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NONCONFORMING TETRAHEDRAL MIXED FINITE ELEMENTS FOR ELASTICITY

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This paper presents a nonconforming finite element approximation of the space of symmetric tensors with square integrable divergence, on tetrahedral meshes. Used for stress approximation together with the full space of piecewise linear vector fields for displacement, this gives a stable mixed finite element method which is shown to be linearly convergent for both the stress and displacement, and which is significantly simpler than any stable conforming mixed finite element method. The method may be viewed as the three-dimensional analogue of a previously developed element in two dimensions. As in that case, a variant of the method is proposed as well, in which the displacement approximation is reduced to piecewise rigid motions and the stress space is reduced accordingly, but the linear convergence is retained.

Keywords: Mixed method; finite element; linear elasticity; nonconforming.

AMS Subject Classification: 65N30, 74S05

1. Introduction

Mixed finite element methods for elasticity simultaneously approximate the displacement vector field and the stress–tensor field. Conforming methods based on the

classical Hellinger–Reissner variational formulation require a finite element space for the stress–tensor that is contained in $H(\mathrm{div}, \Omega; \mathbb{S})$, the space of symmetric $n \times n$ tensor fields which are square integrable with square integrable divergence. For a stable method, this stress space must be compatible with the finite element space used for the displacement, which is a subspace of the vector-valued L^2 function space. It has proven difficult to devise such pairs of spaces. While some stable pairs have been successfully constructed in both two and three dimensions, the resulting elements tend to be quite complicated, especially in three dimensions. For this reason, much attention has been paid to constructing elements which fulfill desired stability, consistency, and convergence conditions, but which relax the requirement that the stress space be contained in $H(\mathrm{div}, \Omega; \mathbb{S})$ in one of two ways: either by relaxing the interelement continuity requirements, which leads to *nonconforming* mixed finite elements, or by relaxing the symmetry requirement, which leads to mixed finite elements with *weak symmetry*. In this paper we construct a new nonconforming mixed finite element for elasticity in three dimensions based on tetrahedral meshes, analogous to a two-dimensional element defined previously.[11] The space Σ_K of shape functions on a tetrahedral element K (which is defined in (3.1) below) is a subspace of the space $\mathcal{P}_2(K; \mathbb{S})$, the space of symmetric tensors with components which are polynomials of degree at most 2. It contains $\mathcal{P}_1(K; \mathbb{S})$ and has dimension 42. The degrees of freedom for $\sigma \in \Sigma_K$ are the integral of σ over K (this is six degrees of freedom, since σ has six components), and the integral and linear moments of σn on each face of K (nine degrees of freedom per face). For the displacements we simply take $\mathcal{P}_1(K, \mathbb{R}^3)$ as the shape functions and use only interior degrees of freedom so as not to impose any interelement degrees of freedom. See the element diagrams in Fig. 1. We note that, since there are no degrees of freedom associated to vertices or edges, only to faces and the interior, our elements may be implemented through hybridization, which may simplify the implementation. See Ref. 5 for the general idea, or Ref. 18 for a case close to the present one.

After some preliminaries in Sec. 2, in Sec. 3 we define the shape function space Σ_K and prove unisolvence of the degrees of freedom. In Sec. 4, we establish the

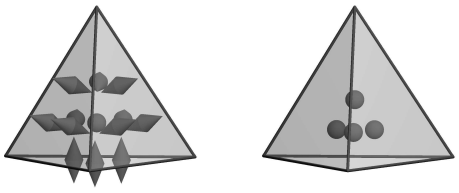

Fig. 1. Degrees of freedom for the stress σ (left) and displacement u (right). The arrows represent moments of σn, which has three components, and so there are 9 degrees of freedom associated to each face. The interior degrees of freedom are the integrals of σ and u, which have 6 and 3 components, respectively.

stability, consistency, and convergence of the resulting mixed method. Finally in Sec. 5 we describe a variant of the method which reduces the displacement space to the space of piecewise rigid motions and reduces the stress space accordingly. The results of this paper were announced previously.[13]

As mentioned, conforming mixed finite elements for elasticity tend to be quite complicated. The earliest elements, which worked only in two dimensions, used composite elements for stress.[7,22] Much more recently, elements using polynomial shape functions were developed for simplicial meshes in two[10] and three dimensions[1,4] and for rectangular meshes.[3,14] Heuristics[4,10] indicate that it is not possible to construct significantly simpler elements with polynomial shape functions and which preserve both the conformity and symmetry of the stress. Many authors have developed mixed elements with weak symmetry,[2,6,8,9,15,17,19,20,24,26–28] which we will not pursue here. For nonconforming methods with strong symmetry, which is the subject of this paper, there have been several elements proposed for rectangular meshes,[12,21,23,29,30] but very little work on simplicial meshes. A two-dimensional nonconforming element of low degree was developed by two of the present authors.[11] As shape functions for stress it uses a 15-dimensional subspace of the space of all quadratic symmetric tensors, while for the displacement it uses piecewise linear vector fields. A second element was also introduced,[11] for which the stress shape function space was reduced to dimension 12 and the displacement functions reduced to the piecewise rigid motions. Gopalakrishnan and Guzmán[18] developed a family of simplicial elements, in both two and three dimensions. As shape functions they used the space of all symmetric tensors of polynomial degree at most $k+1$, paired with piecewise polynomial vector fields of dimension k, for $k \geq 1$. Thus, in two dimensions and in the lowest degree case, they use an 18-dimensional space of shape functions for stress, while in three dimensions, the space has dimension 60. Gopalakrishnan and Guzmán also proposed a reduced variant of their space, in which the displacement space remains the full space of piecewise polynomials of degree k, but the dimension of the stress space is reduced to 15 in two dimensions and to 42 in three dimensions. However, their reduced spaces have a drawback, in that they are not uniquely defined, but for each edge of the triangulation require a choice of a favored endpoint of the edge. In particular, in two dimensions, the reduced space of Ref. 18 uses the same displacement space as the non-reduced space of Ref. 11, uses a stress space of the same dimension, and uses identical degrees of freedom, but the two spaces do *not* coincide (since the space of Ref. 11 does not require a choice of favored edge endpoints).

The elements introduced here may be regarded as the three-dimensional analogue of the element in Ref. 11. Again, they have the same displacement space and the same degrees of freedom as the reduced three-dimensional elements of Ref. 18, but the stress spaces do not coincide. Also, as in the two-dimensional case, our reduced space is of lower dimension than any that has been heretofore proposed.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. We denote by $\mathbb{S}$ the space of 3×3 symmetric matrices and by $L^2(\Omega;\mathbb{R}^3)$ and $L^2(\Omega;\mathbb{S})$ the space of square-integrable vector fields and symmetric matrix fields on Ω, respectively. The space $H(\operatorname{div},\Omega;\mathbb{S})$ consists of matrix fields $\tau \in L^2(\Omega;\mathbb{S})$ with row-wise divergence, $\operatorname{div}\tau$, in $L^2(\Omega;\mathbb{R}^3)$. The Hellinger–Reissner variational formulation seeks $(\sigma,u) \in H(\operatorname{div},\Omega;\mathbb{S}) \times L^2(\Omega;\mathbb{R}^3)$ such that

$$\begin{aligned} \int_\Omega (A\sigma : \tau + \operatorname{div}\tau \cdot u)dx = 0, &\qquad \tau \in H(\operatorname{div},\Omega;\mathbb{S}) \\ \int_\Omega \operatorname{div}\sigma \cdot v\, dx = \int_\Omega f \cdot v\, dx, &\quad v \in L^2(\Omega;\mathbb{R}^n). \end{aligned} \tag{2.1}$$

Here $\sigma : \tau$ denotes the Frobenius inner products of matrices σ and τ, and $A = A(x) : \mathbb{S} \to \mathbb{S}$ denotes the compliance tensor, a linear operator which is bounded and symmetric positive definite uniformly for $x \in \Omega$. The solution u solves the Dirichlet problem for the Lamé equations and so belongs to $\mathring{H}^1(\Omega;\mathbb{R}^3)$. If the domain Ω is smooth and the compliance tensor A is smooth, then $(\sigma,u) \in H^1(\Omega;\mathbb{S}) \times H^2(\Omega;\mathbb{R}^3)$ and

$$\|\sigma\|_1 + \|u\|_2 \le c\|f\|_0, \tag{2.2}$$

with a constant c depending on Ω and A. The same regularity holds if the domain is a convex polyhedron, at least in the isotropic homogeneous case.[25]

We shall also use spaces of the form $H^k(\Omega;X)$ where X is a finite-dimensional vector space and k is a nonnegative integer, the Sobolev space of functions $\Omega \to X$ for which all derivatives of order at most k are square integrable. The norm is denoted by $\|\cdot\|_{\Omega,k}$ or $\|\cdot\|_k$.

To discretize (2.1), we choose finite-dimensional subspaces $\Sigma_h \subset L^2(\Omega;\mathbb{S})$ and $V_h \subset L^2(\Omega;\mathbb{R}^3)$. Assuming that Σ_h consists of matrix fields which are piecewise polynomial with respect to some mesh $\mathcal{T}_h$ of Ω, we define $\operatorname{div}_h\tau \in L^2(\Omega;\mathbb{R}^3)$ by applying the (row-wise) divergence operator piecewise. A mixed finite element approximation of (2.1) is then obtained by seeking $(\sigma_h,u_h) \in \Sigma_h \times V_h$ such that:

$$\begin{aligned} \int_\Omega (A\sigma_h : \tau + \operatorname{div}_h\tau \cdot u_h)dx = 0, &\qquad \tau \in \Sigma_h \\ \int_\Omega \operatorname{div}_h\sigma_h \cdot v\, dx = \int_\Omega f \cdot v_h dx, &\quad v \in V_h. \end{aligned} \tag{2.3}$$

If $\Sigma_h \subset H(\operatorname{div},\Omega;\mathbb{S})$ this is a conforming method, otherwise, as for the elements developed below, it is nonconforming. We recall that a piecewise smooth matrix field τ belongs to $H(\operatorname{div})$ if and only if whenever two tetrahedra in $\mathcal{T}_h$ meet in a common face, the jump $[\![\tau n]\!]$ of the normal components τn across the face vanishes.

3. Definition of the New Elements

We define the finite element spaces Σ_h and V_h in the usual way, by specifying spaces of shape functions and degrees of freedom. The space V_h is simply the space

of all piecewise linear vector fields with respect to the given tetrahedral mesh $\mathcal{T}_h$ of Ω (which we therefore assume is polyhedral). Thus the shape function space on an element $K \in \mathcal{T}_h$ is simply $V_K = \mathcal{P}_1(K;\mathbb{R}^3)$, the space of polynomial vector fields on K of degree at most 1. For degrees of freedom we choose the moments $v \mapsto \int_K v \cdot w\, dx$ with weights $w \in V_K$. Since no degrees of freedom are associated with the proper subsimplices of K, no interelement continuity is imposed on V_h. The associated projection $P_h : L^2(\Omega;\mathbb{R}^3) \to V_h$ is the L^2 projection.

To define the space Σ_h we introduce some notation. If u is a unit vector, let $Q_u : \mathbb{R}^3 \to u^\perp$ be the orthogonal projection onto the plane orthogonal to u. Then Q_u is given by the symmetric matrix $I - uu'$. For a tetrahedron K, let $\Delta_k(K)$ denote the subsimplices of dimension k (vertices, edges, faces and tetrahedra) of K. For an edge $e \in \Delta_1(K)$ let s_e be a unit vector parallel to e and, for a face $f \in \Delta_2(K)$, let n_f be its outward unit normal. We can then define the shape function space

$$\Sigma_K = \{\sigma \in \mathcal{P}_2(K;\mathbb{S}) \,|\, Q_{s_e}\sigma Q_{s_e}|_e \in \mathcal{P}_1(e;\mathbb{S})\, \forall\, e \in \Delta_1(K)\}. \tag{3.1}$$

For $\sigma \in \mathcal{P}_2(K;\mathbb{S})$, $Q_{s_e}\sigma Q_{s_e}|_e$ is a quadratic polynomial on e taking values in the three-dimensional subspace $Q_{s_e}\mathbb{S}Q_{s_e}$ of $\mathbb{S}$. As illustration, for $s_e = (0,0,1)'$ and $\sigma = (\sigma_{ij})_{i,j=1,\dots,3} \in \mathbb{S}$, we have

$$Q_{s_e}\sigma Q_{s_e} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the requirement that $Q_{s_e}\sigma Q_{s_e}|_e$ belong to $\mathcal{P}_1$ represents three linear constraints on σ, and so $\dim \Sigma_K \geq 60 - 3 \times 6 = 42$. We shall now specify 42 degrees of freedom (linear functionals) and show unisolvence, i.e. that if all the degrees of freedom vanish for some $\sigma \in \Sigma_K$, then σ vanishes. This will imply that $\dim \Sigma_K \leq 42$, and so the dimension is exactly 42.

The degrees of freedom we take are:

$$\int_f \sigma n_f \cdot v\, ds, \quad v \in \mathcal{P}_1(f;\mathbb{R}^3), \;\; f \in \Delta_2(K), \qquad \text{(36 degrees of freedom)}, \tag{3.2}$$

$$\int_K \sigma\, dx, \qquad \text{(6 degrees of freedom)}. \tag{3.3}$$

The following lemma will be used in the proof of unisolvence.

Lemma 3.1. *Let f_i and f_j be the faces of K opposite two distinct vertices v_i and v_j and let e be their common edge, with endpoints v_k and v_l. Given $\beta, \gamma \in \mathbb{R}$, there exists a unique $p \in \mathcal{P}_2(K)$ satisfying the following four conditions* (*see Fig.* 2):

(1) $p|_e \in \mathcal{P}_1(e)$,
(2) $p(v_k) = \beta$, $p(v_l) = \gamma$,

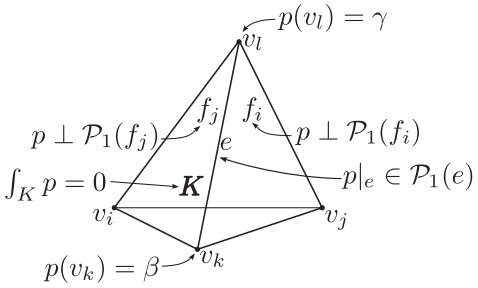

Fig. 2. The conditions of Lemma 3.1.

(3) $p|_{f_i} \perp_{L^2} \mathcal{P}_1(f_i)$, $p_{f_j} \perp_{L^2} \mathcal{P}_1(f_j)$,
(4) $\int_K p\,dx = 0$.

Moreover, $p(v_i) = p(v_j) = 3(\beta + \gamma)/2$.

Proof. For uniqueness we must show that if $p \in \mathcal{P}_2(K)$ satisfies (1)–(4) with $\beta = \gamma = 0$, then p vanishes. Certainly, from (1) and (2), p vanishes on e, and then, using (3), p vanishes on f_i and f_j. Therefore $p = c\lambda_i\lambda_j$ where $\lambda_i \in \mathcal{P}_1(K)$ is the barycentric coordinate function equal to 0 on f_i and 1 at v_i, similarly for λ_j, and c is a constant. Integrating this equation over K and invoking (4) we conclude that p does indeed vanish.

To show the existence of $p \in \mathcal{P}_2(K)$, we simply exhibit its formula in terms of barycentric coordinates:

$$p = \beta\lambda_k^2 + (\beta+\gamma)\lambda_k\lambda_l + \gamma\lambda_l^2 + \frac{3}{2}(\beta+\gamma)(\lambda_i^2+\lambda_j^2) + (-5\beta-\gamma)(\lambda_i+\lambda_j)\lambda_k + (-\beta-5\gamma)(\lambda_i+\lambda_j)\lambda_l + 3(\beta+\gamma)\lambda_i\lambda_j.$$

That this function satisfies (1)–(4) follows from the elementary formula

$$\int_T \lambda^\alpha = \frac{\alpha_1!\cdots\alpha_{d+1}!d!}{(|\alpha|+d)!}|T|, \quad \alpha \in \mathbb{N}_0^{d+1},$$

for the integral of a barycentric monomial over a simplex T of dimension d, which can be established by induction.[16] □

We are now ready to prove the claimed unisolvence result.

Theorem 3.1. *The degrees of freedom given by* (3.2) *and* (3.3) *are unisolvent for the shape function space* Σ_K *defined by* (3.1): *if the degrees of freedom all vanish for some* $\sigma \in \Sigma_K$, *then* $\sigma = 0$.

Proof. Let $g_i = \operatorname{grad}\lambda_i$ be the gradient of the ith barycentric coordinate function. Thus g_i is an inward normal vector to the face f_i with length $1/h_i$ where h_i is the

distance from the ith vertex to f_i. Note that any three of the g_i form a basis for $\mathbb{R}^3$ and that $\sum_i g_i = 0$.

For $\sigma \in \Sigma_K$, define $\sigma_{ij} = \sigma_{ji} = g_i' \sigma g_j \in \mathcal{P}_2(K)$. We shall show that if $\sigma \in \Sigma_K$ and all the degrees of freedom vanish, then $\sigma_{ij} \equiv 0$ on K for all $i \neq j$. This is sufficient, since, fixing j and varying i, we conclude that $\sigma g_j \equiv 0$, and, then, since this holds for each j, that $\sigma \equiv 0$.

If e is an edge of the faces f_i and f_j of K, which may or may not coincide, then $\sigma_{ij} = g_i' \sigma g_j = g_i' Q_s \sigma Q_s g_j$. Thus, from the definition (3.1) of the space Σ_K, σ_{ij} is linear on e. In particular, σ_{ii} is linear on each edge of f_i. Thus $p := \sigma_{ii}|_{f_i}$ is a quadratic polynomial on f_i whose restriction to each edge of f_i is linear. Therefore, on the boundary of f_i, p coincides with its linear interpolant, and, since a quadratic function on a triangle is determined by its boundary values, p is linear. Thus σ_{ii} is actually a linear polynomial on f_i, and, in view of the degrees of freedom (3.2), we conclude that σ_{ii} vanishes on f_i.

For any pair (l, k) of distinct indices (that is, $1 \leq l, k \leq 4$ and $l \neq k$), define

$$\beta_{lk} = \sigma_{ij}(v_k), \quad \beta_{kl} = \sigma_{ij}(v_l), \tag{3.4}$$

where i, j are the two indices unequal to l and k. Now $\sigma_{ij} \in \mathcal{P}_2(K)$ is linear on the common edge e of f_i and f_j, and, because of the vanishing degrees of freedom of σ, σ_{ij} is orthogonal to $\mathcal{P}_1$ on f_i and on f_j and has integral 0 on K. Therefore, by Lemma 3.1 applied with $p = \sigma_{ij}$, it is sufficient to show that β_{lk} and β_{kl} both vanish in order to conclude that σ_{ij} vanishes. In fact, we shall show that the 12 quantities β_{lk}, corresponding to the 12 pairs of distinct indices, satisfy a nonsingular homogeneous system of 12 equations, and so vanish.

The lemma also tells us that $\sigma_{ij}(v_j) = 3(\beta_{lk} + \beta_{kl})/2$. Interchanging j and k gives

$$\sigma_{ik}(v_k) = \frac{3}{2}(\beta_{lj} + \beta_{jl}).$$

Also, by definition,

$$\beta_{jk} = \sigma_{il}(v_k). \tag{3.5}$$

Combining (3.4)–(3.5) gives

$$\sigma_{ij}(v_k) + \sigma_{ik}(v_k) + \sigma_{il}(v_k) = \frac{3}{2}(\beta_{lj} + \beta_{jl}) + (\beta_{lk} + \beta_{jk}).$$

But $\sigma_{ij} + \sigma_{ik} + \sigma_{il} = -\sigma_{ii}$, which vanishes on f_i and so, in particular, at the vertex v_k. Thus we have established the equation

$$a(\beta_{lj} + \beta_{jl}) + b(\beta_{lk} + \beta_{jk}) = 0, \tag{3.6}$$

where $a = 3$, $b = 2$.

For each of the 12 pairs (i, k) of distinct indices, we let j and l be the remaining indices and consider Eq. (3.6). In this way we obtain a system of 12 linear equations in 12 unknowns. If we number the pairs of distinct indices lexographically, the

matrix of the system is:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & 0 & b & a \\
0 & 0 & 0 & 0 & b & a & 0 & 0 & 0 & 0 & a & b \\
0 & 0 & 0 & 0 & a & b & 0 & a & b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & a & b & 0 & a \\
0 & b & a & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & b \\
0 & a & b & 0 & 0 & 0 & a & 0 & b & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 & a & 0 & 0 & 0 & b & a & 0 \\
b & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 \\
a & 0 & b & a & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b & a & 0 & b & a & 0 & 0 & 0 & 0 \\
b & a & 0 & 0 & 0 & 0 & a & b & 0 & 0 & 0 & 0 \\
a & b & 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Its determinant is $16(2a-b)^2b^6(a+b)^4$, as may be verified with a computer algebra package. In particular, when $a = 3$, $b = 2$, the system is nonsingular. Thus all the β_{ij} vanish as claimed, and the proof is complete. □

Having established unisolvency, the assembled finite element space Σ_h is defined as the set of all matrix fields τ such that $\tau|_K \in \Sigma_K$ for all $K \in \mathcal{T}_h$ and for which the degrees of freedom (3.2) have a common value when a face f is shared by two tetrahedra in $\mathcal{T}_h$. If $\tau \in \Sigma_h$, then the jump $[\![\tau n_f]\!]$ of τn_f across such an interior face f need not vanish, but it is orthogonal to $\mathcal{P}_1(f;\mathbb{R}^3)$. The normal component $[\![n_f'\tau n_f]\!]$ is, by the definition of the shape function space, linear on each edge of f so belongs to $\mathcal{P}_1(f)$, and thus

$$[\![n_f'\tau n_f]\!] = 0 \quad \text{on } f, \tag{3.7}$$

for any interior face of the triangulation.

4. Error Analysis

In this section, we show that the pair of spaces Σ_h, V_h give a convergent finite element method. The argument follows the one given in Ref. 11 for the two-dimensional case. As usual, we suppose that we are given a sequence of tetrahedral meshes $\mathcal{T}_h$ indexed by a parameter h which decreases to zero and represents the maximum tetrahedron diameter. We assume that the sequence is shape regular (the ratio of the diameter of a tetrahedron to the diameter of its inscribed ball is bounded), and the constants c which appear in the estimates below may depend on this bound, but are otherwise independent of h.

We start by observing that, by construction,

$$\operatorname{div}_h \Sigma_h \subset V_h. \tag{4.1}$$

The degrees of freedom determine an interpolation operator $\Pi_h : H^1(\Omega;\mathbb{S}) \to \Sigma_h$ by

$$\int_f (\Pi_h\tau - \tau)n \cdot v \, ds = 0, \quad v \in \mathcal{P}_1(f), \;\; f \in \Delta_1(\mathcal{T}_h),$$

$$\int_K (\Pi_h\tau - \tau)dx = 0, \quad K \in \mathcal{T}_h,$$

where $\Delta_k(\mathcal{T}_h) = \bigcup_{K\in\mathcal{T}_h} \Delta_k(K)$. Since

$$\int_K (\operatorname{div}\Pi_h\tau - \operatorname{div}\tau)\cdot v \, dx = -\int_K (\Pi_h\tau - \tau) : \epsilon(v)dx + \int_{\partial K} (\Pi_h\tau - \tau)n \cdot v \, ds = 0,$$

for $\tau \in H^1(K;\mathbb{S})$, $v \in V_K$, $K \in \mathcal{T}_h$, we have the commutativity property

$$\operatorname{div}_h \Pi_h\tau = P_h \operatorname{div}\tau, \quad \tau \in H^1(\Omega;\mathbb{S}). \tag{4.2}$$

Since div maps $H^1(\Omega;\mathbb{S})$ onto $L^2(\Omega;\mathbb{R}^3)$, (4.2) implies that div_h maps Σ_h onto V_h. An immediate consequence is that the finite element method system (2.3) is nonsingular. Indeed, if $f = 0$, then the choice of test functions $\tau = \sigma_h$ and $v = u_h$ implies that $\sigma_h \equiv 0$ and then, choosing τ with $\operatorname{div}_h \tau = u_h$, we get $u_h \equiv 0$.

For the error analysis we also need the approximation and boundedness properties of the projections P_h and Π_h. Obviously, for the L^2 projection, we have

$$\|v - P_h v\|_0 \le ch^m \|v\|_m, \quad 0 \le m \le 2. \tag{4.3}$$

Since Π_h is defined element-by-element and preserves piecewise linear matrix fields, we may scale to a reference element of unit diameter using translation, rotation, and dilation, and use a compactness argument, to obtain

$$\|\tau - \Pi_h\tau\|_0 \le ch^m \|\tau\|_m, \quad m = 1, 2, \tag{4.4}$$

where the constant c depends only on the shape regularity of the elements. See, e.g. Ref. 10 for details. Taking $m = 1$ and using the triangle inequality establishes H^1 boundedness of Π_h:

$$\|\Pi_h\tau\|_0 \le c\|\tau\|_1. \tag{4.5}$$

The final ingredient we need for the convergence analysis is a bound on the consistency error arising from the nonconformity of the elements. Define

$$E_h(u,\tau) = \int_\Omega [\epsilon(u) : \tau + \operatorname{div}_h \tau \cdot u]dx, \quad u \in \mathring{H}^1(\Omega;\mathbb{R}^3), \;\; \tau \in \Sigma_h + H(\operatorname{div},\Omega;\mathbb{S}). \tag{4.6}$$

If $\tau \in H(\operatorname{div},\Omega;\mathbb{S})$, then $E_h(u,\tau) = 0$, by integration by parts. In general,

$$E_h(u,\tau) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \tau n_K \cdot u \, ds = \sum_{f\in\Delta_2(\mathcal{T}_h)} \int_f [\![\tau n_f]\!] \cdot u \, ds,$$

where, again, $[\![\tau n_f]\!]$ denotes the jump of τn_f across the face f. Only the interior faces enter the sum, since u vanishes on $\partial\Omega$. Now $\tau n_f = Q_{n_f}(\tau n_f) + (n_f'\tau n_f)n_f$, so

$$\begin{aligned} E_h(u,\tau) &= \sum_{f\in\Delta_2(\mathcal{T}_h)} \left\{ \int_f [\![Q_{n_f}(\tau n_f)]\!]\cdot u\, ds + \int_f [\![n_f'\tau n_f]\!](n_f'u) ds \right\} \\ &= \sum_{f\in\Delta_2(\mathcal{T}_h)} \int_f [\![Q_{n_f}(\tau n_f)]\!]\cdot u\, ds, \end{aligned}$$

where the last equality follows from (3.7).

We let $W_h \subset V_h$ be the subspace of the displacement space V_h consisting of continuous functions which are zero on the boundary of Ω. In other words, W_h is the standard piecewise linear subspace of $\mathring{H}^1(\Omega;\mathbb{R}^3)$. For any $\tau \in \Sigma_h$ the jumps, $[\![\tau n_f]\!]$, are orthogonal to $\mathcal{P}_1(f;\mathbb{R}^3)$, so $E_h(w,\tau) = 0$ for any $w \in W_h$.

Lemma 4.1. *We may bound the consistency error*

$$|E_h(u,\tau)| \le ch(\|\tau\|_0 + h\|\operatorname{div}_h \tau\|_0)\|u\|_2, \quad \tau\in\Sigma_h,\ \ u\in \mathring{H}^1(\Omega;\mathbb{R}^3)\cap H^2(\Omega;\mathbb{R}^3). \tag{4.7}$$

Furthermore, for any $\rho \in H^1(\Omega;\mathbb{S})$

$$|E_h(u,\Pi_h\rho)| \le ch^2\|\rho\|_1\|u\|_2, \quad u\in \mathring{H}^1(\Omega;\mathbb{R}^3)\cap H^2(\Omega;\mathbb{R}^3). \tag{4.8}$$

Proof. For any $\tau \in \Sigma_h$ we have $E_h(u,\tau) = E_h(u-u_h^I,\tau)$, where $u_h^I \in W_h$ is the piecewise linear interpolant of u. Referring to the definition (4.6), we obtain

$$\begin{aligned} |E_h(u,\tau)| &\le c(\|\operatorname{div}_h\tau\|_0\|u-u_h^I\|_0 + \|\tau\|_0\|\epsilon(u-u_h^I)\|_0 \\ &\le ch(\|\tau\|_0 + h\|\operatorname{div}_h\tau\|_0)\|u\|_2, \end{aligned}$$

which is (4.7). For the second estimate we use that $E_h(u,\Pi_h\rho) = E_h(u-u_h^I,\Pi_h\rho) = E_h(u-u_h^I,\Pi_h\rho-\rho)$, which implies that

$$E_h(u,\Pi_h\rho) = \sum_{K\in\mathcal{T}_h}\int_K \operatorname{div}_h(\Pi_h\rho-\rho)\cdot(u-u_h^I)dx + \int_K (\Pi_h\rho-\rho):\epsilon(u-u_h^I)dx.$$

Utilizing the estimate (4.4), the bound

$$|E_h(u,\Pi_h\rho)| \le c(\|\operatorname{div}\rho\|_0\|u-u_h^I\|_0 + \|\Pi_h\rho-\rho\|_0\|\epsilon(u-u_h^I)\|_0 \le ch^2\|\rho\|_1\|u\|_2$$

is an immediate consequence. □

Remark 4.1. The consistency error estimate (4.7) holds for any $u \in \mathring{H}^1(\Omega;\mathbb{R}^3)$ satisfying $u|_K \in H^2(K,\mathbb{R}^3)$ for each $K \in \mathcal{T}_h$, provided one replaces $\|u\|_2$ with the broken H^2 norm $(\sum_{K\in\mathcal{T}_h}\|u\|^2_{H^2(K,\mathbb{R}^3)})^{1/2}$.

With these ingredients assembled, error bounds for the finite element method now follow in a straightforward fashion.

Theorem 4.1. *Let* (σ, u) *be the solution of* (2.1) *and* (σ_h, u_h) *the solution of* (2.3). *Then*

$$\begin{aligned} \|\sigma - \sigma_h\|_0 &\le ch\|u\|_2, \\ \|\operatorname{div}\sigma - \operatorname{div}_h \sigma_h\|_0 &\le ch^m \|\operatorname{div}\sigma\|_m, \quad 0 \le m \le 2, \\ \|u - u_h\|_0 &\le ch\|u\|_2. \end{aligned} \tag{4.9}$$

Furthermore, if problem (2.1) *admits full elliptic regularity, such that the estimate* (2.2) *holds, then*

$$\|u - u_h\|_0 \le ch^2\|u\|_2.$$

Proof. Subtracting the first equations of (2.1) and (2.3) and invoking the definition (4.6) of the consistency error, we get the error equation

$$\int_\Omega [A(\sigma - \sigma_h) : \tau + (u - u_h) \cdot \operatorname{div}_h \tau] dx = E_h(u, \tau), \quad \tau \in \Sigma_h. \tag{4.10}$$

Comparing the second equations in (2.1) and (2.3), we obtain $\operatorname{div}_h \sigma_h = P_h \operatorname{div}\sigma$, which immediately gives the claimed error estimate on $\operatorname{div}\sigma$. Using the commutativity (4.2), we find that $\operatorname{div}_h(\Pi_h\sigma - \sigma_h) = 0$. Choosing $\tau = \Pi_h\sigma - \sigma_h$ in (4.10), we get

$$\int_\Omega A(\sigma - \sigma_h) : (\Pi_h\sigma - \sigma_h) dx = E_h(u, \Pi_h\sigma - \sigma_h),$$

which implies that

$$\|\sigma - \sigma_h\|_A^2 \le \|\sigma - \Pi_h\sigma\|_A^2 + 2E_h(u, \Pi_h\sigma - \sigma_h),$$

where $\|\tau\|_A^2 := \int A\tau : \tau \, dx$. Combining with (4.4) and (4.7) we conclude that

$$\|\sigma - \sigma_h\| \le ch(\|\sigma\|_1 + \|u\|_2) \le ch\|u\|_2,$$

which is the desired error estimate for σ.

To get the error estimate for u, we choose $\rho \in H^1(\Omega, \mathbb{S})$ such that $\operatorname{div}\rho = P_h u - u_h$ and $\|\rho\|_1 \le c\|P_h u - u_h\|_0$. Then, in light of the commutativity property (4.2) and the bound (4.5), $\tau := \Pi_h\rho \in \Sigma_h$ satisfies $\operatorname{div}_h \tau = P_h u - u_h$ and $\|\tau\|_0 \le c\|P_h u - u_h\|_0$. Hence, using (4.1), (4.10) and (4.7), we get

$$\begin{aligned} \|P_h u - u_h\|_0^2 &= \int_\Omega \operatorname{div}_h \tau \cdot (P_h u - u_h) dx = \int_\Omega \operatorname{div}_h \tau \cdot (u - u_h) dx \\ &= -\int_\Omega A(\sigma - \sigma_h) : \tau dx + E_h(u, \tau) \\ &\le c(\|\sigma - \sigma_h\|_0 + h\|u\|_2)\|P_h u - u_h\|_0. \end{aligned} \tag{4.11}$$

This gives $\|P_h u - u_h\|_0 \le ch\|u\|_2$, and then, by the triangle inequality and (4.3), the error estimate for u can be found.

To establish the final quadratic estimate for $\|u-u_h\|_0$ in the case of full regularity, we use a duality argument. Let $\rho = A^{-1}\,\epsilon(w)$, where $w \in \mathring{H}^1(\Omega;\mathbb{R}^3) \cap H^2(\Omega;\mathbb{R}^3)$ solves the problem $\operatorname{div} A^{-1}\,\epsilon(w) = P_h u - u_h$. It follows from (2.2) that

$$\|\rho\|_1 + \|w\|_2 \le c\|P_h u - u_h\|_0. \tag{4.12}$$

By introducing $w_h^I \in W_h$ as the piecewise linear interpolant of w, we now obtain from (4.11) that

$$\begin{aligned}\|P_h u - u_h\|_0^2 &= -\int_\Omega A(\sigma-\sigma_h) : \Pi_h\rho dx + E_h(u, \Pi_h\rho)\\ &= -\int_\Omega A(\sigma-\sigma_h) : (\Pi_h\rho - \rho) dx + E_h(u, \Pi_h\rho)\\ &\quad -\int_\Omega (\sigma-\sigma_h) : \epsilon(w - w_h^I) dx,\end{aligned}$$

where the final equality follows since

$$\int_\Omega (\sigma-\sigma_h) : \epsilon(w_h^I) dx = -\sum_{K\in\mathcal{T}_h}\int_K \operatorname{div}_h(\sigma-\sigma_h)\cdot w_h^I dx + E_h(w_h^I, \sigma-\sigma_h) = 0.$$

However, by utilizing (4.4), (4.8), the estimate for $\|\sigma-\sigma_h\|_0$ given in (4.9), combined with the approximation property of the interpolant w_h^I, we obtain from the representation of $\|P_h u - u_h\|_0^2$ above that

$$\begin{aligned}\|P_h u - u_h\|_0^2 &\le c(h^2\|\rho\|_1\|u\|_2 + \|\sigma-\sigma_h\|\,\|\,\epsilon(w-w_h^I)\|_0)\\ &\le ch^2\|u\|_2(\|\rho\|_1 + \|w\|_2) \le ch^2\|u\|_2\|P_h u - u_h\|_0,\end{aligned}$$

where we have used (4.12) to obtain the final inequality. This gives $\|P_h u - u_h\|_0 \le ch^2\|u\|_2$. As above, the desired estimate for $\|u-u_h\|_0$ now follows from (4.3) and the triangle inequality. □

Remark 4.2. Although $\|\sigma - \Pi_h\sigma\|_0 = O(h^2)$, we have only shown first order convergence of the finite element solution: $\|\sigma-\sigma_h\|_0 = O(h)$. The lower rate of convergence is due to the consistency error estimated in (4.7).

5. The Reduced Element

As for the two-dimensional element,[11] there is a variant of the element using smaller spaces. Let

$$\mathbb{T}(K) = \{v \in \mathcal{P}_1(K;\mathbb{R}^3)\,|\,v(x) = a + b\times x,\ a,b\in\mathbb{R}^3\},$$

be the space of rigid motions on K. In the reduced method we take $\tilde{V}_K := \mathbb{T}(K)$ instead of $V_K = P_1(K;\mathbb{R}^3)$ as the space of shape functions for displacement, so the dimension is reduced from 12 to 6. As shape functions for stress we take

$$\tilde{\Sigma}_K = \{\tau \in \Sigma_K\,|\,\operatorname{div}_h\tau \in \mathbb{T}\},$$

so $\dim \tilde{\Sigma}_K = 36$. As degrees of freedom for $\tilde{\Sigma}_K$ we take the face moments (3.2) but dispense with the interior degrees of freedom (3.3).

Let us see how the unisolvence argument adapts to these elements. If $\tau \in \tilde{\Sigma}_K$ with vanishing degrees of freedom, then $\operatorname{div} \tau \in \mathbb{T}(K)$, and for all $v \in \mathbb{T}(K)$,

$$\int_K (\operatorname{div} \tau) v \, dx = -\int_K \tau : \epsilon(v) dx + \int_{\partial K} \tau n v \, ds = 0,$$

using the degrees of freedom and the fact that $\epsilon(v) = 0$. Thus $\operatorname{div} \tau = 0$ on K and for all $v \in \mathcal{P}_1(K; \mathbb{R}^3)$,

$$\int_K \tau : \epsilon(v) dx = -\int_K (\operatorname{div} \tau) v \, dx + \int_{\partial K} \tau n v \, ds = 0.$$

This shows that $\int_K \tau \, dx = 0$, so all degrees of freedom (3.3) vanish as well. Therefore the previous unisolvence result applies, and gives $\tau \equiv 0$.

A similar argument establishes the commutativity of the projection into $\tilde{\Sigma}_h$ (the analogue of (4.2)), and the analogue of the inclusion (4.1) obviously holds. The space $\tilde{\Sigma}_K$ still contains $\mathcal{P}_1(K; \mathbb{S})$ so the approximability (4.4) still holds, but the approximability of $\tilde{V}_K$ is of one order lower, i.e. in (4.3) m can be at most 1. As a result, the error estimates given by (4.9) in Theorem 4.1 carry over, except that m is limited to 1 in the error estimate for $\operatorname{div} \sigma$.

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