfrak{g}.W \subset W$. It is obvious that the trivial subspaces $\{0\}$ and V are invariant. The $\mathfrak{g}$-module V is called *simple* if V contains no nontrivial invariant subspaces. If for each invariant subspace W_1 there exists a complementary invariant subspace W_2, then the $\mathfrak{g}$-module V is called *semisimple.*

Now let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an arbitrary pair. Then $\bar{\mathfrak{g}}$ can be regarded as a $\mathfrak{g}$-module, where

$$x.\bar{x} = [x, \bar{x}]$$

for $x \in \mathfrak{g}$, $\bar{x} \in \bar{\mathfrak{g}}$. It is obvious that $\mathfrak{g} \subset \bar{\mathfrak{g}}$ is an invariant subspace of the $\mathfrak{g}$-module $\bar{\mathfrak{g}}$. If there exists an invariant subspace complementary to $\mathfrak{g}$, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *reductive.* The subalgebra $\mathfrak{g}$ is said to be *reductive* if the $\mathfrak{g}$-module $\bar{\mathfrak{g}}$ is semisimple.

Exercise. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair. Prove that

a) if $\mathfrak{g}$ is a reductive subalgebra, then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is reductive;

b) if $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a reductive pair, then $(\bar{\mathfrak{g}}, \mathfrak{g})$ is isotropically-faithful.

Suppose that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is reductive and $\mathfrak{m}$ is an invariant subspace complementary to $\mathfrak{g}$. Then $\mathfrak{m}$ can be identified with $\bar{\mathfrak{g}}/\mathfrak{g}$ and the action of $\mathfrak{g}$ on $\bar{\mathfrak{g}}/\mathfrak{g}$ can be written as

$$x.m = [x, m]$$

for $x \in \mathfrak{g}$, $m \in \mathfrak{m}$.

Theorem. *Each isotropically-faithful pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2 is reductive except those equivalent to one of the following pairs from the classification theorem:*

$$7.1(3), \ 10.1, \ 16.1.$$

Proof. It is sufficient to consider only isotropically-faithful pairs described in the previous theorem and for each pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of this kind to check if there exists an invariant subspace complementary to $\mathfrak{g}$.

Corollary. *Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair of codimension 2 such that $\mathfrak{g}$ is a reductive subalgebra. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one and only one of the following pairs:*

$$1.1, 2.1(\lambda), 2.2(\lambda), 3.1, 3.2, 4.1(x-1),$$

$$11.1, 11.2, 11.3, 14.1, 15.1.$$

Proof. From conditions of the corollary it follows that the subalgebra $\mathfrak{g}$ is either commutative or non-solvable. Moreover, the isotropic representation of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ must be semisimple. It can be easily checked the pairs mentioned in the corollary are the only pairs that satisfy these conditions. It remains to note that for all of these pairs, $\mathfrak{g}$ is reductive.

5.2. Inclusions. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair and $\bar{\mathfrak{p}}$ a subalgebra of $\bar{\mathfrak{g}}$ such that $\bar{\mathfrak{p}} + \mathfrak{g} = \bar{\mathfrak{g}}$. Put $\mathfrak{p} = \bar{\mathfrak{p}} \cap \mathfrak{g}$.

Proposition.

1) The pair $(\bar{\mathfrak{p}}, \mathfrak{p})$ is effective.

2) The pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{p}}, \mathfrak{p})$ have the same codimension.

3) If the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is isotropically-faithful, then the pair $(\bar{\mathfrak{p}}, \mathfrak{p})$ is also isotropically-faithful.

We say that the pair $(\bar{\mathfrak{p}}, \mathfrak{p})$ is a *restriction* of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is an *extension* of the pair $(\bar{\mathfrak{p}}, \mathfrak{p})$. We also say that a restriction (extension) is maximal if $\bar{\mathfrak{p}}$ is a maximal subalgebra of $\bar{\mathfrak{g}}$.

All maximal restrictions for pairs of codimension 2 are given in appendix C.

Theorem.

1) All maximal pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2 are equivalent to one of the following:

$$5.1(p), \ 12.1, \ 12.2, \ 13.1, \ 18.1(n), \ (n \geqslant 2).$$

2) All minimal pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2 are equivalent to one of the following:

$$1.1, \ 4.1(x-1), \ 4.1((x-\lambda)^2+1), \ 11.3.$$

Proof. It immediatly follows from the list of all maximal inclusions.

CHAPTER III

INVARIANTS

§1. Tensor invariants

Suppose that an arbitrary group G acts on a set M. A point $x \in M$ is called an *invariant* of the action if $g.x = x$ for all $g \in G$. Now let $(\overline{G}, M)$ be a homogeneous space. Since $\overline{G}$ acts transitively on M, we see that the action has no invariants. However, we can consider natural prolongations of the action of the group $\overline{G}$ to different objects on M such as the tangent bundle. Actions obtained this way may have invariants.

Let TM denote the tangent bundle on the manifold M. Each element $g \in \overline{G}$ gives the diffeomorphism of M $x \mapsto g.x$ and therefore the isomorphism of tangent spaces

$$dg : T_xM \to T_{g.x}M \quad \text{for all } x \in M.$$

Similarly, each element $g \in \overline{G}$ gives the isomorphism $T^*_xM \to T^*_{g^{-1}.x}M$. This allows to prolong the action of $\overline{G}$ to the space $T^m_n(TM)$ of tensor fields of valence (n, m) on M. For $g \in \overline{G}$, by g^* denote the corresponding transformation. Each tensor field $\omega \in T^m_n(TM)$ can be regarded as a family of tensors of valence (n, m)

$$\{\omega_x \in T^m_n(T_xM)\}_{x\in M}$$

which depends smoothly on x. A tensor field ω is an invariant if

$$g^*.\omega_x = \omega_{g.x}$$

for all $x \in M$, $g \in \overline{G}$.

Let x_0 be an arbitrary point of M and put $G = \overline{G}_{x_0}$. Note that each invariant tensor field ω is uniquely determined by the tensor $\omega_{x_0} \in T^m_n(T_{x_0}M)$. Let g be an element of G. Then the mapping dg is an automorphism of the space $T_{x_0}M$. Therefore, dg gives an action of the group G on $T^m_n(T_{x_0}M)$. This action is called *isotropic*. Since ω is invariant, we see that the tensor ω_{x_0} is invariant under the action of G on $T^m_n(T_{x_0}M)$. Conversely, suppose $\omega_{x_0} \in T^m_n(T_{x_0}M)$ is an arbitrary tensor invariant under the action of G. Putting

$$\omega_{g.x} = g^*\omega_{x_0},$$

we obtain a tensor field ω on M invariant under the action of $\overline{G}$. So, we have proved the following

Proposition. *There is a one-to-one correspondence between the set of tensor fields of valence (n, m) on M invariant under the action of $\overline{G}$ and the set of tensors from $T^m_n(T_{x_0}M)$ invariant under the isotropic action of the Lie group $G = \overline{G}_{x_0}$.*

Note that in order to describe tensors invariant under the isotropic action of G, it is sufficient to know the action of G on the tangent space $T_{x_0}M$. Since this action is linear, we obtain the homomorphism of Lie groups $f : G \to \mathrm{GL}(T_{x_0}M)$.

Theorem. *Let $(\overline{G}, M)$ be a homogeneous space, x an arbitrary point of M, and $G = \overline{G}_{x_0}$. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be the pair of Lie algebras corresponding to the pair $(\overline{G}, G)$.*

a) The tangent space $T_{x_0}M$ can be naturally identified with the factor space $\bar{\mathfrak{g}}/\mathfrak{g}$.

b) The homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(\bar{\mathfrak{g}}/\mathfrak{g})$ defined by the homomorphism of Lie groups $\overline{G} \to \mathrm{GL}(T_{x_0}M)$ coincides with the isotropic representation of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Let $f : G \to \mathrm{GL}(V)$ be an arbitrary homomorphism of Lie groups. The homomorphism f gives a linear action of the Lie group G on V. Then we can naturally define a linear action of G on V^*:

$$(g.\alpha)(v) = \alpha(g^{-1}.v)$$

for $g \in G$, $\alpha \in V^*$, $v \in V$. This allows to prolong the action of G to an arbitrary tensor space:

$$g(v_1 \otimes \cdots \otimes v_n \otimes v^1 \otimes \cdots \otimes v^m) = (g.v_1) \otimes \cdots \otimes (g.v_n) \otimes (g.v^1) \otimes \cdots \otimes (g.v^m)$$

for $g \in G$, $v_1, \ldots, v_n \in V$, $v^1, \ldots, v^m \in V^*$. There exists a homomorphism of Lie groups $G \to \mathrm{GL}(T^n_m(V))$ corresponding to this action. The corresponding homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(T^n_m(V))$ turns the space $T^n_m(V)$ into a $\mathfrak{g}$-module.

Proposition.

a) Let V denote the $\mathfrak{g}$-module corresponding to the linear action of G on V. Then the $\mathfrak{g}$-module V^* corresponding to the action of G on V^* can be given by

$$(x.\alpha)(v) = -\alpha(x.v)$$

for $\alpha \in V^*$, $v \in V$, $x \in \mathfrak{g}$.

b) The $\mathfrak{g}$-module $T^n_m(V)$ corresponding to the action of G on the space $T^n_m(V)$ can be given by

$$\begin{aligned} x.(v_1 \otimes \cdots \otimes v_n \otimes v^1 \otimes \cdots \otimes v^m) = \\ = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes (x.v_i) \otimes \cdots \otimes v_n \otimes v^1 \otimes \cdots \otimes v^m + \\ + \sum_{i=1}^{m} v^1 \otimes \cdots \otimes v_n \otimes v^1 \otimes \cdots \otimes (x.v^i) \otimes \cdots \otimes v^m. \end{aligned}$$

c) A tensor $\omega \in T^n_m(V)$ is invariant under the action of G if and only if $x.\omega = 0$ for all $x \in \mathfrak{g}$.

d) A subspace $W \subset V$ is invariant under the action of G if and only if W is an invariant subspace of the $\mathfrak{g}$-module V.

Thus, description of tensors invariant under the isotropic action of the Lie group G can be reduced to description of tensor invariants of the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$. Note that if we study tensor invariants of a $\mathfrak{g}$-module V, it is important to consider not the Lie algebra $\mathfrak{g}$ itself but its image by the mapping $\mathfrak{g} \to \mathfrak{gl}(V)$, $x \mapsto x_V$. We shall first of all be interested in invariant bilinear forms, operators, vectors and covectors (linear forms) as well as invariant subspaces.

Let us write out conditions for these objects to be invariant.

1. A vector $v \in V$ is invariant $\Leftrightarrow x.v = 0\ \forall x \in \mathfrak{g}$.
2. A covector $\alpha \in V^*$ is invariant $\Leftrightarrow \alpha(x.v) = 0\ \forall v \in V,\ x \in \mathfrak{g}$ or

$$x.v \subset \operatorname{Ker}\alpha\ \forall x \in \mathfrak{g}.$$

3. A bilinear form $b \in \operatorname{Bil}(V)$ is invariant if

$$b(x.v_1, v_2) + b(v_1, x.v_2) = 0 \qquad \forall v_1, v_2 \in V,\ x \in \mathfrak{g}.$$

4. An operator $\varphi \in \mathfrak{gl}(V)$ is invariant if

$$x.\varphi(v) - \varphi(x.v) = 0 \qquad \forall v \in V,\ x \in \mathfrak{g}$$

or

$$x_V\varphi = \varphi_V \qquad \forall x \in \mathfrak{g}.$$

5. A subspace $W \subset V$ is invariant if $x.W \subset W\ \forall x \in \mathfrak{g}$.

To each subalgebra $\mathfrak{a} \subset \mathfrak{gl}(2,\mathbb{R})$ we can assign the natural $\mathfrak{a}$-module $\mathbb{R}^2$. Any two-dimensional faithful module can be represented as a module of this form. Therefore, in order to describe all invariants of a $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$, it is sufficient to do it for the corresponding $\mathfrak{a}$-modules $\mathbb{R}^2$, where $\mathfrak{a} \subset \mathfrak{gl}(2,\mathbb{R})$. Note that if two pairs are isotropically equivalent, then the corresponding $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ have the same invariants.

Invariants of $\mathfrak{a}$-modules $\mathbb{R}^2$ such that $\mathfrak{a}$ is one of the subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ obtained earlier are tabulated in appendix C.

Some corollaries.

1. Only the subalgebras I and III(0) have an invariant positive definite symmetric form. Therefore, the only homogeneous spaces to allow an invariant Riemannian metric are those whose corresponding pairs of Lie algebras are equivalent to one of the following pairs:

$$1.1,\ 2.2(0),\ 11.2,\ 11.3.$$

2. The only homogeneous spaces to allow an invariant pseudo-Riemannian metric are those whose the corresponding pairs of Lie algebras are equal to one of the following pairs:

$$1.1,\ 2.1(-1),\ 11.1.$$

§2. Jet spaces and differential invariants

2.1. Jet spaces.

Let a be an arbitrary point on the line.

Definition. Two functions $f, g \in C^\infty(\mathbb{R})$ are called *k-equivalent* at the point a if

$$f(x) - g(x) = o\left((x-a)^k\right) \text{ as } x \to a.$$

From the definition it follows that two functions f and g are k-equivalent if and only if $f(a) = g(a)$ and $f^{(i)}(a) = g^{(i)}(a),\ 1 \leqslant i \leqslant k$.

It is easy to check that the k-equivalence relation is indeed an equivalence relation. The class of functions k-equivalent to f at a point $a \in \mathbb{R}$ is called the k-jet of f at the point a and is denoted by $[f]_a^k$. In particular, $[f]_a^0$ is the set of all smooth functions such that $g(a) = f(a)$. By J_a^k denote the set of k-equivalence classes at a point $a \in \mathbb{R}$. To every class $[f]_a^k$ we can assign a collection of numbers

$$(y_0, y_1, \dots, y_k), \text{ where } y_i = f^{(i)}(a),\ 0 \leqslant i \leqslant k,$$

which uniquely determine the class.

Exercise. Show that for any collection of numbers $(y_0, y_1, \dots, y_k)$ there exists a smooth function f such that $y_i = f^{(i)}(a),\ 0 \leqslant i \leqslant k$.

Thus, for every point $a \in \mathbb{R}$, the set J_a^k can be identified with $\mathbb{R}^{k+1}$.

By J^k denote the union of all J_a^k:

$$J^k = \bigcup_{a \in \mathbb{R}} J_a^k.$$

It follows that we can consider J^k as $\mathbb{R}^{k+2}$. Namely, to each element $[f]_x^k \in J^k$ assign the set of $k+2$ numbers

$$(x, y_0, y_1, \dots, y_k), \text{ where } y_i = f^{(i)}(x).$$

To every function $f \in C^\infty(\mathbb{R})$ it is possible to assign the curve in J^k:

$$s_k(f): t \mapsto (t,\ [f]_t^k).$$

It is called the *k-jet of the function f*. For example, if we identity J^k with $\mathbb{R}^{k+2}$, then the curve corresponding to the function $f(x) = x^2$ has the form:

$$t \mapsto (t, t^2, 2t, 2, 0, \dots, 0).$$

Exercise. Let $s: t \mapsto (t, y_0(t), y_1(t), \dots, y_n(t))$ be some curve in the space J^k. Show that it has the form $s_k(f)$ if and only if $y_i(t) \in C^\infty(\mathbb{R})$ and $y_{i+1} = y_i'(t)$ for all $t \in \mathbb{R},\ 0 \leqslant i \leqslant k-1$.

Consider now the Lie algebra $\mathcal{D}(\mathbb{R}^2)$ of all vector fields on the plane. Let us recall that every vector field $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$ generates a local one-parameter group of diffeomorphisms $\{\varphi_t\}$. It is uniquely determined by:

$$s_a'(t) = \mathbf{v}_{s_a(t)}, \qquad s_a(o) = a,$$

where $s_a(t) = \varphi_t(a)$ for all $a \in \mathbb{R}$.

Now with every vector field $\mathbf{v}$ on the plane we associate a certain vector field $\mathbf{v}^{(k)}$ on the space J^k. To do it, first, we construct the local one-parameter group of diffeomorphisms $\{\varphi_t\}$ corresponding to the vector field $\mathbf{v} \in \mathcal{D}(\mathbb{R}^2)$. Further, let $(a,\ [f]_a^k)$ be some element of J^k and let functions $g_t, h_t \in C^\infty(\mathbb{R})$ be given by

$$(g_t(x),\ h_t(x)) = \varphi_t(x,\ f(x)) \text{ for all } x \in \mathbb{R}.$$

Then put

$$\varphi_t^{(k)}(a, [f]_a^k) = \left(g_t(a),\ [h_t \circ g_t^{-1}]_{g_t(a)}^k\right).$$

For a fixed point $(a, [f]_a^k) \in J^k$ all the constructions described above will be well-defined for sufficiently small t. In particular, if $t = 0$, then $g_0 = \mathrm{Id}_{\mathbb{R}^2}$, $h_0 = f$, and consequently $\varphi_0^{(k)} = \mathrm{Id}_{J^k}$.

Conversely, for each $t \in \mathbb{R}$ the diffeomorphism $\varphi_t^{(k)}$ of the space J^k will be defined on a certain open domain. Moreover, from the definition it immediately follows that $\varphi_{t_1}^{(k)} \circ \varphi_{t_2}^{(k)} = \varphi_{t_1+t_2}^{(k)}$ is defined everywhere, where this equality makes sense. Thus, we have obtained a local one-parameter transformation group of J^k. The corresponding vector field $\mathbf{v}^k$ on the space J^k is called the k-th prolongation of the vector field $\mathbf{v}$.

Theorem 1. *The map $\mathbf{v} \to \mathbf{v}^{(k)}$ is a homomorphism of the Lie algebra $\mathcal{D}(\mathbb{R}^2)$ into the Lie algebra of all vector fields on J^k.*

Examples.

1) Suppose $k = 0$. Then an element $(a, [f]_a^0)$ of the space J^0 can be identified with the point $(a, f(a))$ on the plane. The functions g_t and h_t are defined by the equality

$$(g_t(x),\ h_t(x)) = \varphi_t(x,\ f(x)).$$

Then

$$\varphi_t^{(0)} \colon (a, b) \mapsto \left(g_t(a),\ [h_t \circ g_t^{-1}]_{g_t(a)}^0\right) =$$
$$\left(g_t(a),\ h_t \circ g_t^{-1}(g_t(a))\right) = (g_t(a),\ h_t(a)) = \varphi_t(a, b).$$

So, in our case $\varphi_t^{(0)} = \varphi_t$. Hence, $\mathbf{v}^{(0)} = \mathbf{v}$. This enables us to identify the spaces J^0 and $\mathbb{R}^2$. In agreement with the identification, we shall denote coordinates on the plane by (x, y_0).

2) Let $\mathbf{v} = y_0 \frac{\partial}{\partial y_0}$ be a vector field on the plane. Find the vector field $\mathbf{v}^{(1)}$. The one-parameter transformation group $\{\varphi_t\}$ corresponding to the vector field $\mathbf{v}$ has the form:

$$\varphi_t \colon (x, y_0) \mapsto (x, e^t y_0).$$

Let us identify the spaces J^1 and $\mathbb{R}^3$. Suppose (a, y_0, y_1) is an arbitrary point in $\mathbb{R}^3$. The corresponding point in J^1 is $\left(a,\ [y_0 + y_1(x-a)]_a^1\right)$. Then the functions g_t and h_t are determined by

$$(g_t(x),\ h_t(x)) = \varphi_t(x,\ y_0 + y_1(x-a)) = (x,\ e^t y_0 + e^t y_1(x-a)),$$

so that

$$g_t(x) = x, \qquad h_t(x) = e^t y_1 x + e^t(y_0 - y_1 a).$$

Then

$$\varphi_t^{(1)}(a, y_0, y_1) = \varphi_t^{(1)}\left(a,\ [y_0 + y_1(x-a)]_a^1\right) = \left(a,\ [h_t]_a^{(1)}\right) = (a, e^t y_0, e^t y_1).$$

Hence

$$\mathbf{v}^{(1)} = y_0 \frac{\partial}{\partial y_0} + y_1 \frac{\partial}{\partial y_1}.$$

As we can see, it is rather difficult to find prolongations of vector fields using only definitions. However, there exist simple formulas to do it. Some of them allow to do it without finding the corresponding one-parameter transformation groups.

Theorem 2. *Let* $\mathbf{v} = A(x, y_0)\frac{\partial}{\partial x} + B_0(x, y_0)\frac{\partial}{\partial y_0}$ *be a vector field on the plane, then the vector field* $\mathbf{v}^{(k)}$ *has the form:*

$$\mathbf{v}^{(k)} = A(x, y_0)\frac{\partial}{\partial x} + \sum_{i=0}^{k} B_i(x, y_0, \dots, y_i)\frac{\partial}{\partial y_i},$$

where

$$B_{i+1} = \frac{dB_i}{dx} - y_{i+1}\frac{dA}{dx},$$

and

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \dots + y_{i+1}\frac{\partial}{\partial y_i}.$$

Example. Let $\mathbf{v} = y_0\frac{\partial}{\partial x} - x\frac{\partial}{\partial y_0}$ be a vector field on the plane. Then, from theorem 2 it follows that

$$\mathbf{v}^{(1)} = y_0\frac{\partial}{\partial x} - x\frac{\partial}{\partial y_0} + B_1\frac{\partial}{\partial y_1},$$

where

$$B_1 = \frac{dB_0}{dx} - y_1\frac{dA}{dx} = -1 - y_1^2.$$

Thus,

$$\mathbf{v}^{(1)} = y_0\frac{\partial}{\partial x} - x\frac{\partial}{\partial y_0} - (1 + y_1^2)\frac{\partial}{\partial y_1}.$$

2.2. Differential invariants of homogeneous spaces. Suppose now $\mathfrak{g}$ is a Lie algebra of vector fields on the plane. Denote by $\mathfrak{g}^{(k)}$ its image by the mapping $\mathbf{v} \mapsto \mathbf{v}^{(k)}$. From theorem 1 it follows that $\mathfrak{g}^{(k)}$ is a Lie algebra of vector fields on the space J^k.

Definition. A *differential invariant of the k-th order of a Lie algebra* $\mathfrak{g} \in \mathcal{D}(\mathbb{R}^2)$ is a function $F \in C^\infty(J^k)$ such that

$$\mathbf{v}^{(k)}(F) = 0 \text{ for all } \mathbf{v} \in \mathfrak{g}.$$

The set of all differential invariants of the k-th order of a Lie algebra $\mathfrak{g} \in \mathcal{D}(\mathbb{R}^2)$ is denoted by $I_k(\mathfrak{g})$.

Using the properties of the action of vector fields on functions, one can easily prove the following

Theorem 3.

1. *The set $I_k(\mathfrak{g})$ is a subalgebra of the algebra $C^\infty(J^k)$.*
2. *If $f_1, \dots, f_n \in I_k(\mathfrak{g})$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \dots, f_n) \in I_k(\mathfrak{g})$.*

Definition. We say that a set of differential invariants $f_1, \dots, f_r$ is a (*local*) *basis* of the Lie algebra $I_k(\mathfrak{g})$, if

(1) the functions are functionally independent (in some neighborhood);
(2) for any differential invariant $f \in I_k(\mathfrak{g})$ there exists a smooth function F such that f can be (locally) written as $f = F(f_1, \dots, f_r)$.

For example, let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ be the Lie algebra of vector fields on $\mathbb{R}^2$ with basis

$$X_1 = \frac{\partial}{\partial y_0}, \quad X_2 = y_0 \frac{\partial}{\partial y_0}, \quad X_3 = y_0^2 \frac{\partial}{\partial y_0}.$$

Then direct calculation shows that the function x is a basis of the algebra of invariants $I_k(\mathfrak{g})$ whenever $k = 0, 1, 2$. If $k = 3$, there exists a basis $\{f_1,\ f_2\}$ of $I_3(\mathfrak{g})$, where

$$f_1 = x, \quad f_2 = \frac{2y_1 y_3 - 3y_2^2}{2y_1^2}.$$

The restriction of the differential invariant f_2 to the 3-jet of a function $\varphi(x)$

$$\frac{2\varphi'\varphi''' - 3(\varphi'')^2}{2(\varphi')^2}$$

is called a *Schwartz derivative* or *Schwartzian* of the function $\varphi(x)$.

In appendix C, for each transitive Lie algebra $\mathfrak{g}$ of vector fields on the plane, we give a nontrivial differential invariant of the least order.

CHAPTER IV

APPLICATIONS TO DIFFERENTIAL EQUATIONS

§1. Ordinary differential equations

1.1. Geometrical interpretation of differential equations. Any ordinary differential equation of the n-th order can be written as

$$F(f^{(n)},\dots,f',f,x)=0,$$

where $F\colon\mathbb{R}^{n+2}\to\mathbb{R}$ is some smooth function.

From the definition of the space of jets J^n we immediately obtain the following interpretation of ordinary differential equations.

An ordinary differential equation of the n-th order is a smooth function F on the space J^n. A solution of this equation is a function $f\colon\mathbb{R}\to\mathbb{R}$ such that $F(s_n(f))\equiv 0$.

Let $\mathcal{E}$ be a surface in J^n given by the equation $F=0$. Note that different functions F can define the same surface $\mathcal{E}$. Besides, solutions of the corresponding differential equations coincide. It shows that in order to define a differential equation it is sufficient to give a surface $\mathcal{E}$ defined by a function F, but the knowledge of F is not necessary.

Hence, it makes sense to reformulate the interpretation of an ordinary differential equation in the following way.

An ordinary differential equation of the n-th order is a hypersurface $\mathcal{E}$ in the space J^n. A solution of this equation is a function $f\colon\mathbb{R}\to\mathbb{R}$ such that $s_n(f)\subset\mathcal{E}$.

Solutions of a differential equation can be characterized internally, i.e. in terms of the space J^n itself. Namely, let $(x,p_0,\dots,p_n)$ be the standard coordinates in J^n. Consider the set of n differential forms $\omega_1,\omega_2,\dots,\omega_n$ on J^n:

$$\omega_i=dp_i-p_i dx.$$

These forms are called the *Cartan forms*. They are really very important because their restrictions to any curve $s_n(f)$, where $f\in C^\infty(\mathbb{R})$, are equal to zero. Indeed, any curve $s_n(f)$ has the form:

$$s_n(f)\colon t\mapsto(t,f(t),f'(t),\dots,f^{(n)}(t)).$$

Then

$$s_n(f)^*(\omega_i)=d(f^{(i-1)}-f^{(i)}dt=f^{(i)}(t)dt-f^{(i)}dt=0.$$

Conversely, suppose

$$s\colon t\mapsto(t,p_0(t),p_1(t),\dots,p_n(t))$$

is a curve in J^n such that $s^*(\omega_i)=0$ for all $1\leqslant i\leqslant n$. It means that

$$d(p_{i-1}(t))-p_i(t)dt=(p'_{i-1}(t)-p_i(t))dt=0,$$

i.e. $p_i(t) = p'_{i-1}(t)$ for all $1 \leqslant i \leqslant n$.

It immediately follows that $p_i(t) = p_0^{(i)}(t)$ for all $1 \leqslant i \leqslant n$. Thus, $s = s_n(p_0(t))$.

Note that we are interested not in curves $s: \mathbb{R} \to J^n$ themselves, but in their images in the space J^n, which are one-dimensional submanifolds $\{s(t) \mid t \in \mathbb{R}\}$. In addition, if f is a solution of a differential equation $\mathcal{E} \subset J^n$, then the corresponding submanifold

$$L_f = \{ \ s_n(f)(t) \mid t \in \mathbb{R} \ \}$$

has the following properties:

1) $L_f \subset \mathcal{E}$;
2) $\omega_{i|_{L_f}} = 0$ for all $1 \leqslant i \leqslant n$;
3) the projection of L_f to the x-axis has no singularities.

The latest condition means that the restriction of the projection

$$\pi: (x, p_0, \ldots, p_n) \mapsto x$$

to the manifold L_f is a smooth diffeomorphism. Moreover, any one-dimensional submanifold $L \subset J^n$ satisfying properties 1)–3) uniquely determines a solution of the equation $\mathcal{E}$. Indeed, since the projection of L onto the x-axis has no singularities, we see that it can be uniquely represented as

$$L = \{ \ (t, p_0(t), p_1(t), \ldots, p_n(t)) \mid t \in \mathbb{R} \ \}.$$

It follows that $p_0(t)$ is a solution of the differential equation $\mathcal{E}$ and $L = L_{p_0}$. If we omit condition 3), then we arrive at the concept of a generalized solution of a differential equation $\mathcal{E}$.

A *generalized solution of a differential equation* $\mathcal{E} \subset J^n$ is a one-dimensional submanifold $L \subset J^n$ such that the following conditions hold:

1) $L_f \subset \mathcal{E}$;
2) $\omega_i|_{L_f} = 0$ for all $1 \leqslant i \leqslant n$.

Those of them that can be projected to the x-axis without singularities determine classical solutions $f \in C^\infty(\mathbb{R})$.

1.2 Symmetries and algorithm of integrating. Now we come to the concept of a symmetry of a differential equation. Symmetries play very important role in the procedure of solution of equations.

Definition. A vector field $\mathcal{D}(\mathbb{R}^2)$ is called a *(point) symmetry* of a differential equation $\mathcal{E} \subset J^n$ if the vector field $\mathbf{v}^{(n)}$ is tangent to the surface $\mathcal{E}$, i.e.

$$\mathbf{v}_a^{(n)} \subset T_a\mathcal{E} \text{ for all } a \in \mathcal{E}.$$

The following proposition follows directly from the definition.

Proposition.

1. The set of all symmetries of an equation $\mathcal{E} \subset J^n$ is a Lie algebra and is denoted by $\mathrm{Sym}(\mathcal{E})$.

2. Let us assume that $\mathcal{E}$ is given by the equation $F = 0$ for some smooth function $F \in \mathcal{C}^\infty(J^n)$. Then $\mathbf{v} \in \mathrm{Sym}(\mathcal{E})$ if and only if

$$\mathbf{v}^{(n)}(F) = \lambda F$$

for a certain smooth function $\lambda \in \mathcal{C}^\infty(J^n)$.

Corollary. *Suppose $\mathfrak{g}$ is a Lie algebra of vector fields on the plane, $F \in I^n(\mathfrak{g})$, and $\mathcal{E}$ is an arbitrary differential equation of the n-th order given by the function F. Then $\mathfrak{g} \subset \mathrm{Sym}(\mathcal{E})$.*

Thus, all ordinary differential equations given by differential invariants of a Lie algebra $\mathfrak{g} \subset \mathcal{D}(\mathbb{R}^2)$ have $\mathfrak{g}$ as an algebra of their point symmetries.

In their works Lie and Bianchi showed that it is useful to consider symmetries in theory of differential equations.

Theorem([6],[9]). *Suppose $\mathfrak{g}$ is an n-dimensional solvable symmetry algebra of an equation $\mathcal{E} \subset J^n$. Then this equation is solvable by quadratures.*

This theorem is constructive. The following algorithm allows to integrate an equation $\mathcal{E}$ starting from its *symmetry algebra* $\mathfrak{g}$.

1. Choose a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ of the Lie algebra $\mathfrak{g}$ so that for every $i \in \mathbb{N}$ there exists $k_i \in \mathbb{N}$ such that the vector fields $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k_i}$ form a basis of the Lie algebra $\mathcal{D}^{(i)}\mathfrak{g}$.

2. Consider the restrictions of the forms ω_i to the surface $\mathcal{E}$ and construct the matrix

$$\Omega = \left(\omega_i(\mathbf{v}_j^{(n)})\right).$$

3. Assume that it is non-singular (otherwise, see item 8). Find the forms ω_i' from the equality

$$\begin{pmatrix} \omega_1' \\ \vdots \\ \omega_n' \end{pmatrix} = \Omega^{-1} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}.$$

4. The form ω_n' is closed. Let us find a function F_n such that $\omega_n' = dF_n$:

$$F_n = \int \omega_n'.$$

5. Restrict all forms $\omega_1', \dots \omega_{n-1}'$ to the level surface $F_n = c_n$, where c_n is an arbitrary real number.

6. The form ω_{n-1}' is closed. Let us integrate it, i.e. let us find a function F_{n-1} such that $dF_{n-1} = \omega_{n-1}'$. Restrict the rest of the forms to the level surface

$$\{F_n = c_n,\ F_{n-1} = c_{n-1}\}, \text{ where } c_{n-1} \in \mathbb{R}.$$

7. Continuing in the same way we obtain the functions $F_1, F_2, \ldots, F_n$. Then any solution L can be given as an intersection of some surface $\{F_1 = c_1, \ldots, F_n = c_n\}$ and the manifold $\mathcal{E}$.

8. In the case of singular matrix Ω there exists a vector field $\mathbf{v} \in \mathfrak{g}$ such that $\omega_i(\mathbf{v}) = 0$ for all $1 \leqslant i \leqslant n$. In such a situation all solutions of the equation $\mathcal{E}$ are trajectories of the one-parameter transformation group $\{= \varphi_t\}$ corresponding to the vector field $\mathbf{v}^{(n)}$:

$$L = \{ \varphi_t(a) \mid t \in \mathbb{R} \}, \text{ where } a \in \mathcal{E}.$$

Example. Suppose $\mathfrak{g} = \langle \frac{\partial}{\partial x}; \frac{\partial}{\partial p_0} \rangle$. The functions p_1 and p_2 form a basis of the algebra of invariant $I^2(\mathfrak{g})$. Hence, all equations of the form $\overline{F}(p_1, p_2) = 0$ have $\mathfrak{g}$ as an algebra of point symmetries. Now we assume that the equation $\overline{F}(p_1, p_2)$ is solvable with respect to p_2 and can be written as $p_2 = F(p_1)$ or, in the classical notation, as $f'' = F(f')$. Let us apply the algorithm described above to equations of this type.

1. Since $\mathcal{D}\mathfrak{g} = \{0\}$, we choose the following basis: $\mathbf{v}_1 = \frac{\partial}{\partial x}$; $\mathbf{v}_2 = \frac{\partial}{\partial p_0}$. One can easily check that $\mathbf{v}_1^{(1)} = \frac{\partial}{\partial x}$, $\mathbf{v}_2^{(1)} = \frac{\partial}{\partial p_0}$.

2. Take (x, p_0, p_1) for coordinates on $\mathcal{E}$. Then the restrictions of the forms ω_1 and ω_2 to $\mathcal{E}$, in the coordinates (x, p_0, p_1), have the form:

$$\omega_1 = dp_0 - p_1 dx, \qquad \omega_2 = dp_1 - F(p_1)dx.$$

The matrix Ω is:

$$\Omega = \begin{pmatrix} \omega_1(\mathbf{v}_1^{(1)}) & \omega_1(\mathbf{v}_2^{(1)}) \\ \omega_2(\mathbf{v}_1^{(1)}) & \omega_2(\mathbf{v}_2^{(1)}) \end{pmatrix} = \begin{pmatrix} -p_1 & 1 \\ -F(p_1) & 0 \end{pmatrix}.$$

3. This matrix is singular only if $F(p_1) = 0$. Otherwise

$$\Omega^{-1} = \frac{1}{F(p_1)} \cdot \begin{pmatrix} 0 & -1 \\ F(p_1) & -p_1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{F(p_1)} \\ 1 & \frac{-p_1}{F(p_1)} \end{pmatrix}.$$

Consequently,

$$\omega_1' = -\frac{dp_1}{F(p_1)} + dx,$$
$$\omega_2' = (dp_0 - p_1 dx) - \frac{p_1 dp_1}{F(p_1)} + p_1 dx = dp_0 - \frac{p_1 dp_1}{F(p_1)}.$$

4. The forms ω_1' and ω_2' are closed. In addition

$$\int \omega_1' = x - \int \frac{dp_1}{F(p_1)}; \qquad \int \omega_2' = p_0 - \int \frac{p_1 dp_1}{F(p_1)}.$$

Hence, any solution of the initial differential equation can be found from the functional equations

$$\begin{cases} x - \displaystyle\int \frac{dp_1}{F(p_1)} & = C_1, \\ p_0 - \displaystyle\int \frac{p_1 dp_1}{F(p_1)} & = C_2 \end{cases} \tag{1}$$

Classical solutions are those of the obtained ones which can be projected to the x-axis without singularities.

For instance, let $F(p_1) = p_1^2$. Then system (1) has the form:

$$\begin{cases} x + \dfrac{1}{p_1} & = C_1, \\ p_0 - \ln p_1 & = C_2. \end{cases}$$

Solving this system, we obtain

$$p_1 = \frac{1}{C_1 - x}; \; p_0 = C_2 + \ln p_1 = C_2 - \ln |C_1 - x|.$$

Finally, every solution has the form

$$L = \left\{ \left(x; \; C_2 - ln|C_1 - x|; \; \frac{1}{C_1 - x}; \; \frac{1}{(C_1 - x)^2} \right) \right\} \subset J^2.$$

Thus, classical solutions of the equation $f'' = (f')^2$ have the form:

$$f(x) = C_2 - \ln |C_1 - x|.$$

§2. Superposition principle

Let $(\overline{G}, M)$ be a real homogeneous space, where the Lie group $\overline{G}$ acts effectively on the manifold M. Let $\overline{\mathfrak{g}}$ be the Lie algebra of $\overline{G}$ and $\mathcal{D}(M)$ the Lie algebra of all vector fields on M. Consider the homomorphism of Lie algebras

$$\rho : \overline{\mathfrak{g}} \to \mathcal{D}(M)$$

corresponding to the action of $\overline{G}$ on M. Since ρ is an injection, it is possible to identify $\overline{\mathfrak{g}}$ with a certain subalgebra of $\mathcal{D}(M)$.

An ordinary differential equation of the first order on the manifold M can be regarded as a smooth mapping $\lambda : \mathbb{R} \to \mathcal{D}(M)$. A solution of this equation is a smooth mapping $\varphi : \mathbb{R} \to M$ such that the tangent vector to the curve φ at any point $t \in \mathbb{R}$ is equal to $\lambda(t)|_{\varphi(t)}$.

Definition. An ordinary differential equation of the first order $\lambda : \mathbb{R} \to \mathcal{D}(M)$ is called *automorphic* if $\lambda(t) \in \overline{\mathfrak{g}}$ for all $t \in \mathbb{R}$.

Fix a basis $X_1, \dots, X_n$ of $\overline{\mathfrak{g}}$ ($n = \dim \overline{\mathfrak{g}}$). Then any automorphic equation is determined by smooth functions $a_i : \mathbb{R} \to \mathbb{R}, 1 \leqslant i \leqslant n$, such that

$$\lambda(t) = a_1(t)X_1 + \dots + a_n(t)X_n.$$

Theorem([6],[9]). *Let $g(t)$ be a curve in $\overline{G}$ that can be uniquely determined as a solution of the equation*

$$\dot{g}(t) = g(t)X(t)$$

with initial condition $g(0) = e$. Then any curve

$$\varphi(t) = g(t).\varphi(0), \quad \varphi(0) = m_0 \in M, \tag{1}$$

in the manifold M is a solution of the equation

$$\lambda : \mathbb{R} \to \mathcal{D}(M).$$

To prove this it is sufficient to represent $\varphi(t)$ as a composition of the following mappings

$$\mathbb{R} \xrightarrow{g} \overline{G} \xrightarrow{\pi} \overline{G} \times M \xrightarrow{\alpha} M,$$

where π is the embedding $x \mapsto (x, m_0)$, and α is the action of $\overline{G}$ on M.

Since any solution is uniquely defined by initial condition, we see that formula (1) gives the set of all solutions of the equation $\lambda : \mathbb{R} \to \mathcal{D}(M)$.

Let us explain how to find this mapping g. We shall construct the function of superposition F, which allows to construct the mapping g for an equation $\lambda : \mathbb{R} \to \mathcal{D}(M)$ starting from the number k (known beforehand) of particular solutions of the equation.

Now we introduce the following concept.

Definition. The *stiffness* of the homogeneous space $(\overline{G}, M)$ is the least natural number k for which there exist points $x_1, \dots, x_k$ of M such that the group $\bigcap_{i=1}^k \overline{G}_{x_i}$ is discrete.

Assume that the stiffness of $(\overline{G}, M)$ is equal to k and there exist points $x_1, \dots, x_k$ of M such that the group $\bigcap_{i=1}^k \overline{G}_{x_i}$ is trivial. Consider the action of $\overline{G}$ on the manifold $M \times M \times \dots \times M$($k$ times) defined by

$$g.(m_1, \dots, m_k) = (g.m_1, \dots, g.m_k),$$

where $g \in \overline{G}, m_i \in M, 1 \leqslant i \leqslant k$. Then the action of $\overline{G}$ on the orbit $O(x_1, \dots, x_k)$ of the point $(x_1, \dots, x_k)$ is simply transitive. Consider the *function of superposition*

$$F : O(x_1, \dots, x_k) \to \overline{G},$$

where $F(y_1, \dots, y_k)$ is the element of $\overline{G}$ such that

$$F(y_1, \dots, y_k).(x_1, \dots, x_k) = (y_1, \dots, y_k)$$

In the general case (when the group $\bigcap_{i=1}^{k} \overline{G}_{x_i}$ is discrete), it is also possible to define the function of superposition but, generally speaking, it will be many-valued.

Now suppose $\lambda : \mathbb{R} \to \mathcal{D}(M)$ is an arbitrary automorphic differential equation. Assume that $\varphi_1, \dots, \varphi_k$ are its particular solutions with initial conditions $\varphi_i(0) = x_i, 1 \leqslant i \leqslant k$. Then it is clear that $(\varphi_1(t), \dots, \varphi_k(t)) \in O(x_1, \dots, x_k)$ for $t \in \mathbb{R}$.

The mapping $g : \mathbb{R} \to \overline{G}$ is determined by $g(t) = F(\varphi_1(t), \dots, \varphi_k(t))$. Thus the knowledge of k particular solutions of the equation λ with definite initial conditions allows to write out the general solution:

$$\varphi(t) = F(\varphi_1(t), \dots, \varphi_k(t)).\varphi(0).$$

Example 1. Consider the homogeneous space $(\overline{G}, M)$, where $\overline{G} = \mathrm{GL}(2, \mathbb{R})$ and $M = \mathbb{R} \times S^1$:

$$\overline{G} = \mathrm{GL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc \neq 0 \right\},$$

$$M = \mathbb{R} \times S^1 \approx \mathbb{R}^2 \backslash \{0\}$$

Therefore, coordinates on M coincide with coordinates (x,y) on $\mathbb{R}^2$. The group $\overline{G}$ acts naturally on M.

Consider the following basis of the corresponding Lie algebra of vector fields on M:

$$X_1 = x\frac{\partial}{\partial x}, \quad X_2 = y\frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial y}, \quad X_4 = y\frac{\partial}{\partial y}.$$

The corresponding system of differential equations has the form:

$$\begin{cases} \dot{x} = a_1 x + a_2 y \\ \dot{y} = a_3 x + a_4 y \end{cases}$$

where a_i $(i = 1, \dots, 4)$ are arbitrary functions of t.

Let us now find the stiffness and the superposition function for the homogeneous space $(\overline{G}, M)$.

The stiffness k equals 2. Indeed, for the points $a_1 = (1, 0)$ and $a_2 = (0, 1)$, the intersection

$$\overline{G}_{a_1} \cap \overline{G}_{a_2}$$

is discrete, whereas the stabilizers of the points a_1, a_2 themselves are not discrete. The superposition function for the points $a_1 = (1, 0)$, $a_2 = (0, 1)$ has the form:

$$F\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

The function F is defined only for those pairs (x_1, y_1), (x_2, y_2) whose coordinates satisfy the condition

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \neq 0.$$

Now assume that $u_1(t) = (x_1(t), y_1(t))$ and $u_2(t) = (x_2(t), y_2(t))$ are two particular solutions with the following initial conditions

$$u_1(0) = (1, 0),$$
$$u_2(0) = (0, 1).$$

Then the general solution with initial condition $u(0) = (c_1, c_2)$ has the form:

$$u(t) = (x(t), y(t)) = (c_1 x_1(t) + c_2 x_2(t), c_1 y_1(t) + c_2 y_2(t)).$$

These results are in agreement with theory of differential equations.

Example 2. Consider the action of the group

$$\overline{G} = \{(A, s, f) | A \in \mathrm{SL}(2, \mathbb{R}), s \in \mathbb{R}_+^*, f \in \mathbb{R}_n[x, y]\}$$

(where by $\mathbb{R}_n[x, y]$ we denote the set of all *homogeneous polynomials* in two variables x and y) on the manifold

$$M = \{(x, y, z) |\, x^2 + y^2 \neq 0\} / \sim,$$

where

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda^n z),$$

and $\lambda \in \mathbb{R}^*$, and $n \geqslant 0$, $n \in \mathbb{N}$.

The action of $\overline{G}$ on M is given by:

$$(A, s, f).(x, y, z) = ((x, y)A^t, sz + f((x, y)A^t)).$$

Let us consider the local chart on M which contains all points (x, y, z) such that $y \neq 0$. Since a point (x, y, z) is equivalent to one and only one point $(\frac{x}{y}, 1, \frac{z}{y^n})$ such that the second coordinate is equal to 1, the local coordinates (p, q) on M have the form:

$$p = \frac{x}{y}, \qquad q = \frac{z}{y^n}.$$

Consider the following basis of the corresponding Lie algebra of vector fields on M:

$$X_1 = \frac{\partial}{\partial p}, \quad X_2 = p\frac{\partial}{\partial p}, \quad X_3 = p^2\frac{\partial}{\partial p} + npq\frac{\partial}{\partial q}, \quad X_4 = q\frac{\partial}{\partial q}, \quad X_{i+5} = p^i\frac{\partial}{\partial q}, \quad 0 \leqslant i \leqslant n.$$

The corresponding differential equation has the form:

$$\begin{cases} \dot{x} = a_1 + a_2 x + a_3 x^2, \\ \dot{y} = a_3 nxy + a_4 y + \sum_{i=0}^{n} a_{i+5} x^i, \end{cases}$$

where a_i $(1 \leqslant i \leqslant n+5)$ are arbitrary functions of t.

Here the stiffness is:

a) $n = 0$ or $n = 1$: $k = 3$. The points such that the group $\cap_{i=1}^{k} \overline{G}_{a_i}$ is discrete are: $a_1 = (0,1,0)$, $a_2 = (1,1,0)$, $a_3 = (2,1,1)$;

b) $n > 1$: $k = n+2$. The points such that the group $\cap_{i=1}^{k} \overline{G}_{a_i}$ is discrete are: $a_1 = (0,1,0)$, $a_2 = (1,1,0)$, $\dots$, $a_{n+1} = (n,1,0)$, $a_{n+2} = (0,1,1)$.

Proof. It is known that the action of $\mathrm{SL}(2,\mathbb{R})$ on the triples of projectively independent points of $\mathbb{R}\mathrm{P}^1$ is simply transitive. Then the stiffness can not be less than 3.

Assume that $n > 1$ and k is the stiffness of the homogeneous space $(\overline{G}, M)$. Let $a_1 = (x_1, y_1, z_1), \dots, a_k = (x_k, y_k, z_k)$ be points of M such that the group $\cap_{i=1}^{k} \overline{G}_{a_i}$ is discrete. For each point a_i its stabilizer $\overline{G}_{a_i}$ is the set of all elements of $\overline{G}$ satisfying the following equations:

$$\begin{cases} (x_i, y_i) \cdot A^t = (\lambda_i x_i, \lambda_i y_i), \\ f((x_i, y_i)A^t) = z_i - \frac{s z_i}{\lambda_i^n}, \lambda_i \in \mathbb{R}^*, \end{cases}$$

where $f(x,y) = \sum_{i=0}^{n} \alpha_i x^i y^{n-i}$. So, we have got $n+2$ variables: $\alpha_i, 0 \leqslant i \leqslant n$ and k. Consequently, the stiffness can not be less than $n+2$ if $n > 1$.

The function of superposition has the form:

$$F((x_1, y_1, z_1), \dots, (x_k, y_k, z_k)) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, s, f \right) \in \overline{G},$$

where

$$a = \mu x_2 - \lambda x_1, \qquad b = \lambda x_1,$$
$$c = \mu y_2 - \lambda y_1, \qquad d = \lambda y_1,$$

$$\lambda = \pm \sqrt{\frac{-2\Delta_{23}}{\Delta_{12}\Delta_{13}}}, \qquad \mu = \pm \sqrt{\frac{-\Delta_{13}}{2\Delta_{12}\Delta_{23}}},$$

$$\Delta_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix},$$

and

$$f(x,y) = \sum_{i=0}^{n} \alpha_i x^i y^{n-i},$$

where $\alpha_i,\ 0 \leqslant i \leqslant n$, can be calculated from:

$$L_n(x) = \sum_{k=1}^{n+1} \frac{\prod_{i\neq k}\left(x - \frac{x_i}{y_i}\right)}{\prod_{i\neq k}\left(\frac{x_k}{y_k} - \frac{x_i}{y_i}\right)} \cdot \frac{z_k}{y_k^n} = \sum_{i=0}^{n} \alpha_i x^i.$$

For the number s we have:
a) $n > 1$

$$s = d^n \left(\frac{z_{n+2}}{y_{n+2}^n} - L_n\left(\frac{x_{n+2}}{y_{n+2}}\right)\right);$$

b) $n = 0, 1$

$$s = (2c+d)^n \left(\frac{z_3}{y_3^n} - L_n\left(\frac{x_3}{y_3}\right)\right).$$

So, if we know k particular solutions of this differential equation with the following initial conditions
a) $n = 0, 1$: $k = 3$

$$\begin{aligned} u_1(0) &= (0,0), \\ u_2(0) &= (1,0), \\ u_3(0) &= (2,1); \end{aligned}$$

b) $n > 1$: $k = n+2$

$$\begin{aligned} u_1(0) &= (0,0), \\ u_2(0) &= (1,0), \\ &\dots \\ u_{n+1}(0) &= (n,0), \\ u_{n+2}(0) &= (0,1); \end{aligned}$$

we can write out the general solution with initial condition $u(0) = (c_1, c_2)$:

$$u(t) = (x(t), y(t)) = \left(\frac{a(t)c_1 + b(t)}{c(t)c_1 + d(t)}, \frac{s(t)c_2 + f(a(t)c_1 + b(t), c(t)c_1 + d(t))}{(c(t)c_1 + d(t))^n} \right),$$

where

$$\left(\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, s(t), f \right) = \\ F((x_1(t), 1, y_1(t)),\ (x_2(t), 1, y_2(t)),\ \dots,\ (x_{n+2}(t), 1, y_{n+2}(t))).$$

APPENDIX A

EXPONENTIAL MAPPING

Let us consider the complex-number series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots. \tag{1}$$

Let us find its radius of convergence. It is possible to do it using the Cauchy radical test:

$$R = 1/l \text{ , where } l = \overline{\lim_{n\to\infty}} \sqrt[n]{|1/n!|}.$$

It is possible to show that

$$\lim_{n\to\infty} \sqrt[n]{|1/n!|} = \lim_{n\to\infty} (n!)^{-1/n} = 0.$$

Hence, series (1) converges at each $z \in \mathbb{C}$.

Remark. It is possible to show this directly: for each $z \in \mathbb{C}$ we can find a natural number N such that for any $n \geqslant N$ $(n \in N)$ the following condition holds:

$$|z/n| \leqslant 1/2.$$

It follows that

$$|z^n/n!| \leqslant |z^N/N!| \cdot |1/2|^{n-N}$$

and therefore series (1) is absolutely convergent for all $z \in \mathbb{C}$. By e^z denote the sum of series (1) at a point $z \in \mathbb{C}$. The following fact is proved in theory of complex functions.

Theorem (for term-by-term differentiation). *Let the radius of convergence of a series*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{2}$$

be equal to $R \geqslant 0$. Then in the domain $|z| < R$, this series is term-by-term differentiable and the series obtained from series (2) by differentiation have the same radius of convergence as series (2).

Theorem 1 (on properties of the function e^z).

1. $f(z) = e^z$ is infinitely continuously differentiable. Besides, $(e^z)' = e^z$.

2. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ for all $z_1,\ z_2 \in \mathbb{R}$ and $e^0 = 1$.

In other words, the map $z \mapsto e^z$ is a homomorphism of the additive group $\mathbb{C}$ into the multiplicative group $\mathbb{C}^*$.

3. $e^z \neq 0$ for all $z \in \mathbb{C}$.

Proof.

1. Since the radius of convergence of series (1) is infinite, we see that the function e^z is infinitely differentiable and its derivative can be found from the equality

$$(e^z)' = \sum_{n=0}^{\infty} \left(\frac{z^n}{n!}\right)' = \sum_{n=1}^{\infty} \left(\frac{z^{n-1}}{(n-1)!}\right)' = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

2. Since series (1) is absolutely convergent at any $z \in \mathbb{C}$, we see that the number $e^{z_1} \cdot e^{z_2}$ can be found as the product of series:

$$e^{z_1} \cdot e^{z_2} = \left(\sum_{n=0}^{\infty} \frac{(z_1)^n}{n!}\right) \cdot \left(\sum_{m=0}^{\infty} \frac{(z_2)^m}{m!}\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{{z_1}^i}{i!} \cdot \frac{{z_2}^{n-i}}{(n-i)!}\right) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} \cdot {z_1}^i {z_2}^{n-i}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{i=0}^{n} C_n^i z_1^i z_2^{n-i}\right) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \cdot (z_1 + z_2)^n = \sum_{n=0}^{\infty} \frac{(z_1 + z_2)^n}{n!} = e^{z_1+z_2}.$$

3. It immediately follows from 2.

For the further study of the function e^z we have to pass to functions of a real variable. Let $z = x + iy$, where $x,\ y \in \mathbb{R}$. Then $e^z = e^x \cdot e^{iy}$. We shall study the functions e^x and e^{ix} separately. Consider the restriction of e^z to real numbers. Since all coefficients of series (1) are real, we obtain the smooth real function e^x.

Theorem 2. *The function e^x is*

1° *an isomorphism of the groups $\mathbb{R}$ and $\mathbb{R}^*_+$;*

2° *a monotonically increasing function and* $\lim\limits_{x\to-\infty} e^x = 0$, $\lim\limits_{x\to\infty} e^x = \infty$.

Proof. The equalities $e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$ and $e^0 = 1$ follow from theorem 1. Let us show that $e^x \geqslant 0$ for all $x \in \mathbb{R}$. Indeed, let $x \geqslant 0$. Since all terms of series (1) are positive, we see that $x \geqslant 0$. If $x \leqslant 0$, then we have $e^x = \frac{1}{e^{-x}} \geqslant 0$. Moreover, if $x \geqslant 0$ then $e^x \geqslant 1$ and if $x \leqslant 0$ then $0 \leqslant e^x \leqslant 1$. Let $x,\ y$ be real numbers such that $x \geqslant y$. Since $e^x = e^y \cdot e^{x-y} \geqslant e^y$, we see that e^x is monotonic. It is obvious that for any $x \geqslant 0$ the following condition holds: $e^x \geqslant 1 + x$. Hence, $\lim\limits_{x\to\infty} e^x = \infty$ and

$\lim\limits_{x\to-\infty} e^x = \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$. This yields the result. Now consider the function e^{ix}, where $x \in \mathbb{R}$. We have:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!} =$$
$$\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{(2k+1)!}.$$

Since the series $\sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$ is convergent, we see that its real and imaginary parts are also convergent. By definition put

$$\cos x = \Re e^{ix},$$
$$\sin x = \Im e^{ix}.$$

Since the function e^{ix} is smooth, we see that the functions $\cos x$ and $\sin x$ are also smooth and can be written as

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Theorem 3. *For each $x \in \mathbb{R}$ the following condition holds*

$$\sin^2 + \cos^2 = 1.$$

Proof. Indeed, $e^{ix} = \cos x + i \sin x$. Then $\cos x - i \sin x = \overline{e^{ix}} = e^{\overline{ix}}$. Therefore, $\sin^2 + \cos^2 = (\cos x + i \sin x)(\cos x - i \sin x) = e^{ix} \cdot e^{-ix} = 1$.

Corollary. *The functions $\sin x$ and $\cos x$ are bounded on $\mathbb{R}$. Moreover, $|\sin x| \leqslant 1$ and $|\cos x| \leqslant 1$ for all $x \in \mathbb{R}$.*

So, the function e^{ix} takes real numbers into the circle $|z| = 1$. We shall see it later that e^{ix} is surjective.

Theorem 4.

1° $(\cos x)' = \sin x$, $(\sin x)' = \cos x$;

2° $\cos 0 = 1$, $\sin 0 = 0$;

3° $\sin(x + y) = \sin x \cos y + \sin y \cos x$,

$\cos(x + y) = \cos x \cos y - \sin x \sin y$;

4° $\sin x$ *is an odd function,* $\cos x$ *is an even one.*

Proof. The desired formulas immediately follow from:

1°
$$(\sin x)' + i(\cos x)' = (e^{ix})' = ie^{ix} = i(\cos x + i \sin x) = (-\sin x) + i \cos x;$$

2° $e^{i\cdot 0} = 1$;

3° $e^{i(x+y)} = e^{ix}e^{iy}$;

4° $\overline{e^{ix}} = e^{-ix}$.

Corollary. *There exists an* $\epsilon > 0$ *such that the function* $\sin x$ *is positive on the interval* $(0, \epsilon)$.

Proof. Since $\cos 0 = 1$, we see that there exists an $\epsilon > 0$ such that $\cos x > 0$ on the interval $(-\epsilon, \epsilon)$. Hence, $\sin x$ strictly increases on $(-\epsilon, \epsilon)$. Using the condition $\sin 0 = 0$, we obtain $\sin x > 0$ for $x \in (0, \epsilon)$.

Theorem 5. $\sin 4 < 0$.

Proof. We have

$$\sin 4 = 4 - \frac{4^3}{3!} + \frac{4^5}{5!} - \frac{4^7}{7!} + \frac{4^9}{9!} - \sum_{k=3}^{\infty} \left(\frac{4^{4k-1}}{(4k-1)!} - \frac{4^{4k+1}}{(4k+1)!} \right).$$

But

$$4 - \frac{4^3}{3!} + \frac{4^5}{5!} - \frac{4^7}{7!} + \frac{4^9}{9!} =$$
$$= \frac{1451520 - 3870720 + 3096576 - 1179648 + 262144}{9!} = \frac{-240128}{9!} < 0,$$

and

$$\left(\frac{4^{4k-1}}{(4k-1)!} - \frac{4^{4k+1}}{(4k+1)!} \right) = \frac{4^{4k-1}}{(4k-1)!} \left(1 - \frac{16}{(4k+1)4k} \right) > 0$$

for $k \geqslant 3$. Hence, $\sin 4 < 0$.

Corollary. *The set of positive solutions of the equation* $\sin x = 0$ *is not empty.*

Proof. Indeed, since $\sin 4 < 0$ and there exists $\epsilon > 0$ such that $\sin x > 0$ for all $x \in (0, \epsilon)$, we see that there exists a number a such that $\sin a = 0$.

By R denote the set of all positive solutions of the equation $\sin x = 0$. Since $R = \{x | \sin x = 0\} \cap [\epsilon, +\infty)$ for some $\epsilon > 0$, we see that R is closed. Therefore, R has the smallest element. We denote this element by π.

Theorem 6.

1. *The table of the values of* $\sin x$ *and* $\cos x$ *at the points* $\frac{\pi}{2}$; π; $\frac{3\pi}{2}$; 2π *has the form:*

x	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	*1*	*0*	*-1*	*0*
$\cos x$	*0*	*-1*	*0*	*1*

.

2. *The functions* $\sin x$ *and* $\cos x$ *are periodic with period* $T = 2\pi$.
3. *The set of all solutions of the equation* $\sin x = 0$ *has the form* $R = \{\pi k \mid k \in \mathbb{Z}\}$.
4. *The mapping* $x \mapsto e^{ix}$ *is a bijection of the half-open interval* $[0, 2\pi)$ *onto the circle* $\{z \in \mathbb{C} \mid |z| = 1\}$.

Proof. Let us calculate $\cos \pi$. If $\sin \pi = 0$, then $\cos \pi = \pm 1$. But $\sin x > 0$ for all $x \in (0, \pi)$ and therefore $\cos x$ is a strictly decreasing function on $(0, \pi)$, $\cos \pi = -1$.

Furthermore, we have $0 = \sin\pi = \sin(\frac{\pi}{2} + \frac{\pi}{2}) = 2\sin\frac{\pi}{2}\cdot\cos\frac{\pi}{2}$. But $\sin\frac{\pi}{2} > 0$. Thus, $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$. Similarly, we can find the values of $\sin x$ and $\cos x$ at the other points.

2. Note that

$$\sin(x+\pi) = \sin x\cos\pi + \sin\pi\cos x = -\sin x,$$
$$\cos(x+\pi) = \cos x\cos\pi - \sin x\sin\pi = -\cos x.$$

It follows that $\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$, i.e. $\sin x$ and $\cos x$ are periodic with period $T = 2\pi$.

3. It is clear that $\sin\pi k = 0$ for all $k \in \mathbb{Z}$. Moreover, if $\sin\alpha = 0$ then $\sin(\alpha+\pi k) = 0$ for all $k \in \mathbb{Z}$. Since π is the smallest positive solution of the equation $\sin x = 0$, we see that each solution x_0 of $\sin x = 0$ has the form $x_0 = \pi k$, where k is some integer.

4. Since $\cos x$ monotonically decreases from 1 to -1 on $[0,\pi]$ and $\sin x > 0$ on $(0,\pi)$, we see that the mapping $x \mapsto e^{ix}$ is a bijection of the segment $[0,\pi]$ onto the semicircle $\{z \in \mathbb{C} \mid |z| = 1,\ \Im z \geqslant 0\}$. This and the formulas $\sin(x+\pi) = -\sin x$ and $\cos(x+\pi) = -\cos\pi$ yield the result.

Corollary. The function e^z is periodic with period $T = 2\pi i$ and is a surjection onto $\mathbb{C}^*$.

In the sequel, as a rule, we shall speak about the exponential of a real argument: $x \mapsto e^x$, $x \in \mathbb{R}$.

Theorem 7. *A solution of the differential equation $f' = f$, where f is real-valued function, with initial condition $f(0) = 1$ is unique and has the form:*

$$f(x) = e^x.$$

Proof. From theory of differential equations it follows that a solution exists and is unique. It remains to note that the function $f(x) = e^x$ is a solution of our differential equation.

Corollary. *The set of all solution of the differential equation $f' = f$ has the form $\{f(x) \mid f(x) = Ce^x,\ C \in \mathbb{R}\}$.*

Theorem 8. *Any smooth homomorphism of the group $\mathbb{R}$ into the group $\mathbb{R}^*_+$ has the form:*

$$x \mapsto e^{\lambda x},\ \lambda \in \mathbb{R}.$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}^*_+$ be a smooth homomorphism of groups. Then $f(0) = 1$ and $f(x+y) = f(x)f(y)$. Put $\lambda = f'(0)$. We have:

$$f'(x) = \lim_{y\to 0}\frac{f(x+y) - f(x)}{y} = f(x)\cdot\lim_{y\to 0}\frac{f(y) - f(0)}{y} = f'(0)f(x) = \lambda f(x).$$

In the case of $\lambda = 0$ we get $f(x) = 1 = e^{0\cdot x}$ for all $x \in \mathbb{R}$. Suppose $\lambda \neq 0$. Put $g(x) = f(\frac{x}{\lambda})$. Then $g(0) = f(0) = 1$ and $g'(x) = \frac{1}{\lambda}f'(\frac{x}{\lambda}) = g(x)$. From theorem 7 it follows that $g(x) = e^x$ and $f(x) = e^{\lambda x}$.

Remark. It is possible to show that the theorem is true when f is a continuous homomorphism.

APPENDIX B

ORDINARY DIFFERENTIAL EQUATIONS

In this appendix we shall give basic definitions and formulate some basic results of theory of ordinary differential equations.

Definition 1. *An ordinary differential equation of the first order with respect to a vector-valued function* $f: \mathbb{R} \to \mathbb{R}^n$ is an equation of the form:

$$f' = F(f, t), \tag{1}$$

where $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is some fixed function.

The basic result of theory of first-order differential equations is as follows:

Theorem 1. *Let* $a \in \mathbb{R}$ *and* $f_0 \in \mathbb{R}^n$ *be some fixed point on the line and vector in* $\mathbb{R}^n$ *respectively. If* F *is a differentiable function, then a solution of equation* (1) *with initial condition* $f(a) = f_0$ *exists and is uniquely determined in a certain neighborhood of the point* a.

If the solution of equation (1) can be determined on the whole line for any initial conditions, then equation (1) is called *globally solvable.*

Examples.

1) The ordinary differential equation $f' = f$ with respect to a real-valued function f is globally solvable. Indeed, for all $a \in \mathbb{R}$, $C \in \mathbb{R}$ the function $f(t) = Ce^{x-a}$ is the solution of this equation with initial condition $f(a) = C$.

2) Let us consider the ordinary differential equation

$$f' = f^2.$$

Its solution with initial condition $f(0) = C$ has the form:

$$f(t) = \frac{C}{1 - tC}.$$

It is easy to see that for $C \neq 0$ this solution is not completely defined. So, the equation is not globally solvable.

Definition 2. *An ordinary differential equation of the n-th order with respect to a real-valued function* f is an equation of the form

$$f^{(n)} = F(f^{(n-1)}, \dots, f', f, t), \tag{2}$$

where $F: \mathbb{R}^{n+1} \to \mathbb{R}$ is some function.

Any differential equation (2) can be reduced to equation (1) by introducing additional variables $f_i = f^{(i)}$, $1 \leqslant i \leqslant n-1$. Define the function $g\colon \mathbb{R} \to \mathbb{R}^n$ as

$$g(t) = (\, f(t),\, f_1(t),\, \dots ,\, f_{n-1}(t)\,).$$

Then the function g satisfies the ordinary differential equation

$$g'(t) = G(g,\, t),$$

where

$$G\big(\, x_0,\, x_1,\, \dots ,\, x_{n-1},\, t\big) = \big(\, x_1,\, \dots ,\, x_{n-1},\, F(\, x_0,\, x_1,\, \dots,\, x_{n-1},\, t\,)\big).$$

Consequently, equation (2) with initial conditions

$$f(a) = C_0,\;\; f'(a) = C_1,\;\; \dots\;\; ,\, f^{(n-1)} = C_{n-1}$$

for some fixed a, $C_0, \dots , C_{n-1} \in \mathbb{R}$ is also uniquely solvable in a certain neighborhood of the point a.

In the sequel we shall be particularly interesting in so-called *homogeneous linear ordinary differential equations with constant coefficients.* They are n-th order ordinary differential equations of the form

$$f^{(n)} = F(f^{(n-1)}, \dots , f', f).$$

where $F\colon \mathbb{R}^n \to \mathbb{R}$ is some fixed linear mapping. They are usually written as

$$f^{(n)} + a_{n-1} f^{(n-1)} + \dots + a_1 f' + a_0 f = 0, \tag{3}$$

where a_0, a_1, $\dots$, a_{n-1} are some constants. To every equation of this form it is possible to assign the polynomial

$$p(x) = x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

which is called *the characteristic polynomial* of equation (3).

Suppose $\mathcal{F}$ is the set of all solutions of equation (3).

Theorem 2.

1°. The set $\mathcal{F}$ is an n-dimensional vector space.

2°. Assume that

$$p(x) = (x-\alpha_1)^{r_1} \cdot \dots \cdot (x-\alpha_m)^{r_m} \cdot ((x-\beta_1)^2 + \gamma_1^2)^{s_1} \cdot \dots \cdot ((x-\beta_l)^2 + \gamma_l^2)^{s_l}$$

is the prime decomposition of the characteristic polynomial of equation (3)*. Then the functions*

$$x^i e^{\alpha_j x}, \qquad 0 \leqslant i \leqslant r_j - 1,\; 1 \leqslant j \leqslant m;$$

$$x^i \sin \gamma_j x \cdot e^{\beta_j x}, \quad x^i \cos \gamma_j x \cdot e^{\beta_j x}, \qquad 0 \leqslant i \leqslant s_j - 1,\; 1 \leqslant j \leqslant l,$$

form a basis of the space $\mathcal{F}$.

3°. The space $\mathcal{F}$ is invariant under linear operators $L_a\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $a \in \mathbb{R}$, where

$$\big(L_a f\big)(t) = f(t-a)$$

for all $f \in C^\infty(\mathbb{R})$.

As a matter of fact, the latest property is defining for the solution space of equations (3).

Theorem 3. *Let $\mathcal{F}$ be a finite-dimensional subspace of the vector space $C^\infty(\mathbb{R})$ invariant under all linear operators of the form L_a, $a \in \mathbb{R}$. Then $\mathcal{F}$ is exactly the solution space of some equation of form* (3).

APPENDIX C

LISTS

List 1. Transitive finite-dimensional Lie algebras of vector fields on the plane

1.1.

$$\langle\frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}\rangle.$$

2.1.

$$\langle\frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; x_1\frac{\partial}{\partial x_1} + \lambda x_2\frac{\partial}{\partial x_2}\rangle, \quad |\lambda| \leqslant 1.$$

2.2.

$$\langle\frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; (\lambda x_1 - x_2)\frac{\partial}{\partial x_1} + (x_1 + \lambda x_2)\frac{\partial}{\partial x_2}\rangle, \quad \lambda \geqslant 0.$$

3.1.

$$\langle\frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; x_1\frac{\partial}{\partial x_1}; x_2\frac{\partial}{\partial x_2}\rangle.$$

3.2.

$$\langle\frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}; -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\rangle.$$

4.1.

$$\langle\frac{\partial}{\partial x_1}; \varphi_1(x_1)\frac{\partial}{\partial x_2}; \dots; \varphi_n(x_1)\frac{\partial}{\partial x_2}\rangle,$$

where functions $\varphi_1, \dots, \varphi_n$ form a basis of solutions of some homogeneous linear differential equation

$$f^{(n)} + c_{n-1}f^{(n-1)} + \dots + c_0 f = 0$$

with constant coefficients.

5.1.

$$\langle\frac{\partial}{\partial x_1}; x_2\frac{\partial}{\partial x_2}; \varphi_1(x_1)\frac{\partial}{\partial x_2}; \dots; \varphi_n(x_1)\frac{\partial}{\partial x_2}\rangle,$$

where functions $\varphi_1, \dots, \varphi_n$ are the same as in 4.1.

6.1.

$$\langle\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1}+\lambda x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-1}\frac{\partial}{\partial x_2}\rangle,\qquad \lambda\neq n-1.$$

7.1.

$$\langle\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1}+(n-1)\frac{\partial}{\partial x_2}+\frac{x_1^{n-1}}{(n-1)!}\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-2}\frac{\partial}{\partial x_2}\rangle.$$

8.1.

$$\langle\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-1}\frac{\partial}{\partial x_2}\rangle.$$

9.1.

$$\langle\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1}+(n-1)x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\dots;x_1^{n-1}\frac{\partial}{\partial x_2}\rangle.$$

10.1.

$$\langle\frac{\partial}{\partial x_1};\ 2x_1\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}\rangle.$$

11.1.

$$\langle\frac{\partial}{\partial x_1}+x_2^2\frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\rangle.$$

11.2.

$$\langle x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1};\ (1-x_1^2+x_2^2)\frac{\partial}{\partial x_1}-2x_1x_2\frac{\partial}{\partial x_2};\ -2x_1x_2\frac{\partial}{\partial x_1}+(1+x_1^2-x_2^2)\frac{\partial}{\partial x_2}\rangle.$$

11.3.

$$\langle x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1};\ (1+x_1^2-x_2^2)\frac{\partial}{\partial x_1}+2x_1x_2\frac{\partial}{\partial x_2};\ 2x_1x_2\frac{\partial}{\partial x_1}+(1-x_1^2+x_2^2)\frac{\partial}{\partial x_2}\rangle.$$

12.1.

$$\langle\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1};\ x_1^2\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_2};\ x_2^2\frac{\partial}{\partial x_2}\rangle.$$

12.2.

$$\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2};\ -x_2\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial x_2};$$
$$(x_1^2-x_2^2)\frac{\partial}{\partial x_1}+2x_1x_2\frac{\partial}{\partial x_2};\ -2x_1x_2\frac{\partial}{\partial x_1}+(x_1^2-x_2^2)\frac{\partial}{\partial x_2}\rangle.$$

13.1.

$$\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1}+x_1x_2\frac{\partial}{\partial x_2};\ x_1x_2\frac{\partial}{\partial x_1}+x_2^2\frac{\partial}{\partial x_2}\rangle.$$

14.1.

$$\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_1}\rangle.$$

15.1.

$$\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\ x_2\frac{\partial}{\partial x_1}\rangle.$$

16.1.

$$\langle\frac{\partial}{\partial x_1};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_1};\ x_1^2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}\rangle.$$

17.1.

$$\langle\frac{\partial}{\partial x_1};\ 2x_1\frac{\partial}{\partial x_1}+nx_2\frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1}+nx_1x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\ldots;x_1^n\frac{\partial}{\partial x_2}\rangle.$$

18.1.

$$\langle\frac{\partial}{\partial x_1};\ x_1\frac{\partial}{\partial x_1};\ x_2\frac{\partial}{\partial x_2};\ x_1^2\frac{\partial}{\partial x_1}+nx_1x_2\frac{\partial}{\partial x_2};\ \frac{\partial}{\partial x_2};\ x_1\frac{\partial}{\partial x_2};\ldots;x_1^n\frac{\partial}{\partial x_2}\rangle.$$

List 2. TRANSITIVE LIE TRANSFORMATION GROUPS OF THE PLANE

1.1. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto (x_1 + a_1, x_2 + a_2) \mid a_1, a_2 \in \mathbb{R} \,\}.$$

2.1. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto (ax_1 + b_1, a^\lambda x_2 + b_2) \mid a \in \mathbb{R}^*_+,\ b_1, b_2 \in \mathbb{R} \,\}.$$

2.2. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto \big(\mathrm{e}^{\lambda a}(x_1 \cos a + x_2 \sin a) + b_1,\\ \mathrm{e}^{\lambda a}(-x_1 \sin a + x_2 \cos a) + b_2\big) \mid a, b_1, b_2 \in \mathbb{R} \,\}.$$

3.1. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto (a_1 x_1 + b_1, a_2 x_2 + b_2) \mid a_1, a_2 \in \mathbb{R}^*_+,\ b_1, b_2 \in \mathbb{R} \,\}.$$

3.2. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto \big(a(x_1 \cos b + x_2 \sin b) + c_1,\\ a(-x_1 \sin b + x_2 \cos b) + c_2\big) \mid a \in \mathbb{R}^*_+,\ b, c_1, c_2 \in \mathbb{R} \,\}.$$

4.1. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto \big(x_1 + a, x_2 + f(x_1 + a)\big) \mid a \in \mathbb{R},\ f \in \mathcal{F} \,\},$$

where $\mathcal{F}$ is the solution space of the differential equation

$$f^{(n)} + c_{n-1} f^{(n-1)} + \cdots + c_1 f' + c_0 = 0, \qquad c_0, c_1, c_{n-1} \in \mathbb{R}.$$

5.1. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto \big(x_1 + a, bx_2 + f(x_1 + a)\big) \mid a \in \mathbb{R},\ b \in \mathbb{R}^*_+,\ f \in \mathcal{F} \,\},$$

where space $\mathcal{F}$ is the same as in 4.1.

6.1. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto \big(ax_1 + b, a^\lambda x_2 + f(ax_1 + b)\big) \mid a \in \mathbb{R}^*_+,\ b \in \mathbb{R},\ f \in \mathbb{R}_{n-1}[x] \,\},\\ \lambda \neq n - 1.$$

7.1. $S = \mathbb{R}^2$.

$$G = \{\, (x_1, x_2) \longmapsto \big(ax_1 + b, a^{n-1} x_2 + (ax_1 + b)^{n-1} + f(ax_1 + b)\big) \mid\\ a \in \mathbb{R}^*_+,\ b \in \mathbb{R},\ f \in \mathbb{R}_{n-2}[x] \,\}.$$

8.1. $S=\mathbb{R}^2$.

$$G=\{\,(x_1,x_2)\longmapsto \big(ax_1+b, cx_2+f(ax_1+b)\big)\mid a,c\in\mathbb{R}_+^*,\ b\in\mathbb{R},\ f\in\mathbb{R}_{n-1}[x]\,\}.$$

9.1. $S=\mathbb{R}^2$.

$$G=\{\,(x_1,x_2)\longmapsto \big(ax_1+b, a^{n-1}x_2+f(ax_1+b)\big)\mid a\in\mathbb{R}_+^*,\ b\in\mathbb{R},\ f\in\mathbb{R}_{n-1}[x]\,\}.$$

10.1. $S=\mathbb{R}^2\backslash\{0\}$.

$$G=\left\{\,(y_1,y_2)\longmapsto(a_{11}y_1+a_{12}y_2,a_{21}y_1+a_{22}y_2)\,\middle|\,\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\in\mathrm{SL}(2,\mathbb{R})\,\right\}.$$

Parametrization $\pi:(x_1,x_2)\mapsto(x_1\,\mathrm{e}^{x_2},\mathrm{e}^{x_2})$:

$$G=\left\{\,(x_1,x_2)\longmapsto\left(\frac{a_{11}x_1+a_{12}x_2}{a_{21}x_1+a_{22}x_2},\ x_2+\ln(a_{21}x_1+a_{22})\right)\,\right\}.$$

11.1. $S=\mathbb{R}P^1\times\mathbb{R}P^1$.

$$G=\Big\{\,(y_1:y_2,z_1:z_2)\longmapsto\big((a_{11}y_1+a_{12}y_2):(a_{21}y_1+a_{22}y_2),(a_{22}z_1-a_{21}z_2):\\ (-a_{12}z_1+a_{11}z_2)\big)\,\Big|\,\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\in\mathrm{SL}(2,\mathbb{R})\,\Big\}.$$

Parametrization $\pi:(x_1,x_2)\mapsto(x_1:1,x_2:1)$:

$$G=\left\{\,(x_1,x_2)\longmapsto\left(\frac{a_{11}x_1+a_{12}}{a_{21}x_1+a_{22}},\ \frac{a_{22}x_2-a_{21}}{-a_{12}x_1+a_{11}}\right)\,\right\}.$$

11.2. $S=\{\,(y_1,y_2,y_3)\mid y_1^2+y_2^2=y_3^2-1,\ y_3>0\,\}$.

$$G=\Big\{\,(y_1,y_2,y_3)\longmapsto(a_{11}y_1+a_{12}y_2+a_{13}y_3,\ a_{21}y_1+a_{22}y_2+a_{23}y_3,\\ a_{31}y_1+a_{32}y_2+a_{33}y_3)\,\Big|\,\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\in\mathrm{SO}(2,1)\,\Big\}.$$

Parametrization $\pi:(x_1,x_2)\mapsto\left(\frac{x_1}{\sqrt{1-x_1^2-x_2^2}};\ \frac{x_2}{\sqrt{1-x_1^2-x_2^2}};\ \frac{1}{\sqrt{1-x_1^2-x_2^2}}\right)$:

$$G=\left\{\,(x_1,x_2)\longmapsto\left(\frac{a_{11}x_1+a_{12}x_2+a_{13}}{a_{31}x_1+a_{32}x_2+a_{33}};\ \frac{a_{21}x_1+a_{22}x_2+a_{23}}{a_{31}x_1+a_{32}x_2+a_{33}}\right)\,\right\}.$$

11.3. $S=S^2=\{\,(y_1,y_2,y_3)\mid y_1^2+y_2^2+y_3^2=1\,\}$.

$$G=\Big\{\,(y_1,y_2,y_3)\longmapsto(a_{11}y_1+a_{12}y_2+a_{13}y_3,\ a_{21}y_1+a_{22}y_2+a_{23}y_3,\\ a_{31}y_1+a_{32}y_2+a_{33}y_3)\,\Big|\,\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\in\mathrm{SO}(3)\,\Big\}.$$

Parametrization $\pi : (x_1, x_2) \mapsto \left(\frac{x_1}{\sqrt{1+x_1^2+x_2^2}}; \frac{x_2}{\sqrt{1+x_1^2+x_2^2}}; \frac{1}{\sqrt{1+x_1^2+x_2^2}}\right)$:

$$G = \left\{ (x_1, x_2) \longmapsto \left(\frac{a_{11}x_1 + a_{12}x_2 + a_{13}}{a_{31}x_1 + a_{32}x_2 + a_{33}}, \frac{a_{21}x_1 + a_{22}x_2 + a_{23}}{a_{31}x_1 + a_{32}x_2 + a_{33}} \right) \right\}.$$

12.1. $S = \mathbb{R}P^1 \times \mathbb{R}P^1$.

$$G = \Big\{ (y_1 : y_2, z_1 : z_2) \longmapsto \big((a_{11}y_1 + a_{12}y_2) : (a_{21}y_1 + a_{22}y_2), (b_{11}z_1 + b_{12}z_2) : (b_{21}z_1 + b_{22}z_2)\big) \Big| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \Big\}.$$

Parametrization $\pi : (x_1, x_2) \mapsto (x_1 : 1, x_2 : 1)$:

$$G = \left\{ (x_1, x_2) \longmapsto \left(\frac{a_{11}x_1 + a_{12}}{a_{21}x_1 + a_{22}}, \frac{b_{11}x_1 + b_{12}x_2}{b_{21}x_1 + b_{22}x_2} \right) \right\}.$$

12.2. $S = \mathbb{C}P^1 = \left(\mathbb{C}^2 \backslash \{0\}\right) / \sim$, where

$$(z_1, z_2) \sim (\lambda z_1, \lambda z_2), \qquad \lambda \in \mathbb{C}^*.$$

$$G = \left\{ [z_1 : z_2] \longmapsto \big[(a_{11}z_1 + a_{12}z_2) : (a_{21}z_1 + a_{22}z_2)\big] \middle| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \right\}.$$

Parametrization $\pi : z = x_1 + ix_2 \mapsto [z : 1]$:

$$G = \left\{ z \longmapsto \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} \right\}.$$

13.1. $S = \mathbb{R}P^2$.

$$G = \Big\{ [y_1 : y_2 : y_3] \longmapsto \big[(a_{11}y_1 + a_{12}y_2 + a_{13}y_3) : (a_{21}y_1 + a_{22}y_2 + a_{23}y_3) : (a_{31}y_1 + a_{32}y_2 + a_{33}y_3)\big] \Big| \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R}) \Big\}.$$

Parametrization $\pi : (x_1, x_2) \mapsto [x_1 : x_2 : 1]$:

$$G = \left\{ (x_1, x_2) \longmapsto \left(\frac{a_{11}x_1 + a_{12}x_2 + a_{13}}{a_{31}x_1 + a_{32}x_2 + a_{33}}; \frac{a_{21}x_1 + a_{22}x_2 + a_{23}}{a_{31}x_1 + a_{32}x_2 + a_{33}} \right) \right\}.$$

14.1. $S = \mathbb{R}^2$.

$$G = \Big\{ (x_1, x_2) \longmapsto (a_{11}x_1 + a_{12}x_2 + b_1, a_{21}x_1 + a_{22}x_2 + b_2) \Big| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),\ b_1, b_2 \in \mathbb{R} \Big\}.$$

15.1. $S = \mathbb{R}^2$.

$$G = \Big\{ (x_1, x_2) \longmapsto (a_{11}x_1 + a_{12}x_2 + b_1, a_{21}x_1 + a_{22}x_2 + b_2) \Big| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),\ b_1, b_2 \in \mathbb{R} \Big\}.$$

16.1. $S = \mathbb{R}^2 \backslash \{0\}$.

$$G = \left\{ (y_1, y_2) \longmapsto (a_{11}y_1 + a_{12}y_2,\ a_{21}y_1 + a_{22}y_2) \,\middle|\, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R}) \right\}.$$

Parametrization $\pi : (x_1, x_2) \mapsto (x_1\,\mathrm{e}^{x_2}, \mathrm{e}^{x_2})$:

$$G = \left\{ (x_1, x_2) \longmapsto \left(\frac{a_{11}x_1 + a_{12}x_2}{a_{21}x_1 + a_{22}x_2},\ x_2 + \ln(a_{21}x_1 + a_{22}) \right) \right\}.$$

17.1. $S = \{ (y_1, y_2, z) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 \neq 0 \} / \sim$, where

$$(y_1, y_2, z) \sim (\lambda y_1, \lambda y_2, \lambda^n z), \qquad \lambda \in \mathbb{R}^*;$$

$$G = \Big\{ (y_1, y_2, z) \longmapsto \big(a_{11}y_1 + a_{12}y_2,\ a_{21}y_1 + a_{22}y_2,\ z + f(a_{11}y_1 + a_{12}y_2, a_{21}y_1 + a_{22}y_2)\big) \,\Big|\, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2.\mathbb{R}),\ f \in \mathbb{R}^n[y_1, y_2] \Big\}.$$

Parametrization $\pi : (x_1, x_2) \mapsto (x_1, 1, x_2)$:

$$G = \left\{ (x_1, x_2) \longmapsto \left(\frac{a_{11}x_1 + a_{12}}{a_{21}x_1 + a_{22}},\ \frac{x_2}{(a_{21}x_1 + a_{22})^n} + \tilde{f}\left(\frac{a_{11}x_1 + a_{12}}{a_{21}x_1 + a_{22}} \right) \right) \,\middle|\, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),\ \tilde{f} \in \mathbb{R}_n[x] \right\}.$$

18.1. $S = \{ (y_1, y_2, z) \in \mathbb{R}^3 \mid y_1^2 + y_2^2 \neq 0 \} / \sim$, where

$$(y_1, y_2, z) \sim (\lambda y_1, \lambda y_2, \lambda^n z), \qquad \lambda \in \mathbb{R}^*;$$

$$G = \Big\{ (y_1, y_2, z) \longmapsto \big(a_{11}y_1 + a_{12}y_2, a_{21}y_1 + a_{22}y_2, bz + f(a_{11}y_1 + a_{12}y_2, a_{21}y_1 + a_{22}y_2)\big) \,\Big|\, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), b \in \mathbb{R}_+^*, f \in \mathbb{R}^n[x, y] \Big\}.$$

Parametrization $\pi : (x_1, x_2) \mapsto (x_1, 1, x_2)$:

$$G = \left\{ (x_1, x_2) \longmapsto \left(\frac{a_{11}x_1 + a_{12}}{a_{21}x_1 + a_{22}},\ \frac{bx_2}{(a_{21}x_1 + a_{22})^n} + \tilde{f}\left(\frac{a_{11}x_1 + a_{12}}{a_{21}x_1 + a_{22}} \right) \right) \,\middle|\, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),\ b \in \mathbb{R}_+^*,\ \tilde{f} \in \mathbb{R}_n[x] \right\}.$$

Remark. In all the cases except 11.1 the group G is a transitive transformation group of the surface S. As to case 11.1 the group G acts transitively on the set

$$\{ (y_1 : y_2, z_1 : z_2) \in \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1 \mid y_1 z_1 + y_2 z_2 \neq 0 \}.$$

List 3. EFFECTIVE PAIRS $(\bar{\mathfrak{g}}, \mathfrak{g})$ OF CODIMENSION 2

Here we use the following notation:

(1) for the subalgebras of $\mathfrak{gl}(2, \mathbb{R})$:

$$\mathfrak{t}(2,\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\},$$
$$\mathfrak{st}(2,\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} = \mathfrak{t}(2,\mathbb{R}) \cap \mathfrak{sl}(2,\mathbb{R});$$

(2) for the Lie algebra of orthogonal matrices

$$\mathfrak{so}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) | A + {}^t A = 0\};$$

(3) for the n-by-n matrices

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad S_n = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & -2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -n \end{pmatrix}.$$

(4) for Frobenius matrix

$$F(p) = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix},$$

where $p(x) = (-1)^n \cdot (x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0)$ is its characteristic polynomial.

1.1

$$\bar{\mathfrak{g}} = \mathbb{R}^2, \quad \mathfrak{g} = \{0\}.$$

2.1(λ)

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & \lambda x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}, \ |\lambda| \leqslant 1.$$

2.2(λ)

$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} \lambda x & -x \\ x & \lambda x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}, \lambda \geqslant 0.$$

3.1
$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}.$$

3.2
$$\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{a} \times \{0\}, \text{ where } \mathfrak{a} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}.$$

4.1(p)
$$\bar{\mathfrak{g}} = \{xF(p) | x \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$
$$\mathfrak{g} = \{0\} \times \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix} \middle| x_i \in \mathbb{R} \right\}, \text{ where } p \in \mathbb{R}[x], n = \deg p \geqslant 1;$$

in addition, if $p = \alpha x^n$, then $n \geqslant 2$.

5.1(p)
$$\bar{\mathfrak{g}} = \{xE_n + yF(p) | x, y \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$
$$\mathfrak{g} = \{xE_n | x \in \mathbb{R}\} \times \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix} \middle| x_i \in \mathbb{R} \right\}, \text{ where } p \in \mathbb{R}[x], n = \deg p \geqslant 2.$$

6.1(n, λ)
$$\bar{\mathfrak{g}} = \{x(\lambda E_n + S_n) + yN_n | x, y \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$
$$\mathfrak{g} = \{x(\lambda E_n + S_n) | x \in \mathbb{R}\} \leftthreetimes \left\{ \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2, \lambda \in \mathbb{R}, \lambda \neq n.$$

7.1(n)
$$\bar{\mathfrak{g}} = \left\{ \left(x_n(nE_n + S_n) + yN_n, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \middle| x_i \in \mathbb{R}, y \in \mathbb{R} \right\} \subset 9.1(n),$$
$$\mathfrak{g} = \left\{ \left(x_n(nE_n + S_n), \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right) \middle| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2.$$

8.1(n)

$$\bar{\mathfrak{g}} = \{xE_n + yS_n + zN_n | x, y, z \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$

$$\mathfrak{g} = \{xE_n + yS_n | x, y \in \mathbb{R}\} \times \left\{ \left. \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2.$$

9.1(n)

$$\bar{\mathfrak{g}} = \{x(nE_n + S_n) + yN_n | x, y \in \mathbb{R}\} \leftthreetimes \mathbb{R}^n,$$

$$\mathfrak{g} = \{x(nE_n + S_n) | x \in \mathbb{R}\} \times \left\{ \left. \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right| x_i \in \mathbb{R} \right\}, \text{ where } n \geqslant 2.$$

10.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}), \quad \mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\}.$$

11.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}), \quad \mathfrak{g} = \left\{ \left. \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \right| x \in \mathbb{R} \right\}.$$

11.2

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}), \quad \mathfrak{g} = \left\{ \left. \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \right| x \in \mathbb{R} \right\} = \mathfrak{so}(2).$$

11.3

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) = \left\{ \left. \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}, \quad \mathfrak{g} = \left\{ \left. \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \right| x \in \mathbb{R} \right\}.$$

12.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}), \quad \mathfrak{g} = \mathfrak{st}(2, \mathbb{R}) \times \mathfrak{st}(2, \mathbb{R}).$$

12.2

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}, \quad \mathfrak{g} = \mathfrak{st}(2, \mathbb{C})_{\mathbb{R}}.$$

13.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(3, \mathbb{R}), \quad \mathfrak{g} = \left\{ \left. \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \right| A \in \mathfrak{gl}(2, \mathbb{R}), B \in \mathrm{Mat}_{1 \times 2}(\mathbb{R}) \right\}.$$

14.1

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}) \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \times \{0\}.$$

15.1

$$\bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2, \quad \mathfrak{g} = \mathfrak{gl}(2,\mathbb{R}) \times \{0\}.$$

16.1

$$\bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}.$$

17.1(n)

$$\begin{aligned} \bar{\mathfrak{g}} &= \mathfrak{sl}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \mathbb{R}^n, \\ \mathfrak{g} &= \mathfrak{st}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \{a_0x^n + \dots + a_{n-1}x^{n-1}y | a_i \in \mathbb{R}\}, \text{ where } n \geqslant 0. \end{aligned}$$

18.1(n)

$$\begin{aligned} n \geqslant 1: \quad & \bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \mathbb{R}^n, \\ & \mathfrak{g} = \mathfrak{t}(2,\mathbb{R}) \leftthreetimes_{\pi_n} \{\, a_0x^n + \dots + a_{n-1}x^{n-1}y \mid a_i \in \mathbb{R} \,\}; \\ n = 0: \quad & \bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times (\mathbb{R}_+^* \leftthreetimes \mathbb{R}), \\ & \mathfrak{g} = \mathfrak{st}(2,\mathbb{R}) \times (\mathbb{R}_+^* \times \{0\}). \end{aligned}$$

List 4. All two-dimensional homogeneous spaces

4. Comlpete list of two-dimensional spaces

1.1

$$\overline{G} = \mathbb{R}^2, \quad G = \{0\}.$$
$$M = \mathbb{R}^2.$$
$$(u, v).(p, q) = (p + u, q + v).$$
$$C = \mathbb{R}^2.$$

Nontrivial discrete subgroups have the form:

$$\{\,(n, m) \mid n, m \in \mathbb{Z}\,\} \text{ – the torus;}$$
$$\{\,(n, 0) \mid n \in \mathbb{Z}\,\} \qquad \text{ – the cylinder.}$$

2.1(λ)

$$\overline{G} = A \leftthreetimes \mathbb{R}^2, \quad G = A \times \{0\},$$
$$\text{where } A = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{\lambda} \end{pmatrix} \middle| \; x \in \mathbb{R}_+^* \right\}.$$
$$M = \mathbb{R}^2.$$
$$\big(x,\ (u, v)\big).(p, q) = (xp + u, x^{\lambda} q + v).$$
$$C = \begin{cases} \{\,(0, 0)\,\}, & \lambda \neq 0; \\ \{\,(0, p) \mid p \in \mathbb{R}\,\}, & \lambda = 0. \end{cases}$$

If $\lambda = 0$ then nontrivial discrete subgroups have the form:

$$\{\,(0, n) \mid n \in \mathbb{Z}\,\} \text{ – the cylinder.}$$

2.2(λ)

$$\overline{G} = A \leftthreetimes \mathbb{R}^2, \quad G = A \times \{0\},$$
$$\text{where } A = \left\{ \mathrm{e}^{\lambda x} \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}.$$
$$M = \mathbb{R}^2.$$
$$(x, (u, v)).(p, q) = (\mathrm{e}^{\lambda x}(p\cos x - q\sin x) + u, \mathrm{e}^{\lambda x}(p\sin x + q\cos x) + v).$$
$$C = \{(0, 0)\}.$$

There are no any nontrivial discrete subgroups.

3.1

$$\overline{G} = A \leftthreetimes \mathbb{R}^2, \quad G = A \times \{0\},$$

$$\text{where } A = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| \, x, y \in \mathbb{R}_+^* \right\}.$$

$$M = \mathbb{R}^2.$$

$$(x, y, (u, v)).(p, q) = (xp + u, yq + v).$$

$$C = \{(0,0)\}.$$

There are no any nontrivial discrete subgroups.

3.2

$$\overline{G} = A \leftthreetimes \mathbb{R}^2, \quad G = A \times \{0\},$$

$$\text{where } A = \left\{ x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix} \middle| \, x \in \mathbb{R}_+^*, y \in \mathbb{R} \right\}.$$

$$M = \mathbb{R}^2.$$

$$(x, y, (u, v)).(p, q) = (x(p\cos y - q \sin y) + u, x(p \sin y + q \cos y) + v).$$

$$C = \{(0,0)\}.$$

There are no any nontrivial discrete subgroups.

4.1 (p)

$$\overline{G} = \{\, (x, f) \mid x \in \mathbb{R},\ f \in \mathcal{F}_p \,\},$$

where $\mathcal{F}_p$ is the solution space of the homogeneous linear equation, corresponding to the polynomial p.

$$G = \{\, (0, f) \mid f(0) = 0 \,\}.$$

$$M = \mathbb{R}^2.$$

$$(x, f).(p, q) = \big(p + x, q + f(p + x)\big).$$

The set C has one of the following forms.

$$C = \{\, (\pi k, q) \mid k \in \mathbb{N},\ q \in \mathbb{R} \,\} \text{ in special cases:}$$

$$\begin{aligned}
a) \quad p(x) &= \prod_{i=1}^{r} \big((x-\lambda)^2 + b_i^2\big), && \lambda \in \mathbb{R},\ b_i \in \mathbb{Z}; \\
b) \quad p(x) &= \prod_{i=1}^{r} \big((x-\lambda)^2 + b_i^2\big), && \lambda \in \mathbb{R},\ b_i \in 2\mathbb{Z}+1; \\
c) \quad p(x) &= (x-\lambda) \prod_{i=1}^{r} \big((x-\lambda)^2 + b_i^2\big), && \lambda \in \mathbb{R},\ b_i \in 2\mathbb{Z};
\end{aligned}$$

$$C = \{\, (0, q) \ \mid\ q \in \mathbb{R} \,\} \text{ in nonspecial cases.}$$

All nontrivial discrete subgroups have the form:

$$
\begin{array}{lll}
a), c) & \{ (\pi nk, 0) \mid k \in \mathbb{Z} \}, & n \in \mathbb{N} \text{ – the cylinder,} \\
 & \{ (0, k) \mid k \in \mathbb{Z} \}, & \text{ – the cylinder,} \\
\lambda = 0: & \{ (2\pi nk, m) \mid k, m \in \mathbb{Z} \}, & n \in \mathbb{N} \text{ – the torus;} \\
b) & \{ (2\pi nk, 0) \mid k \in \mathbb{Z} \}, & n \in \mathbb{N} \text{ – the cylinder,} \\
 & \{ (0, k) \mid k \in \mathbb{Z} \}, & \text{ – the cylinder,} \\
 & \{ \big((2n-1)\pi k, 0\big) \mid k \in \mathbb{Z} \}, & n \in \mathbb{N} \text{ – the Möbius strip,} \\
\lambda = 0: & \{ (2\pi nk, m) \mid k, m \in \mathbb{Z} \}, & n \in \mathbb{N} \text{ – the torus,} \\
\lambda = 0: & \{ \big((2n-1)\pi k, m\big) \mid k, m \in \mathbb{Z} \}, & n \in \mathbb{N} \text{ – the Klein bottle;}
\end{array}
$$

in nonspecial cases $C = \{ (0, k) \mid k \in \mathbb{Z} \}$ – the cylinder.

5.1 (p) (Here we use the notation introduced in item 4.1).

$$
\begin{aligned}
&\overline{G} = \{(x, y, f) | x \in \mathbb{R}, y \in \mathbb{R}^*_+, f \in \mathcal{F}_p\}, \\
&G = \{(0, y, f) | y \in \mathbb{R}^*_+, f(0) = 0\}. \\
&M = \mathbb{R}^2.
\end{aligned}
$$

$$(x, y, f).(p, q) = (p + x, yq + f(p + x)).$$

$$
\begin{aligned}
&C = \{(0, 0)\} \text{ in inexceptional cases and} \\
&C = \{(\pi k, 0) | k \in \mathbb{Z}\} \text{ in exceptional cases.}
\end{aligned}
$$

Nontrivial discrete subgroups exist only in exceptional cases and have the form:

$$
\begin{array}{lll}
a), c): & \{(\pi nk, 0) | k \in \mathbb{Z}\}, & n \in \mathbb{N}\text{—the cylinder;} \\
b): & \{(2\pi nk, 0) | k \in \mathbb{Z}\}, & n \in \mathbb{N}\text{—the cylinder;} \\
 & \{((2n-1)\pi k, 0) | k \in \mathbb{Z}\}, & n \in \mathbb{N}\text{—the Möbius strip.}
\end{array}
$$

6.1 (λ, n)

$$
\begin{aligned}
&\overline{G} = \{(x, y, f) | x \in \mathbb{R}^*_+, y \in \mathbb{R}, f \in \mathbb{R}_{n-1}[x]\}, \\
&G = \{(x, 0, f) | x \in \mathbb{R}^*_+, f(0) = 0\}. \\
&M = \mathbb{R}^2.
\end{aligned}
$$

$$(x, y, f).(p, q) = (xp + y, x^{\lambda - 1} q + f(xp + y)).$$

$$
\begin{aligned}
&C = \{(0, 0)\} \text{ if } \lambda \neq 1 \text{ and} \\
&C = \{(0, a) | a \in \mathbb{R}\} \text{ if } \lambda = 1.
\end{aligned}
$$

In the case $\lambda = 1$:

$$(0, a).(p, q) = (p, q + a).$$

Nontrivial discrete subgroups:

$$\{(0, k)|k \in \mathbb{Z}\} \text{ —the cylinder.}$$

7.1 (n)

$$\begin{aligned}\overline{G} &= \{(x, y, f)|x \in \mathbb{R}_+^*, y \in \mathbb{R}, f \in \mathbb{R}_{n-2}[x]\},\\ G &= \{(x, 0, f)|x \in \mathbb{R}_+^*, f(0) = 0\}.\\ M &= \mathbb{R}^2.\end{aligned}$$

$$(x, y, f).(p, q) = \big(xp + y, x^{n-1}q + (xp + y)^{n-1} + f(xp + y)\big).$$

$$C = \{(0, 0)\}.$$

There are no any nontrivial discrete subgroups.

8.1 (n)

$$\begin{aligned}\overline{G} &= \{(x, y, z, f)|x, z \in \mathbb{R}_+^*, y \in \mathbb{R}, f \in \mathbb{R}_{n-1}[x]\},\\ G &= \{(x, 0, z, f)|x, z \in \mathbb{R}_+^*, f(0) = 0\}.\\ M &= \mathbb{R}^2.\end{aligned}$$

$$(x, y, z, f).(p, q) = (xp + y, zq + f(xp + y)).$$

$$C = \{(0, 0)\}.$$

There are no any nontrivial discrete subgroups.

9.1 (n) See $6.1(n, \lambda)$ where $\lambda = n$.

10.1

$$\overline{G} = \widetilde{\mathrm{SL}(2, \mathbb{R})}, \quad G = \{\, (1, y, 0) \mid y \in \mathbb{R}\,\}.$$

$$M = \mathbb{R}_+^* \times \mathbb{R}.$$

$$(x, y, z).(p, q) = \big(pX(x, y, q),\ Z(x, y, q) + z\big).$$

$$C = \{\, (p, \pi k) \mid k \in \mathbb{Z},\ p \in \mathbb{R}_+^*\,\}.$$

Nontrivial discrete subgroups have the form:

$$\begin{array}{lll}\{(1, \pi n k) \mid k \in \mathbb{Z}\}, & n \in \mathbb{N} & \text{– the cylinder;}\\ \{(\alpha^k, \pi n k) \mid k \in \mathbb{Z}\}, & n \in \mathbb{N},\ \alpha \in \mathbb{R}_+^* & \text{– the cylinder;}\\ \{(\alpha^k, \pi l n) \mid k, l \in \mathbb{Z}\}, & n \in \mathbb{N},\ \alpha \in \mathbb{R}_+^* & \text{– the torus.}\end{array}$$

11.1

$$\overline{G} = \widetilde{\mathrm{SL}(2,\mathbb{R})}^{*}, \quad G = \{\, (x,0,0) \mid y \in \mathbb{R}^{*}_{+} \,\}.$$

$$M = \mathbb{R}^2.$$

$$(x,y,z).(p,q) = \big(Y(x,y,q) + X(x,y,q)p,\ z + Z(x,y,q)\big).$$

$$C = \{\, (0,\pi k/2) \mid k \in \mathbb{Z},\ p \in \mathbb{R}^{*}_{+} \,\}.$$

Nontrivial discrete subgroups have the form:

$\{(0,\pi n k) \mid k \in \mathbb{Z}\}$, $n \in \mathbb{N}$ – the cylinder;
$\{(0,(2n-1)\pi n k/2) \mid k \in \mathbb{Z}\}$, $n \in \mathbb{N}$ – the Möbius strip.

11.2

$$\overline{G} = \mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1 \right\},$$

$$G = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \middle| \, \alpha \in \mathbb{R} \right\}.$$

$$M = \{z \in \mathbb{C} | \mathrm{Im} z > 0\} \approx \mathbb{R}^2.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}.$$

$$C = \{i\}.$$

There are no any nontrivial discrete subgroups.

11.3

$$\overline{G} = \mathrm{SO}(3) = \{A \in \mathrm{SL}(3,\mathbb{R}) |\ {}^{t}\!AA = E\},$$

$$G = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, \alpha \in \mathbb{R} \right\}.$$

$$M = S^2 = \{v \in \mathbb{R}^3 |\ |v| = 1\}.$$

$$A.v = Av.$$

$$C = \{(0,0,\pm 1)\}.$$

$$(0,0,\pm 1).v = \pm v.$$

Nontrivial discrete subgroups have the form:

$$\{(0,0,\pm 1)\} \text{ —the projective plane } \mathbb{R}\mathrm{P}^2 .$$

12.1

$$\begin{aligned}
&\overline{G} = \widetilde{\mathrm{SL}(2,\mathbb{R})} \times \widetilde{\mathrm{SL}(2,\mathbb{R})} = (\mathbb{R}^*_+ \times \mathbb{R}^2) \times (\mathbb{R}^*_+ \times \mathbb{R}^2),\\
&G = \{(x,y,0) | x \in \mathbb{R}^*_+, y \in \mathbb{R}\} \times \{(u,v,0) | u \in \mathbb{R}^*_+, v \in \mathbb{R}\}.\\
&M = \mathbb{R}^2.
\end{aligned}$$

$$(x,y,z,u,v,w).(p,q) = (Z(x,y,p)+z, Z(u,v,q)+w).$$

$$C = \{(\pi n, \pi m) | n,m \in \mathbb{Z}\}.$$

$$(\pi n, \pi m).(p,q) = (p+\pi n, q+\pi m).$$

Nontrivial discrete subgroups have the form:

$$\begin{array}{lll}
\{(\pi nk, 0) | k \in \mathbb{Z}\}, & n \in \mathbb{N} & \text{— the cylinder;}\\
\{(\pi nk, \pi mk) | k \in \mathbb{Z}\}, & n,m \in \mathbb{N}, n \geqslant m & \text{— the cylinder;}\\
\{(\pi nk, \pi ml) | k,l \in \mathbb{Z}\}, & n,m \in \mathbb{N}, n \geqslant m & \text{— the torus.}
\end{array}$$

12.2

$$\begin{aligned}
&\overline{G} = \mathrm{SL}(2,\mathbb{C})/\{\pm E\},\\
&G = \mathrm{ST}(2,\mathbb{C})/\{\pm E\}.\\
&M = \mathbb{C}\mathrm{P}^1 \approx S^2.
\end{aligned}$$

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}.(p:q) = ((xp+yq):(zp+tq)).$$

$$C = \{(1:0)\}.$$

There are no any nontrivial discrete subgroups.

13.1

$$\begin{aligned}
&\overline{G} = \mathrm{SL}(3,\mathbb{R}),\\
&G = \left\{ \begin{pmatrix} (\det A)^{-1} & B \\ 0 & A \end{pmatrix} \middle| A \in \mathrm{GL}_+(2,\mathbb{R}), B \in \mathrm{Mat}_{1\times 2}(\mathbb{R}) \right\}.\\
&M = S^2 = \{v \in \mathbb{R}^3 | \ |v| = 1\}.
\end{aligned}$$

$$A.v = \frac{Av}{|Av|}.$$

$$C = \{(\pm 1, 0, 0)\}.$$

$$(\pm 1, 0, 0).v = \pm v.$$

Nontrivial discrete subgroups have the form:

$$\{(\pm 1, 0, 0)\} \text{ — the projective plane } \mathbb{R}\,\mathrm{P}^2 .$$

14.1

$$\begin{aligned}\overline{G} &= \mathrm{SL}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2,\\ G &= \mathrm{SL}(2,\mathbb{R}) \times \{0\}.\\ M &= \mathbb{R}^2.\end{aligned}$$

$$(A,v).w = Aw + v, \text{ where } (A,v) \in \overline{G}, w \in \mathbb{R}^2.$$

$$C = \{0\}.$$

There are no any nontrivial discrete subgroups.

15.1

$$\begin{aligned}\overline{G} &= \mathrm{GL}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2,\\ G &= \mathrm{GL}(2,\mathbb{R}) \times \{0\}.\\ M &= \mathbb{R}^2.\end{aligned}$$

$$(A,v).w = Aw + v, \text{ where } (A,v) \in \overline{G}, w \in \mathbb{R}^2.$$

$$C = \{0\}.$$

There are no any nontrivial discrete subgroups.

16.1

$$\begin{aligned}\overline{G} &= \widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathbb{R}^*_+,\\ G &= \{(t^{-1}, y, 0, t) | y \in \mathbb{R}, t \in \mathbb{R}^*_+\}.\\ M &= \mathbb{R}^2.\end{aligned}$$

$$(x,y,z,t).(p,q) = (p(tX(x,y,q))^{1/2}, z + Z(x,y,q)).$$

The group C and all nontrivial discrete subgroups here the same as in 10.1.

17.1

$$\overline{G} = \widetilde{\mathrm{SL}(2,\mathbb{R})} \leftthreetimes \mathbb{R}^n[x,y].$$

$$G = \{\, (x,y,0) \mid x \in \mathbb{R}^*_+,\ y \in \mathbb{R} \,\} \leftthreetimes \{\, f \in \mathbb{R}^n[x,y] \mid f(0,1) = 0 \,\}.$$

$$M = \mathbb{R}^2$$

$$\big(x,y,z,f\big).(p,q) = \big(\alpha,\ X^{-\frac{n}{2}} q + f(-\sin\alpha, \cos\alpha)\big),$$

where $\alpha = z + Z(x, y, p)$, $X = X(x, y, p)$.

$$C = \begin{cases} \{\,(\pi k, q) \mid k \in \mathbb{Z},\ q \in \mathbb{R}\,\}, & n = 0 \\ \{\,(\pi k, 0) \mid k \in \mathbb{Z}, & n \geqslant 1. \end{cases}$$

Nontrivial discrete subgroups have the form:

$$\begin{array}{llll} n = 0: & \{\,(\pi m k, 0) \mid k \in \mathbb{Z}\,\}, & m \in \mathbb{N} & \text{– the cylinder;} \\ & \{\,(0, k) \mid k \in \mathbb{Z}\,\}, & & \text{– the cylinder;} \\ & \{\,(\pi m k, l) \mid k, l \in \mathbb{Z}\,\}, & m \in \mathbb{N} & \text{– the torus;} \\ n \geqslant 1: & \{\,(\pi n k, 0) \mid k \in \mathbb{Z}\,\}, & n \in \mathbb{N} & \begin{cases} \text{the Möbius strip if } nm \text{ is odd,} \\ \text{the cylinder if } nm \text{ is even.} \end{cases} \end{array}$$

18.1 (Here we use the notation of item 17.1).

$$\begin{aligned} &\overline{G} = (\widetilde{\mathrm{SL}(2, \mathbb{R})} \times \mathbb{R}_+^*) \leftthreetimes \mathbb{R}_n[x, y] = \widetilde{\mathrm{GL}_+(2, \mathbb{R})} \leftthreetimes \mathbb{R}_n[x, y], \\ &G = \{(x, y, 0, t) | x, t \in \mathbb{R}_+^*, y \in \mathbb{R}\} \times \{f \in \mathbb{R}_n[x, y] | f(0, 1) = 0\}. \\ &M = \mathbb{R}^2. \end{aligned}$$

$$(x, y, z, t, f).(p, q) = (\alpha, tX^{-n/2}(x, y, p)q + f(-\sin\alpha, \cos\alpha)),$$

where $\alpha = z + Z(x, y, p)$.

$$C = \{(\pi k, 0) | k \in \mathbb{Z}\}.$$

Nontrivial discrete subgroups have the form:

$$\{(\pi m k, 0) | k \in \mathbb{Z}\}, m \in \mathbb{N}.$$

If mn is even then factormanifold is a cylinder, else — a Möbius strip.

List 5. SUBALGEBRAS OF THE LIE ALGEBRA $\mathfrak{gl}(2,\mathbb{R})$.

I	$\{0\}$;	VII	$\left\{\left.\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\right\| x,y\in\mathbb{R}\right\}$;
$II(\lambda)$	$\left\{\left.\begin{pmatrix} x & 0 \\ 0 & \lambda x \end{pmatrix}\right\| x\in\mathbb{R}\right\}$, $\lvert\lambda\rvert\leqslant 1$;	$VIII(\lambda)$	$\left\{\left.\begin{pmatrix} \lambda x & y \\ 0 & x \end{pmatrix}\right\| x,y\in\mathbb{R}\right\}$;
$III(\lambda)$	$\left\{\left.\begin{pmatrix} \lambda x & -x \\ x & \lambda x \end{pmatrix}\right\| x\in\mathbb{R}\right\}$, $\lambda\geqslant 0$;	IX	$\left\{\left.\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right\| x,y\in\mathbb{R}\right\}$;
IV	$\left\{\left.\begin{pmatrix} x & x \\ 0 & x \end{pmatrix}\right\| x\in\mathbb{R}\right\}$;	X	$\left\{\left.\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}\right\| x,y,z\in\mathbb{R}\right\}$;
V	$\left\{\left.\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right\| x\in\mathbb{R}\right\}$;	XI	$\mathfrak{sl}(2,\mathbb{R})$;
VI	$\left\{\left.\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right\| x,y\in\mathbb{R}\right\}$;	XII	$\mathfrak{gl}(2,\mathbb{R})$.

List 6. ISOTROPIC REPRESENTATIONS OF TWO-DIMENSIONAL HOMOGENEOUS SPACES

1.1 — I;
2.1(λ) — $II(\lambda)$;
2.2(λ) — $III(\lambda)$;
3.1 — VI;
3.2 — VII;
4.1(p), $\deg p = 1$ — I;
4.1(p), $\deg p \geqslant 2$ — V;
5.1(p) — IX;
6.1(n,λ) — $VIII(\lambda-1)$;
7.1(2) — IV;
7.1(n), $n\geqslant 3$ — $VIII(n-1)$;
8.1(n) — X;
9.1(n) — $VIII(n-1)$;
10.1 — V;
11.1 — $II(-1)$;
11.2 — $III(0)$;
11.3 — $III(0)$;
12.1 — VI;
12.2 — VII;
13.1 — XII;
14.1 — XI;
15.1 — XII;
16.1 — $VIII(0)$;
17.1(0) — $II(0)$;
17.1(n), $n\geqslant 1$ — $VIII(n/2)$;
18.1(0) — VI;
18.1(n), $n\geqslant 1$ — X.

List 7. MAXIMAL INCLUSIONS OF TWO-DIMENSIONAL HOMOGENEOUS SPACES

$2.1 \supset 1.1.$

$2.2 \supset 1.1.$

$3.1 \supset 2.1(\lambda).$

$3.2 \supset 1.1,\ 2.2(\lambda).$

$4.1(p(x)) \supset 4.1(q(x))$, where q is a maximal divisor of p.

$5.1(p(x)) \supset 4.1(p(x+\alpha))$, $(\alpha \in \mathbb{R})$;
$5(q(x))$, where q is a maximal divisor of p.

$6.1(n,\lambda) \supset 4.1(x^n)$; $4.1(p)$, where $p(x) = (x-\lambda+1)\cdot\ldots\cdot(x-\lambda+n)$;
$6.1(n-1,\lambda)$.

$7.1(n) \supset 4.1(x^{n-1})$; $4.1(p)$ where $p(x) = (x+1)\cdot\ldots\cdot(x+n-1)$.

$8.1(n) \supset 5.1(x^n)$; $5.1(p)$, where $p(x) = (x+1)\cdot\ldots\cdot(x+n)$; $6.1(n,\lambda)$; $8.1(n-1)$.

$9.1(n) \supset 4.1(x^n)$; $4.1(p)$, where $p = x(x+1)\ldots(x+n-1)$; $6.1(n-1,n)$; $7.1(n)$.

$10.1 \supset 4.1(x-1).$

$11.1 \supset 4.1(x-1).$

$11.2 \supset 4.1(x-1).$

$12.1 \supset 11.1;\ 18.1(0);\ 17.1(0).$

$12.2 \supset 3.2;\ 11.2;\ 11.3.$

$13.1 \supset 11.1;\ 11.2;\ 11.3;\ 18.1(1).$

$14.1 \supset 6.1(2,3/2);\ 10.1.$

$15.1 \supset 3.2;\ 8.1(2);\ 14.1;\ 16.1.$

$16.1 \supset 10.1.$

$17.1(n) \supset 6.1(n+1, n/2+1)$; $4.1(p)$, where

$$p(x) = \prod_{i=0}^{k}(x^2+(n-2i)^2) \text{ for } n = 2k+1,$$

$$p(x) = x\prod_{i=0}^{k-1}(x^2+(n-2i)^2) \text{ for } n = 2k.$$

$18.1(n) \supset 8.1(n)$; 16.1; $17.1(n)$; $5.1(p)$, where

$$p(x) = \prod_{i=0}^{k}(x^2+(n-2i)^2) \text{ for } n = 2k+1,$$

$$p(x) = x\prod_{i=0}^{k-1}(x^2+(n-2i)^2) \text{ for } n = 2k.$$

Remark. Here we identify the following pairs:

$4.1(x) \cong 1.1;$	$5.1(x-\lambda) \cong 2.1(0);$	$6.1(1,\lambda) \cong 2.1(\lambda-1);$
$6.1(n,n) \cong 9.1(n)$	$9.1(1) \cong 2.1(0);$	$8.1(1) \cong 3.1).$

List 8. TENSOR INVARIANTS OF TWO-DIMENSIONAL HOMOGENEOUS SPACES

Invariants of $\mathfrak{a}$-modules $\mathbb{R}^2$ such that $\mathfrak{a}$ is one of the subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ obtained earlier are tabulated below. Here e_1, e_2 denote a basis of $\mathbb{R}^2$ and e^1, e^2 the corresponding dual basis. Bilinear forms are given by their Gram matrices in the standard basis.

	V	V^*	$\mathrm{Bil}(V)$	$\mathfrak{gl}(V)$	Subspaces
I	$xe_1 + ye_2$	$xe^1 + ye^2$	$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$	$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$	$\forall\ W \subset V$
$II(\lambda),\ 0 < \lvert\lambda\rvert < 1$	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\mathbb{R}e_1,\ \ \mathbb{R}e_2$
$II(0)$	xe_2	xe^2	$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\mathbb{R}e_1,\ \ \mathbb{R}e_2$
$II(1)$	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & y \\ z & t \end{pmatrix}$	$\forall\ W \subset V$
$II(-1)$	$\{0\}$	$\{0\}$	$\begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\mathbb{R}e_1,\ \ \mathbb{R}e_2$
$III(\lambda),\ \ \lambda\ >\ 0$	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$	—
$III(0)$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$	$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$	—
IV	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
V	xe_1	xe^2	$\begin{pmatrix} x & y \\ -y & 0 \end{pmatrix}$	$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
VI	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\mathbb{R}e_1,\ \ \mathbb{R}e_2$
VII	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$	—
$VIII(\lambda),\ \lambda \neq 0, \pm 1$	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
$VIII(0)$	$\{0\}$	xe_2	$\{0\}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
$VIII(-1)$	$\{0\}$	$\{0\}$	$\begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
$VIII(1)$	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
IX	xe_1	$\{0\}$	$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
X	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\mathbb{R}e_1$
XI	$\{0\}$	$\{0\}$	$\begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	—
XII	$\{0\}$	$\{0\}$	$\{0\}$	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	—

List 9. FIRST DIFFERENTIAL INVARIANTS OF TWO-DIMENSIONAL HOMOGENEOUS SPACES

1.1.

$$y_1$$

2.1.

$$\lambda \neq 1: \quad \frac{y_1^{\lambda-2}}{y_2^{\lambda-1}};$$
$$\lambda = 1: \quad y_1$$

2.2.

$$\frac{(1+y_1^2)^{\frac{3}{2}} e^{\lambda \tan^{-1} y_1}}{y_2}.$$

3.1.

$$\frac{y_3 y_1}{y_2^2}.$$

3.2.

$$\frac{3y_1 y_2^2 - (1+y_1^2) y_3}{y_2^2}.$$

4.1.

$$y_n - c_{n-1} y_{n-1} - \cdots - c_1 y_1 - c_0 y_0.$$

5.1.

$$\frac{y_{n+1} - c_{n-1} y_n - \cdots - c_1 y_2 - c_0 y_1}{y_n - c_{n-1} y_{n-1} - \cdots - c_1 y_1 - c_0 y_0}.$$

6.1.

$$\frac{y_n^{\lambda-n-1}}{y_{n+1}^{\lambda-n}}.$$

7.1.

$$y_n e^{y_{n-1}}.$$

8.1.

$$\frac{y_n y_{n+2}}{y_{n+1}^2}.$$

9.1.

$$\frac{y_n^2}{y_{n+1}}.$$

10.1.

$$(y_1^2 - y_2) e^{-4y_0}.$$

11.1.

$$y_1^{-\frac{3}{2}} (y_2 + 2y_0 y_1 + 2x y_1^2 + x y_0 y_2).$$

11.2.

$$e^{-\frac{3}{2}y_1^2}(2(1+y_1^2)(xy_1-y_0)+y_2(1-x^2-y_0^2)).$$

11.3.

$$e^{-\frac{3}{2}y_1^2}(2(1+y_1^2)(xy_1-y_0)-y_2(1+x^2+y_0^2)).$$

12.1.

$$k=\frac{27y_1^3(20y_3^3-12y_1y_3y_5+18y_2^2y_5-60y_2y_3y_4+15y_1y_4^2)}{(3y_2^2-2y_1y_3)^3}.$$

12.2.

$$\alpha(1+y_1^2)^3y_1^3+\frac{45}{2}y_1^3+27y_1^5+\frac{135}{14}y_1^7,$$

where

$$\begin{aligned}
\alpha&=\gamma\left(\frac{3py_1-y_1^2-1}{3py_1}\right)^3+\beta,\\
\beta&=-\frac{45}{2p}+\frac{5(3-y_1^2)}{2y_1^2p^2}-\frac{5(17y_1^4+2y_1^2+1)}{6y_1^3p^3}-\\
&\quad-\frac{5(1+y_1^2)(y_1^3-16y_1^2+y^1-4)}{9y_1^3p^4}-\frac{8(1+y_1^2)^2(4y_1^2+1)}{27y_1^3p^5},\\
\gamma&=\frac{15p+20y_1}{pq}-\frac{5(y_1^2+1)}{2q^2}+\frac{2(y_1^2+1)-6y_1p}{pr},\\
p&=\frac{y_2^2}{y_3},\quad q=\frac{y_2^3}{y_4},\quad r=\frac{y_2^4}{y_5}.
\end{aligned}$$

13.1.

$$f_3^3f_2^{-8},$$

where

$$f_2=y_2^2y_5-3y_2y_3y_4+2y_3^3,$$

$$f_3=\begin{vmatrix}
y_3 & y_4 & y_5 & y_6 & y_7\\
y_2 & y_3 & y_4 & y_5 & y_6\\
-y_2^2 & 0 & y_3^2 & 2y_3y_4 & 2y_3y_5+y_4^2\\
0 & y_2^2 & 2y_2y_3 & 2y_2y_4+y_3^2 & 2y_2y_5+2y_3y_4\\
0 & 0 & y_2^2 & 3y_2y_3 & 3y_3^2+3y_2y_4
\end{vmatrix}.$$

14.1.

$$\frac{5y_3^2-3y_2y_4}{y_2^{8/3}}.$$

15.1.

$$\frac{(6y_2y_4-10y_3^2)^{3/2}}{9y_2^2y_5-45y_2y_3y_4+40y_3^3}.$$

16.1.

$$\frac{(y_1^2 - y_2)^3}{(4y_1^3 - 6y_1y_2 + y_3^2)^2}.$$

17.1.

$$(n+2)^2 y_{n+3} y_{n+1}^{-\frac{n+6}{n+2}} - 2(n+3)^2 y_{n+2}^{\frac{n+2}{n+3}} y_{n+1}^{-\frac{n+4}{n+2}}.$$

18.1.

$$\frac{(n+2)(n+3)p_2^5p_4 - 3(n+3)(n+4)p_2^3p_3^3 - 2(n+2)(n+4)p_2^5p_3^2}{p_3^2((n+3)p_3 - (n+2)p_2^2)^{5/2}},$$

where

$$p_i = \frac{y_{n+i}}{y_{n+1}}, \quad 2 \leqslant i \leqslant 4.$$

List 10. AUTOMORPHIC DIFFERENTIAL EQUATIONS CORRESPONDING TO TWO-DIMENSIONAL HOMOGENEOUS SPACES

Now for each two-dimensional homogeneous space we list:

1) the corresponding differential equation;

2) initial conditions for particular solutions $u_i(t) = (x_i(t), y_i(t)), 1 \leqslant i \leqslant k$, where k is the stiffness of homogeneous space;

3) the general solution.

1.1.

$$\begin{cases} \dot{x} = a_1 \\ \dot{y} = a_2 \end{cases};$$

$u_1(0) = (0,0)$

$$u(t) = (x(t), y(t)) = (c_1 + x_1(t), c_2 + y_1(t)).$$

2.1.

$$\begin{cases} \dot{x} = a_1 x + a_2 \\ \dot{y} = \lambda a_1 y + a_3 \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1,0)$.

$$u(t) = (x(t), y(t)) = \big(c_1(x_2(t) - x_1(t)) + x_1(t), c_2(x_2(t) - x_1(t))^\lambda + y_1(t)\big).$$

2.2.

$$\begin{cases} \dot{x} = a_1(\lambda x - y) + a_2 \\ \dot{y} = a_1(\lambda y + x) + a_3 \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1,0)$.

$$u(t) = (x(t), y(t)) = \big(e^{\lambda x}(c_1 \cos x - c_2 \sin x) + x_1(t), \\ e^{\lambda x}(c_1 \sin x + c_2 \cos x) + y_1(t)\big),$$

where

$$x = \begin{cases} \lambda = 0 : \begin{cases} x_2(t) - x_1(t) = 0 : x = \frac{\pi}{2} \\ x_2(t) - x_1(t) \neq 0 : x = \operatorname{arctg} \frac{y_2(t) - y_1(t)}{x_2(t) - x_1(t)}, \end{cases} \\ \lambda \neq 0 : x = \frac{1}{2\lambda} \ln\big((x_2(t) - x_1(t))^2 + (y_2(t) - y_1(t))^2\big). \end{cases}$$

3.1.

$$\begin{cases} \dot{x} = a_1 x + a_3 \\ \dot{y} = a_2 y + a_4 \end{cases};$$

$u_1(0) = (0,0)$,

$u_2(0) = (1, 1)$.

$$u(t) = (x(t), y(t)) = ((1 - c_1)x_1(t) + c_1 x_2(t), (1 - c_2)y_1(t) + c_2 y_2(t)).$$

3.2.

$$\begin{cases} \dot{x} = a_1 x - a_2 y + a_3 \\ \dot{y} = a_1 y + a_2 x + a_4 \end{cases};$$

$u_1(0) = (0, 0)$,
$u_2(0) = (1, 0)$.

$$u(t) = (x(t), y(t)) = (e^x(c_1 \cos y - c_2 \sin y) + x_1(t), e^x(c_1 \sin y + c_2 \cos y) + y_1(t)),$$

where

$$x = \frac{1}{2} \ln\left((x_2(t) - x_1(t))^2 + (y_2(t) - y_1(t))^2\right),$$

$$y = \begin{cases} x_2(t) - x_1(t) = 0 : y = \frac{\pi}{2}, \\ x_2(t) - x_1(t) \neq 0 : y = \operatorname{arctg} \frac{y_2(t) - y_1(t)}{x_2(t) - x_1(t)} \end{cases}.$$

4.1.

$$\begin{cases} \dot{x} = a_1 \\ \dot{y} = \sum_{i=0}^{n-1} a_{i+2}\omega^{(i)}(x) \end{cases},$$

where
$\omega^{(i)}(x)$, $0 \leqslant i \leqslant n-2$ is a basis of the solution space of the equation

$$f^{(n)} + c_{n-1}f^{(n-1)} + \cdots + c_1 f' + c_0 = 0, \qquad c_0, c_1, c_{n-1} \in \mathbb{R}.$$

$u_1(0) = (p_1, 0)$,
$u_2(0) = (p_2, 0)$,
$\vdots$
$u_n(0) = (p_n, 0)$, where p_j, $1 \leqslant j \leqslant n$, such that $\det A \neq 0$; here

$$A = \left(\omega^{(i)}(p_j)\right)_{\substack{0 \leqslant i \leqslant n-1 \\ 1 \leqslant j \leqslant n}}.$$

$$u(t) = (x(t), y(t)) = (c_1 - p_1 + x_1(t), c_2 + f(c_1 - p_1 + x_1(t))),$$

where

$$f = Y^t (A^{-1})^t W,$$

$$Y = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad W = \begin{pmatrix} \omega^{(0)} \\ \vdots \\ \omega^{(n-1)} \end{pmatrix}, \quad A = \left(\omega^{(i)}(x_j(t))\right)_{\substack{0 \leqslant i \leqslant n-1 \\ 1 \leqslant j \leqslant n}}.$$

5.1.
$$\begin{cases} \dot{x} = a_1 \\ \dot{y} = a_2 y + \sum_{i=0}^{n-1} a_{i+3}\omega^{(i)}(x) \end{cases},$$

where $\omega^{(i)}(x)$, $0 \leqslant i \leqslant n-1$, are the same as in 4.1.

$u_1(0) = (p_1, 0)$,

$\vdots$

$u_n(0) = (p_n, 0)$,

$u_{n+1} = (0, 1)$, where p_j, $1 \leqslant j \leqslant n$, are also the same as in 4.1.

$$u(t) = (x(t), y(t)) = (c_1 + x_{n+1}(t), (y_{n+1}(t) - f(x_{n+1}(t)))c_2 + f(c_1 + x_{n+1}(t))),$$

where
$$f = Y^t (A^{-1})^t W,$$
$$Y = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad W = \begin{pmatrix} \omega^{(0)} \\ \vdots \\ \omega^{(n-1)} \end{pmatrix}, \quad A = \left(\omega^{(i)}(x_j(t))\right)_{\substack{0 \leqslant i \leqslant n-1 \\ 1 \leqslant j \leqslant n}}.$$

6.1.
$$\begin{cases} \dot{x} = a_1 x + a_2 \\ \dot{y} = \lambda a_1 y + \sum_{i=0}^{n-1} a_{i+3} x^i \end{cases};$$

$u_1(0) = (0, 0)$,

$u_2(0) = (1, 0)$,

$\vdots$

$u_n(0) = (n-1, 0)$.

$$\begin{aligned} u(t) = (x(t), y(t)) = ((x_2(t) - x_1(t))c_1 + x_1(t), \\ (x_2(t) - x_1(t))^\lambda c_2 + f((x_2(t) - x_1(t))c_1 + x_1(t))), \end{aligned}$$

$$f(x) = \sum_{k=1}^{n} \frac{\prod_{i \neq k}(x - x_i(t))}{\prod_{i \neq k}(x_k(t) - x_i(t))} y_k(t).$$

7.1.
$$\begin{cases} \dot{x} = a_1 x + a_2 \\ \dot{y} = a_1(x^{n-1} + (n-1)y) + \sum_{i=0}^{n-2} a_{i+3} x^i \end{cases};$$

$u_1(0) = (0, 0)$,

$u_2(0) = (1, 0)$,

$\vdots$

$u_{n-1}(0) = (n-2, 0)$.

$$\begin{aligned} u(t) = (x(t), y(t)) = ((x_2(t) - x_1(t))c_1 + x_1(t), \\ (x_2(t) - x_1(t))^{n-1}c_2 + \ln(x_2(t) - x_1(t))((x_2(t) - x_1(t))c_1 + \\ + x_1(t))^{n-1} + f((x_2(t) - x_1(t))c_1 + x_1(t))), \end{aligned}$$

$$f(x) = \sum_{k=1}^{n-1} \frac{\prod_{i \neq k}(x - x_i(t))}{\prod_{i \neq k}(x_k(t) - x_i(t))} \left(y_k(t) - \ln(x_2(t) - x_1(t))(x_i(t))^{n-1}\right).$$

8.1.

$$\begin{cases} \dot{x} = a_1 x + a_2 \\ \dot{y} = a_3 y + \sum_{i=0}^{n-1} a_{i+4} x^i \end{cases};$$

$u_1(0) = (0, 0)$,

$\vdots$

$u_n(0) = (n-1, 0)$

$u_{n+1}(0) = (n, 1)$

$$\begin{aligned} u(t) = (x(t), y(t)) = ((x_2(t) - x_1(t))c_1 + x_1(t), \\ (y_{n+1}(t) - f(x_{n+1}(t)))c_2 + f((x_2(t) - x_1(t))c_1 + x_1(t))), \end{aligned}$$

$$f(x) = \sum_{k=1}^{n} \frac{\prod_{i \neq k}(x - x_i(t))}{\prod_{i \neq k}(x_k(t) - x_i(t))} y_k(t).$$

9.1.

$$\begin{cases} \dot{x} = a_1 x + a_2 \\ \dot{y} = n a_1 y + \sum_{i=0}^{n-1} a_{i+3} x^i \end{cases};$$

$u_1(0) = (0, 0)$,

$u_2(0) = (1, 0)$,

$\vdots$

$u_n(0) = (n-1, 0)$

$$\begin{aligned} u(t) = (x(t), y(t)) = ((x_2(t) - x_1(t))c_1 + x_1(t), \\ (x_2(t) - x_1(t))^n c_2 + f((x_2(t) - x_1(t))c_1 + x_1(t))), \end{aligned}$$

$$f(x) = \sum_{k=1}^{n} \frac{\prod_{i \neq k}(x - x_i(t))}{\prod_{i \neq k}(x_k(t) - x_i(t))} y_k(t).$$

10.1.

$$\begin{cases} \dot{x} = a_1 x + a_2 y \\ \dot{y} = a_3 x - a_1 y \end{cases};$$

$u_1(0) = (1,0)$,
$u_2(0) = (0,1)$.

$$u(t) = (x(t), y(t)) = (c_1x_1(t) + c_2x_2(t), c_1y_1(t) + c_2y_2(t)).$$

11.1
$$\begin{cases} \dot{x} = a_1x^2 - a_2x + a_3 \\ \dot{y} = a_3y^2 + a_2y + a_1 \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1,1)$.

$$u(t) = (x(t), y(t)) = \left(\frac{\lambda(t)x_1(t) + \mu(t)c_1}{\lambda(t) - \mu(t)y_1(t)c_1}, \frac{\mu(t)y_1(t) + \lambda(t)c_2}{\mu(t) - \lambda(t)y_1(t)c_2}\right),$$

where
$$\lambda(t) = \sqrt{\frac{1 + y_1(t)x_2(t)}{(x_2(t) - x_1(t))(1 + x_1(t)y_1(t))}},$$
$$\mu(t) = \sqrt{\frac{x_2(t) - x_1(t)}{(1 + y_1(t)x_2(t))(1 + x_1(t)y_1(t))}}.$$

11.2.
$$\begin{cases} \dot{x} = a_1(1 - x^2 + y^2) - 2a_2xy + a_3y \\ \dot{y} = a_2(1 + x^2 - y^2) - 2a_1xy - a_3x \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1/2,0)$.

$$u(t) = (x(t), y(t)) = \left(\mathcal{R}e\left(\frac{\lambda z_1(t) + \overline{\lambda}c}{\lambda + \overline{\lambda}\overline{z}_1c}\right), \mathcal{I}m\left(\frac{\lambda z_1(t) + \overline{\lambda}c}{\lambda + \overline{\lambda}\overline{z}_1c}\right)\right),$$

where
$$\lambda = \sqrt{\frac{\overline{z}_1z_2 - 1}{z_1 - z_2}},$$

$z_j(t) = x_j(t) + \omega\mathbf{i}y_j(t),\ j = 1,2, \quad c = c_1 + \mathbf{i}c_2$.

11.3.
$$\begin{cases} \dot{x} = a_1(1 - x^2 + y^2) + 2a_2xy + a_3y \\ \dot{y} = a_2(1 + x^2 - y^2) + 2a_1xy - a_3x \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1,0)$.

$$u(t) = (x(t), y(t)) = \left(\mathcal{R}e\left(\frac{\lambda z_1(t) + \overline{\lambda}c}{\lambda - \overline{\lambda}\overline{z}_1c}\right), \mathcal{I}m\left(\frac{\lambda z_1(t) + \overline{\lambda}c}{\lambda - \overline{\lambda}\overline{z}_1c}\right)\right),$$

where

$$\lambda = \sqrt{\frac{\overline{z}_1 z_2 + 1}{z_2 - z_1}},$$

$z_j(t) = x_j(t) + \mathbf{i}y_j(t),\ j = 1,2, \quad c = c_1 + \mathbf{i}c_2.$

12.1.

$$\begin{cases} \dot{x} = a_1x^2 + a_2x + a_3 \\ \dot{y} = a_4y^2 + a_5y + a_6 \end{cases};$$

$u_1(0) = (0,0),$
$u_2(0) = (1,1),$
$u_3(0) = (2,2).$

$$u(t) = (x(t), y(t)) = \left(\frac{(2-c_1)(x_3(t)-x_2(t))x_1(t) + c_1(x_2(t)-x_1(t))x_3(t)}{(2-c_1)(x_3(t)-x_2(t)) + c_1(x_2(t)-x_1(t))},\right.$$
$$\left.\frac{(2-c_1)(y_3(t)-y_2(t))y_1(t) + c_1(y_2(t)-y_1(t))y_3(t)}{(2-c_1)(y_3(t)-y_2(t)) + c_1(y_2(t)-y_1(t))}\right).$$

12.2.

$$\begin{cases} \dot{x} = a_1(x^2 - y^2) + a_2x + a_3 \\ \dot{y} = a_4xy + a_5y + a_6 \end{cases};$$

$u_1(0) = (0,0),$
$u_2(0) = (1,0),$
$u_3(0) = (0,1).$

$$u(t) = (x(t), y(t)) = (\mathcal{R}e\,\lambda, \mathcal{I}m\,\lambda),$$

where

$$\lambda = \frac{(z_3(t)-z_2(t))z_1(t) + \mathbf{i}c(z_3(t)-z_1(t))z_2(t) + c(z_2(t)-z_1(t))z_3(t)}{z_3(t)-z_2(t) + \mathbf{i}c(z_3(t)-z_1(t)) + c(z_2(t)-z_1(t))},$$

$z_j(t) = x_j(t) + \mathbf{i}y_j(t),\ j = 1,2,3, \quad c = c_1 + \mathbf{i}c_2.$

13.1.

$$\begin{cases} \dot{x} = a_1x^2 + a_2xy + a_3x + a_5y + a_7 \\ \dot{y} = a_2y^2 + a_1xy + a_4x + a_6y + a_8 \end{cases};$$

$u_1(0) = (0,0),$
$u_2(0) = (1,0),$
$u_3(0) = (0,1),$
$u_4(0) = (1,1).$

$$u(t) = (x(t), y(t)) = \left(\frac{(1-c_1-c_2)\Delta_{234}(t)x_1(t) + c_1\Delta_{134}(t)x_2(t) - c_2\Delta_{124}(t)x_3(t)}{(1-c_1-c_2)\Delta_{234}(t) + c_1\Delta_{134}(t) - c_2\Delta_{124}(t)},\right.$$
$$\left.\frac{(1-c_1-c_2)\Delta_{234}(t)y_1(t) + c_1\Delta_{134}(t)y_2(t) - c_2\Delta_{124}(t)y_3(t)}{(1-c_1-c_2)\Delta_{234}(t) + c_1\Delta_{134}(t) - c_2\Delta_{124}(t)}\right),$$

where

$$\Delta_{ijk}(t) = \begin{vmatrix} 1 & 1 & 1 \\ x_i(t) & x_j(t) & x_k(t) \\ y_i(t) & y_j(t) & y_k(t) \end{vmatrix}.$$

14.1.

$$\begin{cases} \dot{x} = a_1x + a_2y + a_4 \\ \dot{y} = a_3x - a_1y + a_5 \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1,0)$,
$u_3(0) = (0,1)$.

$$u(t) = (x(t), y(t)) = ((1 - c_1 - c_2)x_1(t) + c_1x_2(t) + c_2x_3(t), \\ (1 - c_1 - c_2)y_1(t) + c_1y_1(t) + c_2y_2(t)).$$

15.1.

$$\begin{cases} \dot{x} = a_1x + a_2y + a_5 \\ \dot{y} = a_3x + a_4y + a_6 \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1,0)$,
$u_3(0) = (0,1)$.

$$u(t) = (x(t), y(t)) = ((1 - c_1 - c_2)x_1(t) + c_1x_1(t) + c_2x_2(t), \\ (1 - c_1 - c_2)y_2(t) + c_1y_1(t) + c_2y_2(t)).$$

16.1. (Here we use the standard coordinates on the plane.)

$$\begin{cases} \dot{x} = a_1x + a_2y \\ \dot{y} = a_3x + a_4y \end{cases};$$

$u_1(0) = (1,0)$,
$u_2(0) = (0,1)$.

$$u(t) = (x(t), y(t)) = (c_1x_1(t) + c_2x_2(t), c_1y_1(t) + c_2y_2(t)).$$

17.1.

$$\begin{cases} \dot{x} = a_1x^2 + 2a_2x + a_3 \\ \dot{y} = a_1nxy + na_2y + \sum_{i=0}^{n} a_{i+4}x^i \end{cases};$$

$u_1(0) = (0,0)$,
$u_2(0) = (1,0)$,

$\vdots$
$u_{n+1}(0) = (n, 0)$.

$$u(t) = (x(t), y(t)) = \left(\frac{a(t)c_1 + b(t)}{c(t)c_1 + d(t)}, \frac{c_2 + f(a(t)c_1 + b(t), c(t)c_1 + d(t))}{(c(t)c_1 + d(t))^n} \right),$$

$$a(t) = \mu(t)x_2(t) - \lambda(t)x_1(t), \quad b(t) = \lambda(t)x_1(t), \quad c(t) = \mu(t) - \lambda(t), \quad d(t) = \lambda(t),$$

$$\lambda(t) = \pm\sqrt{\frac{2(x_2(t) - x_3(t))}{(x_2(t) - x_1(t))(x_1(t) - x_3(t))}},$$

$$\mu(t) = \pm\sqrt{\frac{x_1(t) - x_3(t)}{2(x_2(t) - x_1(t))(x_2(t) - x_3(t))}},$$

$$f(x, y) = \sum_{i=0}^{n} \alpha_i x^i y^{n-i},$$

and $\alpha_i, 0 \leqslant i \leqslant n$, can be calculated from:

$$L_n(x) = \sum_{k=1}^{n+1} \frac{\prod_{i \neq k} (x - x_i(t))}{\prod_{i \neq k} (x_k(t) - x_i(t))} \cdot y_k(t) = \sum_{i=0}^{n} \alpha_i x^i.$$

18.1.
$$\begin{cases} \dot{x} = a_1 x^2 + a_2 x + a_3 \\ \dot{y} = a_1 n x y + a_4 y + \sum_{i=0}^{n} a_{i+5} x^i \end{cases};$$

a) $n > 1$
$u_1(0) = (0, 0)$,
$u_2(0) = (1, 0)$,
$\vdots$
$u_{n+1}(0) = (n, 0)$,
$u_{n+2}(0) = (0, 1)$.
b) $n = 0, 1$
$u_1(0) = (0, 0)$,
$u_2(0) = (1, 0)$,
$u_3(0) = (2, 1)$.

$$u(t) = (x(t), y(t)) = \left(\frac{a(t)c_1 + b(t)}{c(t)c_1 + d(t)}, \frac{s(t)c_2 + f(a(t)c_1 + b(t), c(t)c_1 + d(t))}{(c(t)c_1 + d(t))^n} \right),$$

$$\begin{aligned} a(t) &= \mu(t)x_2(t) - \lambda(t)x_1(t), \qquad & b(t) &= \lambda(t)x_1(t), \\ c(t) &= \mu(t) - \lambda(t), & d(t) &= \lambda(t), \end{aligned}$$

$$\lambda(t) = \pm\sqrt{\frac{2(x_2(t) - x_3(t))}{(x_2(t) - x_1(t))(x_1(t) - x_3(t))}},$$

$$\mu(t) = \pm\sqrt{\frac{x_1(t) - x_3(t)}{2(x_2(t) - x_1(t))(x_2(t) - x_3(t))}},$$

$$f(x, y) = \sum_{i=0}^{n} \alpha_i x^i y^{n-i},$$

and $\alpha_i, 0 \leqslant i \leqslant n$, can be calculated from:

$$L_n(x) = \sum_{k=1}^{n+1} \frac{\prod_{i \neq k} (x - x_i(t))}{\prod_{i \neq k} (x_k(t) - x_i(t))} \cdot y_k(t) = \sum_{i=0}^{n} \alpha_i x^i.$$

a) $n > 1$

$$k(t) = (d(t))^n \big(y_{n+2}(t) - f(x_{n+2}(t), 1)\big);$$

b) $n = 0, 1$

$$k(t) = (2c(t) + d(t))^n \big(y_3(t) - f(y_3(t), 1)\big).$$