Topics in Polynomial Interpolation Theory

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Summary

This thesis studies two aspects of polynomial interpolation theory. The first part sets forth explicit formulas for the coefficients of polynomial interpolants to implicit functions. A formula for the higher-order derivatives of implicit functions appears as a limiting case of these formulas. The second part delves into certain configurations of points in space — generalized principal lattices — that are well suited as interpolation points. Applying the theory of real algebraic curves then allows the construction of many new examples of such configurations.

Acknowledgments

It is no secret that pursuing a doctoral study can be a hard and daunting task, and my case was certainly no exception to this. But once things started to go well, it has been exceptionally rewarding and worth the hardship. Needless to say, this would not have been possible if it had not been for many people helping me out.

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Blindern, December 2010, Georg Muntingh

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CHAPTER 1

Introduction to polynomial interpolation

It is an old problem in mathematics to estimate the value of a function at certain points from known values at other points. In classical times, such methods were used to construct tables with values of complicated functions, like the logarithm, trigonometric functions, and more recently statistical density functions. These *lookup tables*, as they are now called, had a wide range of practical applications, from computations in celestial mechanics to nautical navigation.

More generally, the process of reconstructing a curve, surface, or any other geometric object from certain known data is called *interpolation*, a word that is derived from the Latin word *interpolare* which means "to refurbish" or "to patch" [76]. In the context of mathematics, the word interpolation was used by J. Wallis as early as 1655 [2, 83].

These days, applications range over many different fields of pure and applied mathematics. Besides the evaluation of transcendental functions mentioned above, interpolation theory finds applications in numerical differentiation, numerical integration of differential equations, typography, and the computer-aided geometric design of cars, ships, and airplanes.

In some sense, polynomials are the simplest type of interpolants to work with, as their definition only involves a finite number of additions, subtractions, and multiplications. The fact that polynomial interpolants can suffer from *Runge's phenomenon* (see Figure 1.1) [30, 69] has given them a slightly bad reputation. Their simplicity, however, makes them perfectly suitable to be used as the building blocks of other interpolating functions with better behaviour.

A highly popular example are the *splines*, which are defined piecewise by polynomials. These functions can avoid Runge's phenomenon, have good convergence, can be evaluated accurately, and are flexible with respect to manipulation of the data. After splines were used for the first time by Schoenberg in 1946 for the purpose of approximating functions [70–72], the literature on splines has grown to a vast theory that is very popular in the industry.

Another example is that of *blending* interpolants, where local interpolants are combined into a single global interpolant. A classical case of this is Shepard's method [73],

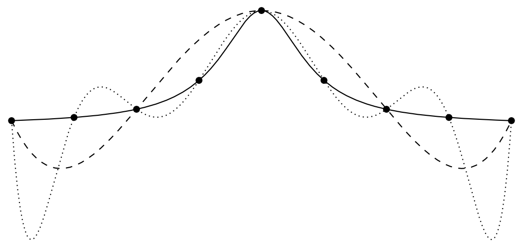


Figure 1.1: *The figure shows fourth order (drawn dashed) and eighth order (drawn dotted) polynomial interpolants to the* Runge function $f(x) = \frac{1}{1+x^2}$ *(drawn solid) at equidistant data points in the interval* $[-5, 5]$*. The Runge phenomenon describes the peculiarity that the higher the degree of the interpolating polynomial, the more wildly it oscillates at the boundary of the interpolation interval (and therefore the more it differs from* f *in the supremum norm).*

where a global interpolant at a point x is formed by taking weighted linear combinations of known values $\{y_i\}$ at the data points $\{x_i\}$, the weights being the reciprocals of some distance function between x and the $\{x_i\}$. More recently, Floater and Hormann showed how weighted linear combinations of polynomial interpolants give rise to (a family of) rational interpolants that admit a barycentric form, have a high rate of approximation, and are without real poles [34].

This thesis discusses a few known results, and several new results, in polynomial interpolation theory. This chapter introduces some of the basic concepts in interpolation theory, while preparing for the material of the other chapters. Each of the next two chapters comprises a paper, one accepted for publication by Mathematics of Computation and the other ready for submission, on formulas for divided differences of implicit functions. While the presentation of the first paper was discussed in detail with my co-author Prof. Michael Floater, I developed and drafted both papers. A formula for the higher-order derivatives of implicit functions appears as a limiting case of these formulas. Insofar as the number of terms in these formulas describes a previously unknown pattern, I published these sequences in The On-Line Encyclopedia of Integer Sequences. The final chapter shows explicitly how algebraic curves of a certain type give rise to generalized principal lattices in higher-dimensional space, yielding many new examples of such meshes.

1.1 Univariate polynomial interpolation

In this section we introduce polynomial interpolation in its simplest form, namely as the study of interpolating polynomials on the real line. Already in this case, we encounter several important notions needed in the following chapters.

Suppose that we are given $n + 1$ distinct real numbers $x_0 < x_1 < \cdots < x_n$, which we refer to as the *data points*, and corresponding real numbers $f_0, f_1, \ldots, f_n$, called the *(data) values*. We wish to find a function p that passes through these points, in the

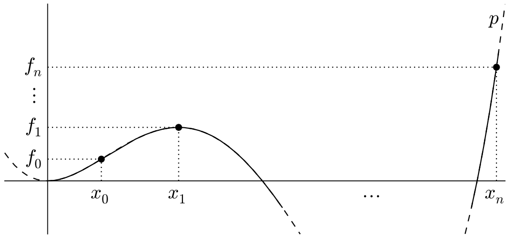


Figure 1.2: *The figure shows points $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$ in the plane together with an interpolating function p.*

sense that

$$p(x_i) = f_i, \qquad \text{for } i = 0, 1, \ldots, n. \tag{1.1}$$

This *univariate interpolation problem* is shown geometrically in Figure 1.2.

Let Π_n be the vector space of univariate polynomials with real coefficients and of degree at most n. If we require $p \in \Pi_n$, then any polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ satisfies (1.1) if and only if it satisfies the linear system

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}. \tag{1.2}$$

The above matrix is a *Vandermonde matrix*, and, by induction on n, its determinant can be shown to be $\prod_{0 \le i < j \le n}(x_j - x_i)$. As long as the data points are distinct, therefore, the interpolation problem (1.1) can be seen to have a unique solution within Π_n. Note that this uniqueness only depends on the configuration of the data points $\{x_i\}$ and not on the values $\{f_i\}$. This phenomenon becomes very complex in the multivariate case, and it forms the foundation for our study of the generalized principal lattices in Chapter 4.

Although inverting the linear system (1.2) provides a theoretical solution to the univariate interpolation problem (1.1), this method is rarely used in practice. One reason for this is that the condition number of the Vandermonde matrix may be very large [39, 40], in which case Gaussian elimination typically leads to large numerical errors. Secondly, standard Gaussian elimination requires $\mathcal{O}(n^3)$ operations, which is often too slow in practice.

The next two sections introduce classical methods much better suited for constructing the univariate interpolating polynomial. In each case, the trick is to choose a basis of Π_n that is convenient for the given configuration of data points.

1.1.1 Newton form of the interpolation polynomial

Suppose we are given the univariate interpolation problem (1.1). Let us try to construct the solution $p_n \in \Pi_n$ recursively from a supposedly known polynomial p_{n-1} of degree

$n-1$ with values $f_0, f_1, \ldots, f_{n-1}$ at the points $x_0, x_1, \ldots, x_{n-1}$.

The unique interpolation polynomial of degree 0 with value f_0 at x_0 is of course $p_0(x) = b_0 := f_0$. Adding any linear term of the form $b_1(x - x_0)$ will not change the value at x_0. Solving $b_0 + b_1(x_1 - x_0) = f_1$ for b_1, we obtain a polynomial $p_1(x) = b_0+b_1(x-x_0)$ that satisfies (1.1) for $n = 1$. Next, adding any quadratic term of the form $b_2(x-x_0)(x-x_1)$ will not change the values at x_0 and x_1. Solving $b_0+b_1(x_2-x_0)+b_2(x_2-x_0)(x_2-x_1) = f_2$ for b_2 yields a polynomial $p_2(x) = b_0+b_1(x-x_0)+b_2(x-x_0)(x-x_1)$ that satisfies (1.1) for $n = 2$. Continuing this process, we arrive at the solution $p = p_n \in \Pi_n$ to (1.1) in the *Newton form*

$$p(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)\cdots(x - x_{n-1}). \quad (1.3)$$

Once the coefficients $b_0, b_1, \ldots, b_n$ are known, the Newton form of the interpolating polynomial can be evaluated efficiently akin to Horner's rule for nested multiplication, by writing

$$p_n(x) = b_0 + (x - x_0)(b_1 + (x - x_1)(b_2 + \cdots + (x - x_{n-1})b_n)\cdots). \quad (1.4)$$

Note that the Newton form expresses the unique solution $p \in \Pi_n$ to (1.1) in terms of the *Newton basis*

$$\big\{1, (x - x_0), (x - x_0)(x - x_1), \ldots, (x - x_0)\cdots(x - x_{n-1})\big\} \quad (1.5)$$

of Π_n. In this coordinate system, the linear system (1.2) becomes the triangular system

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_1 - x_0 & 0 & \cdots & 0 \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_1 - x_0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_0 & (x_n - x_0)(x_n - x_1) & \cdots & (x_n - x_0)\cdots(x_n - x_{n-1}) \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix},$$

which can be solved directly. In practice, however, one computes the coefficients as follows. Denoting the kth coefficient b_k by $[x_0, x_1, \ldots, x_k]f$, it can be shown (see [47, Section 6.1]) that these coefficients can be obtained recursively by setting $[x_0]f := f_0$ and

$$[x_0, x_1, \ldots, x_k]f := \frac{[x_1, x_2, \ldots, x_k]f - [x_0, x_1, \ldots, x_{k-1}]f}{x_k - x_0}, \qquad k = 1, 2, \ldots, n. \quad (1.6)$$

See Table 1.1 for a graphical depiction of the recursion process. For this reason, the kth coefficient is called the *kth order divided difference of f at the points $x_0, x_1, \ldots, x_k$*. See [6] for a recent survey.

One immediately finds the expression $[x_0, x_1]f = \frac{f_1-f_0}{x_1-x_0}$ for the first order divided difference of f at x_0, x_1. In case the values $f_0, f_1, \ldots, f_n$ represent the values of an underlying smooth function $f(x)$ at the data points $x_0, x_1, \ldots, x_n$, one sees that $[x_0, x_1]f$ represents the slope of the secant through the points $\big(x_0, f(x_0)\big)$ and $\big(x_1, f(x_1)\big)$. In particular, one has the limiting case $\lim_{x_0,x_1\to x}[x_0, x_1]f = f'(x)$. More generally, we have the following theorem from [47, p. 248].

data		0th order		1st order		2nd order		3rd order
x_0, f_0	$\rightarrow$	$[x_0]f$						
			$\searrow$					
				$[x_0, x_1]f$				
			$\nearrow$		$\searrow$			
x_1, f_1	$\rightarrow$	$[x_1]f$				$[x_0, x_1, x_2]f$		
			$\searrow$		$\nearrow$		$\searrow$	
				$[x_1, x_2]f$				$[x_0, x_1, x_2, x_3]f$
			$\nearrow$		$\searrow$		$\nearrow$	
x_2, f_2	$\rightarrow$	$[x_2]f$				$[x_1, x_2, x_3]f$		
			$\searrow$		$\nearrow$			
				$[x_2, x_3]f$				
			$\nearrow$					
x_3, f_3	$\rightarrow$	$[x_3]f$						

Table 1.1: *The recursive process from (1.6) for computing the third order divided difference $[x_0, x_1, x_2, x_3]f$ from the data points x_0, x_1, x_2, x_3 and corresponding values f_0, f_1, f_2, f_3.*

Theorem 1 (mean value theorem for divided differences)**.** *Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points and let f have a continuous derivative of order n in the interval $(\min\{x_0, \ldots, x_n\}, \max\{x_0, \ldots, x_n\})$. Then, for some point ξ in this interval,*

$$[x_0, \ldots, x_n]f = \frac{1}{k!} f^{(k)}(\xi).$$

As a direct consequence, one finds

$$\lim_{x_0, x_1, \ldots, x_n \to x} [x_0, x_1, \ldots, x_n]f = \frac{1}{n!} f^{(n)}(x).$$

In this sense, divided differences can be thought of as a [discrete] generalization of derivatives. One can therefore set out to generalize the theory of differential calculus to the divided difference setting. In this manner, a divided difference version of a Leibniz rule was found by Popoviciu [66,67] and Steffensen [79]. More recently, divided difference versions of univariate chain rules were found in [35,84] and multivariate chain rules in [33,85], analogously to Faà di Bruno's formula for derivatives. In [36], one of these univariate chain rules was applied to find a formula for the (higher-order) divided differences of the inverse of a function. In the paper of Chapter 2, we generalize this formula to one that expresses the divided differences of a function y, implicitly defined by a function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ in the sense that $g\big(x, y(x)\big) = 0$ and $\frac{\partial g}{\partial y}\big(x, y(x)\big) \neq 0$ in some domain in $\mathbb{R}^2$, in terms of divided differences of g.

For a given function f, let p_n be the unique polynomial of degree at most n interpolating f at the data points $x_0, x_1, \ldots, x_n$. As described earlier in this section, adding a term $[x_0, \ldots, x_n, x_{n+1}]f \cdot (x - x_0) \cdots (x - x_n)$ to p_n yields the interpolant of degree at most $n+1$ with, in addition, value $f(x_{n+1})$ at x_{n+1}. In particular, taking $x = x_{n+1}$,

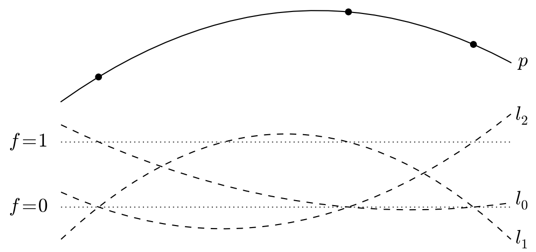


Figure 1.3: *Drawn dashed, are the Lagrange polynomials l_0, l_1, l_2 for the data $(x_0, f_0) = (1, 2), (x_1, f_1) = (3, 3)$, and $(x_2, f_2) = (4, 2.5)$. Note that $l_j(x_i) = \delta_{i,j}$. The Lagrange form of the interpolating polynomial p, drawn solid, is found by taking the linear combination $p(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x)$.*

we find the *exact* expression for the error

$$f(x) - p_n(x) = [x_0, \ldots, x_n, x]f \cdot (x - x_0) \cdots (x - x_n)$$

(cf. [47, p. 248]). Having such an exact expression for the error (and not just a bound) has the advantage that it can be used to derive error bounds of more complicated interpolants built up from polynomial interpolants. In [34], for instance, this expression was used to bound the error of a rational interpolant made by blending polynomial interpolants.

1.1.2 Lagrange and barycentric form of the interpolation polynomial

Besides the Newton form, there is another commonly used representation of the solution to the univariate interpolation problem (1.1). The idea is based upon another convenient basis of Π_n, namely the one with elements

$$l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}, \qquad j = 0, \ldots, n.$$

As these *Lagrange polynomials* have the property that $l_j(x_j) = 1$ and $l_j(x_i) = 0$ for $i \neq j$, the solution $p \in \Pi_n$ to (1.1) can be expressed in the *Lagrange form*

$$p(x) = f_0 l_0(x) + f_1 l_1(x) + \cdots + f_n l_n(x). \tag{1.7}$$

The basis $\{l_0, l_1, \ldots, l_n\}$ of Π_n is called the *Lagrange basis* associated to the univariate interpolation problem.

As the coefficients of p in the Lagrange basis are simply the data values of the interpolation problem, it is trivial to solve the linear system (1.2) in this basis. Clearly, the costly part in using the Lagrange form is not to find these coefficients. Evaluating Expression 1.7 naively, on the other hand, will set you back $\mathcal{O}(n^2)$ floating point operations (flops), as opposed to the $\mathcal{O}(n)$ flops required for evaluating Equation 1.4 once the divided differences are known. It is known, however, that the Lagrange form can be altered to a form that can be evaluated in $\mathcal{O}(n)$ operations as well [3].

Let $l(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$. Then the Lagrange polynomials can be rewritten as

$$l_j(x) = l(x) \frac{w_j}{x - x_j}, \qquad \text{where } w_j := \frac{1}{\prod_{i \neq j} (x_j - x_i)}, \qquad j = 0, 1, \ldots, n,$$

are called the *barycentric weights* associated to the univariate interpolation problem. The interpolating polynomial $p \in \Pi_n$ satisfying (1.1) can then be brought into *barycentric form*

$$p(x) = l(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j.$$

After one has computed the barycentric weights $w_0, w_1, \ldots, w_n$, this formula can be evaluated in $\mathcal{O}(n)$ flops. Moreover, after adding a new data point (x_{n+1}, f_{n+1}) to the interpolation problem, the weights can be updated in $\mathcal{O}(n)$ flops by first dividing w_j by $x_j - x_{n+1}$ for $j = 0, 1, \ldots, n$ and then computing w_{n+1} from scratch.

1.2 Multivariate polynomial interpolation

Compared to the field of univariate interpolation theory, which dates back to at least the 17th century, the field of *multivariate* interpolation theory is fairly recent. While initial publications on the topic appeared in the second half of the 19th century [42], the number of publications has grown substantially with the advent of computers.

Many of the problems encountered in multivariate polynomial interpolation have no direct analogue in the univariate case, and need to be solved with mathematics outside of the fields of approximation theory and numerical analysis. This makes the multivariate case much more complex than the univariate case. One of the issues that has been attacked from many different angles is the problem of finding configurations of points for which any interpolation problem has a unique solution in some given function space.

1.2.1 Configurations of points

As remarked in Section 1.1, the univariate interpolation problem (1.1) has a unique solution $p \in \Pi_n$ as long as the data points $x_0, x_1, \ldots, x_n$ are distinct. We proceed to show that the situation is more complicated in the multivariate setting.

Given m points $x_1, x_2, \ldots, x_m$ in $\mathbb{R}^N$ and corresponding values $f_1, f_2, \ldots, f_m$ in $\mathbb{R}$, we wish to find a function p that passes through these points, in the sense that

$$p(x_i) = f_i, \qquad \text{for } i = 1, 2, \ldots, m. \tag{1.8}$$

We shall refer to this as the *multivariate interpolation problem.* Write $X = \{x_1, x_2, \ldots, x_m\}$ and consider any vector space F of functions $\mathbb{R}^N \longrightarrow \mathbb{R}$. We call X *unisolvent*

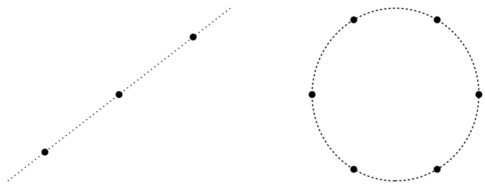


Figure 1.4: *The configuration of points to the left is not unisolvent in Π_1^2, as the three points all lie on a line. The configuration of points to the right, formed by the vertices of a hexagon, is not unisolvent in Π_2^2, as all points lie on a circle.*

in F if for any choice of the values $f_1, f_2, \ldots, f_m$ there exists a unique function $f \in F$ satisfying (1.8).

Let Π_d^N denote the set of polynomials in N variables with real coefficients and of total degree at most d. Then Π_d^N forms a vector space of dimension $\binom{d+N}{N}$. It is natural to study the case where the number of equations m matches the dimension of Π_d^N. That is, when $m = \binom{d+N}{N}$. The simplest nontrivial case is that of three given data points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ in $\mathbb{R}^2$ and corresponding values f_1, f_2, f_3 in $\mathbb{R}$. The vector space of bivariate polynomials $p(x, y)$ with real coefficients and total degree at most 1 has dimension 3, and a basis is given by $\{1, x, y\}$. Any polynomial $p(x, y) = a + bx + cy$ satisfies $p(x_i, y_i) = f_i$ for $i = 1, 2, 3$ if and only if

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

The above matrix is singular if and only if the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear, in which case there does not exist a unique polynomial $p(x, y) = a + bx + cy$ that interpolates the data as required. Note that this is not surprising: If $p(x, y) \in \Pi_1^2$ is a solution to (1.8) and $l(x, y) \in \Pi_1^2$ is a linear affine polynomial representing the line through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, then for any $\alpha \in \mathbb{R}$ the polynomial $p(x, y) + \alpha l(x, y)$ is a solution to (1.8) as well.

More generally, any finite set X of points in $\mathbb{R}^N$ is unisolvent in Π_d^N if and only if

$$\#X = \dim \Pi_d^N = \binom{N+d}{d}$$

and X is not contained in an algebraic hypersurface of degree d (see Figure 1.4). Although this condition gives a precise characterization of the configurations of points in space that are unisolvent in Π_d^N, checking this criterion requires about the same effort as trying to solve (1.8) directly. It is therefore of interest to find special configurations of points in space that are unisolvent in some function space and retain other convenient properties from the univariate case. The next two sections introduce two such configurations.

1.2.2 Rectangular meshes

The most direct manner to generalize the univariate interpolation problem to the multivariate setting is by means of rectangular meshes. It seems that these meshes were first considered in the early 20th century [42, 60, 61].

Let us focus on the 2-dimensional case, which simplifies notation and generalizes readily to the higher-dimensional case. A *rectangular mesh* X is a Cartesian product

$$X = \{x_0, \ldots, x_m\} \times \{y_0, \ldots, y_n\} = \big\{(x_i, y_j) \,:\, i = 0, \ldots, m,\ j = 0, \ldots, n\big\}, \quad (1.9)$$

where $x_0 < x_1 < \cdots < x_m$ and $y_0 < y_1 < \cdots < y_n$ for some $m, n \geq 0$. See Figure 1.5a. For any set of values $\{f_{ij} \,:\, i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n\}$ corresponding to such a rectangular mesh, the multivariate interpolation problem (1.8) takes on the form

$$p(x_i, y_j) = f_{ij}, \qquad \text{for } i = 0, 1, \ldots, m, \qquad j = 0, 1, \ldots, n. \quad (1.10)$$

It is natural to look for solutions to (1.10) in the vector space $\Pi^2_{m,n}$ of bivariate polynomials $p(x, y)$ with real coefficients and bidegree at most (m, n) (that is, of degree at most m in x and of degree at most n in y). Note that the dimension of $\Pi^2_{m,n}$ coincides with the number of points in X. The rectangular mesh X is sometimes called the *tensor product mesh*, because the vector space $\Pi^2_{m,n}$ is isomorphic to the tensor product of the vector spaces Π^2_m and Π^2_n. Because of the linearity of the univariate interpolation problem and the bilinear nature of the tensor product, many of the constructions for univariate interpolation problems trivially carry over to constructions on rectangular meshes.

To illustrate this connection, let be given data points $x_0 < x_1$ and $y_0 < y_1$ and real numbers f_0, f_1, g_0, g_1. Suppose we are given the two univariate interpolation problems

$$p(x_0) = f_0, \qquad p(x_1) = f_1,$$

$$q(y_0) = g_0, \qquad q(y_1) = g_1,$$

where $p(z) = a_0 + a_1 z$ and $q(z) = b_0 + b_1 z$ are both elements of Π_1. In matrix form, these problems can be written as

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & y_0 \\ 1 & y_1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix}.$$

Equating the tensor product of the left hand sides with the tensor product of the right hand sides yields an equation

$$\left(\begin{bmatrix} 1 & y_0 \\ 1 & y_1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \right) = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \otimes \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}.$$

Using the *mixed product rule* $(AC) \otimes (BD) = (A \otimes B)(C \otimes D)$ to interchange the tensor product and the matrix product, this equation is equivalent to the system

$$\begin{bmatrix} 1 & x_0 & y_0 & x_0 y_0 \\ 1 & x_1 & y_0 & x_1 y_0 \\ 1 & x_0 & y_1 & x_0 y_1 \\ 1 & x_1 & y_1 & x_1 y_1 \end{bmatrix} \begin{bmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{bmatrix} = \begin{bmatrix} g_0 f_0 \\ g_0 f_1 \\ g_1 f_0 \\ g_1 f_1 \end{bmatrix},$$

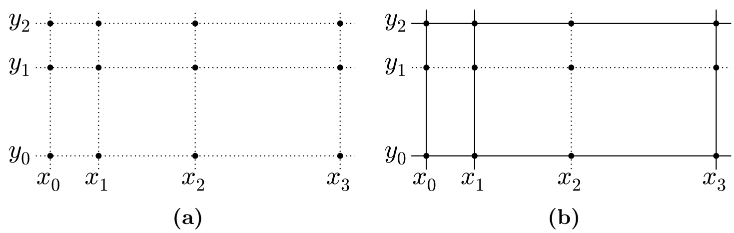


Figure 1.5: *The figure to the left shows a rectangular mesh as in (1.9) for* $(m,n) = (2,1)$*. The figure to the right shows the zeroset of the Lagrange polynomial* $l_{2,1}(x,y)$ *as a union of lines (drawn solid) passing through all points* (x_i, y_j) *except* (x_2, y_1)*.*

which represents the bivariate interpolation problem on the rectangular mesh $\{x_0, x_1\} \times \{y_0, y_1\}$ in $\Pi^2_{1,1}$.

Conversely, writing $p(x,y) = \sum_{i=0}^m \sum_{j=0}^n a_{ij} x^i y^j$, the bivariate interpolation problem on a rectangular mesh from (1.10) can be brought into the matrix form

$$\begin{bmatrix} 1 & y_0 & \cdots & y_0^n \\ 1 & y_1 & \cdots & y_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_n & \cdots & y_n^n \end{bmatrix} \otimes \begin{bmatrix} 1 & x_0 & \cdots & x_0^m \\ 1 & x_1 & \cdots & x_1^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^m \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ \vdots \\ a_{m0} \\ a_{01} \\ a_{11} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} f_{00} \\ f_{10} \\ \vdots \\ f_{m0} \\ f_{01} \\ f_{11} \\ \vdots \\ f_{mn} \end{bmatrix}.$$

It can be shown that any tensor product $A \otimes B$ is singular precisely when A or B is singular. As the Vandermonde matrices in the above equation are nonsingular, it follows that the rectangular mesh (1.9) is unisolvent in $\Pi^2_{m,n}$.

The Lagrange basis from Section 1.1.2 generalizes to the *Lagrange basis of* $\Pi^2_{m,n}$ with elements

$$l_{ij}(x,y) = \prod_{\substack{k=0\\k\neq i}}^m \frac{x - x_k}{x_i - x_k} \prod_{\substack{l=0\\l\neq j}}^n \frac{y - y_l}{y_j - y_l}, \qquad i = 0,1,\ldots,m, \qquad j = 0,1,\ldots,n.$$

Each *Lagrange polynomial* $l_{ij}(x,y)$ is of bidegree (m,n) and satisfies $l_{ij}(x_k, y_l) = \delta_{ik}\delta_{jl}$ for any i,j,k,l. It follows that the solution $p \in \Pi^2_{m,n}$ to (1.10) can be written in the *Lagrange form*

$$p(x,y) = \sum_{i=0}^m \sum_{j=0}^n f_{ij} l_{ij}(x,y).$$

Analogously to Equation 1.3, the interpolating polynomial p can be expressed in terms of the *Newton basis*

$$\left\{ \prod_{k=0}^{i-1}(x - x_k) \prod_{l=0}^{j-1}(y - y_l) \;:\; i = 0, 1, \ldots, m,\; j = 0, 1, \ldots, n \right\}$$

of $\Pi^2_{m,n}$ (an empty product is considered to be 1), yielding the *Newton form*

$$p(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{ij} \prod_{k=0}^{i-1}(x - x_k) \prod_{l=0}^{j-1}(y - y_l).$$

Let us denote the coefficient b_{ij} by $[x_0, x_1, \ldots, x_i; y_0, y_1, \ldots, y_j]f$ and refer to it as the *(bivariate) divided difference of order* (i, j). It can be shown (see [47, Section 6.6]) that these coefficients can be obtained recursively as follows. For $i = j = 0$ one sets $[x_0; y_0]f = f_{00}$. If $i > 0$ one defines

$$[x_0, \ldots, x_i; y_0, \ldots, y_j]f := \frac{[x_1, \ldots, x_i; y_0, \ldots, y_j]f - [x_0, \ldots, x_{i-1}; y_0, \ldots, y_j]f}{x_i - x_0},$$

and if $j > 0$ one defines

$$[x_0, \ldots, x_i; y_0, \ldots, y_j]f := \frac{[x_0, \ldots, x_i; y_1, \ldots, y_j]f - [x_0, \ldots, x_i; y_0, \ldots, y_{j-1}]f}{y_j - y_0}.$$

If both $i > 0$ and $j > 0$, the divided difference $[x_0, \ldots, x_i; y_0, \ldots, y_j]f$ is uniquely defined by either recursion formula.

For a given bivariate function f, let $p \in \Pi^2_{m,n}$ be the polynomial interpolating f at the rectangular mesh (1.9). From [47, Section 6.5], we have the exact expression

$$\begin{aligned} f(x, y) - p(x, y) = [x_0, \ldots, x_m, x; y]f \prod_{i=0}^{m}(x - x_i) + [x; y_0, \ldots, y_n, y]f \prod_{j=0}^{n}(y - y_j) \\ - [x_0, \ldots, x_m, x; y_0, \ldots, y_n, y]f \prod_{i=0}^{m}(x - x_i) \prod_{j=0}^{n}(y - y_j) \end{aligned}$$

for the error in the point (x, y).

Divided differences of functions with more than two variables are defined similarly. Let $y(x_1, \ldots, x_N)$ be a function that is implicitly defined by a function $g : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ via the relations

$$g\big(x_1, \ldots, x_N, y(x_1, \ldots, x_N)\big) = 0, \qquad \frac{\partial g}{\partial y}\big(x_1, \ldots, x_N, y(x_1, \ldots, x_N)\big) \neq 0$$

that hold in some domain in $\mathbb{R}^N$. In the paper of Chapter 3, we find a formula that expresses the divided differences of y in terms of the divided differences of g, which is a generalization of the paper of Chapter 2. Moreover, letting the points in the rectangular grid coalesce to a single point, we find a formula that expresses the derivatives of y (of any order) in terms of the derivatives of g.

1.2.3 Geometric characterization

In this section we discuss a characterization of meshes in $\mathbb{R}^N$ for which the interpolant in Π_d^N can be brought into a form similar to the Lagrange form.

Let us generalize the notion of a Lagrange form. Consider the ordered set of data points $X = \{x_1, \ldots, x_m\} \subset \mathbb{R}^N$ with $m = \binom{N+d}{d}$ for some nonnegative integer d. A polynomial $l \in \Pi_d^N$ is called a *Lagrange polynomial associated to* $x \in X$ if $l(x) = 1$ and $l(y) = 0$ for any $y \in X \backslash \{x\}$. An ordered basis $\{l_1, \ldots, l_m\}$ of Π_d^N is called a *Lagrange basis* for X, if $l_i(x_j) = \delta_{ij}$ for $1 \leq i, j \leq m$. If, in addition, each Lagrange polynomial splits into a product of real linear factors, we call $\{l_1, \ldots, l_m\}$ a *simple* Lagrange basis for X. Whenever one has a Lagrange basis for X, the multivariate interpolation problem (1.8) admits a unique solution

$$p(x) = \sum_{i=1}^{m} f_i l_i(x) \in \Pi_d^N.$$

In [24], Chung and Yao formulated these ideas in a geometric manner.

Definition 2. As above, let $X = \{x_1, \ldots, x_m\} \subset \mathbb{R}^N$ with $m = \binom{N+d}{d}$ for some nonnegative integer d. Then X is said to satisfy the *geometric characterization*, if for each data point x_i there exist d distinct hyperplanes $H_{ij} : h_{ij}(x) = 0$, with $j = 1, 2, \ldots, d$, such that

(i) x_i does not lie on any of these hyperplanes, and

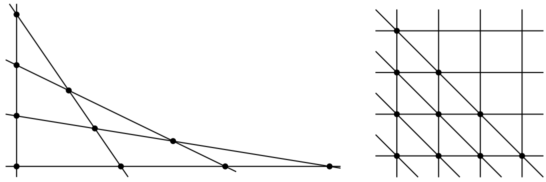
(ii) all other data points in $X \backslash \{x_i\}$ lie on at least one of these hyperplanes.

Let $X = \{x_1, \ldots, x_m\}$ be a set satisfying the geometric characterization with corresponding hyperplanes $H_{ij} : h_{ij}(x) = 0$, with $j = 1, \ldots, d$, associated to each $x_i \in X$. For any $x_i \in X$, the polynomial

$$l_i(x) = \prod_{j=1}^{d} \frac{h_{ij}(x)}{h_{ij}(x_i)}$$

is a Lagrange polynomial associated to $x_i \in X$, and the collection $\{l_1, \ldots, l_m\}$ forms a simple Lagrange basis for X. It follows that there exists a simple Lagrange basis for any set X satisfying the geometric characterization. Conversely, if a set X admits a simple Lagrange basis of Π_d^N, it will satisfy the geometric characterization [24, Theorem 1].

As a first example, suppose we are given $N + d$ hyperplanes $H_1, H_2, \ldots, H_{N+d} \subset \mathbb{R}^N$, any N of which intersect in precisely one point. If all of these $\binom{N+d}{N}$ points are distinct, we call the set X of these points a *natural lattice* (see Figure 1.6a for an example in the plane). Any natural lattice satisfies the geometric characterization [24, Theorem 3]. Another example is formed by the generalized principal lattices, of which the triangular meshes form are the simplest cases, see Figure 1.6b. Any generalized principal lattice is formed by intersecting collections of hyperplanes in a manner very similar to how a triangular mesh is formed by intersecting collections of hyperplanes. A precise definition of generalized principal lattices in the plane will be given in Section 4.1 and a general definition in Section 4.4.1.



(a) (b)

Figure 1.6: *To the left: A natural lattice corresponding to* $2+3$ *lines in* $\mathbb{R}^2$ *that intersect generically. To the right: A triangular mesh.*

Although the geometric characterization is a beautiful way to look at Lagrange interpolation, it does not show us how all such meshes can be constructed. Moreover, their incidence structures are not well understood [7], as is shown by the following conjecture, made in [41].

Conjecture 3 (Gasca, Maeztu)**.** *Let* $X = \{x_1, x_2, \ldots, x_m\} \subset \mathbb{R}^2$*, with* $m = \binom{d+2}{2}$*, satisfy the geometric characterization with lines* $L_{i1}, \ldots, L_{id}$ *associated to* x_i *for* $i = 1, 2, \ldots, m$*. Then one of these lines contains* $d+1$ *points of* X*.*

Despite the simple nature of this statement, it has shown difficult to prove. It is only known to be true for $d \leq 4$ [9, 11], and the complexity of the proofs increases with the degree d.

To gain better insight into the incidence structures of meshes satisfying the geometric characterization, Carnicer, Gasca, and Godes set out to classify the meshes in the plane that satisfy the geometric characterization according to their *defect* [12, 19, 21, 22]. Note that each of the $d+2$ lines in Figure 1.6a passes through $d+1$ points of X. Carnicer and Gasca refer to the natural lattices in the plane as meshes with defect 0 (called *default* in [12]), as these represent the generic case of $d+2$ lines intersecting in the plane. More generally, let $X \subset \mathbb{R}^2$ be a configuration of points satisfying the geometric characterization for which there are precisely k lines passing through $d+1$ points of X. Such a set is said to have defect $d+2-k$.

In [13, Theorem 4.1], it is shown that if Conjecture 3 holds for all degrees d up to some D, then, for any set $X \subset \mathbb{R}^2$ satisfying the geometric characterization and with $\#X = \binom{d'+2}{2}$ elements for some $d' \leq D$, there are at least *three* such lines with $d+1$ points of X. This would imply that the defect can be at most $d-1$. Note that for the triangular mesh in Figure 1.6b only three of the lines contain $d+1$ points of X. In this sense, the natural lattices and the generalized principal lattices represent opposite ends in the classification of Carnicer and Gasca (compare [18, Theorem 3.6]). This makes studying generalized principal lattices a worthy goal.

In Section 4.1–4.3, we show how generalized principal lattices are a natural generalization of triangular meshes and exhibit in detail a classification of generalized principal lattices in the plane by Carnicer and Gasca [15, 16]. The final section of Chapter 4 studies generalized principal lattices in higher-dimensional space, which were introduced by Carnicer, Gasca, and Sauer in [17]. Each generalized principal lattice constructed in this article corresponds to a parameterized curve. After converting these curves to implicit form, we realized that these are all real algebraic curves in $\mathbb{P}^n$ of degree $n+1$ and arithmetic genus 1. Moreover, for $n = 3$ these curves can be conceived very concretely as the complete intersection of two quadric surfaces. With the help of a technical tool introduced in [17], it is shown in Sections 4.4.5 – 4.4.9 that all curves of this type can be used to define a generalized principal lattice in 3-dimensional projective space. The resulting classification is summarized in Table 4.4.

CHAPTER 2

Paper 1: Divided Differences of Univariate Implicit Functions

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Abstract

Under general conditions, the equation $g(x, y) = 0$ implicitly defines y locally as a function of x. In this article, we express divided differences of y in terms of bivariate divided differences of g, generalizing a recent result on divided differences of inverse functions.

2.1 Introduction

Divided differences can be viewed as a discrete analogue of derivatives and are commonly used in approximation theory, see [6] for a survey.

Recently, the second author and Lyche established two univariate chain rules for divided differences [35], both of which can be viewed as analogous to Faà di Bruno's formula for differentiating composite functions [31, 48]. One of these formulas was simultaneously discovered by Wang and Xu [85]. In a follow-up preprint, the other chain rule was generalized to the composition of vector-valued functions of several variables [33], yielding a formula analogous to a multivariate version of Faà di Bruno's formula [27].

In [36], the univariate chain rule was applied to find a formula for divided differences of the inverse of a function. In Theorem 4, the Main Theorem of this paper, we use the multivariate chain rule to prove a similar formula for divided differences of implicitly defined functions. Equation 2.16 shows that the formula for divided differences of inverse functions in [36] follows as a special case.

More precisely, let y be a function that is defined implicitly by a function $g : \mathbb{R}^2 \to \mathbb{R}$ via $g\big(x, y(x)\big) = 0$ and $\frac{\partial g}{\partial y}\big(x, y(x)\big) \neq 0$, for every x in an open interval $U \subset \mathbb{R}$. Then the Main Theorem states that for any

$$x_0, \ldots, x_n \in U, \qquad y_0 := y(x_0), \ldots, y_n := y(x_n) \in y(U)$$

we can express $[x_0, \ldots, x_n]y$ as a sum of terms involving the divided differences

$$[x_{i_0}, \ldots, x_{i_s}; y_{i_s}, \ldots y_{i_r}]g,$$

with $0 \leq i_0 < i_1 < \cdots < i_r \leq n$.

In Section 3.2, we define these divided differences and explain our notation. In Section 3.3, we apply the multivariate chain rule to derive a formula that recursively expresses divided differences of y in terms of divided differences of g and lower order divided differences of y. Finally, in Section 2.4, we solve this recursive formula to obtain a formula that expresses divided differences of y solely in terms of divided differences of g. We end the section with applying the Main Theorem in some special cases.

2.2 Divided differences

Let $[x_0, \ldots, x_n]f$ denote the *divided difference* of a function $f : (a, b) \to \mathbb{R}$ at the points $x_0, \ldots, x_n$, with $a < x_0 \leq \cdots \leq x_n < b$. If all inequalities are strict, this notion is recursively defined by $[x_0]f := f(x_0)$ and

$$[x_0, \ldots, x_n]f = \frac{[x_1, \ldots, x_n]f - [x_0, \ldots, x_{n-1}]f}{x_n - x_0} \qquad \text{if } n > 0.$$

If some of the $\{x_i\}$ coincide, we define $[x_0, \ldots, x_n]f$ as the limit of this formula when the distances between these $\{x_i\}$ become arbitrary small, provided f is sufficiently smooth there. In particular, when $x_0 = \cdots = x_n$, one can show that $[x_0, \ldots, x_n]f = f^{(n)}(x_0)/n!$. For given $i_0, \ldots, i_k$ satisfying $i_0 \leq i_1 \leq \cdots \leq i_k$, we shall sometimes shorten notation to

$$[i_0 i_1 \cdots i_k]f := [x_{i_0}, x_{i_1}, \ldots x_{i_k}]f. \tag{2.1}$$

The above definitions generalize to bivariate divided differences as follows. Let $f : U \to \mathbb{R}$ be defined on some 2-dimensional interval

$$U = (a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2.$$

Suppose we are given $m, n \geq 0$ and points $x_0, \ldots, x_m \in (a_1, b_1)$ satisfying $x_0 < \cdots < x_m$ and $y_0, \ldots, y_m \in (a_2, b_2)$ satisfying $y_0 < \cdots < y_m$. The Cartesian product

$$\{x_0, \ldots, x_m\} \times \{y_0, \ldots, y_n\}$$

defines a rectangular grid of points in U. The *(bivariate) divided difference* of f at this grid, denoted by

$$[x_0, \ldots, x_m; y_0, \ldots, y_n]f, \tag{2.2}$$

can be defined recursively as follows. If $m = n = 0$, the grid consists of only one point (x_0, y_0), and we define $[x_0; y_0]f := f(x_0, y_0)$ as the value of f at this point. In case $m > 0$, we can define (2.2) as

$$\frac{[x_1, \ldots, x_m; y_0, \ldots, y_n]f - [x_0, \ldots, x_{m-1}; y_0, \ldots, y_n]f}{x_m - x_0},$$

or if $n > 0$, as

$$\frac{[x_0, \ldots, x_m; y_1, \ldots, y_n]f - [x_0, \ldots, x_m; y_0, \ldots, y_{n-1}]f}{y_n - y_0}.$$

If both $m > 0$ and $n > 0$ the divided difference (2.2) is uniquely defined by either recursion formula.

As for univariate divided differences, we can let some of the points coalesce by taking limits, as long as f is sufficiently smooth. In particular when $x_0 = \cdots = x_m$ and $y_0 = \cdots = y_n$, this legitimates the notation

$$[x_0, \ldots, x_m; y_0, \ldots, y_n]f := \frac{1}{m!n!}\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(x_0, y_0).$$

Similarly to Equation 2.1, we shall more often than not shorten the notation for bivariate divided differences to

$$[i_0 i_1 \cdots i_s; j_0 j_1 \cdots j_t]f := [x_{i_0}, x_{i_1}, \ldots, x_{i_s}; y_{j_0}, y_{j_1}, \ldots, y_{j_t}]f. \tag{2.3}$$

2.3 A recursive formula for implicit functions

Let y be a function implicitly defined by $g\big(x, y(x)\big) = 0$ as in Section 3.1. The first step in expressing divided differences of y in terms of those of g is to express those of g in terms of those of y. This link is provided by a special case of the the multivariate chain rule of [33]. Let $\mathbb{R} \xrightarrow{\mathbf{f}} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}$ be a composition of sufficiently smooth functions $\mathbf{f} = (\phi, \psi)$ and g. In this case, the formula of [33] for $n \geq 1$ is

$$[x_0, x_1, \ldots, x_n](g \circ \mathbf{f}) = \sum_{k=1}^{n} \sum_{0=i_0<i_1<\cdots<i_k=n} \sum_{s=0}^{k} \tag{2.4}$$
$$[\phi(x_{i_0}), \phi(x_{i_1}), \ldots, \phi(x_{i_s}); \psi(x_{i_s}), \psi(x_{i_{s+1}}), \ldots, \psi(x_{i_k})]g \times$$
$$\prod_{l=1}^{s}[x_{i_{l-1}}, x_{i_{l-1}+1}, \ldots, x_{i_l}]\phi \prod_{l=s+1}^{k} [x_{i_{l-1}}, x_{i_{l-1}+1}, \ldots, x_{i_l}]\psi.$$

Now we choose $\mathbf{f}$ to be the graph of a function y, i.e., $\mathbf{f} : x \mapsto \big(\phi(x), \psi(x)\big) = \big(x, y(x)\big)$. Then the divided differences of ϕ of order greater than one are zero, implying that the summand is zero unless $(i_0, i_1, \ldots, i_s) = (0, 1, \ldots, s)$; below this condition is realized by restricting the third sum in Equation 2.4 to integers s that satisfy $s = i_s - i_0$. Since additionally divided differences of ϕ of order one are one, we obtain

$$[x_0, x_1, \ldots, x_n]g\big(\cdot, y(\cdot)\big) = \sum_{k=1}^{n} \sum_{0=i_0<i_1<\cdots<i_k=n} \sum_{\substack{s=0 \\ s=i_s-i_0}}^{k} \tag{2.5}$$

$$[x_0, x_1, \ldots, x_s; y_{i_s}, y_{i_{s+1}}, \ldots, y_{i_k}]g \prod_{l=s+1}^{k} [x_{i_{l-1}}, x_{i_{l-1}+1}, \ldots, x_{i_l}]y,$$

where $y_j := y(x_j)$ for $j = 0, 1, \ldots, n$. For example, when $n = 1$ this formula becomes

$$[x_0, x_1]g\big(\cdot, y(\cdot)\big) =$$

$$[x_0; y_0, y_1]g\,[x_0, x_1]y + \qquad (k, s) = (1, 0)$$

$$[x_0, x_1; y_1]g, \qquad (k, s) = (1, 1)$$

and when $n = 2$,

$$[x_0, x_1, x_2]g\big(\cdot, y(\cdot)\big) =$$

$$[x_0; y_0, y_2]g\,[x_0, x_1, x_2]y + \qquad (k, s) = (1, 0)$$

$$[x_0; y_0, y_1, y_2]g\,[x_0, x_1]y\,[x_1, x_2]y + \qquad (k, s) = (2, 0)$$

$$[x_0, x_1; y_1, y_2]g\,[x_1, x_2]y + \qquad (k, s) = (2, 1)$$

$$[x_0, x_1, x_2; y_2]g. \qquad (k, s) = (2, 2)$$

In case y is implicitly defined by $g\big(x, y(x)\big) = 0$, the left hand side of Equation 2.5 is zero. In the case $n = 1$, therefore, we see that

$$[01]y = -\frac{[01; 1]g}{[0; 01]g}, \tag{2.6}$$

where we now used the shorthand notation from Equations 2.1 and 3.6. For $n \geq 2$, the highest order divided difference of y present in the right hand side of Equation 2.5 appears in the term $[0; 0n]g\,[01 \cdots n]y$. Moving this term to the left hand side and dividing by $-[0; 0n]g$, one finds a formula that expresses $[01 \cdots n]y$ recursively in terms of lower order divided differences of y and divided differences of g,

$$[01 \cdots n]y = -\sum_{k=2}^{n} \sum_{0=i_0<\cdots<i_k=n} \sum_{\substack{s=0 \\ s=i_s-i_0}}^{k} \tag{2.7}$$

$$\frac{[01 \cdots s; i_s i_{s+1} \cdots i_k]g}{[0; 0n]g} \prod_{l=s+1}^{k} [i_{l-1}(i_{l-1}+1) \cdots i_l]y.$$

We shall now simplify Equation 2.7. By Equation 2.6, the first order divided differences of y appearing in the product of Equation 2.7 can be expressed as quotients of divided differences of g. To separate, for every sequence $(i_0, i_1, \ldots, i_k)$ appearing in Equation 2.7, the divided differences of g from those of y, we define an expression involving only divided differences of g,

$$\{i_0 \cdots i_k\}g := -\sum_{\substack{s=0 \\ s=i_s-i_0}}^{k} \frac{[i_0 \cdots i_s; i_s \cdots i_k]g}{[i_0; i_0 i_k]g} \prod_{\substack{l=s+1 \\ i_l-i_{l-1}=1}}^{k} \left(-\frac{[i_{l-1} i_l; i_l]g}{[i_{l-1}; i_{l-1} i_l]g}\right). \tag{2.8}$$

Note that if a sequence $(i_0, \ldots, i_k)$ starts with precisely s consecutive integers, the expression $\{i_0 \cdots i_k\}g$ will comprise s terms. For instance,

$$\{023\}g = \frac{[0;023]g}{[0;03]g}\frac{[23;3]g}{[2;23]g},$$

$$\{013\}g = \frac{[0;013]g}{[0;03]g}\frac{[01;1]g}{[0;01]g} - \frac{[01;13]g}{[0;03]g},$$

$$\{012\}g = -\frac{[0;012]g}{[0;02]g}\frac{[01;1]g}{[0;01]g}\frac{[12;2]g}{[1;12]g} + \frac{[01;12]g}{[0;02]g}\frac{[12;2]g}{[1;12]g} - \frac{[012;2]g}{[0;02]g}.$$

The remaining divided differences $[i_{l-1} \cdots i_l]y$ in the product of Equation 2.7 are those with $i_l - i_{l-1} \geq 2$, and each of these comes after any s satisfying $s = i_s - i_0$. We might therefore as well start the product of these remaining divided differences at $l = 1$ instead of at $l = s+1$, which has the advantage of making it independent of s. Equation 2.7 can thus be rewritten as

$$[0 \cdots n]y = \sum_{k=2}^{n} \sum_{0=i_0<\cdots<i_k=n} \{i_0 \cdots i_k\}g \prod_{\substack{l=1 \\ i_l - i_{l-1} \geq 2}}^{k} [i_{l-1} \cdots i_l]y. \tag{2.7$'$}$$

For $n = 2, 3, 4$ this expression amounts to

$$[012]y = \{012\}g, \tag{2.9}$$

$$[0123]y = \{0123\}g + \{023\}g\,[012]y + \{013\}g\,[123]y, \tag{2.10}$$

$$\begin{aligned}[01234]y = {} & \{01234\}g + \{0134\}g\,[123]y + \{034\}g\,[0123]y \\ & + \{0124\}g\,[234]y + \{0234\}g\,[012]y + \{014\}g\,[1234]y \\ & + \{024\}g\,[012]y\,[234]y. \end{aligned} \tag{2.11}$$

2.4 A formula for divided differences of implicit functions

In this section we shall solve the recursive formula from Equation 2.7′. Repeatedly applying Equation 2.7′ to itself yields

$$[012]y = \{012\}g, \tag{2.12}$$

$$[0123]y = \{0123\}g + \{023\}g\,\{012\}g + \{013\}g\,\{123\}g, \tag{2.13}$$

$$\begin{aligned}[01234]y = {} & \{01234\}g + \{0134\}g\,\{123\}g + \{034\}g\,\{013\}g\,\{123\}g \\ & + \{034\}g\,\{0123\}g + \{034\}g\,\{023\}g\,\{012\}g + \{0124\}g\,\{234\}g \\ & + \{0234\}g\,\{012\}g + \{014\}g\,\{134\}g\,\{123\}g + \{014\}g\,\{1234\}g \\ & + \{014\}g\,\{124\}g\,\{234\}g + \{024\}g\,\{012\}g\,\{234\}g. \end{aligned} \tag{2.14}$$

Examining these examples, one finds that each term in the right hand sides of the above formulas corresponds to a partition of a convex polygon in a manner we shall now make precise.

With a sequence of labels $0, 1, \ldots, n$ we associate the ordered vertices of a convex polygon. A *partition of a convex polygon* is the result of connecting any pairs of nonadjacent vertices with straight line segments, none of which intersect. We refer to these line segments as the *inner edges* of the partition. We denote the set of all such partitions of a polygon with vertices $0, 1, \ldots, n$ by $\mathcal{P}(0, 1, \ldots, n)$. Every partition $\pi \in \mathcal{P}(0, 1, \ldots, n)$ is described by its set $F(\pi)$ of (oriented) faces. Each face $f \in F(\pi)$ is defined by some increasing sequence of vertices $i_0, i_1, \ldots, i_k$ of the polygon, i.e., $f = (i_0, i_1, \ldots, i_k)$. We denote the set of edges in π by $E(\pi)$.

Let y be a function implicitly defined by $g\big(x, y(x)\big) = 0$ and $(x_0, y_0), \ldots,$ (x_n, y_n) be as in Section 3.1. Equations 2.12–2.14 suggest the following theorem.

Theorem 4 (Main Theorem). *For y and g defined as above and sufficiently smooth and for $n \geq 2$,*

$$[0 \cdots n]y = \sum_{\pi \in \mathcal{P}(0,\ldots,n)} \prod_{(v_0,\ldots,v_r) \in F(\pi)} \{v_0 \cdots v_r\}g, \tag{2.15}$$

where $\{v_0 \cdots v_r\}g$ is defined by Equation 2.8.

Before we proceed with the proof of this theorem, we make some remarks. For $n = 2, 3, 4$ this theorem reduces to the statements of Equations 2.12–2.14. To prove Theorem 4, our plan is to use Equation 2.7′ recursively to express $[01 \cdots n]y$ solely in terms of divided differences of g. We have found it helpful to assign some visual meaning to Equation 2.7′. Every sequence $\mathbf{i} = (i_0, i_1, \ldots, i_k)$ that appears in Equation 2.7′ induces a partition $\pi_{\mathbf{i}} \in \mathcal{P}(0, 1, \ldots, n)$ whose set of faces comprises an *inner face* $(i_0, i_1, \ldots, i_k)$ and *outer faces* $(i_j, i_j + 1, \ldots, i_{j+1})$ for every $j = 0, \ldots, k-1$ with $i_{j+1} - i_j \geq 2$. We denote by $\mathcal{P}_{\mathbf{i}}$ the set of all partitions of the disjoint union of these outer faces. An example of such a sequence $\mathbf{i}$, together with its inner face, outer faces, and partition set $\mathcal{P}_{\mathbf{i}}$ is given in Figure 2.1.

We shall now associate divided differences to these geometric objects. To each outer face $(i_j, i_j + 1, \ldots, i_{j+1})$ we associate the divided difference $[i_j(i_j + 1) \cdots i_{j+1}]y$, and to each inner face $(i_0, i_1, \ldots, i_k)$ we associate the expression $\{i_0 \cdots i_k\}g$. For any sequence $\mathbf{i}$ that appears in the sum of Equation 2.7′, the corresponding inner face therefore represents that part of Equation 2.7′ that can be written solely in terms of divided differences of g, while the outer faces represent the part that is still expressed as a divided difference of y.

Repeatedly applying Equation 2.7′ yields a recursion tree, in which each node represents a product of divided difference expressions associated to inner and outer faces. These recursion trees are depicted in Figure 2.2 for $n = 2, 3, 4$. Equation 2.7′ roughly states that the expression of any nonleaf vertex is equal to the sum of the expressions of its descendants.

Proof of the Main Theorem. This theorem is a generalization of Theorem 1 in [36], and the proofs are analogous. We prove the formula by induction on the order n of the divided difference of y.

By the above discussion, the formula holds for $n = 2, 3, 4$. For $n \geq 5$, assume the formula holds for all smaller n. Consider the recursive formula from Equation 2.7′. For every sequence $\mathbf{i}$ that appears in this equation, the corresponding outer faces have

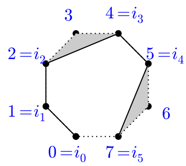

Figure 2.1: *For $n = 7$, the sequence $\mathbf{i} = (0, 1, 2, 4, 5, 7)$ gives rise to the two outer faces $(2, 3, 4)$ and $(5, 6, 7)$, which are drawn shaded in the figure. The set $\mathcal{P}_{\mathbf{i}}$ contains in this case just $1 \times 1 = 1$ partition, namely the union of the unique partitions $\{(2, 3, 4)\}$ and $\{(5, 6, 7)\}$ of the outer faces.*

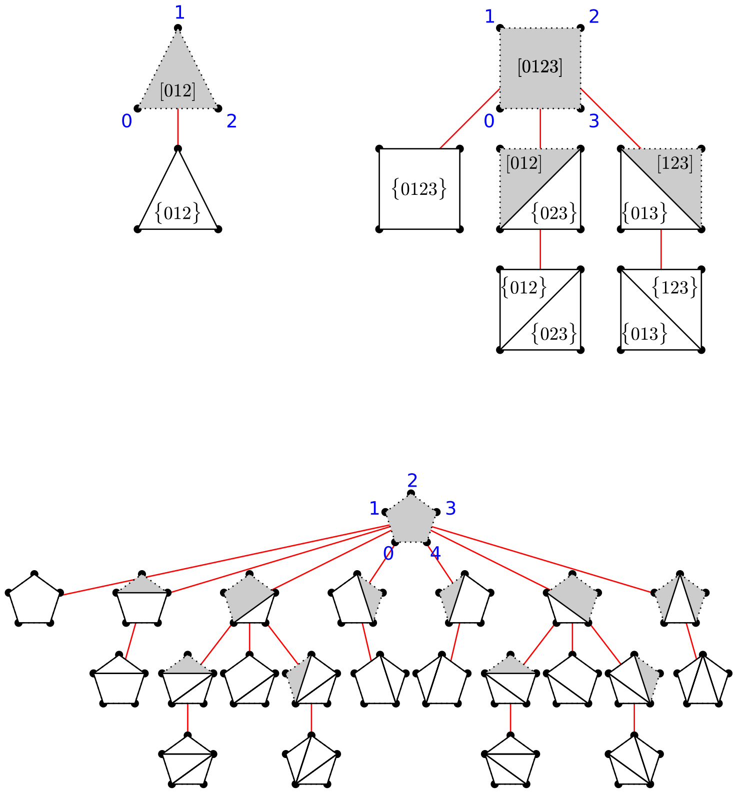

Figure 2.2: *For $n = 2, 3, 4$, the figure depicts the recursion trees obtained by repeatedly applying Equation 2.7′. The top levels of these recursion trees correspond to Equations 2.9–2.11.*

fewer vertices than the full polygon. By the induction hypothesis, we can therefore replace each divided difference $[i_l \cdots i_{l+1}]y$ appearing in the product of Equation 2.7′ by an expression involving only divided differences of g.

As before, let $\mathcal{P}_{\mathbf{i}}$ denote the set of all partitions of the disjoint union of the outer faces induced by $\mathbf{i}$. Then, by the induction hypothesis, the product in Equation 2.7′ is equal to

$$\sum_{\pi \in \mathcal{P}_{\mathbf{i}}} \prod_{(v_0,\ldots,v_r)\in F(\pi)} \{v_0 \cdots v_r\}g.$$

For a given inner face $\mathbf{i}$, the set $\mathcal{P}_{\mathbf{i}}$ can be identified with $\{\pi \in \mathcal{P}(0,\ldots,n) : \mathbf{i} \in F(\pi)\}$ by the bijection $F(\pi) \mapsto F(\pi) \cup \{\mathbf{i}\}$. Substituting the above expression into Equation 2.7′ then yields

$$\begin{aligned}
[0 \cdots n]y &= \sum_{\substack{\text{inner faces}\\ \mathbf{i}=(i_0,\ldots,i_k)}} \{i_0 \cdots i_k\}g \sum_{\pi\in\mathcal{P}_{\mathbf{i}}} \prod_{(v_0,\ldots,v_r)\in F(\pi)} \{v_0 \cdots v_r\}g \\
&= \sum_{\substack{\text{inner faces}\\ \mathbf{i}=(i_0,\ldots,i_k)}} \sum_{\substack{\pi\in\mathcal{P}(0,\ldots,n)\\ \mathbf{i}\in F(\pi)}} \prod_{(v_0,\ldots,v_r)\in F(\pi)} \{v_0 \cdots v_r\}g \\
&= \sum_{\pi\in\mathcal{P}(0,\ldots,n)} \prod_{(v_0,\ldots,v_r)\in F(\pi)} \{v_0 \cdots v_r\}g. \qquad \square
\end{aligned}$$

Intuitively, this proof can be expressed in terms of the recursion tree as follows. As remarked in the previous section, Equation 2.7′ states that the expression of any nonleaf vertex is equal to the sum of the expressions of its descendants. By induction, the expression $[01 \cdots n]y$ of the root vertex is therefore equal to the sum of the expressions of the leaves, which, by construction, correspond to partitions of the full polygon.

Example 1. Let us apply Theorem 4 to find a simple expression for divided differences of the function $y(x) = \sqrt{1-x^2}$ defined on the interval $(-1,1)$. This function is implicitly defined by the polynomial $g(x,y) = x^2 + y^2 - 1 = 0$. For any knots x_a, x_b, x_c, x_d satisfying $-1 < x_a \le x_b \le x_c \le x_d < 1$ and corresponding function values y_a, y_b, y_c, y_d, one finds

$$[x_a, x_b; y_c]g = x_a + x_b, \qquad [x_a, x_b, x_c; y_d]g = 1,$$

$$[x_a; y_b, y_c]g = y_b + y_c, \qquad [x_a; y_b, y_c, y_d]g = 1,$$

and all other divided differences of g of nonzero order are zero. In particular, every divided difference of g of total order at least three is zero, which means that the sum in Equation 2.15 will only be over *triangulations* (i.e., partitions in which all faces are triangles). For a polygon with vertices $0, 1, \ldots, n$, Exercise 6.19a of [?Stanley2] states that the number of such triangulations is given by the Catalan number

$$C(n-1) = \frac{1}{n}\binom{2n-2}{n-1}.$$

Consider, for a given triangulation $\pi \in \mathcal{P}(0,1,\ldots,n)$, a face $(a,b,c) \in F(\pi)$ from the product in Equation 2.15. As any divided difference of the form $[x_a, x_b; y_b, y_c]g$ is zero for this g, Equation 2.8 expresses $\{abc\}g$ as a sum of at most two terms. There are four cases.

$$\{abc\}g = \begin{cases} \dfrac{-1}{y_a+y_c}\left[1+\dfrac{x_a+x_b}{y_a+y_b}\cdot\dfrac{x_b+x_c}{y_b+y_c}\right] & a,b,c \text{ consecutive;} \\ \dfrac{1}{y_a+y_c}\cdot\dfrac{x_a+x_b}{y_a+y_b} & \text{only } a,b \text{ consecutive;} \\ \dfrac{1}{y_a+y_c}\cdot\dfrac{x_b+x_c}{y_b+y_c} & \text{only } b,c \text{ consecutive;} \\ \dfrac{-1}{y_a+y_c} & \text{otherwise.} \end{cases}$$

For example, when $n=3$, our convex polygon is a quadrilateral, which admits $C(3-1)=2$ triangulations π_1 and π_2 with sets of faces

$$F(\pi_1)=\{(0,1,2),(0,2,3)\}, \qquad F(\pi_2)=\{(0,1,3),(1,2,3)\}.$$

One finds

$$\begin{aligned} [x_0,x_1,x_2,x_3]\sqrt{1-x^2} = \{012\}g\,\{023\}g + \{013\}g\,\{123\}g = \\ \frac{-1}{(y_0+y_3)(y_0+y_2)}\left[1+\frac{x_0+x_1}{y_0+y_1}\cdot\frac{x_1+x_2}{y_1+y_2}\right]\cdot\frac{x_2+x_3}{y_2+y_3}+ \\ \frac{-1}{(y_0+y_3)(y_1+y_3)}\left[1+\frac{x_1+x_2}{y_1+y_2}\cdot\frac{x_2+x_3}{y_2+y_3}\right]\cdot\frac{x_0+x_1}{y_0+y_1}. \end{aligned}$$

Example 2. Next we show that Theorem 4 is a generalization of Theorem 1 of [36], which gives a similar formula for inverse functions. To see this, we apply Theorem 4 to a function y implicitly defined by a function $g(x,y)=x-h(y)$. Referring to Equation 2.8, we need to compute $[i_0\cdots i_s; i_s\cdots i_k]g$ for this choice of g and various indices $i_0,\dots,i_k$ and $s\in\{0,\dots,k\}$. Applying the recursive definition of bivariate divided differences, one obtains

$$[i_0\cdots i_s; i_s\cdots i_k]x = \begin{cases} x_{i_0} & \text{if } s=0, s=k; \\ 1 & \text{if } s=1, s=k; \\ 0 & \text{otherwise,} \end{cases}$$

$$[i_0\cdots i_s; i_s\cdots i_k]h(y) = \begin{cases} [i_s\cdots i_k]h & \text{if } s=0; \\ 0 & \text{otherwise.} \end{cases}$$

Consider a face $f=(v_0,\dots,v_r)$ of a given partition $\pi\in\mathcal{P}(0,\dots,n)$ in Equation 2.15. Since $r\geq 2$, the divided difference $[v_0\cdots v_s; v_s\cdots v_r]\big(x-h(y)\big)$ is zero for $s\geq 1$. Using this, Equation 2.8 expresses $\{v_0\cdots v_r\}g$ as a single term

$$\begin{aligned} \{v_0\cdots v_r\}g &= -\frac{[v_0; v_0\cdots v_r]g}{[v_0; v_0 v_r]g}\prod_{\substack{l=1\\ v_l-v_{l-1}=1}}^{r}\left(-\frac{[v_{l-1}v_l; v_l]g}{[v_{l-1}; v_{l-1}v_l]g}\right) \\ &= -\frac{[v_0\cdots v_r]h}{[v_0 v_r]h}\prod_{\substack{l=1\\ v_l-v_{l-1}=1}}^{r}\frac{1}{[v_{l-1}v_l]h}. \end{aligned}$$

Taking the product over all faces in the partition π, the denominators of the factors in the above equation correspond to the edges of the partition, while the numerators correspond to the faces of the partition. As there is a minus sign for each face in the partition, we arrive at the formula

$$[01\cdots n]y = \sum_{\pi\in\mathcal{P}(0,\ldots,n)} (-1)^{\#F(\pi)} \frac{\prod\limits_{(v_0,\ldots,v_r)\in F(\pi)} [v_0v_1\cdots v_r]h}{\prod\limits_{(v_0,v_1)\in E(\pi)} [v_0v_1]h}, \tag{2.16}$$

which appears as Equation 11 in [36].

Note that the inverse of the algebraic function $y = \sqrt{1-x^2}$ in Example 1 is again an algebraic function. Equation 2.16 would therefore not have been of much help to find a simple expression for divided differences of y. In fact, Example 1 can be thought of as one of the simplest examples for which Theorem 4 improves on Equation 2.16, as it concerns a polynomial g with bidegree as low as (2,2).

Example 3. In this example we shall derive a quotient rule for divided differences. That is, we shall find a formula that expresses divided differences of the quotient $y = P(x)/Q(x)$ in terms of divided differences of P and of Q. Let $g(x,y) = Q(x)y - P(x)$. Then, in Equation 2.8,

$$[i_0\cdots i_s; i_s\cdots i_k]g = \begin{cases} y_{i_s}[i_0\cdots i_s]Q - [i_0\cdots i_s]P & \text{if } s = k; \\ [i_0\cdots i_s]Q & \text{if } s = k-1; \\ 0 & \text{otherwise.} \end{cases} \tag{2.17}$$

In Equation 2.15, therefore, the only partitions with a nonzero contribution are those whose faces have all their vertices consecutive, except possibly the final one. In particular, the inner face with vertices $0 = i_0 < \cdots < i_k = n$ should either be the full polygon, or should have a unique inner edge (i_{k-1}, n). By induction, it follows that the partitions with a nonzero contribution to Equation 2.15 are precisely those for which all inner edges end at n. These partitions correspond to subsets $I \subset \{1,2,\ldots,n-2\}$, including the empty set, by associating with any such I the partition with inner edges $\{(i,n) : i \in I\}$. Equation 2.15 becomes

$$\begin{aligned} [0\cdots n]\frac{P}{Q} ={}& \{0\cdots n\}g + \\ & \sum_{r=1}^{n-2}\sum_{k=1}^{r}\sum_{0=i_0<i_1<\cdots<i_k=r} \{r\cdots n\}g \prod_{j=1}^{k}\{i_{j-1}\cdots i_j n\}g, \end{aligned} \tag{2.18}$$

where the dots represent consecutive nodes and an empty product is understood to be one. A long but straightforward calculation involving Equations 2.8, 2.17, and 2.18 yields

$$\begin{aligned} [0\cdots n]\frac{P}{Q} ={}& \frac{[0\cdots n]P}{Q_0} + \\ & \sum_{r=1}^{n}\frac{[r\cdots n]P}{Q_r}\sum_{k=1}^{r}(-1)^k \sum_{0=i_0<i_1<\cdots<i_k=r}\prod_{j=1}^{k}\frac{[i_{j-1}\cdots i_j]Q}{Q_{i_{j-1}}}, \end{aligned}$$

where $Q_i := Q(x_i)$ for $i = 0, \dots, n$. Alternatively, this equation can be found by applying a univariate chain rule to the composition $x \mapsto Q(x) \mapsto 1/Q(x)$, as described in Section 4 of [35].

Finally, we note that taking the limit $x_0, \dots, x_n \to x$ in Equations 2.6, 2.8, 2.12, and 2.13 yields

$$
\begin{aligned}
y'(x) &= -\frac{g_{10}}{g_{01}}, \\
y''(x) &= -\frac{g_{20}}{g_{01}} + 2\frac{g_{11}g_{10}}{g_{01}^2} - \frac{g_{02}g_{10}^2}{g_{01}^3}, \\
y'''(x) &= -\frac{g_{30}}{g_{01}} + 3\frac{g_{21}g_{10}}{g_{01}^2} + 3\frac{g_{20}g_{11}}{g_{01}^2} - 3\frac{g_{20}g_{10}g_{02}}{g_{01}^3} - 3\frac{g_{12}g_{10}^2}{g_{01}^3} \\
&\quad - 6\frac{g_{11}^2g_{10}}{g_{01}^3} + \frac{g_{10}^3g_{03}}{g_{01}^4} + 9\frac{g_{11}g_{10}^2g_{02}}{g_{01}^4} - 3\frac{g_{10}^3g_{02}^2}{g_{01}^5},
\end{aligned}
$$

where we introduced the shorthand

$$
g_{st} := \frac{\partial^{s+t} g}{\partial x^s \partial y^t}\big(x, y(x)\big).
$$

These formulas agree with the examples given in [26], [25, Page 153] and with a formula stated as Equation 7 in [87].

Acknowledgment

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CHAPTER 3

Paper 2: Divided Differences of Multivariate Implicit Functions

Georg Muntingh

Abstract

Under general conditions, the equation $g(x^1, \ldots, x^q, y) = 0$ implicitly defines y locally as a function of $x^1, \ldots, x^q$. In this article, we express divided differences of y in terms of divided differences of g, generalizing a recent formula for the case where y is univariate. The formula involves a sum over a combinatorial structure whose elements can be viewed either as polygonal partitions or as planar trees. Through this connection, we prove as a corollary a formula for derivatives of y in terms of derivatives of g. The relation between these formulas yields a generating function for the number of terms in the formula for divided differences of implicit functions.

3.1 Introduction

Divided differences can be viewed as a discrete analogue of derivatives and are commonly used in approximation theory, see [6] for a survey.

Recently, Floater and Lyche introduced a multivariate chain rule for divided differences [33], analogous to a multivariate form of Faá di Bruno's formula for derivatives [27, 31, 48]. In Theorem 1 in [56], this chain rule was applied to find an expression for divided differences of univariate implicit functions, thereby generalizing a formula by Floater and Lyche for divided differences of the inverse of a function [36].

In Theorem 5, the Main Theorem of this paper, we generalize Theorem 1 in [56] to divided differences of multivariate implicit functions. More precisely, for some open box $U \subset \mathbb{R}^q$ and open interval $V \subset \mathbb{R}$, let $y : U \longrightarrow V$ be a function that is implicitly defined by a function $g : U \times V \longrightarrow \mathbb{R}$ via

$$g\big(\mathbf{x}, y(\mathbf{x})\big) = 0, \qquad \frac{\partial g}{\partial y}\big(\mathbf{x}, y(\mathbf{x})\big) \neq 0 \qquad \forall\, \mathbf{x} \in U. \quad (3.1)$$

Then the Main Theorem states that, for any rectangular grid

$$\{x_0^1, \ldots, x_{n_1}^1\} \times \cdots \times \{x_0^q, \ldots, x_{n_q}^q\} \subset U,$$

we can express the divided difference $[x_0^1, \ldots, x_{n_1}^1; \cdots; x_0^q, \ldots, x_{n_q}^q]y$ as a sum of terms involving the divided differences of g.

In the next section, we define these divided differences and explain our notation. In Section 3.3, we apply the multivariate chain rule to derive a formula that recursively expresses divided differences of y in terms of divided differences of g and lower-order divided differences of y. This recursive formula is solved in Section 3.4, yielding a formula that expresses divided differences of y solely in terms of divided differences of g. This formula is stated in the Main Theorem as a sum over polygonal partitions. It is shown in Section 3.5, that such polygonal partitions correspond to planar trees of a certain type, giving rise to an alternative form of the Main Theorem. Switching between these combinatorial structures, we are able to prove as a special case in Section 3.6 a generalization of a formula by Comtet, Fiolet, and Wilde for the *derivatives* of y in terms of the derivatives of g. In the final section, this connection is used to find a generating function for the number of terms appearing in the Main Theorem.

3.2 Divided differences

Consider a function $y : U \longrightarrow \mathbb{R}$ defined on some open box

$$U = (a_1, b_1) \times \cdots \times (a_q, b_q) \subset \mathbb{R}^q. \tag{3.2}$$

Suppose that, for some integers $n_1, \ldots, n_q \geq 0$ and all $j = 1, \ldots, q$, we are given points $x_0^j, \ldots, x_{n_j}^j \in (a_j, b_j)$ satisfying $a_j < x_0^j < \cdots < x_{n_j}^j < b_j$. The Cartesian product

$$\{x_0^1, \ldots, x_{n_1}^1\} \times \cdots \times \{x_0^q, \ldots, x_{n_q}^q\} \tag{3.3}$$

defines a rectangular grid of points in U. The *divided difference* of y at this grid, denoted by

$$[x_0^1, \ldots, x_{n_1}^1; \cdots; x_0^q, \ldots, x_{n_q}^q]y, \tag{3.4}$$

can be defined recursively as follows. If $n_1 = \cdots = n_q = 0$, the grid consists of only one point $(x_0^1, \ldots, x_0^q)$, and we define $[x_0^1; \cdots; x_0^q]y := y(x_0^1, \ldots, x_0^q)$ as the value of y at this point. In case $n_j > 0$ for some $1 \leq j \leq q$, we can define (3.4) recursively by

$$\begin{aligned} (x_{n_j}^j - x_0^j)[x_0^1, \ldots, x_{n_1}^1; \cdots; x_0^q, \ldots, x_{n_q}^q]y = \\ [x_0^1, \ldots, x_{n_1}^1; \cdots; \widehat{x_0^j}, x_1^j, \ldots, x_{n_j-1}^j, x_{n_j}^j; \cdots; x_0^q, \ldots, x_{n_q}^q]y - \\ [x_0^1, \ldots, x_{n_1}^1; \cdots; x_0^j, x_1^j, \ldots, x_{n_j-1}^j, \widehat{x_{n_j}^j}; \cdots; x_0^q, \ldots, x_{n_q}^q]y, \end{aligned} \tag{3.5}$$

where the hat signifies omission of a symbol. If several of the n_j are greater than zero, the divided difference (3.4) is uniquely defined by any of these recursive formulas. We refer to the size of the grid $(n_1, \ldots, n_q)$ as the *order* of the divided difference in Equation 3.4.

For any $\mathbf{a} = (a_1, \ldots, a_q), \mathbf{b} = (b_1, \ldots, b_q) \in \mathbb{N}^q$, write $\mathbf{a} \leq \mathbf{b}$ whenever $a_j \leq b_j$ for every $1 \leq j \leq q$. Additionally, we write $\mathbf{a} < \mathbf{b}$ whenever $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. In this manner, the symbol $\leq$ defines a partial order on $\mathbb{R}^q$. We use the notation

$$[\mathbf{x} : \mathbf{a}, \mathbf{b}]y := \big[x^1_{a_1}, x^1_{a_1+1}, \ldots, x^1_{b_1};\ \cdots\ ; x^q_{a_q}, x^q_{a_q+1}, \ldots, x^q_{b_q}\big]y$$

for the divided difference of y with respect to the grid of all points with indices "between $\mathbf{a}$ and $\mathbf{b}$".

Divided differences of the function $g : U \times V \longrightarrow \mathbb{R}$ in Equation 3.1 are defined similarly. For these functions, however, we stress the distinction between the variables $x^1, \ldots, x^q$ and the variable y by replacing the final semi-colon by a bar in our notation.

As the notation of Equation 3.4 quickly grows cumbersome, we shall more often than not shorten the notation for divided differences to one that involves just the indices,

$$\big[i_{1,0} \cdots i_{1,s_1}; \cdots ; i_{q,0} \cdots i_{q,s_q}\big]y := \big[x^1_{i_{1,0}}, \ldots, x^1_{i_{1,s_1}}; \cdots ; x^q_{i_{q,0}}, \ldots, x^q_{i_{q,s_q}}\big]y, \tag{3.6}$$

$$\big[i_{1,0} \cdots i_{1,s_1}; \cdots ; i_{q,0} \cdots i_{q,s_q}\big| j_0 \cdots j_t\big]g := \big[x^1_{i_{1,0}}, \ldots, x^1_{i_{1,s_1}}; \cdots ; x^q_{i_{q,0}}, \ldots, x^q_{i_{q,s_q}}\big| y_{j_0}, \ldots, y_{j_t}\big]g. \tag{3.7}$$

We can let some of the points coalesce by taking limits, as long as y is sufficiently smooth. In particular, letting all points in the grid coalesce to a single point $\mathbf{x}_0 = (x^1_0, \ldots, x^q_0)$ yields, for any tuple $\mathbf{n} = (n_1, \ldots, n_q) \in \mathbb{N}^q$, the notation

$$\big[\underbrace{x^1_0, \ldots, x^1_0}_{n_1+1}; \cdots ; \underbrace{x^q_0, \ldots, x^q_0}_{n_q+1}\big]y = \frac{1}{\mathbf{n}!}\frac{\partial^{|\mathbf{n}|}y}{\partial \mathbf{x}^{\mathbf{n}}}(\mathbf{x}_0).$$

Here the derivatives are written in multi-index notation, $|\mathbf{n}| := n_1 + \cdots + n_q$, and $\mathbf{n}! := n_1! \cdots n_q!$. Letting, in addition, the y-values coalesce to a single point y_0 yields the notation

$$\big[\underbrace{x^1_0, \ldots, x^1_0}_{n_1+1}; \cdots ; \underbrace{x^q_0, \ldots, x^q_0}_{n_q+1}\big|\underbrace{y_0, \ldots, y_0}_{m+1}\big]g = \frac{1}{\mathbf{n}!m!}\frac{\partial^{|\mathbf{n}|+m}g}{\partial \mathbf{x}^{\mathbf{n}}\partial y^m}(\mathbf{x}_0, y_0).$$

3.3 A recursive formula

Let y and g be related as in Equation 3.1. In this section, we derive a formula that expresses divided differences of y recursively as divided differences of g and lower-order divided differences of y.

Consider a composition of functions $\mathbb{R}^q \xrightarrow{\mathbf{f}} U \times V \xrightarrow{g} \mathbb{R}$, where we write $\mathbf{f} : \mathbf{x} \longmapsto \big(f^1(\mathbf{x}), \ldots, f^q(\mathbf{x}), f'(\mathbf{x})\big)$. Here f' does not signify the derivative of some function f but is simply some coordinate function. Let be given a nonzero vector $\mathbf{n} = (n_1, \ldots, n_q) \in \mathbb{N}^q$ and a grid of points

$$\big\{\mathbf{x}_{\mathbf{i}} : \mathbf{0} \leq \mathbf{i} \leq \mathbf{n}\big\} = \big\{x^1_0, \ldots, x^1_{n_1}\big\} \times \cdots \times \big\{x^q_0, \ldots, x^q_{n_q}\big\} \subset U,$$

where $x^j_0 \leq \cdots \leq x^j_{n_j}$ for $j = 1, \ldots, q$. Note that we allow for these coordinates to coincide. Let $f'_{\mathbf{i}} := f'(\mathbf{x}_{\mathbf{i}})$ and $f^j_{\mathbf{i}} := f^j(\mathbf{x}_{\mathbf{i}})$ for $j = 1, \ldots, q$ and $\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}$. From

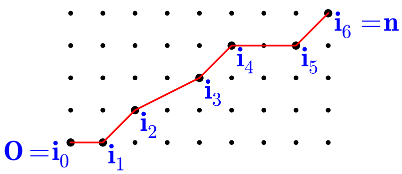


Figure 3.1: *Any sequence* $\mathbf{0} = \mathbf{i}_0 < \mathbf{i}_1 < \cdots < \mathbf{i}_k = \mathbf{n}$ *represents a lattice path from* $\mathbf{0}$ *to* $\mathbf{n}$.

[33, Theorem 2], we have the following *multivariate chain rule* in case $\mathbf{f}$ and g are sufficiently smooth,

$$[\mathbf{x} : \mathbf{0}, \mathbf{n}](g \circ \mathbf{f}) = \sum_{k=1}^{|\mathbf{n}|} \sum_{\mathbf{0}=\mathbf{i}_0<\cdots<\mathbf{i}_k=\mathbf{n}} \sum_{0=j_0\leq\cdots\leq j_{q+1}=k} \tag{3.8}$$

$$[f^1_{\mathbf{i}_{j_0}}, f^1_{\mathbf{i}_{j_0+1}}, \ldots, f^1_{\mathbf{i}_{j_1}}; \cdots ; f^q_{\mathbf{i}_{j_{q-1}}}, f^q_{\mathbf{i}_{j_{q-1}+1}}, \ldots, f^q_{\mathbf{i}_{j_q}} | f'_{\mathbf{i}_{j_q}}, f'_{\mathbf{i}_{j_q+1}}, \ldots, f'_{\mathbf{i}_{j_{q+1}}}]g$$

$$\times \left(\prod_{r=1}^{q} \prod_{j=j_{r-1}+1}^{j_r} [\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j] f^r \right) \left(\prod_{j=j_q+1}^{j_{q+1}} [\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j] f' \right),$$

where an empty product is considered to be one. (The formula in [33] includes a term for $k = 0$, but this term doesn't show up because we chose $\mathbf{n} \neq \mathbf{0}$.) Here one should think of $\mathbf{0} = \mathbf{i}_0 < \cdots < \mathbf{i}_k = \mathbf{n}$ as a lattice path from $\mathbf{0}$ to $\mathbf{n}$ and of $0 = j_0 \leq \cdots \leq j_{q+1} = k$ as indices of points along this path; see Figure 3.1.

Next, let $\mathbf{f} : \mathbf{x} = (x^1, \ldots, x^q) \longmapsto \big(\mathbf{x}, y(\mathbf{x})\big)$ define the graph of a function y that is implicitly defined by g as in Equation 3.1. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$ denote the standard basis of $\mathbb{R}^q$, and let $1 \leq j \leq k$ and $1 \leq r \leq q$ be as in Equation 3.8. It follows directly from Equation 3.5 that the divided difference $[\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j]f^r$ of the coordinate function $f^r : (x^1, \ldots, x^q) \longmapsto x^r$ is equal to one whenever $\mathbf{i}_j - \mathbf{i}_{j-1} = \mathbf{e}_r$, and zero otherwise. The only choices of $0 = j_0 \leq \cdots \leq j_{q+1} = k$ that yield a nonzero term in Equation 3.8 are therefore those satisfying

$$\mathbf{i}_j - \mathbf{i}_{j-1} = \mathbf{e}_r, \qquad \text{for} \quad j = j_{r-1} + 1, \ldots, j_r, \quad r = 1, \ldots, q. \tag{3.9}$$

Alternatively, let $(s_1, \ldots, s_q, t) := (j_1 - j_0, \ldots, j_q - j_{q-1}, j_{q+1} - j_q)$ be the sequence of *jumps* in the sequence $(j_0, \ldots, j_{q+1})$. In terms of these jumps, Equation 3.9 is equivalent to the statement that the path $\mathbf{i}_0 < \cdots < \mathbf{i}_k$ starts with

s_1 steps of $\mathbf{e}_1$, followed by

s_2 steps of $\mathbf{e}_2$, followed by

$\vdots$

s_q steps of $\mathbf{e}_q$, followed by

t arbitrary steps.

Let us call any tuple $(s_1, \ldots, s_q, t)$ with this property *compatible with* $(\mathbf{i}_0, \ldots, \mathbf{i}_k)$, or simply *compatible* if it is clear which sequence $(\mathbf{i}_0, \ldots, \mathbf{i}_k)$ is referred to. Note that such a tuple forms an integer partition $k = s_1 + \cdots + s_q + t$. Equation 3.8 thus implies

$$[\mathbf{x} : \mathbf{0}, \mathbf{n}]g\big(\cdot, y(\cdot)\big) = \sum_{k=1}^{|\mathbf{n}|} \sum_{\mathbf{0}=\mathbf{i}_0<\cdots<\mathbf{i}_k=\mathbf{n}} \sum_{\substack{\text{compatible} \\ (s_1,\ldots,s_q,t)}} \tag{3.10}$$

$$[0\,1\cdots s_1;\ \cdots\ ;0\,1\cdots s_q|\mathbf{i}_{|\mathbf{s}|}\,\mathbf{i}_{|\mathbf{s}|+1}\cdots\mathbf{i}_{|\mathbf{s}|+t}]g \prod_{j=|\mathbf{s}|+1}^{|\mathbf{s}|+t} [\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j]y,$$

where we used the shorthand notation

$$\begin{aligned}&[0\,1\cdots s_1;\ \cdots\ ;0\,1\cdots s_q|\mathbf{i}_{|\mathbf{s}|}\,\mathbf{i}_{|\mathbf{s}|+1}\cdots\mathbf{i}_{|\mathbf{s}|+t}]g \\ =\ &\big[x_0^1, x_1^1, \ldots, x_{s_1}^1;\ \cdots\ ;x_0^q, x_1^q, \ldots, x_{s_q}^q\big|y_{\mathbf{i}_{|\mathbf{s}|}}, y_{\mathbf{i}_{|\mathbf{s}|+1}}, \ldots, y_{\mathbf{i}_{|\mathbf{s}|+t}}\big]g, \qquad y_{\mathbf{i}} := y(\mathbf{x}_{\mathbf{i}}),\end{aligned}$$

from Equation 3.7 for the divided differences of g.

If y is implicitly defined by g as in Equation 3.1, the left hand side of Equation 3.10 is zero. Suppose $\mathbf{n}$ has length $|\mathbf{n}| = 1$. Then $\mathbf{n} = \mathbf{e}_r$ for some $1 \le r \le q$. In this case, the right hand side of Equation 3.10 consists of two terms with $k = 1$ and $\mathbf{0} = \mathbf{i}_0 < \mathbf{i}_1 = \mathbf{e}_r$. One finds

$$0 = [\mathbf{x} : \mathbf{0}, \mathbf{e}_r]g\big(\cdot, y(\cdot)\big) =$$

$$[0; \cdots ; 0|\mathbf{0}\,\mathbf{e}_r]g\,[\mathbf{x} : \mathbf{0}, \mathbf{e}_r]y + [\underbrace{0; \cdots ; 0}_{r-1}; 0\,1; 0; \cdots ; 0|\mathbf{e}_r]g,$$

or, equivalently,

$$[\mathbf{x} : \mathbf{0}, \mathbf{e}_r]y = -\frac{[\overbrace{0; \cdots ; 0}^{r-1}; 0\,1; 0; \cdots ; 0|\mathbf{e}_r]g}{[0; \cdots ; 0|\mathbf{0}\,\mathbf{e}_r]g}, \qquad \text{for } r = 1, \ldots, q. \tag{\textbf{R1}}$$

For example when y is a function of $q = 2$ variables, this equation represents the two formulas

$$[0\,1; 0]y = -\frac{[0\,1; 0|\mathbf{e}_1]g}{[0; 0|\mathbf{0}\,\mathbf{e}_1]g}, \qquad [0; 0\,1]y = -\frac{[0; 0\,1|\mathbf{e}_2]g}{[0; 0|\mathbf{0}\,\mathbf{e}_2]g}. \tag{3.11}$$

Now suppose $\mathbf{n}$ has length $|\mathbf{n}| > 1$. There is only one term in the right hand side of Equation 3.10 with $k = 1$. This term is given by

$$\mathbf{0} = \mathbf{i}_0 < \mathbf{i}_1 = \mathbf{n}, \qquad s_1 = \cdots = s_q = 0, \qquad t = 1$$

and contains the highest order divided difference of y. Isolating this divided difference yields a formula that recursively expresses divided differences of y in terms of divided differences of g and lower-order divided differences of y,

$$[\mathbf{x} : \mathbf{0}, \mathbf{n}]y = \sum_{k=2}^{|\mathbf{n}|} \sum_{\mathbf{0}=\mathbf{i}_0<\cdots<\mathbf{i}_k=\mathbf{n}} \sum_{\substack{\text{compatible} \\ (s_1,\ldots,s_q,t)}} \tag{\textbf{R2}}$$

$$\left(-\frac{[0\,1\cdots s_1;\ \cdots\ ;0\,1\cdots s_q|\mathbf{i}_{|\mathbf{s}|}\mathbf{i}_{|\mathbf{s}|+1}\cdots\mathbf{i}_{|\mathbf{s}|+t}]g}{[0;\cdots;0|\mathbf{0}\,\mathbf{n}]g}\right)\prod_{j=|\mathbf{s}|+1}^{|\mathbf{s}|+t}[\mathbf{x}:\mathbf{i}_{j-1},\mathbf{i}_j]y.$$

Let us simplify this formula. The product in Equation **R2** can be split into two products

$$\prod_{\substack{j=|\mathbf{s}|+1\\|\mathbf{i}_j-\mathbf{i}_{j-1}|=1}}^{|\mathbf{s}|+t}[\mathbf{x}:\mathbf{i}_{j-1},\mathbf{i}_j]y\prod_{\substack{j=|\mathbf{s}|+1\\|\mathbf{i}_j-\mathbf{i}_{j-1}|\geq 2}}^{|\mathbf{s}|+t}[\mathbf{x}:\mathbf{i}_{j-1},\mathbf{i}_j]y. \tag{3.12}$$

From Equation **R1** it follows that each divided difference in the first product can be expressed as a quotient of divided differences of g. Our ultimate goal is to express the left hand side of Equation **R2** solely in terms of divided differences of g. To achieve this, it seems natural to split the right hand side into a part that can directly be expressed in terms of divided differences of g and a remaining part involving higher-order divided differences of y. The former part can be expressed by introducing, for every sequence

$$(i_0^1,\ldots,i_0^q)=\mathbf{i}_0<(i_1^1,\ldots,i_1^q)=\mathbf{i}_1<\cdots<(i_k^1,\ldots,i_k^q)=\mathbf{i}_k,$$

a symbol $\{\mathbf{i}_0\cdots\mathbf{i}_k\}g$ for the expression

$$\sum_{\substack{\text{compatible}\\(s_1,\ldots,s_q,t)}}\left(-\frac{\left[i_0^1\cdots(i_0^1+s_1);\ \cdots\ ;i_0^q\cdots(i_0^q+s_q)\middle|\mathbf{i}_{|\mathbf{s}|}\cdots\mathbf{i}_{|\mathbf{s}|+t}\right]g}{\left[i_0^1;\cdots;i_0^q\middle|\mathbf{i}_0\,\mathbf{i}_k\right]g}\right) \tag{3.13}$$

$$\times\prod_{r=1}^{q}\prod_{\substack{j=|\mathbf{s}|+1\\\mathbf{i}_j-\mathbf{i}_{j-1}=\mathbf{e}_r}}^{|\mathbf{s}|+t}\left(-\frac{\left[i_{j-1}^1;\cdots;i_{j-1}^{r-1};i_{j-1}^r\,i_j^r;i_j^{r+1};\cdots;i_j^q\middle|\mathbf{i}_j\right]g}{\left[i_{j-1}^1;\cdots;i_{j-1}^q\middle|\mathbf{i}_{j-1}\,\mathbf{i}_j\right]g}\right)$$

involving only divided differences of g. Whenever it is hard to visually separate the multi-indices $\mathbf{i}_0,\ldots,\mathbf{i}_k$, we write $\{\mathbf{i}_0,\ldots,\mathbf{i}_k\}g$ instead of $\{\mathbf{i}_0\cdots\mathbf{i}_k\}g$.

The divided differences $[\mathbf{x}:\mathbf{i}_{j-1},\mathbf{i}_j]y$ that appear in the second product of Equation 3.12 satisfy $|\mathbf{i}_j-\mathbf{i}_{j-1}|\geq 2$. As Equation 3.9 guarantees that this cannot happen for $j\leq j_q=|\mathbf{s}|$, we might as well start the product of these remaining divided differences at $j=1$ instead of at $j=|\mathbf{s}|+1$. This has the advantage of making the expression independent of $|\mathbf{s}|$. Equation **R2** can therefore be written in the concise form

$$[\mathbf{x}:\mathbf{0},\mathbf{n}]y=\sum_{k=2}^{|\mathbf{n}|}\sum_{\mathbf{0}=\mathbf{i}_0<\cdots<\mathbf{i}_k=\mathbf{n}}\{\mathbf{i}_0\cdots\mathbf{i}_k\}g\prod_{\substack{j=1\\|\mathbf{i}_j-\mathbf{i}_{j-1}|\geq 2}}^{k}[\mathbf{x}:\mathbf{i}_{j-1},\mathbf{i}_j]y. \tag{$\mathbf{R2'}$}$$

Equation **R1** gives a formula for $[\mathbf{x}:\mathbf{0},\mathbf{n}]y$ when $|\mathbf{n}|=1$. Let us consider Equation **R2′** for the case that $|\mathbf{n}|=2$. For such $\mathbf{n}$, either $\mathbf{n}=2\mathbf{e}_r$ with $1\leq r\leq q$, or $\mathbf{n}=\mathbf{e}_r+\mathbf{e}_s$ with $1\leq r<s\leq q$. In the examples below we compute $[\mathbf{x}:\mathbf{0},\mathbf{n}]y$ for these two cases, assuming $q=2$ to simplify notation.

Example 4. Suppose $\mathbf{n}=2\mathbf{e}_1$ (the case $\mathbf{n}=2\mathbf{e}_2$ is similar). The only possible lattice path $\mathbf{0}=\mathbf{i}_0<\cdots<\mathbf{i}_k=\mathbf{n}$ with $k=2$ in Equation **R2′** is given by $\mathbf{0}<\mathbf{e}_1<2\mathbf{e}_1$. As for such a path the product in Equation **R2′** is empty, one has $[\mathbf{x}:\mathbf{0},2\mathbf{e}_1]y=\{\mathbf{0},\mathbf{e}_1,2\mathbf{e}_1\}g$.

To compute $\{\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1\}g$, we need to find out which integer partitions $2 = s_1+s_2+t$ are compatible with this path. These are precisely the triples (s_1, s_2, t) for which

$$\mathbf{e}_1 = \mathbf{i}_1 - \mathbf{i}_0 = \mathbf{i}_2 - \mathbf{i}_1 = \cdots = \mathbf{i}_{s_1} - \mathbf{i}_{s_1-1}, \tag{3.14}$$

$$\mathbf{e}_2 = \mathbf{i}_{s_1+1} - \mathbf{i}_{s_1} = \mathbf{i}_{s_1+2} - \mathbf{i}_{s_1+1} = \cdots = \mathbf{i}_{s_1+s_2} - \mathbf{i}_{s_1+s_2-1}, \tag{3.15}$$

where the first (respectively second) statement is considered to be trivially satisfied whenever $s_1 = 0$ (respectively $s_2 = 0$). As both $\mathbf{i}_1-\mathbf{i}_0$ and $\mathbf{i}_2-\mathbf{i}_1$ are equal to $\mathbf{e}_1$, the first condition is automatically satisfied. As neither $\mathbf{i}_1-\mathbf{i}_0$ nor $\mathbf{i}_2-\mathbf{i}_1$ is equal to $\mathbf{e}_2$, necessarily $s_2 = 0$. It follows that there are three triples $(s_1, s_2, t) = (0, 0, 2)$, $(1, 0, 1)$, $(2, 0, 0)$ compatible with $(\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1)$. Each of these sequences corresponds to a term in $\{\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1\}g$, and we conclude that

$$\begin{aligned}&[\mathbf{x} : \mathbf{0}, 2\mathbf{e}_1]y = \{\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1\}g = \\ &\quad -\frac{[0; 0|\mathbf{0}, \mathbf{e}_1, 2\mathbf{e}_1]g}{[0; 0|\mathbf{0}, 2\mathbf{e}_1]g}\frac{[0\,1; 0|\mathbf{e}_1]g}{[0; 0|\mathbf{0}, \mathbf{e}_1]g}\frac{[1\,2; 0|2\mathbf{e}_1]g}{[1; 0|\mathbf{e}_1, 2\mathbf{e}_1]g} \\ &\quad +\frac{[0\,1; 0|\mathbf{e}_1, 2\mathbf{e}_1]g}{[0; 0|\mathbf{0}, 2\mathbf{e}_1]g}\frac{[1\,2; 0|2\mathbf{e}_1]g}{[1; 0|\mathbf{e}_1, 2\mathbf{e}_1]g} \\ &\quad -\frac{[0\,1\,2; 0|2\mathbf{e}_1]g}{[0; 0|\mathbf{0}, 2\mathbf{e}_1]g}.\end{aligned} \tag{3.16}$$

Example 5. Suppose $\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2$. Equation $\mathbf{R2}'$ is a sum over the two possible lattice paths $\mathbf{0} < \mathbf{e}_1 < \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{0} < \mathbf{e}_2 < \mathbf{e}_1 + \mathbf{e}_2$.

A triple (s_1, s_2, t) is compatible with the path $\mathbf{0} < \mathbf{e}_1 < \mathbf{e}_1 + \mathbf{e}_2$ precisely when Equations 3.14 and 3.15 hold. For this path, the first equation is equivalent to s_1 either being 0 or 1. If $s_1 = 0$, then the second equation implies that $s_2 = 0$. If $s_1 = 1$, on the other hand, the second equation implies that s_2 is either 0 or 1. One finds three triples $(s_1, s_2, t) = (0, 0, 2)$, $(1, 0, 1)$, $(1, 1, 0)$ compatible with $(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2)$, yielding

$$\begin{aligned}&\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2\}g = \\ &\quad -\frac{[0; 0|\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2]g}{[0; 0|\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2]g}\frac{[0\,1; 0|\mathbf{e}_1]g}{[0; 0|\mathbf{0}, \mathbf{e}_1]g}\frac{[1; 0\,1|\mathbf{e}_1 + \mathbf{e}_2]g}{[1; 0|\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2]g} \\ &\quad +\frac{[0\,1; \mathbf{e}_1|\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2]g}{[0; 0|\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2]g}\frac{[1; 0\,1|\mathbf{e}_1 + \mathbf{e}_2]g}{[1; 0|\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2]g} \\ &\quad -\frac{[0\,1; 0\,1|\mathbf{e}_1 + \mathbf{e}_2]g}{[0; 0|\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2]g}.\end{aligned}$$

Similarly, a triple (s_1, s_2, t) is compatible with the path $\mathbf{0} < \mathbf{e}_2 < \mathbf{e}_1 + \mathbf{e}_2$ precisely when Equations 3.14, 3.15 hold. For this path, however, the fact that $\mathbf{i}_2-\mathbf{i}_1 = \mathbf{e}_1$ comes after $\mathbf{i}_1 - \mathbf{i}_0 = \mathbf{e}_2$ implies that $s_1 = 0$. One finds two triples $(s_1, s_2, t) = (0, 0, 2)$, $(0, 1, 1)$ compatible with $(\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)$, yielding

$$\begin{aligned}&\{\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}g = \\ &\quad -\frac{[0; 0|\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2]g}{[0; 0|\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2]g}\frac{[0; 0\,1|\mathbf{e}_2]g}{[0; 0|\mathbf{0}, \mathbf{e}_2]g}\frac{[0\,1; 1|\mathbf{e}_1 + \mathbf{e}_2]g}{[0; 1|\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2]g}\end{aligned}$$

$$+\frac{[0;0\,1|\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2]g}{[0;0|\mathbf{0},\mathbf{e}_1+\mathbf{e}_2]g}\frac{[0\,1;1|\mathbf{e}_1+\mathbf{e}_2]g}{[0;1|\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2]g}.$$

As for both paths the product in Equation $\mathbf{R2}'$ is empty, it follows that

$$[\mathbf{x}:\mathbf{0},\mathbf{e}_1+\mathbf{e}_2]y=\{\mathbf{0},\mathbf{e}_1,\mathbf{e}_1+\mathbf{e}_2\}g+\{\mathbf{0},\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2\}g. \tag{3.17}$$

3.4 A formula for divided differences of implicit functions

Let y be implicitly defined by g as in Equation 3.1. In this section we derive a formula that expresses divided differences of y solely in terms of divided differences of g. For $\mathbf{n}$ with $|\mathbf{n}|=2$, Equation $\mathbf{R2}'$ immediately yields the two formulas

$$[\mathbf{x}:\mathbf{0},2\mathbf{e}_r]y=\{\mathbf{0},\mathbf{e}_r,2\mathbf{e}_r\}g, \tag{3.18}$$

$$[\mathbf{x}:\mathbf{0},\mathbf{e}_r+\mathbf{e}_s]y=\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s\}g+\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s\}g, \tag{3.19}$$

where $1\le r<s\le q$ and the expressions $\{\mathbf{i}_0\cdots\mathbf{i}_k\}$ are defined in Equation 3.13. For $\mathbf{n}$ with $|\mathbf{n}|=3$, one can distinguish three cases: $\mathbf{n}=3\mathbf{e}_r$, $\mathbf{n}=2\mathbf{e}_r+\mathbf{e}_s$, and $\mathbf{n}=\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t$, with $1\le r,s,t\le q$ distinct. Let us compute $[\mathbf{x}:\mathbf{0},\mathbf{n}]$ for these $\mathbf{n}$ to get a feel for what a general formula should be. Repeatedly applying Equation $\mathbf{R2}'$ yields

$$\begin{aligned}&[\mathbf{x}:\mathbf{0},3\mathbf{e}_r]y= \\ &\quad\{\mathbf{0},\mathbf{e}_r,2\mathbf{e}_r,3\mathbf{e}_r\}g+\{\mathbf{0},2\mathbf{e}_r,3\mathbf{e}_r\}g\cdot\{\mathbf{0},\mathbf{e}_r,2\mathbf{e}_r\}g+\{\mathbf{0},\mathbf{e}_r,3\mathbf{e}_r\}g\cdot\{\mathbf{e}_r,2\mathbf{e}_r,3\mathbf{e}_r\}g\end{aligned} \tag{3.20}$$

$$\begin{aligned}&[\mathbf{x}:\mathbf{0},2\mathbf{e}_r+\mathbf{e}_s]y= \\ &\quad\{\mathbf{0},\mathbf{e}_r,2\mathbf{e}_r,2\mathbf{e}_r+\mathbf{e}_s\}g+\{\mathbf{0},2\mathbf{e}_r,2\mathbf{e}_r+\mathbf{e}_s\}g\cdot\{\mathbf{0},\mathbf{e}_r,2\mathbf{e}_r\}g \\ &\qquad\qquad+\{\mathbf{0},\mathbf{e}_r,2\mathbf{e}_r+\mathbf{e}_s\}g\cdot\{\mathbf{e}_r,2\mathbf{e}_r,2\mathbf{e}_r+\mathbf{e}_s\}g \\ &+\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s,2\mathbf{e}_r+\mathbf{e}_s\}g+\{\mathbf{0},\mathbf{e}_r+\mathbf{e}_s,2\mathbf{e}_r+\mathbf{e}_s\}g\cdot\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s\}g \\ &\qquad\qquad+\{\mathbf{0},\mathbf{e}_r,2\mathbf{e}_r+\mathbf{e}_s\}g\cdot\{\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s,2\mathbf{e}_r+\mathbf{e}_s\}g \\ &+\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s,2\mathbf{e}_r+\mathbf{e}_s\}g+\{\mathbf{0},\mathbf{e}_r+\mathbf{e}_s,2\mathbf{e}_r+\mathbf{e}_s\}g\cdot\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s\}g \\ &\qquad\qquad+\{\mathbf{0},\mathbf{e}_s,2\mathbf{e}_r+\mathbf{e}_s\}g\cdot\{\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s,2\mathbf{e}_r+\mathbf{e}_s\}g\end{aligned} \tag{3.21}$$

$$\begin{aligned}&[\mathbf{x}:\mathbf{0},\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t]y= \\ &\quad\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g+\{\mathbf{0},\mathbf{e}_r+\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s\}g \\ &\qquad\qquad+\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g \\ &+\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g+\{\mathbf{0},\mathbf{e}_r+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_t\}g \\ &\qquad\qquad+\{\mathbf{0},\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{e}_r,\mathbf{e}_r+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g \\ &+\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g+\{\mathbf{0},\mathbf{e}_r+\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s\}g \\ &\qquad\qquad+\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\end{aligned} \tag{3.22}$$

$$+\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_s+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g+\{\mathbf{0},\mathbf{e}_s+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_s+\mathbf{e}_t\}g$$
$$+\{\mathbf{0},\mathbf{e}_s,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{e}_s,\mathbf{e}_s+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g$$
$$+\{\mathbf{0},\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g+\{\mathbf{0},\mathbf{e}_r+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{0},\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_t\}g$$
$$+\{\mathbf{0},\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g$$
$$+\{\mathbf{0},\mathbf{e}_t,\mathbf{e}_s+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g+\{\mathbf{0},\mathbf{e}_s+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{0},\mathbf{e}_t,\mathbf{e}_s+\mathbf{e}_t\}g$$
$$+\{\mathbf{0},\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g\cdot\{\mathbf{e}_t,\mathbf{e}_s+\mathbf{e}_t,\mathbf{e}_r+\mathbf{e}_s+\mathbf{e}_t\}g$$

These three formulas exhibit a remarkable pattern. For every choice of the path $\mathbf{0}=\mathbf{p}_0<\mathbf{p}_1<\mathbf{p}_2<\mathbf{p}_3=\mathbf{n}$, we seem to be getting a sum

$$\{\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\}g+\{\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3\}g\cdot\{\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2\}g+\{\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3\}g\cdot\{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\}g.$$

This expression bears a striking resemblance to the right hand side of the univariate formula

$$[0123]y=\{0123\}g+\{023\}g\,\{012\}g+\{013\}g\,\{123\}g$$

established in [56, Theorem 4]. This suggests that, for general $\mathbf{n}=(n_1,\dots,n_q)$, the divided difference $[\mathbf{x}:\mathbf{0},\mathbf{n}]y$ is a sum of $\binom{n_1+\cdots+n_q}{n_1,\dots,n_q}$ univariate formulas, one for each choice of the path $\mathbf{0}=\mathbf{p}_0<\cdots<\mathbf{p}_{|\mathbf{n}|}=\mathbf{n}$. See Figure 3.2a for an example of such a path.

Theorem 5 below casts this suspicion into a precise form. In order to state this theorem, we introduce some notation for polygon partitions. With a sequence of labels $\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n$ we associate the ordered vertices of a convex polygon. A *partition of a convex polygon* is the result of connecting any pairs of nonadjacent vertices with straight line segments, none of which intersect. We refer to these line segments as the *inner edges* of the partition. We denote the set of all partitions of the polygon with vertices $\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n$ by $\mathcal{P}(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$. Every partition $\pi\in\mathcal{P}(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$ is described by its set $F(\pi)$ of (oriented) faces. Each face $f\in F(\pi)$ is represented by a subsequence $f=(\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_k)$ of the sequence $(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$ of length at least three. We let $E(\pi)$ denote the set of edges in π, each of which is represented by a subsequence $(\mathbf{v}_0,\mathbf{v}_1)$ of $(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$ of length two. Figure 3.2b depicts an example of such a polygon partition.

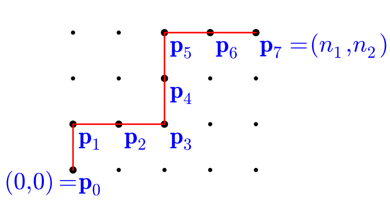
Armed with this notation for polygon partitions, we are now able to state the Main Theorem of this paper.

Theorem 5 (Main Theorem)**.** *For* $\mathbf{n}$ *with* $|\mathbf{n}|\geq 2$*,*

$$[\mathbf{x}:\mathbf{0},\mathbf{n}]y=\sum_{\mathbf{0}=\mathbf{p}_0<\mathbf{p}_1<\cdots<\mathbf{p}_{|\mathbf{n}|}=\mathbf{n}}\ \sum_{\pi\in\mathcal{P}(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_{|\mathbf{n}|})}\ \prod_{(\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_r)\in F(\pi)}\{\mathbf{v}_0\mathbf{v}_1\cdots\mathbf{v}_r\}g. \quad (3.23)$$

To prove Theorem 5, our plan is to use Equation $\mathbf{R2'}$ recursively to express $[\mathbf{x}:\mathbf{0},\mathbf{n}]y$ solely in terms of divided differences of g. Before we proceed with this proof, we assign some visual meaning to Equation $\mathbf{R2'}$ to highlight the backbone of this proof. We call a sequence $\mathbf{i}=(\mathbf{i}_0,\mathbf{i}_1,\dots,\mathbf{i}_k)$ a *subpath* of $\mathbf{p}=(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$ and $\mathbf{p}$ a *superpath* of $\mathbf{i}$, whenever

$$\mathbf{i}_0=\mathbf{p}_{l_0}<\mathbf{p}_{l_0+1}<\cdots<\mathbf{i}_1=\mathbf{p}_{l_1}<\mathbf{p}_{l_1+1}<\cdots<\mathbf{i}_k=\mathbf{p}_{l_k},$$



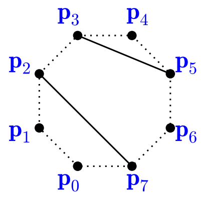


Figure 3.2: *For* $\mathbf{n} = (n_1, n_2) = (4, 3)$*, the figure to the left depicts a choice of a path* $\mathbf{0} = \mathbf{p}_0 < \cdots < \mathbf{p}_{|\mathbf{n}|} = \mathbf{n}$*. The figure to the right shows a partition of the convex polygon corresponding to this path with faces* $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_7), (\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7), (\mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5)$*, inner edges* $(\mathbf{p}_2, \mathbf{p}_7), (\mathbf{p}_3, \mathbf{p}_5)$ *(drawn solid), and outer edges* $(\mathbf{p}_0, \mathbf{p}_1), (\mathbf{p}_1, \mathbf{p}_2), \ldots,$ $(\mathbf{p}_6, \mathbf{p}_7), (\mathbf{p}_0, \mathbf{p}_7)$ *(drawn dotted).*

for some increasing indices $0 = l_0 < l_1 < \cdots < l_k = |\mathbf{n}|$. Every subpath $\mathbf{i}$ of $\mathbf{p}$ induces a partition in $\mathcal{P}(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{|\mathbf{n}|})$ whose set of faces comprises an *inner face* $(\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_k)$ and *outer faces* $(\mathbf{i}_{j-1}, \ldots, \mathbf{i}_j) = (\mathbf{p}_{l_{j-1}}, \mathbf{p}_{l_{j-1}+1}, \ldots, \mathbf{p}_{l_j})$ for every $j = 1, \ldots, k$ with $|\mathbf{i}_j - \mathbf{i}_{j-1}| \geq 2$. See Figure 3.3b for an example.

In general, a sequence $(\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_k)$ has several superpaths $(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$. Let us introduce some notation to consider simultaneously the partitions of the outer faces (each of which is a convex polygon itself) of $\mathbf{i} = (\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_k)$ for all these superpaths. We define

$$\mathcal{P}_{\mathbf{i}} := \prod_{\substack{j=1 \\ m:=|\mathbf{i}_j - \mathbf{i}_{j-1}| \geq 2}}^{k} \coprod_{\mathbf{i}_{j-1} = \mathbf{q}_0 < \cdots < \mathbf{q}_m = \mathbf{i}_j} \mathcal{P}(\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_m),$$

which represents a set of tuples of partitions, each entry in such a tuple corresponding to a partition of a path with steps in $\{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$ from $\mathbf{i}_{j-1}$ to $\mathbf{i}_j$ for some j. For example, for $\mathbf{i} = (\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_5) = \big((0,0), (0,1), (1,1), (2,2), (2,3), (4,3)\big)$, one has $|\mathbf{i}_j - \mathbf{i}_{j-1}| \geq 2$ only for $j = 3, 5$ (see Figure 3.3). There are two paths with steps in $\{\mathbf{e}_1, \mathbf{e}_2\}$ from $\mathbf{i}_2 = (1,1)$ to $\mathbf{i}_3 = (2,2)$ and only one from $\mathbf{i}_4 = (2,3)$ to $\mathbf{i}_5 = (4,3)$. It follows that

$$\begin{aligned}\mathcal{P}_{\mathbf{i}} &= \Big(\mathcal{P}\big(\mathbf{i}_2, (1,2), \mathbf{i}_3\big) \sqcup \mathcal{P}\big(\mathbf{i}_2, (2,1), \mathbf{i}_3\big)\Big) \times \mathcal{P}\big(\mathbf{i}_4, (3,3), \mathbf{i}_5\big) \\ &= \Big\{\Big(\big(\mathbf{i}_2, (1,2), \mathbf{i}_3\big), \big(\mathbf{i}_4, (3,3), \mathbf{i}_5\big)\Big), \Big(\big(\mathbf{i}_2, (2,1), \mathbf{i}_3\big), \big(\mathbf{i}_4, (3,3), \mathbf{i}_5\big)\Big)\Big\}.\end{aligned}$$

We now associate divided differences to these geometric objects. To each outer face $(\mathbf{i}_{j-1}, \ldots, \mathbf{i}_j)$ we associate the divided difference $[\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j]y$, and to each inner face $(\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_k)$ we associate the expression $\{\mathbf{i}_0 \cdots \mathbf{i}_k\}g$. For any sequence $\mathbf{i}$ that appears in the sum of Equation **R2′**, the corresponding inner face therefore represents that part of Equation **R2′** that can be written solely in terms of divided differences of g, while the outer faces represent the part that is still expressed as a divided difference of y.

Proof of Theorem 5. The proof is by induction on $|\mathbf{n}|$. Equations 3.18–3.22 show that the formula holds for $|\mathbf{n}| = 2, 3$. For a fixed $|\mathbf{n}| \geq 4$, suppose the formula holds for all

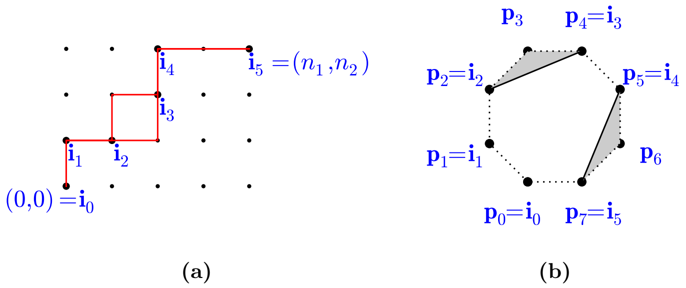


Figure 3.3: *For $\mathbf{n} = (n_1, n_2) = (4,3)$, the figure to the left shows the points in the sequence $\mathbf{i} = (\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_5) = \big((0,0),(0,1),(1,1),(2,2),(2,3),(4,3)\big)$, together with the paths traced out by its two superpaths $\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_7)$. The figure to the right shows the convex polygon corresponding to each of these paths. The sequence $\mathbf{i}$ gives rise to two outer faces $(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ and $(\mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7)$, which are drawn shaded in the figure. Depending on the choice of the superpath $\mathbf{p}$, the former outer face is either $\big(\mathbf{i}_2, (1,2), \mathbf{i}_3\big)$ or $\big(\mathbf{i}_2, (2,1), \mathbf{i}_3\big)$, while the latter is equal to $\big(\mathbf{i}_4, (3,3), \mathbf{i}_5\big)$ for both paths $\mathbf{p}$.*

smaller $|\mathbf{n}|$ (but with $|\mathbf{n}| \geq 2$). Consider the recursion formula from Equation **R2′**. As in each term $k \geq 2$, one has $|\mathbf{i}_j - \mathbf{i}_{j-1}| < |\mathbf{n}|$ for $j = 1, \ldots, k$. By induction, therefore, we can replace each divided difference $[\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j]y$ in Equation **R2′** by an expression involving only divided differences of g. The symbol $\mathcal{P}_\mathbf{i}$ enables us to consider these expressions for $[\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j]y$ simultaneously, yielding

$$\prod_{\substack{j=1 \\ |\mathbf{i}_j - \mathbf{i}_{j-1}| \geq 2}}^{k} [\mathbf{x} : \mathbf{i}_{j-1}, \mathbf{i}_j]y \tag{3.24}$$

$$= \prod_{\substack{j=1 \\ m := |\mathbf{i}_j - \mathbf{i}_{j-1}| \geq 2}}^{k} \; \sum_{\mathbf{i}_{j-1} = \mathbf{q}_0 < \cdots < \mathbf{q}_m = \mathbf{i}_j} \; \sum_{\pi \in \mathcal{P}(\mathbf{q}_0, \ldots, \mathbf{q}_m)} \; \prod_{(\mathbf{v}_0, \ldots, \mathbf{v}_r) \in F(\pi)} \{\mathbf{v}_0 \cdots \mathbf{v}_r\} g$$

$$= \sum_{(\pi_1, \pi_2, \ldots) \in \mathcal{P}_\mathbf{i}} \; \prod_{\substack{j \geq 1 \\ (\mathbf{v}_0, \ldots, \mathbf{v}_r) \in F(\pi_j)}} \{\mathbf{v}_0 \cdots \mathbf{v}_r\} g.$$

For a given sequence $\mathbf{0} = \mathbf{i}_0 < \cdots < \mathbf{i}_k = \mathbf{n}$ with $k \geq 2$, the set $\mathcal{P}_\mathbf{i}$ can be identified with the set

$$\big\{\pi \in \mathcal{P}(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|}) : \mathbf{0} = \mathbf{p}_0 < \cdots < \mathbf{p}_{|\mathbf{n}|} = \mathbf{n},\ \mathbf{i} \in F(\pi)\big\}$$

by the bijection that maps any tuple $(\pi_1, \pi_2, \ldots)$ in $\mathcal{P}_\mathbf{i}$ to the partition π with $F(\pi) = \{\mathbf{i}\} \cup F(\pi_1) \cup F(\pi_2) \cup \cdots$. Applying this bijection to Equation 3.24 and substituting the result into the recursive formula yields

$$[\mathbf{x} : \mathbf{0}, \mathbf{n}]y$$

$$= \sum_{k=2}^{|\mathbf{n}|} \sum_{\mathbf{0}=\mathbf{i}_0<\cdots<\mathbf{i}_k=\mathbf{n}} \{\mathbf{i}_0 \cdots \mathbf{i}_k\}g \sum_{(\pi_1,\pi_2,\ldots)\in\mathcal{P}_{\mathbf{i}}} \prod_{\substack{j\geq 1 \\ (\mathbf{v}_0,\ldots,\mathbf{v}_r)\in F(\pi_j)}} \{\mathbf{v}_0 \cdots \mathbf{v}_r\}g$$

$$= \sum_{k=2}^{|\mathbf{n}|} \sum_{\substack{\mathbf{0}=\mathbf{i}_0<\cdots<\mathbf{i}_k=\mathbf{n} \\ \text{a subpath of} \\ \mathbf{0}=\mathbf{p}_0<\cdots<\mathbf{p}_{|\mathbf{n}|}=\mathbf{n}}} \sum_{\substack{\pi\in\mathcal{P}(\mathbf{p}_0,\ldots,\mathbf{p}_{|\mathbf{n}|}) \\ \mathbf{i}\in F(\pi)}} \prod_{(\mathbf{v}_0,\ldots,\mathbf{v}_r)\in F(\pi)} \{\mathbf{v}_0 \cdots \mathbf{v}_r\}g$$

$$= \sum_{\mathbf{0}=\mathbf{p}_0<\cdots<\mathbf{p}_{|\mathbf{n}|}=\mathbf{n}} \sum_{\pi\in\mathcal{P}(\mathbf{p}_0,\ldots,\mathbf{p}_{|\mathbf{n}|})} \prod_{(\mathbf{v}_0,\ldots,\mathbf{v}_r)\in F(\pi)} \{\mathbf{v}_0 \cdots \mathbf{v}_r\}g. \qquad \square$$

3.5 Polygon partitions and planar trees

While the compact nature of Equation 3.23 is useful to state and prove Theorem 5, it is less appropriate for proving Corollary 8 of Section 3.6 (below). In this section we adapt Equation 3.23 to a form better suited for this purpose.

We recall the following lemma, which appears as Proposition 6.2.1 in [77].

Lemma 6. *For all integers m, n with $m > n \geq 2$, there is a bijection between the following two structures:*

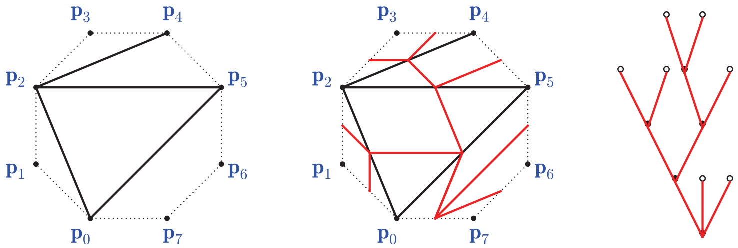
- *Planar trees with m vertices of which n are leaves and all other vertices have at least two descendants.*
- *Partitions with $m - n$ faces of a convex polygon with $n + 1$ vertices.*

We now explicitly describe this bijection. Suppose we are given a polygon partition in $\mathcal{P}(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$ with $m - n$ faces. To the edge $(\mathbf{p}_0, \mathbf{p}_n)$ we associate a vertex that represents the root of our tree. As $(\mathbf{p}_0, \mathbf{p}_n)$ is an outer edge, it belongs to a unique face. The other edges of this face are taken to be the descendants of the root vertex. As we are constructing a *planar* tree, we need to order these descendants; the vertices correspond from left to right to the edges encountered when traversing the border of the face clockwise, starting at $(\mathbf{p}_0, \mathbf{p}_n)$. We then repeat this process for each of the new edges until we are out of edges. This construction yields a rooted planar tree with n leaves corresponding to the outer edges $(\mathbf{p}_0, \mathbf{p}_1), (\mathbf{p}_1, \mathbf{p}_2), \ldots, (\mathbf{p}_{n-1}, \mathbf{p}_n)$ and $m - n$ nonleaf vertices corresponding to the faces of the polygon partition.

For example, for the partition in $\mathcal{P}(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_7)$ with set of faces

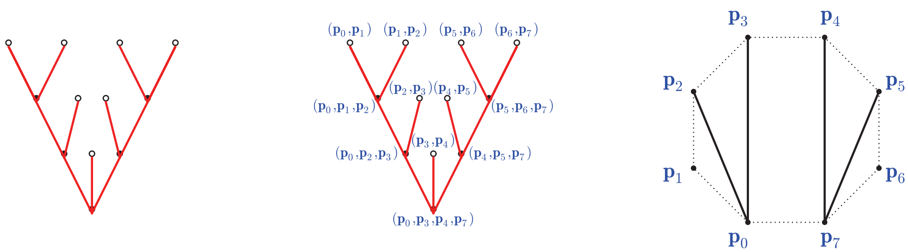
$$\big\{(\mathbf{p}_0, \mathbf{p}_5, \mathbf{p}_6, \mathbf{p}_7), (\mathbf{p}_0, \mathbf{p}_2, \mathbf{p}_5), (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2), (\mathbf{p}_2, \mathbf{p}_4, \mathbf{p}_5), (\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)\big\},$$

this bijection can be visualized as follows.



Here leaves are drawn as circles and nonleaf vertices are drawn as dots.

Conversely, suppose we are given a planar tree with m vertices of which n are leaves, and all the other vertices have at least two descendants. Assign labels $(\mathbf{p}_0, \mathbf{p}_1), (\mathbf{p}_1, \mathbf{p}_2), \ldots, (\mathbf{p}_{n-1}, \mathbf{p}_n)$ to the leaves as they are encountered while traversing the tree depth-first from left to right. Recursively, we assign the label $(\mathbf{p}_{i_0}, \mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_r})$ to any non-leaf vertex whose descendants have labels of the form $(\mathbf{p}_{i_0}, \ldots, \mathbf{p}_{i_1}), (\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_2}), \ldots, (\mathbf{p}_{i_{r-1}}, \ldots, \mathbf{p}_{i_r})$. The labels of the non-leaf vertices then coincide with the faces of a partition in $\mathcal{P}(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$, and the labels of the leaves correspond to the outer edges unequal to $(\mathbf{p}_0, \mathbf{p}_n)$ of the full polygon. The following picture illustrates this construction with an example.



Let $\mathcal{T}(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$ denote the set of rooted planar trees with n leaves for which all nonleaf vertices have at least two descendants. Represent each tree $\tau \in \mathcal{T}(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$ by its set $V(\tau)$ of *non-leaf* vertices that are labeled as above, and label the leaves correspondingly. Note that $V(\tau) = F(\pi)$ whenever a tree τ and polygon partition π are related via the above bijection. From this it follows that Equation 3.23 can equivalently be stated in terms of planar trees as

$$[\mathbf{x} : \mathbf{0}, \mathbf{n}]y = \sum_{\mathbf{0}=\mathbf{p}_0<\mathbf{p}_1<\cdots<\mathbf{p}_{|\mathbf{n}|}=\mathbf{n}} \; \sum_{\tau \in \mathcal{T}(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{|\mathbf{n}|})} \; \prod_{(\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_k) \in V(\tau)} \{\mathbf{i}_0 \mathbf{i}_1 \cdots \mathbf{i}_k\} g. \quad (3.23')$$

We wish to bring this equation into a form where we can distinguish the individual terms in the divided differences of g. For this, we replace $\mathcal{T}(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ by a structure $\mathcal{T}'(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ that encompasses all combinations of all terms in the expressions $\{\mathbf{i}_0 \cdots \mathbf{i}_k\} g$. In particular, we extend each tree in $\mathcal{T}(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ with additional nonleaf vertices for every factor in the second line of Equation 3.13, so that these can be treated the same as the original nonleaf vertices.

Let τ be one of the trees in $\mathcal{T}(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ corresponding to a polygon partition π. Any nonleaf vertex $\mathbf{v} = (\mathbf{i}_0, \ldots, \mathbf{i}_k)$ in $V(\tau)$ defines, together with its direct descendants,

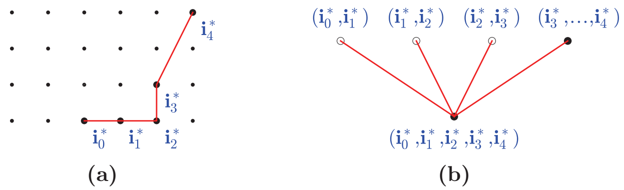


Figure 3.4: *The figure to the left shows the lattice path* $(\mathbf{i}_0^*, \mathbf{i}_1^*, \mathbf{i}_2^*, \mathbf{i}_3^*, \mathbf{i}_4^*) = (2\mathbf{e}_1, 3\mathbf{e}_1, 4\mathbf{e}_1, 4\mathbf{e}_1 + \mathbf{e}_2, 5\mathbf{e}_1 + 3\mathbf{e}_2)$. *The figure to the right shows a star* $*$ *of type* $(s_1^*, s_2^*, t^*) = (2, 1, 1)$ *with root* $(\mathbf{i}_0^*, \mathbf{i}_1^*, \mathbf{i}_2^*, \mathbf{i}_3^*, \mathbf{i}_4^*)$, *with the circles representing leaves and the discs nonleaves.*

a subtree $*_\mathbf{v}$ of τ called a *[rooted planar] star*. Note that the bijection of Lemma 6 induces a bijection between $F(\pi)$ and the set $\mathrm{Stars}(\tau)$ of stars of nonleaf vertices of τ. A star $*$ is said to be *of type* $(s_1^*, \ldots, s_q^*, t^*) = (\mathbf{s}^*, t^*)$, if the sequence of descendants of its root starts with

s_1^* leaves with labels $(\mathbf{a}, \mathbf{b})$ satisfying $\mathbf{b} - \mathbf{a} = \mathbf{e}_1$, followed by

s_2^* leaves with labels $(\mathbf{a}, \mathbf{b})$ satisfying $\mathbf{b} - \mathbf{a} = \mathbf{e}_2$, followed by

$\vdots$

s_q^* leaves with labels $(\mathbf{a}, \mathbf{b})$ satisfying $\mathbf{b} - \mathbf{a} = \mathbf{e}_q$, followed by

t^* nonleaves,

see Figure 3.4. Note that such a type does not exist for every star, as leaves can appear after nonleaves.

For every integer partition $k = s_1 + \cdots + s_q + t$ compatible with $\mathbf{v} = (\mathbf{i}_0, \ldots, \mathbf{i}_k)$, we can extend $*_\mathbf{v}$ to a tree $\tau_\mathbf{v}^{\mathbf{s},t}$ by inserting an edge at the leaves among the final t descendants $(\mathbf{i}_{|\mathbf{s}|}, \mathbf{i}_{|\mathbf{s}|+1}), (\mathbf{i}_{|\mathbf{s}|+1}, \mathbf{i}_{|\mathbf{s}|+2}), \ldots, (\mathbf{i}_{|\mathbf{s}|+t-1}, \mathbf{i}_{|\mathbf{s}|+t})$ of $\mathbf{v}$. That is, we insert an edge for every factor in the second line of Equation 3.13. Note that if there are no such factors, then $\tau_\mathbf{v}^{\mathbf{s},t} = *_\mathbf{v}$. Every star $*$ in $\tau_\mathbf{v}^{\mathbf{s},t}$ is then of some (necessarily unique) type $(\mathbf{s}^*, t^*)$. Using these notions, one can write

$$\{\mathbf{i}_0 \cdots \mathbf{i}_k\} g = \sum_{\substack{\text{compatible}\\ (\mathbf{s},t)}} \prod_{* \in \mathrm{Stars}(\tau_\mathbf{v}^{\mathbf{s},t})} \left(- \frac{\left[i_0^{1*} \cdots (i_0^{1*} + s_1^*);\ \cdots\ ; i_0^{q*} \cdots (i_0^{q*} + s_q^*) \middle| \mathbf{i}^*_{|\mathbf{s}^*|} \cdots \mathbf{i}^*_{|\mathbf{s}^*|+t^*} \right] g}{\left[i_0^{1*}; \cdots ; i_0^{q*} \middle| \mathbf{i}_0^* \, \mathbf{i}_{k^*}^* \right] g} \right),$$

where each star $*$ is of type $(\mathbf{s}^*, t^*) = \left(s_1^*, \ldots, s_q^*, t^*\right)$ and has root $\left(\mathbf{i}_0^*, \ldots, \mathbf{i}_{k^*}^*\right)$, with

$$\left(i_0^{1*}, \ldots, i_0^{q*}\right) = \mathbf{i}_0^* < \left(i_1^{1*}, \ldots, i_1^{q*}\right) = \mathbf{i}_1^* < \cdots < \left(i_{k^*}^{1*}, \ldots, i_{k^*}^{q*}\right) = \mathbf{i}_{k^*}^*.$$

Let $\mathcal{T}'(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ be the set of rooted planar trees obtained by taking a rooted planar tree in $\mathcal{T}(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ and replacing each of its stars $*_\mathbf{v}$ by $\tau_\mathbf{v}^{\mathbf{s},t}$ for some $(\mathbf{s}, t)$

compatible with $\mathbf{v}$. Equivalently, by construction, $\mathcal{T}'(\mathbf{p}_0,\ldots,\mathbf{p}_{|\mathbf{n}|})$ is the set of rooted planar trees with leaves $(\mathbf{p}_0,\mathbf{p}_1),(\mathbf{p}_1,\mathbf{p}_2),\ \ldots,(\mathbf{p}_{|\mathbf{n}|-1},\mathbf{p}_{|\mathbf{n}|})$ and nonleaves labeled accordingly, for which each star $*$ is of some type $(\mathbf{s}^*,t^*)\neq(\mathbf{0},1)$. Equation 3.23′ can then be stated as

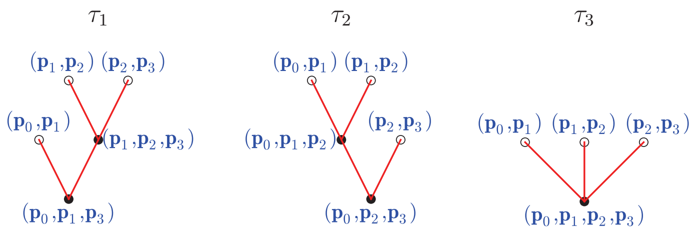
$$[\mathbf{x}:\mathbf{0},\mathbf{n}]y=\sum_{\mathbf{0}=\mathbf{p}_0<\mathbf{p}_1<\cdots<\mathbf{p}_{|\mathbf{n}|}=\mathbf{n}}\ \sum_{\tau'\in\mathcal{T}'(\mathbf{p}_0,\mathbf{p}_1,\ldots,\mathbf{p}_{|\mathbf{n}|})}\ \prod_{*\in\mathrm{Stars}(\tau')} \tag{3.23''}$$

$$\left(-\frac{[i_0^{1*}\cdots(i_0^{1*}+s_1^*);\ \cdots\ ;i_0^{q*}\cdots(i_0^{q*}+s_q^*)|\mathbf{i}^*_{|\mathbf{s}^*|}\cdots\mathbf{i}^*_{|\mathbf{s}^*|+t^*}]g}{[i_0^{1*};\cdots;i_0^{q*}|\mathbf{i}_0^*\,\mathbf{i}_{k^*}^*]g}\right),$$

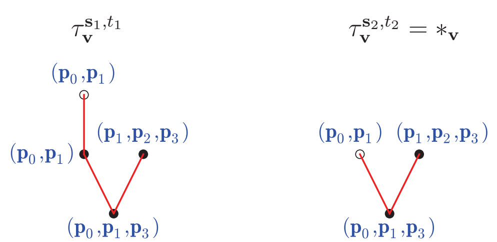
where again each star $*$ is of type $(\mathbf{s}^*,t^*)=(s_1^*,\ldots,s_q^*,t^*)$ and has root $(\mathbf{i}_0^*,\ldots,\mathbf{i}_{k^*}^*)$, with

$$\left(i_0^{1*},\ldots,i_0^{q*}\right)=\mathbf{i}_0^*<\left(i_1^{1*},\ldots,i_1^{q*}\right)=\mathbf{i}_1^*<\cdots<\left(i_{k^*}^{1*},\ldots,i_{k^*}^{q*}\right)=\mathbf{i}_{k^*}^*.$$

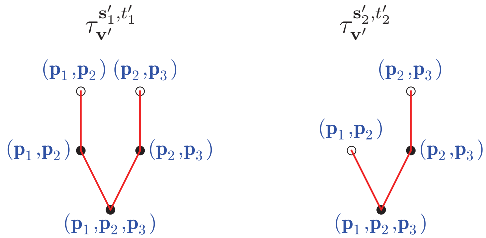
Example 6. To the path $(\mathbf{p}_0,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=(\mathbf{0},\mathbf{e}_1,\mathbf{e}_1+\mathbf{e}_2,2\mathbf{e}_1+\mathbf{e}_2)$ correspond three trees in $\mathcal{T}(\mathbf{p}_0,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$,



Let us consider the first tree τ_1. There are two tuples $(\mathbf{s}_1,t_1)=(0,\ldots,0,2)$, $(\mathbf{s}_2,t_2)=(1,0,\ldots,0,1)$ compatible with the nonleaf vertex $\mathbf{v}=(\mathbf{p}_0,\mathbf{p}_1,\mathbf{p}_3)$, and we can extend $*_\mathbf{v}$ to two corresponding trees



Similarly, there are two tuples $(\mathbf{s}_1',t_1')=(0,\ldots,0,2)$, $(\mathbf{s}_2',t_2')=(0,1,0,\ldots,0,1)$ compatible with the other nonleaf vertex $\mathbf{v}'=(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ of τ_1, and we find two trees



corresponding to $*_{\mathbf{v}'}$. It follows that the tree τ_1 yields $2 \times 2 = 4$ different trees in $\mathcal{T}'(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. Analogously, one can check that τ_2 yields 1×3 trees and τ_3 yields 3 trees.

We end this section with a lemma that appears as Theorem 5.3.10 in [77] and is needed to compute the coefficients in Equation 3.25.

Lemma 7. *For any $k \in \mathbb{N}$, there are*

$$\frac{1}{r_0 + r_1 + \cdots + r_k} \binom{r_0 + r_1 + \cdots + r_k}{r_0, r_1, \ldots, r_k}$$

different planar trees with r_0 vertices with 0 descendants (leaves), r_1 vertices with 1 descendant, …, r_k vertices with k descendants, and no vertices with more than k descendants.

3.6 Higher implicit partial derivatives

Whenever g and y are sufficiently smooth, coalescing the grid points in Equation 3.23′ results in a formula for the derivatives of y in terms of the derivatives of g. We show that this formula generalizes a formula that appears as Equation 7 in [87], which corrects a misprint in an earlier formula by Comtet and Fiolet [26].

The formula, as stated in Equation 3.25, uses some notation for $(q+1)$-dimensional partitions. If $(\mathbf{n}, m) \in \mathbb{N}^q \times \mathbb{N}$ is a nonzero tuple of nonnegative integers, then a $(q+1)$-*dimensional partition* p of $(\mathbf{n}, m)$, denoted by $p \vdash (\mathbf{n}, m)$, is a multiset $\{(\mathbf{s}_1, t_1), \ldots, (\mathbf{s}_r, t_r)\}$ of nonzero tuples in $\mathbb{N}^q \times \mathbb{N}$ that sum to $(\mathbf{n}, m)$ when counting multiplicities. We let $|p| = r$ denote the number of terms in the partition p, counting the multiplicity $\mu_{p;\mathbf{s},t}$ of each tuple $(\mathbf{s}, t)$ in p.

Let y be implicitly defined by g as in Equation 3.1. We introduce the shorthands

$$y_\mathbf{n} = y_\mathbf{n}(\mathbf{x}) := \frac{\partial^{|\mathbf{n}|} y}{\partial \mathbf{x}^\mathbf{n}}(\mathbf{x}), \qquad g_{\mathbf{s},t} = g_{\mathbf{s},t}\big(\mathbf{x}, y(\mathbf{x})\big) := \frac{\partial^{|\mathbf{s}|+t} g}{\partial \mathbf{x}^\mathbf{s} y^t}\big(\mathbf{x}, y(\mathbf{x})\big).$$

As the multiplicities $\mu_{p;\mathbf{s},t}$ sum to $|p|$, it makes sense to consider the multinomial coefficient

$$\binom{|p|}{\ldots, \mu_{p;\mathbf{s},t}, \ldots}$$

for any partition p.

Corollary 8. *If y and g are sufficiently smooth, then, for any $\mathbf{x} \in U$ and nonzero $\mathbf{n} \in \mathbb{N}^q$,*

$$\frac{y_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!} = \sum_{\substack{p \vdash (\mathbf{n}, |p|-1) \\ (\mathbf{0},1) \notin p}} \frac{1}{|p|} \binom{|p|}{\ldots, \mu_{p;\mathbf{s},t}, \ldots} \prod_{(\mathbf{s},t) \in p} \left(-\frac{1}{\mathbf{s}!t!} \frac{g_{\mathbf{s},t}\big(\mathbf{x}, y(\mathbf{x})\big)}{g_{\mathbf{0},1}\big(\mathbf{x}, y(\mathbf{x})\big)} \right). \tag{3.25}$$

Here the product is understood to be of $\mu_{p;\mathbf{s},t}$ copies for every distinct element $(\mathbf{s}, t)$ of the multiset p. The self-referring nature of the summation makes it not directly obvious that there is only a finite number of partitions p of this form for any $\mathbf{n}$. Given such a partition p, only $a \leq |\mathbf{n}|$ of its elements $(\mathbf{s}, t)$ satisfy $\mathbf{s} \neq \mathbf{0}$. Since p does not have $(\mathbf{0}, 1)$ as an element, each of the final coordinates of the b remaining elements of p is at least two. Then $2b \leq |p| - 1 = a + b - 1$ implies that p contains at most $|p| = a + b \leq 2a - 1 \leq 2|\mathbf{n}| - 1$ elements. This bound guarantees that any partition p should sum to a tuple smaller than $(\mathbf{n}, 2|\mathbf{n}| - 1)$, implying that, for given $\mathbf{n}$, there is but a finite number of multisets p of nonzero tuples in $\mathbb{N}^q \times \mathbb{N}$ satisfying $p \vdash (\mathbf{n}, |p| - 1)$ and $(\mathbf{0}, 1) \notin p$.

On the other hand, the existence of partitions of this form can be seen by taking simple examples. For example, for $q = 2$ and $\mathbf{n} = (1, 0), (2, 0), (1, 1)$ one finds partitions

$$\begin{aligned}
\mathbf{n} = (1,0) :\ & \{(1,0,0)\} \vdash (1,0,0), \\
\mathbf{n} = (2,0) :\ & \{(2,0,0)\} \vdash (2,0,0), \\
& \{(1,0,1), (1,0,0)\} \vdash (2,0,1), \\
& \{(0,0,2), (1,0,0), (1,0,0)\} \vdash (2,0,2), \\
\mathbf{n} = (1,1) :\ & \{(1,1,0)\} \vdash (1,1,0), \\
& \{(1,0,0), (0,1,1)\},\ \{(0,1,0), (1,0,1)\} \vdash (1,1,1), \\
& \{(1,0,0), (0,1,0), (0,0,2)\} \vdash (1,1,2).
\end{aligned} \tag{3.26}$$

For these $\mathbf{n}$, the corollary states

$$\begin{aligned}
y_{1,0} &= -\frac{g_{1,0,0}}{g_{0,0,1}}, \\
y_{2,0} &= -\frac{g_{2,0,0}}{g_{0,0,1}} + 2\frac{g_{1,0,1}g_{1,0,0}}{g_{0,0,1}^2} - \frac{g_{0,0,2}g_{1,0,0}^2}{g_{0,0,1}^3}, \\
y_{1,1} &= -\frac{g_{1,1,0}}{g_{0,0,1}} + \frac{g_{1,0,0}g_{0,1,1}}{g_{0,0,1}^2} + \frac{g_{0,1,0}g_{1,0,1}}{g_{0,0,1}^2} - \frac{g_{1,0,0}g_{0,1,0}g_{0,0,2}}{g_{0,0,1}^3}.
\end{aligned} \tag{3.27}$$

Coalescing the grid to a single point $\mathbf{x}_0$ in Equation 3.23″, one finds that

$$\frac{y_{\mathbf{n}}(\mathbf{x}_0)}{\mathbf{n}!} = \sum_{\mathbf{0}=\mathbf{p}_0<\mathbf{p}_1<\cdots<\mathbf{p}_{|\mathbf{n}|}=\mathbf{n}} \ \sum_{\tau' \in \mathcal{T}'(\mathbf{p}_0,\mathbf{p}_1,\ldots,\mathbf{p}_{|\mathbf{n}|})} \tag{3.28}$$

$$\prod_{* \in \mathrm{Stars}(\tau')} \left(-\frac{1}{\mathbf{s}^*!t^*!} \frac{g_{\mathbf{s}^*,t^*}\big(\mathbf{x}_0, y(\mathbf{x}_0)\big)}{g_{\mathbf{0},1}\big(\mathbf{x}_0, y(\mathbf{x}_0)\big)} \right),$$

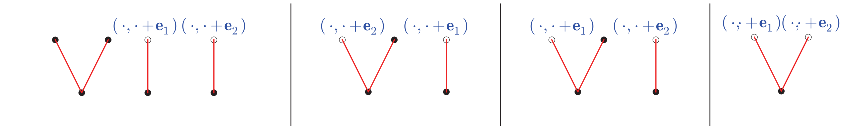
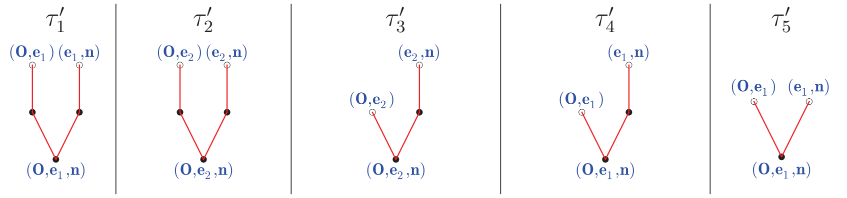
p	$\{(0,0,2),(1,0,0),(0,1,0)\}$	$\{(0,1,1),(1,0,0)\}$	$\{(1,0,1),(0,1,0)\}$	$\{(1,1,0)\}$
$*$				
τ'				

Table 3.1: *For $\mathbf{n} = (1,1)$, the first row lists the multisets p with $(0,0,1) \notin p$ and $p \vdash (\mathbf{n}, |p|-1)$. The second row depicts the stars $*$ associated to each of these multisets. The third row shows the different trees τ' that can be formed by connecting these stars, together with the labels of their vertices.*

where each star $*$ is of type $(\mathbf{s}^*, t^*)$. Clearly Equations 3.25 and 3.28 are in a similar form. The only difference seems to be that in Equation 3.25 equal terms are grouped together into one term with a coefficient. It is not surprising that there are duplicate terms in Equation 3.28, as each term depends only on the types of the stars, not on how these stars are connected to form a tree.

We first provide an example that introduces the flavor of the proof of Corollary 8.

Example 7. Let $\mathbf{n} = (1,1)$. Coalescing the grid to a single point (x_0^1, x_0^2) in Equation 3.17, one finds

$$\begin{aligned} y_{1,1} = &- \frac{1}{2}\frac{g_{0,0,2}g_{1,0,0}g_{0,1,0}}{g_{0,0,1}^3} + \frac{g_{1,0,1}g_{0,1,0}}{g_{0,0,1}^2} - \frac{g_{1,1,0}}{g_{0,0,1}} \\ &- \frac{1}{2}\frac{g_{0,0,2}g_{0,1,0}g_{1,0,0}}{g_{0,0,1}^3} + \frac{g_{0,1,1}g_{1,0,0}}{g_{0,0,1}^2} \end{aligned} \tag{3.29}$$

at this point (x_0^1, x_0^2). Clearly Equations 3.27 and 3.29 are equivalent. In this example we hint at how the terms in these equations are related, suggesting a link that generalizes to the generic construction in the proof of Corollary 8.

First of all note that in both Equations 3.27 and 3.29 the denominators can be determined from their numerators. Taking for granted that the coefficients agree, it therefore suffices to check that, for these equations, the monomials of the numerators of their terms agree.

For each monomial in Equation 3.29, the orders of the derivatives in the numerators form a multiset p of triples in $\mathbb{N}^2 \times \mathbb{N}$ with $(0,0,1) \notin p$ and $p \vdash (1,1,|p|-1)$. It follows that every monomial in Equation 3.29 appears in Equation 3.27 as well.

Conversely, we show that every monomial in Equation 3.27 appears in Equation 3.29 as well, by pointing out which paths $(0,0) = \mathbf{p}_0 < \mathbf{p}_1 < \mathbf{p}_2 = (1,1)$ and trees $\tau' \in \mathcal{T}'(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)$ correspond to it. For a given monomial in Equation 3.27, let p be the corresponding partition in Equation 3.26. To each triple (s_1, s_2, t) in p we associate a star of type (s_1, s_2, t). As can be seen in Table 3.1, one can, for each multiset p, connect these stars to a planar tree τ' with two leaves, and sometimes there are several ways to do this. After demanding the first coordinate of the left leaf to be $(0, 0)$, there is only one way to label the leaves of τ' that agrees with the types of stars of τ'. Thus we find three trees $\tau'_1, \tau'_4, \tau'_5 \in \mathcal{T}'\big((0,0),(1,0),(1,1)\big)$ and two trees $\tau'_2, \tau'_3 \in \mathcal{T}'\big((0,0),(0,1),(1,1)\big)$, each of which corresponds to a term in Equation 3.29.

Proof of Corollary 8. In both Equation 3.25 and 3.28, each term comprises some coefficient and a monomial in the symbols $g_{\mathbf{s},t}$ divided by a power of $g_{\mathbf{0},1}$ of the same total degree. As this monomial uniquely determines the denominator, the following three steps suffice to show that Equations 3.25 and 3.28 are equivalent.

1. Every term in Equation 3.28 appears also in Equation 3.25. Consider an arbitrary term T in the right hand side of Equation 3.28. This term arises from picking a path $\mathbf{0} = \mathbf{p}_0 < \cdots < \mathbf{p}_{|\mathbf{n}|} = \mathbf{n}$ and a tree $\tau' \in \mathcal{T}'(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$. Let $p = \big\{(\mathbf{s}_1, t_1), \ldots, (\mathbf{s}_{|p|}, t_{|p|})\big\}$ be the multiset of types of the stars in τ'. That is, p is the multiset of orders of the derivatives in the numerator of T.

Since for any tree in $\mathcal{T}'(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ the steps made by its leaves sum to $\mathbf{n}$, it follows that $\mathbf{s}_1 + \cdots + \mathbf{s}_{|p|} = \mathbf{n}$. Moreover, with the exception of the root of τ', the root of each star in Stars(τ') connects to one of the $t_1 + \cdots + t_{|p|}$ nonleaf descendants of the stars, implying that $t_1 + \cdots + t_{|p|} = |p| - 1$. As τ' has no vertices with precisely one nonleaf descendant, none of the types in p can be equal to $(\mathbf{0}, 1)$. The orders of the derivatives in the numerator of Equation 3.28 therefore constitute a multiset p with elements in $\mathbb{N}^q \times \mathbb{N}$ for which $(\mathbf{0}, 1) \notin p$ and $p \vdash (\mathbf{n}, |p| - 1)$. We conclude that, up to coefficients, each term in Equation 3.28 appears as a term in Equation 3.25 as well.

2. Every term in Equation 3.25 appears also in Equation 3.28. Suppose we are given a multiset $p = \{(\mathbf{s}_1, t), \ldots, (\mathbf{s}_{|p|}, t_{|p|})\}$ of tuples in $\mathbb{N}^q \times \mathbb{N}$ satisfying $(\mathbf{0}, 1) \notin p$ and $p \vdash (\mathbf{n}, |p| - 1)$ as in Equation 3.25. To each element $(\mathbf{s}, t)$ of p, we associate a star of type $(\mathbf{s}, t)$ whose labels are yet to be determined. Because the number of stars $|p|$ is one more than the sum of the non-leaf descendants $t_1 + \cdots + t_{|p|}$, one can always connect these stars to a planar tree τ' with $|\mathbf{s}_1| + \cdots + |\mathbf{s}_{|p|}| = |\mathbf{n}|$ leaves. Moreover, since each star in τ' is of some type unequal to $(\mathbf{0}, 1)$, there is a unique path $\mathbf{0} = \mathbf{p}_0 < \cdots < \mathbf{p}_{|\mathbf{n}|} = \mathbf{n}$ and labeling of the leaves of τ' such that $\tau' \in \mathcal{T}'(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$. We conclude that, up to coefficients, each term in Equation 3.25 appears as a term in Equation 3.28 as well.

3. Corresponding terms have equal coefficients. Now we have shown that the monomials in Equation 3.25 are the same as those in Equation 3.28, it remains to show that their coefficients agree. Every term in Equation 3.28 corresponding to a tree $\tau' \in \mathcal{T}'(\mathbf{p}_0, \ldots, \mathbf{p}_{|\mathbf{n}|})$ with p the multiset of the types of the stars of τ' will contribute $(-1)^{\#\mathrm{Stars}(\tau')} = (-1)^{|p|}$ to the term in Equation 3.25 corresponding to p. The coefficients can be shown to agree, therefore, by counting, for every multiset p with $(\mathbf{0}, 1) \notin p \vdash (\mathbf{n}, |p| - 1)$, the number of different planar trees that can be formed by connecting the stars of types corresponding to the elements of p.

Let p be as in Equation 3.25. Let us call two elements $(\mathbf{s}_i, t_i), (\mathbf{s}_j, t_j)$ of p *equivalent* whenever $t_i = t_j$. The equivalence classes form a new multiset p' in which each element $[(\mathbf{s}_i, t)]$, or simply t for short, has multiplicity $\mu_t := \sum_{\mathbf{s} \geq \mathbf{0}} \mu_{p;\mathbf{s},t}$. Clearly p' has the same number of elements as p. Associate with each $t \in p'$ a star with t descendants. Here we think of a leaf as a star with 0 descendants. By Lemma 7, one can construct

$$\frac{1}{|p'|}\binom{|p'|}{\mu_0, \mu_1, \ldots}$$

different planar trees from these stars. For the vertices with t descendants of any such planar tree, the multinomial coefficient of $\{\mu_{p;\mathbf{s},t} : \mathbf{s} \geq \mathbf{0}\}$ gives the number of ways to reinsert the leaves. It follows that we can form

$$\frac{1}{|p'|}\binom{|p'|}{\mu_0, \mu_1, \ldots}\prod_{t\geq 0}\binom{\mu_t}{\ldots, \mu_{p;\mathbf{s},t}, \ldots} = \frac{1}{|p|}\binom{|p|}{\ldots, \mu_{p;\mathbf{s},t}, \ldots}$$

different planar trees from the stars corresponding to p. This agrees with the coefficient in Equation 3.25. □

3.7 Analysis of the number of terms

In this section we count the number of terms that appear in Equation 3.23 by means of a generating function. It was obtained by modifying a generating function given by Wilde in Section 7 in [87] for the number of terms in Equation 3.25, which, in its turn, is based upon an erroneous generating function by Comtet and Fiolet in [26] and [25, Page 175]. Comtet and Fiolet do, however, give a table with the correct number of terms for $\mathbf{n} = n_1 = 1, 2, \ldots, 23$, so this error is presumably a repeated misprint.

Let $E := \mathbb{N}^q \times \mathbb{N} - \{(\mathbf{0}, 0), (\mathbf{0}, 1)\}$. For $\mathbf{n} \in \mathbb{N}^q$ with $|\mathbf{n}| = 1$, let $a(\mathbf{n}) = 1$ be the number of terms in Equation **R1**. For $\mathbf{n} \in \mathbb{N}^q$ with $|\mathbf{n}| \geq 2$, let $a(\mathbf{n})$ be the number of terms in the right hand side of Equation 3.23. Thus $a(\mathbf{n})$ represents the number of terms needed to express $[\mathbf{x} : \mathbf{0}, \mathbf{n}]y$ in terms of divided differences of g.

Lemma 9. *For nonzero $\mathbf{n} \in \mathbb{N}^q$, the number of terms $a(\mathbf{n})$ is equal to the coefficient of $\mathbf{x}^{\mathbf{n}} y^{|\mathbf{n}|-1}$ in*

$$h(\mathbf{x}, y) := -\log\left(1 - \sum_{(\mathbf{s},t)\in E} \mathbf{x}^{\mathbf{s}} y^{|\mathbf{s}|+t-1}\right) = \sum_{r=1}^{\infty} \frac{1}{r}\left(\sum_{(\mathbf{s},t)\in E} \mathbf{x}^{\mathbf{s}} y^{|\mathbf{s}|+t-1}\right)^r. \tag{3.30}$$

Proof. For any positive integer r, let

$$\mathcal{M}_r := \left\{\mu : E \longrightarrow \mathbb{N} \;\middle|\; \sum_{e\in E} \mu(e) = r\right\}$$

be a set of weight functions on E with total weight r. That is, $\mathcal{M}_r$ represents all possible ways to assign nonnegative integers to the elements of E that sum to r. For any positive integer r and $\mu \in \mathcal{M}_r$ with support $\{e_1, \ldots, e_l\}$, we can form the multinomial coefficient

$$\binom{r}{\ldots, \mu(e), \ldots} := \binom{r}{\mu(e_1), \ldots, \mu(e_l)}.$$

```
def P(N1,N2,q):
    E = CartesianProduct(range(N1+1), range(N2+1), \
                         range(N1+N2+1))
    E = [(i1,i2,j) for (i1,i2,j) in E if (\
              (i1,i2,j) != (0,0,0)        \
          and (i1,i2,j) != (0,0,1)        \
          and i1 + i2 + j <= N1 + N2      \
          and 2*(i1 + i2) + j - 1 <= 2*(N1+N2) - q)]

    S = sum([X1^i1*X2^i2*Y^(i1+i2+t-1) for (i1,i2,t) in E])
    return S

R.<X1,X2,Y> = PolynomialRing(ZZ,3)

N1, N2 = 6, 6

h = sum([((P(N1,N2,q))^q)/q for q in range(1,2*(N1+N2))])
for n1 in range(0, N1+1):
    for n2 in range(0, n1+1):
        if (n1, n2) != (0, 0):
            a = h.coefficient({X1:n1, X2:n2, Y:n1+n2-1})
            print "a(",n1,",", n2,") = ", a
```

Listing 3.1: *The* Sage *code computes the number of terms* $a(n_1, n_2)$ *for* $q = 2$.

By the *multinomial theorem*, expanding the rth power in the right hand side of Equation 3.30 yields

$$h(\mathbf{x}, y) = \sum_{r=1}^{\infty} \sum_{\mu \in \mathcal{M}_r} \frac{1}{r} \binom{r}{\dots, \mu(e), \dots} \prod_{e:=(\mathbf{s},t) \in E} \left(\mathbf{x}^{\mathbf{s}} y^{|\mathbf{s}|+t-1}\right)^{\mu(e)}. \tag{3.31}$$

For any nonzero $\mathbf{n} \in \mathbb{N}^q$, let us compute the coefficient of $\mathbf{x}^{\mathbf{n}} y^{|\mathbf{n}|-1}$. Let r and μ be as in Equation 3.31. The elements $e \in E$ with $\mu(e) > 0$ define a multiset $p = \{(\mathbf{s}_1, t_1), \dots, (\mathbf{s}_r, t_r)\}$, where we listed $\mu(e)$ copies of each e in p. The term in Equation 3.31 that corresponds to r and μ contributes to $\mathbf{x}^{\mathbf{n}} y^{|\mathbf{n}|-1}$ precisely when

$$\mathbf{s}_1 + \cdots + \mathbf{s}_r = \mathbf{n} \quad \text{and} \quad |\mathbf{s}_1| + t_1 - 1 + \cdots + |\mathbf{s}_r| + t_r - 1 = |\mathbf{n}| - 1,$$

or equivalently when $\mathbf{s}_1 + \cdots + \mathbf{s}_r = \mathbf{n}$ and $t_1 + \cdots + t_r = r - 1$. In other words, precisely when p forms a $(q+1)$-dimensional partition of $(\mathbf{n}, |p| - 1)$. In this case, the term contributes

$$\frac{1}{r} \binom{r}{\mu(\mathbf{s}_1, t_1), \dots, \mu(\mathbf{s}_r, t_r)}$$

to the coefficient of $\mathbf{x}^{\mathbf{n}} y^{|\mathbf{n}|-1}$. By the proof of Theorem 8, this is precisely the number of terms in Equation 3.23 that collapse to the term in Equation 3.25 that corresponds to p. Summing these contributions over all multisets p appearing in Equation 3.25 yields the coefficient of $\mathbf{x}^{\mathbf{n}} y^{|\mathbf{n}|-1}$. □

$a(n_1,n_2)$	0	1	2	3	4	5	6
0	X	1	3	13	71	441	2955
1		5	33	245	1921	15525	127905
2			351	3597	35931	352665	3417975
3				46709	563821	6483285	72009645
4					7963151	104772825	1309699875
5						1550685285	21523435641
6							328234085775

Table 3.2: *The upper triangular part of a symmetric table of the number of terms $a(n_1, n_2)$ in Equation 3.23.*

The coefficient of $\mathbf{x}^{\mathbf{n}} y^{|\mathbf{n}|-1}$ can be computed by brute force, by taking into account only the terms that can contribute. See Listing 3.1 for an implementation of such a procedure in `Sage` [80], using `Singular` [44] as a back-end.

Note that it is trivial to extend this code to any other number q of variables. The program yields the number of terms $a(n_1, n_2)$ for $n_1, n_2 \leq 6$, enlisted in Table 3.2. Note that the number of terms $a(2,0) = 3$ and $a(1,1) = 5$ agree with what was found in Examples 4, 5. More entries in Table 3.2 can be found in The On-Line Encyclopedia of Integer Sequences [57, 58], as can the number of terms in Equation 3.25 for $q = 1, 2$ [59, 88].

Any bivariate divided difference $\big[x_0^1, \ldots, x_{n_1}^1; x_0^2\big] y(\cdot, \cdot)$ is equal to the univariate divided difference $\big[x_0^1, \ldots, x_{n_1}^1\big]$ of the partial function $y\big(\cdot, x_0^2\big)$. The first row in Table 3.2 therefore represents the number of terms that appear in Equation 3.23 when specializing Theorem 5 to $q = 1$, which is the case that was treated in [56].

Although the number of terms $a(\mathbf{n})$ grows exponentially with $\mathbf{n}$, its *growth factor* can be smaller in practical applications. Suppose we are given a function $g(\mathbf{x}, y)$, for which the orders of its nonvanishing derivatives constitute the set $\Delta_g := \{(\mathbf{s}, t) \in \mathbb{N}^q \times \mathbb{N} : g_{\mathbf{s},t} \not\equiv 0\}$ of lattice points. Then any divided difference of g is zero whenever its order is not a member of Δ_g. That is,

$$\big[x_0^1, \ldots, x_{s_1}^1; \cdots; x_0^q, \ldots, x_{s_q}^q \big| y_0, \ldots, y_t\big] g = 0,$$

whenever $(s_1, \ldots, s_q, t) \notin \Delta_g$. The converse is not always true, as for nontrivial $g_{\mathbf{s},t}$ a divided difference of g of order $(\mathbf{s}, t)$ can be zero for a particular choice of the grid. For a generic grid, however, a divided difference of g is zero precisely when its order is not a member of Δ_g.

Let $E_g := E \cap \Delta_g$. For a *generic* $\mathbf{x} \in U$, a partition $p \vdash (\mathbf{n}, |p| - 1)$ with $(\mathbf{0}, 1) \notin p$ yields a nonzero term in Equation 3.25 precisely when all its elements are a member of Δ_g. When counting the number of *nonzero* terms $a_g(\mathbf{n})$ for a given function g and a generic grid, we should therefore consider precisely those multisets in the proof of Lemma 9 for which all elements are a member of E_g. This proves the following theorem.

Theorem 10. *For nonzero $\mathbf{n} \in \mathbb{N}^q$ and a generic choice of the grid the number of*

$a_g(n_1, n_2)$	0	1	2	3	4	5	6
0	X	1	2	4	12	40	144
1		2	8	32	136	592	2624
2			44	232	1216	6304	32416
3				1520	9440	56448	328384
4					67440	454944	2942912
5						3409728	24224256
6							187227264

Table 3.3: *The upper triangular part of a symmetric table of the number of terms $a_g(n_1, n_2)$ in Equation 3.23 for $g(x^1, x^2, y) = (x^1)^2 + (x^2)^2 + y^2 - 1$.*

nonzero terms $a_g(\mathbf{n})$ is the coefficient of $\mathbf{x}^{\mathbf{n}} y^{|\mathbf{n}|-1}$ in

$$h_g(\mathbf{x}, y) := -\log\left(1 - \sum_{(\mathbf{s},t)\in E_g} \mathbf{x}^{\mathbf{s}} y^{|\mathbf{s}|+t-1}\right) = \sum_{r=1}^{\infty} \frac{1}{r}\left(\sum_{(\mathbf{s},t)\in E_g} \mathbf{x}^{\mathbf{s}} y^{|\mathbf{s}|+t-1}\right)^r. \qquad (3.32)$$

As an example, let us see how many terms one gets for a polynomial of low degree. Suppose $g(x^1, x^2, y) := (x^1)^2 + (x^2)^2 + y^2 - 1$. Then

$$\Delta_g = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,0,2), (0,2,0), (2,0,0)\},$$

$$E_g = \{(0,1,0), (1,0,0), (0,0,2), (0,2,0), (2,0,0)\}.$$

The generating function becomes

$$h_g(x^1, x^2, y) = -\log\left(1 - x^1 - x^2 - y - (x^1)^2 y - (x^2)^2 y\right).$$

Running Listing 3.1 with E replaced by E_g, one finds, enlisted in Table 3.3, the number of nonzero terms for this particular function g and a generic grid. Although the number of nonzero terms grows exponentially as well, the table suggests that it does so with a smaller growth factor.

Chapter 4

Generalized Principal Lattices

In this chapter we study the generalized principal lattices briefly mentioned in the introduction. These meshes are unisolvent in some space of polynomials of bounded total degree, play an important role in the classification of the sets satisfying the geometric characterization, and have a simple error formula [17, Theorem 10].

In Section 4.1, we show how generalized principal lattices are a natural generalization of triangular meshes. The section after that briefly summarizes how any planar generalized principal lattice corresponds to a plane cubic curve equipped with a group law that encodes collinearity of its points, which is a result recently developed in a series of articles by Carnicer, García-Esnaola, Gasca, and Godes [10, 14, 15, 20].

Concrete examples in [10, 14] show how this characterization can be applied to construct generalized principal lattices in the plane, and in [15, 16] a classification was derived from Newton's classification of cubic curves in the plane. In Section 4.3 we exhibit this classification, using the Plücker relations to derive some of the claims made in [16] without proof. In addition we give for each curve type in the classification a detailed example, several of which are based upon the ones given by Carnicer, Gasca, and García-Esnaola, that shows concretely how to find generalized principal lattices and what they look like.

The final section of the chapter studies generalized principal lattices in higher-dimensional space, which were introduced by Carnicer, Gasca, and Sauer in [17]. In this article, the theory of Haar systems is applied to construct various examples of generalized principal lattices (of the types (c), (d), (e), (i), (j), (k) in Table 4.4). Earlier, $(n+1)$-*pencil lattices* (type (n) in Table 4.4) were introduced by Lee and Phillips [53], generalizing the *principal lattices* (type (o) in Table 4.4) introduced by Nicolaides [63]. The construction of the generalized principal lattices in [17] can be carried out for any family of parameterized curves satisfying two properties, called **P1** and **P2** in Section 4.4.3.

Because the curves underlying the examples in [17] are given parametrically, it is not immediately obvious that they are of the same type. After converting these curves to implicit form, however, we realized that these are all real algebraic curves in $\mathbb{P}^n$

of degree $n+1$ and arithmetic genus 1. Moreover, for $n = 3$ these curves can be conceived very concretely as the complete intersection of two quadric surfaces, whose classification we recall in Section 4.4.4. This prompted us to investigate whether all such curves satisfy Properties **P1**, **P2** and can thus be used to construct generalized principal lattices in $\mathbb{P}^3$. In Sections 4.4.5 – 4.4.9, we bring each curve type into a normal form by means of a real projective change of coordinates and find a parameterization satisfying Properties **P1**, **P2**. The resulting classification is summarized in Table 4.4.

4.1 From triangular meshes to generalized principal lattices

In this section we show that the notion of a generalized principal lattice naturally arises from the simple notion of a triangular mesh. For simplicity we only consider meshes in the plane, postponing the definition of a generalized principal lattice in higher-dimensional space to a later section.

A *triangular mesh of degree m* is a mesh of points in the plane that is, after scaling by an appropriate factor, of the form

$$S = \{x_{ij} := (i,j) \in \mathbb{N}^2 \,:\, i,j \geq 0,\ i+j \leq m\}. \tag{4.1}$$

Figure 4.1a shows an example of a triangular mesh for $m = 6$. The set in Expression 4.1 can be constructed by taking the points of triple intersection of the 3 families of $m+1$ lines given by the linear forms

$$\begin{array}{cccc} x-0, & x-1, & \ldots, & x-m; \\ y-0, & y-1, & \ldots, & y-m; \\ x+y-0, & x+y-1, & \ldots, & x+y-m. \end{array} \tag{4.2}$$

Triangular meshes have been studied extensively in the literature, appearing as early as 1903 in the work of Otto Biermann [4]. See also the classical textbooks [47, Section 6.6], [49, Sections 11.5, 11.16], and [78, Chapter 19].

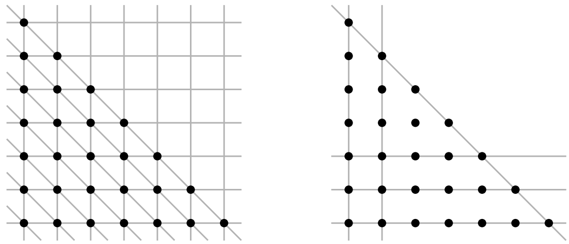
For any choice of real function values $\{f_{ij}\}$ at the points $\{x_{ij}\}$, the lines in Equation 4.2 immediately yield the Lagrange interpolation polynomial

$$L(x) = \sum_{\substack{i,j\geq 0 \\ i+j\leq m}} f_{ij} L_{ij},$$

where the Lagrange polynomials L_{ij} are defined as

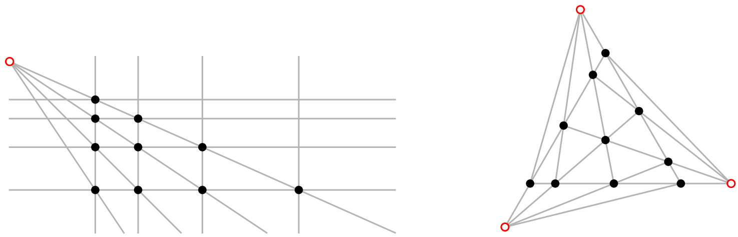
$$L_{ij}(x,y) = \left(\prod_{k=0}^{i-1} \frac{x-k}{i-k}\right) \cdot \left(\prod_{k=0}^{j-1} \frac{y-k}{j-k}\right) \cdot \left(\prod_{k=i+j+1}^{m} \frac{x+y-k}{i+j-k}\right).$$

Note that each L_{ij} is a polynomial of total degree m whose linear factors represent lines from Equation 4.2. The linear factors are normalized such that $L_{ij}(x_{ij}) = 1$, and the lines are chosen such as to contain every point except x_{ij}, see Figure 4.1b and compare with Figure 1.5b.



(a) (b)

Figure 4.1: *For $m = 6$, the figure to the left shows the triangular mesh from Expression 4.1 and the corresponding lines from Equation 4.2. The figure to the right shows only the lines whose linear forms appear in the Lagrange polynomial L_{ij} for $(i, j) = (2, 3)$.*



(a) (b)

Figure 4.2: *To the left, a geometric mesh with $m = 3$ and $q = 1.5$. To the right, a rotation-symmetric 3-pencil lattice with $m = 3$.*

Alternatively, any such interpolating polynomial can be brought into Newton form, which leads to explicit error formulas in terms of bivariate divided differences [47, Section 6.6].

In [63], Nicolaides generalizes the notion of a triangular mesh to that of a *principal lattice* (called *regular mesh* in [52]). To understand his construction, let us rephrase the definition of a triangular mesh as follows. Let $\mathbb{N}_m := \{0, 1, \ldots, m\}$ and consider the simplex $\Delta \subset \mathbb{R}^2$ spanned by the vertices $v_0 = (0, 0), v_1 = (m, 0), v_2 = (0, m)$. In terms of barycentric coordinates, any triangular mesh of degree m can be written as

$$\left\{ x \in \mathbb{R}^2 \,:\, x = \frac{\lambda_0}{m} v_0 + \frac{\lambda_1}{m} v_1 + \frac{\lambda_2}{m} v_2,\ \lambda_i \in \mathbb{N}_m,\ \lambda_0 + \lambda_1 + \lambda_2 = m \right\}.$$

Replacing Δ by a general simplex in $\mathbb{R}^2$, one arrives at Nicolaides' definition of a principal lattice. For principal lattices, one again has explicit formulas for the Lagrange form of any interpolant and its error [63, 64].

Next, Lee and Phillips discovered that instead of using meshes in which the coordinates are evenly spaced (that is, where they form an arithmetic progression), one could more generally consider meshes for which the differences between consecutive coordinates form a geometric progression. In [51, 52], these meshes are called *geometric meshes*. More precisely, they considered, for any $q > 0$, the mesh of points

$$\{x_{ij} = ([i], [j]') \in \mathbb{R}^2 \,:\, i, j \geq 0,\ i + j \leq m\},$$

where $[0] := 0$, $[0]' := 0$, and

$$[i] := 1 + q + q^2 + \cdots + q^{i-1}, \qquad [j]' := 1 + q^{-1} + q^{-2} + \cdots + q^{-j+1}$$

for $i, j > 0$. These points are the triple points of intersection of the lines

$$\begin{aligned} x - [i] &= 0, & i &= 0, \ldots, m; \\ y - [j]' &= 0, & j &= 0, \ldots, m; \\ x + q^{k-1}y - [k] &= 0, & k &= 0, \ldots, m \end{aligned}$$

(taking precisely one line from each family). Note that one recovers a triangular mesh by taking $q = 1$. Analogously to the case of triangular meshes, these lines lead to a simple expression for the Lagrange interpolating polynomial. Using notation from the theory of divided q-differences, one finds a simple expression for the Newton form as well [65, Section 5.4].

The discovery of geometric meshes shed some new light on principal lattices. Let us embed the Euclidean plane $\mathbb{R}^2$ in the projective plane $\mathbb{P}^2$ by the injection $(x, y) \longmapsto [1 : x : y]$. Denote the first coordinate of $\mathbb{P}^2$ by w. The *linear pencil $\mathcal{P}$ of lines in $\mathbb{P}^2$ with vertex* $[a : b : c]$ is defined as the set of lines in $\mathbb{P}^2$ passing through $[a : b : c]$. Note that the lines defined by $x + q^{k-1}y - [k] = 0$ with $0 \leq k \leq m$ all pass through the point $\left(\frac{1}{1-q}, \frac{1}{1-q^{-1}}\right) \in \mathbb{R}^2$. In other words, they are part of a linear pencil of lines. But in the projective plane the same holds for the other two sets of lines, but there the lines meet in a point at infinity: The lines defined by $x - [i] = 0$ meet in the point $[0 : 0 : 1]$, and the lines defined by $y - [j]' = 0$ meet in the point $[0 : 1 : 0]$. Similarly, each family of lines defining a triangular mesh is part of a linear pencil, the vertices being $[0 : 0 : 1], [0 : 1 : 0]$, and $[0 : 1 : -1]$ in this case.

One can construct the Lagrange interpolating polynomial for any mesh defined as the points of triple intersection of three families of $m+1$ lines whose incidence structure equals that of the lines defining a triangular mesh of degree m. This raises the question of how to construct other families of $3 \times (m + 1)$ lines with this incidence structure. Defining the notion of a *3-pencil lattice*, Lee and Phillips gave a flexible construction for lines coming from pencils whose vertices are not collinear [53]. The principal lattices appear as a degenerate case of the vertices spanning the line at infinity. An example of a 3-pencil lattice can be found in Figure 4.2b. We postpone the definition of an $(n + 1)$-pencil lattice to Section 4.4 and refer the reader to [65, Section 5.4] for a friendly introduction for the case $n = 2$.

In [14, 15], Carnicer and Gasca developed the idea of unifying the three linear pencils of lines of a 3-pencil lattice into one cubic pencil of lines. Just as a linear pencil of lines can be defined as the set of lines $L : aw + bx + cy = 0$ whose *line coordinates* $[a : b : c]$ satisfy the homogeneous linear equation $w_1a + x_1b + y_1c = 0$ for some fixed point $[w_1 : x_1 : y_1] \in \mathbb{P}^2$, a *cubic pencil of lines* is defined as the set of lines whose line coordinates satisfy a homogeneous cubic equation $F(a,b,c) = 0$ with no repeating factors. Via the dual correspondence, the lines in a cubic pencil can thus be identified with the points of a reduced cubic curve embedded in the dual projective space $\widehat{\mathbb{P}}^2$. In this manner, the lines defining a 3-pencil lattice with vertices $[w_1 : x_1 : y_1], [w_2 : x_2 : y_2]$, and $[w_3 : x_3 : y_3]$ correspond to points on the curve in $\widehat{\mathbb{P}}^2$ with homogeneous cubic equation

$$F(a,b,c) := (w_1a + x_1b + y_1c) \cdot (w_2a + x_2b + y_2c) \cdot (w_3a + x_3b + y_3c) = 0.$$

Although not all cubic forms can be factored into a product of three linear forms, it turns out that these other cubic forms give rise to meshes similar to principal lattices. To account for these new configurations, Carnicer and Godes introduced, using an abstract combinatorial definition, the concept of a generalized principal lattice [18].

For any nonnegative integer m, we introduce the notation

$$S_m := \big\{(i_0, i_1, i_2) \,:\, i_0, i_1, i_2 \in \{0, 1, \ldots, m\},\ i_0 + i_1 + i_2 = m\big\}.$$

Definition 11 (Generalized Principal Lattice in $\mathbb{P}^2$)**.** Let $m \geq 0$, and let

$$L_0^0, L_1^0, \ldots, L_m^0, \qquad L_0^1, L_1^1, \ldots, L_m^1, \qquad L_0^2, L_1^2, \ldots, L_m^2$$

be three families of $m+1$ lines in $\mathbb{P}^2$ for which any two of the $3(m+1)$ lines are distinct. Suppose that

GPL$_1$ Any intersection $L_{i_1}^{r_1} \cap L_{i_2}^{r_2}$, corresponding to distinct indices $r_1, r_2 \in \{0,1,2\}$, consists of exactly one point.

GPL$_2$ The intersection $L_{i_0}^0 \cap L_{i_1}^1 \cap L_{i_2}^2 \neq \emptyset$ whenever $(i_0, i_1, i_2) \in S_m$.

Under these assumptions, the set of points

$$X := \big\{x_{\mathbf{i}} \,:\, L_{i_0}^0 \cap L_{i_1}^1 \cap L_{i_2}^2 = \{x_{\mathbf{i}}\},\ \mathbf{i} = (i_0, i_1, i_2) \in S_m\big\} \tag{4.3}$$

is a *generalized principal lattice of degree m in the plane* if, additionally,

GPL$_3$ For any $i_0, i_1, i_2 \in \{0, 1, \ldots, m\}$,

$$L_{i_0}^0 \cap L_{i_1}^1 \cap L_{i_2}^2 \cap X \neq \emptyset \quad \Longrightarrow \quad (i_0, i_1, i_2) \in S_m.$$

Note that Definition 11 is weaker than one that replaces the conditions **GPL**$_1$, **GPL**$_2$, and **GPL**$_3$ by the condition

$$\big|L_{i_0}^0 \cap L_{i_1}^1 \cap L_{i_2}^2\big| = 1 \qquad \Longleftrightarrow \qquad (i_0, i_1, i_2) \in S_m,$$

as was Carnicer and Gasca's initial attempt at generalizing the concept of a principal lattice [14, 15]. Figure 4.3 shows a generalized principal lattice in the plane defined by a configuration of lines $\{L_i^r\}$ for which there exists a point P, not belonging to the generalized principal lattice, in which three lines $L_{i_0}^0, L_{i_1}^1, L_{i_2}^2$ meet for which $i_0+i_1+i_2 \neq m$ (compare [18, Remark 2.3]).

A disadvantage of Definition 11 is that it is not immediately clear how to obtain such configurations of lines $\{L_i^r\}$. For this purpose, Carnicer and Gasca introduced a characterization of generalized principal lattices in terms of its associated cubic pencil. This is the topic of the next section.

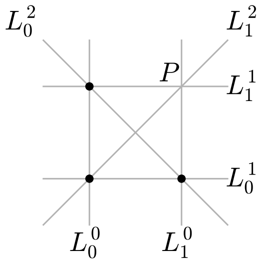


Figure 4.3: *A generalized principal lattice with $m = 1$, for which there is an additional point P where the three lines L_1^0, L_1^1, L_1^2 intersect.*

4.2 A characterization of planar generalized principal lattices

Let Λ be a cubic pencil of lines defined by a homogeneous cubic equation $F(a,b,c) = 0$, and let $C \subset \widehat{\mathbb{P}}^2$ be the associated projective cubic curve. Denote the nonsingular part of C by C_{ns}, and let Λ_{ns} denote the corresponding subset of Λ. Define V to be the set of vertices of the linear pencils contained in Λ. The number of such vertices depends on the factorization of F in $\mathbb{R}[a,b,c]$:

- $|V| = 0$ if F is irreducible,
- $|V| = 1$ if F is the product of a linear and an irreducible quadratic form,
- $|V| = 3$ if F is the product of three linear forms.

One cannot just select any three families of $m+1$ lines from Λ and expect them to define a generalized principal lattice. For instance, permuting the lines of a triangular mesh will typically not yield a configuration of lines that defines a generalized principal lattice, even though all lines are still part of a cubic pencil. In order to select three appropriate families of lines from Λ, Carnicer and Gasca encode the concurrency properties of Definition 11 in terms of a binary operation $\oplus$ on Λ_{ns}. More precisely, from [16, Prop. 3] we have the following theorem.

Theorem 12 (Carnicer and Gasca)**.** *As above, let Λ be a cubic pencil of lines with nonsingular part Λ_{ns} and set of vertices V. Then there exists a binary operation $\oplus : \Lambda_{\mathrm{ns}} \times \Lambda_{\mathrm{ns}} \longrightarrow \Lambda_{\mathrm{ns}}$ and line $O \in \Lambda_{\mathrm{ns}}$ such that $(\Lambda_{\mathrm{ns}}, \oplus, O)$ is an Abelian group satisfying the following property.*

♣ *Any three lines $L_1, L_2, L_3 \in \Lambda_{\mathrm{ns}}$ such that $L_1 \cap L_2 \cap L_3 \cap V = \emptyset$ are concurrent if and only if $L_1 \oplus L_2 \oplus L_3 = O$.*

In this group, let us denote by $\ominus L$ the inverse of a line L and by kL the sum $L \oplus \cdots \oplus L$ of k lines L. When C is an elliptic curve, the binary operation $\oplus$ is the well-known group law on an elliptic curve. When C is a *singular* irreducible curve, the binary operation $\oplus$ satisfying Property ♣ is quite known as well [5, Ex. 10.19–10.22]

[38, Ex. 5.35]. In [10, Prop. 4.1] and [15], a similar binary operation $\oplus$ satisfying Property ♣ was established for cubic pencils coming from reducible reduced cubic curves. At the moment, the existence of such a binary operation is more important than its precise form. In Section 4.3, we shall explicitly describe these groups for various cubic curves.

How can the group law $\oplus$ be used to construct a generalized principal lattice? In [20, Thm. 2.4], one finds the following characterization of generalized principal lattices in the plane.

Theorem 13 (Carnicer and Gasca)**.** *Let Λ be a cubic pencil of lines with associated group $(\Lambda_{\rm ns}, \oplus, O)$ as above.*

1. *Let H, K_1, K_2 be three lines of $\Lambda_{\rm ns}$. Then the $3(m+1)$ lines*

$$L_i^0 := K_1 \oplus iH, \qquad i = 0, \ldots, m \quad (4.4)$$

$$L_j^1 := K_2 \oplus jH, \qquad j = 0, \ldots, m$$

$$L_k^2 := \ominus K_1 \ominus K_2 \oplus (k-m)H, \qquad k = 0, \ldots, m$$

are distinct if and only if

$$H, 2H, \ldots, mH \neq O, \qquad \text{and} \quad (4.5)$$

$$K_1 \ominus K_2 \oplus mH,\ \ominus 2K_1 \ominus K_2,\ \ominus K_1 \ominus 2K_2 \notin \{O, H, 2H, \ldots, 2mH\}.$$

Moreover, if $(i,j,k) \in S_m$ then $L_i^0 \oplus L_j^1 \oplus L_k^2 = O$.

2. *Let H, K_1, K_2 be three lines of $\Lambda_{\rm ns}$ satisfying Conditions 4.5 and let $\{L_i^r\}$ be the lines defined by Equation 4.4. If $(i,j,k) \in S_m$, then the lines L_i^0, L_j^1, L_k^2 are concurrent. Let X be defined from the lines $\{L_i^r\}$ as in Equation 4.3. If $X \cap V = \emptyset$, then X is a generalized principal lattice of degree m.*

3. *Let X be a generalized principal lattice of degree m in the plane defined by lines $\{L_i^r\}$ in $\Lambda_{\rm ns}$. Then there exist lines $H, K_1, K_2 \in \Lambda_{\rm ns}$ such that Equation 4.4 holds.*

Conditions 4.5 are computationally useful, because the group structures on $\Lambda_{\rm ns}$ satisfying Property ♣ are all of a particularly simple form; they are the direct product of either the additive group $\mathbb{R}$ or the circle group $\mathbb{S}^1$ with one or two of the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ [15]. For such groups one easily finds triples (H, K_1, K_2) satisfying Conditions 4.5.

The following theorem is a combination of Theorem 3.5 and Corollary 3.6 in [20]. It states that any generalized principal lattice comes from a cubic pencil of lines and that this pencil is unique whenever the mesh has enough points.

Theorem 14 (Carnicer and Godes)**.** *Let X be a generalized principal lattice of degree m in the plane defined by a family $\{L_i^r\}$ of lines. There exists a cubic pencil Λ of lines containing all lines in $\{L_i^r\}$. Moreover, Λ is unique if and only if $m \geq 4$.*

4.3 A classification of planar generalized principal lattices

As cubic pencils of lines correspond to cubic curves, it is possible to carry over the projective classification of real cubic curves to a classification of cubic pencils of lines. Newton was the first to propose a classification of reduced real cubic curves [62], and a gentle account can be found in [5]. In [16], Carnicer and Gasca carried out a parallel classification of the generalized principal lattices in the plane, giving explicit examples and a geometric interpretation for each class.

In this section we shall present the classification by Carnicer and Gasca, giving additional details and derivations for each type of generalized principal lattice. The classification is summarized in Table 4.1. In Section 4.3.1 we state some results about the duality of curves in the projective plane, which shall be used throughout the remainder of the section. In Sections 4.3.2 and 4.3.3, we shall see that the principal lattices and 3-pencil lattices correspond to the cubic pencils whose associated curve is a union of three lines. This was shown explicitly in [15]. Next, Sections 4.3.4 – 4.3.6 contain a discussion of the generalized principal lattices corresponding to the union of a conic and a line. Finally, Sections 4.3.7 – 4.3.10 discuss generalized principal lattices corresponding to pencils whose associated curve is irreducible.

4.3.1 The dual correspondence for curves

The dual correspondence for curves is a natural extension of the dual correspondence between points and lines in the projective plane.

Let $C \subset \mathbb{P}^2$ be an irreducible projective curve defined by an equation $F(w, x, y) = 0$ of degree at least two. The tangent line to a smooth point $P = [w_0 : x_0 : y_0]$ of C is given by $\frac{\partial F}{\partial w}(P)w + \frac{\partial F}{\partial x}(P)x + \frac{\partial F}{\partial y}(P)y = 0$. Using the dual correspondence for points and lines, one obtains a rational map

$$\phi : C \dashrightarrow \widehat{\mathbb{P}}^2, \qquad P \longmapsto \left[\frac{\partial F}{\partial w}(P) : \frac{\partial F}{\partial x}(P) : \frac{\partial F}{\partial y}(P)\right]$$

that sends any smooth point P of C to its tangent line in dual space. The Zariski closure of $\phi(C)$ is an irreducible curve in $\widehat{\mathbb{P}}^2$ called the *dual curve* $\widehat{C}$ of C [82]. The dual of $\widehat{C}$ is isomorphic to C [82]. It follows that C and $\widehat{C}$ are birationally equivalent and in particular that they have equal geometric genus.

Following [32], we call an irreducible curve $C \subset \mathbb{P}^2$ of degree at least two a *Plücker curve*, if C and $\widehat{C}$ have no other singularities than simple nodes and simple cusps. Irreducible conics and cubics are examples of Plücker curves. Let $C \subset \mathbb{P}^2$ be a Plücker curve of degree d and geometric genus g with δ nodes and κ cusps. Of the dual curve $\widehat{C}$, denote the genus by $\widehat{g}$, the number of nodes by $\widehat{\delta}$, and the number of cusps by $\widehat{\kappa}$. We recall the following *Plücker formulas* [32, 82]:

$$\widehat{d} = d(d-1) - 2\delta - 3\kappa, \tag{4.6}$$

$$\widehat{\kappa} = 3d(d-2) - 6\delta - 8\kappa. \tag{4.7}$$

Moreover, by *Clebsch's genus formula*,

$$\frac{1}{2}(d-1)(d-2)-\delta-\kappa = g = \widehat{g} = \frac{1}{2}(\widehat{d}-1)(\widehat{d}-2)-\widehat{\delta}-\widehat{\kappa}. \tag{4.8}$$

Any line that is tangent to two points on a curve is called a *bitangent* of this curve. Any node on C corresponds to a bitangent on $\widehat{C}$ and any inflection point on C corresponds to a cusp on $\widehat{C}$ (and vice versa). In the remainder of this section, this correspondence is used to determine the form of the curves dual to reduced irreducible cubics, culminating in the visual description of planar generalized principal lattices presented in Table 4.1 (see Section 4.3.10 below).

According to [82, Theorem III.6.3], the inflection points of any projective curve $C : F(w,x,y) = 0$ are the nonsingular points of C in the intersection of C with its *Hessian curve*

$$H_C : \det \begin{bmatrix} \frac{\partial^2 F}{\partial w^2} & \frac{\partial^2 F}{\partial w \partial x} & \frac{\partial^2 F}{\partial w \partial y} \\ \frac{\partial^2 F}{\partial x \partial w} & \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial w} & \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{bmatrix} = 0.$$

The Hessian curve passes through the singularities of C. By Bezout's Theorem, one finds that a cubic can have up to nine (possibly nonreal) inflection points. How many of these are real depends on the particular curve C. Any nonsingular irreducible cubic has precisely three real inflection points [5, Ex. III.12.8], while any singular irreducible cubic has either one or three real inflection points [5, Ex. III.12.18].

Examples with explicit descriptions of the group structures on C_{ns} can be found in [14–16], and several of the examples in Sections 4.3.2 – 4.3.10 are based upon these. The classification of the planar generalized principal lattices is summarized in Table 4.1; it will serve as a model for our results in higher-dimensional space.

4.3.2 C reducible, the union of three concurrent lines

Suppose that C is a union of three lines L_1, L_2, L_3 that meet in a single point P. The dual to C is a union of three vertices v_1, v_2, v_3 lying on the line L_P dual to P. Picking an integer $m \geq 0$ and three lines $H, K_1, K_2 \neq L_P$ as in Theorem 13.(2), one finds a generalized principal lattice.

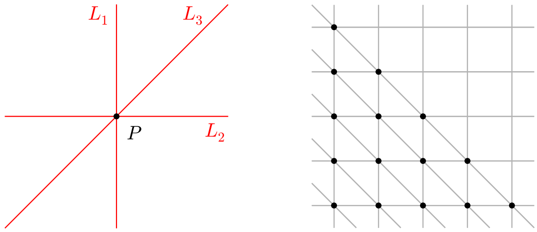
For instance, the lines $L_1 : x = 0, L_2 : y = 0, L_3 : x - y = 0$ meet in the point $P = [1:0:0]$. The line dual to P is the line at infinity $L_P : w = 0$. Dual to the lines L_1, L_2, L_3, one has the vertices $v_1 = [0:1:0], v_2 = [0:0:1], v_3 = [0:1:-1]$. The bijection

$$\phi : \mathbb{R} \times \mathbb{Z}_3 \longrightarrow C_{\text{ns}}, \qquad (t,a) \longmapsto \begin{cases} [-t:1:0] & \text{if } a = 0; \\ [-t:0:1] & \text{if } a = 1; \\ [\ \ t:1:1] & \text{if } a = 2 \end{cases}$$

induces a group structure on C_{ns} satisfying Property ♣ [16]. Consider the points

$$\begin{aligned} P_H &:= \phi(1,0) = [-1:1:0], \\ P_{K_1} &:= \phi(0,1) = [\ \ 0:0:1], \\ P_{K_2} &:= \phi(0,2) = [\ \ 0:1:1] \end{aligned}$$

on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) for $m = 4$, one finds a triangular mesh.



4.3.3 C reducible, the union of three nonconcurrent lines

Suppose that C is a union of three lines L_1, L_2, L_3 that are not concurrent. Write $\{P_k\} := L_i \cap L_j$ whenever $\{i, j, k\} = \{1, 2, 3\}$. The dual to C is the union of three vertices v_1, v_2, v_3 that do not lie on a line. Picking an integer $m \geq 0$ and three lines $H, K_1, K_2 \neq P_1, P_2, P_3$ as in Theorem 13.(2), one finds a generalized principal lattice.

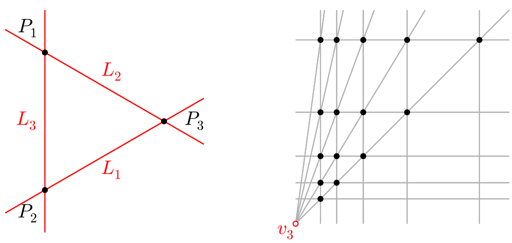
For instance, the lines $L_1 : x = 0$, $L_2 : y = 0$, $L_3 : w = 0$ are not concurrent. Equivalently, the points of intersection $P_1 = [0 : 1 : 0]$, $P_2 = [0 : 0 : 1]$, $P_3 = [1 : 0 : 0]$ do not coincide. Via the dual correspondence, the lines L_1, L_2 correspond to vertices v_1, v_2 at infinity, while the line L_3 corresponds to a vertex v_3 in the finite plane. The bijection $\phi : \mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow C_{\mathrm{ns}}$ given by

$$(t, a, b) \longmapsto \begin{cases} [\ -(-1)^a \mathrm{e}^{+t} : \ 1 : \ 0\ ] & \text{if } b = 0; \\ [\ -(-1)^a \mathrm{e}^{-t} : \ 0 : \ 1\ ] & \text{if } b = 1; \\ [\ 0 : \ (-1)^a \mathrm{e}^{t} : \ -1\ ] & \text{if } b = 2 \end{cases}$$

induces a group structure on C_{ns} that satisfies Property ♣ [16]. Consider the points

$$\begin{aligned} P_H &:= \phi(1, 0, 0) = [-\mathrm{e} : 1 : \ \ 0], \\ P_{K_1} &:= \phi(0, 0, 1) = [-1 : 0 : \ \ 1], \\ P_{K_2} &:= \phi(0, 0, 2) = [\ \ 0 : 1 : -1] \end{aligned}$$

on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) for $m = 4$, one finds the following 3-pencil lattice.



4.3.4 C reducible, the disjoint union of a conic and a line

Suppose that C is the union of a conic C_1 and a line L that do not intersect (not at infinity either). In dual space, the line L corresponds to a vertex v. By Equation 4.6, the dual $\widehat{C}_1$ of the conic C_1 is a conic as well. Since C_1 and L do not intersect, no line passing through v is a tangent of $\widehat{C}_1$. This is only possible when v is lying *inside of* $\widehat{C}_1$. Picking an integer $m \geq 0$ and three lines H, K_1, K_2 as in Theorem 13.(2), one finds three families of $m+1$ lines; one of these families is contained in the linear pencil through v, while the other two are tangents to the conic $\widehat{C}_1$. Together, these $3(m+1)$ lines define a generalized principal lattice.

Let $C = C_1 \cup L$ be the union of the conic $C_1 : F(w,x,y) = x^2 + y^2 - w^2 = 0$ and the line $L : w = 0$ at infinity. The tangent line to the point $R = [w_0 : x_0 : y_0]$ on C_1 is given by

$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y = -2w_0 w + 2x_0 x + 2y_0 y = 0.$$

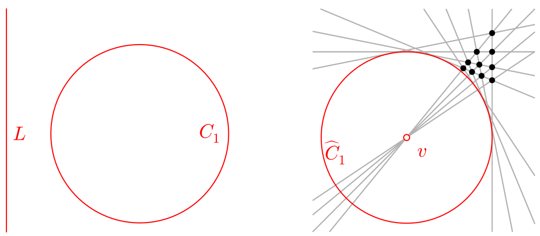
It follows that the triples $[a : b : c] = [-2w_0 : 2x_0 : 2y_0]$ form the conic $\widehat{C}_1 : b^2 + c^2 - a^2 = 0$ in the dual plane. The line L corresponds to the vertex $v = [1 : 0 : 0]$. The bijection $\phi : \mathbb{S}^1 \times \mathbb{Z}_2 \longrightarrow C_{\mathrm{ns}}$ given by

$$(t,a) \longmapsto \begin{cases} [\ \ 0 : \sin(t/2) : \cos(t/2)\ ] & \text{if } a = 0; \\ [\ -1 : \cos(t) : \sin(t)\ ] & \text{if } a = 1 \end{cases}$$

induces a group structure on C_{ns} that satisfies Property ♣ [14, 16]. Consider the points

$$\begin{aligned} P_H &:= \phi(\pi/16, 0) = [0 : \sin(\pi/32) : \cos(\pi/32)], \\ P_{K_1} &:= \phi(3\pi/8, 1) = [1 : -\cos(3\pi/8) : -\sin(3\pi/8)], \\ P_{K_2} &:= \phi(0, 1) = [1 : -1 : 0] \end{aligned}$$

on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) for $m = 3$, one finds the following generalized principal lattice.



4.3.5 C reducible, the union of a conic and a tangent line

Suppose that C is the union of a conic C_1 and a line L tangent to C_1 at the point P. Such a curve C is sometimes called a *tangential cubic*. In dual space, the line L corresponds to a vertex v and the point P to a line L_P. The dual to C_1 is a conic $\widehat{C}_1$. Since C_1 has degree two, it intersects L only in the point P (but with multiplicity 2).

The line L_P is therefore the only tangent of $\widehat{C}_1$ that passes through the point v. This is only possible when v is a point *on* $\widehat{C}_1$. Picking an integer $m \geq 0$ and three lines $H, K_1, K_2 \neq L_P$ as in Theorem 13.(2), one finds a generalized principal lattice.

For instance, the conic $C_1 : F(w,x,y) = wy - x^2 = 0$ and the line $L : y = 0$ intersect at the point $P = [1:0:0]$ with multiplicity 2. The tangent line to the point $R = [w_0 : x_0 : y_0]$ of C_1 is given by

$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y = y_0 w - 2x_0 x + w_0 y = 0.$$

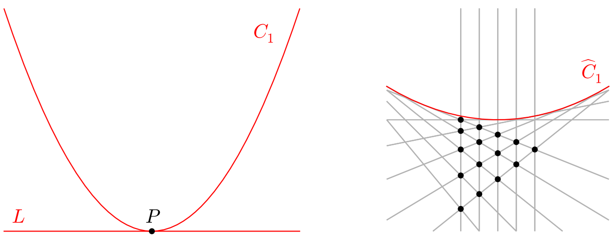
It follows that the triples $[a:b:c] = [y_0 : -2x_0 : w_0]$ form the conic $\widehat{C}_1 : b^2 - 4ac = 0$ in the dual plane. The line L corresponds to the vertex $v = [0:0:1]$ at infinity. The bijection $\phi : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow C_{\mathrm{ns}}$ given by

$$(t,a) \longmapsto \begin{cases} [t \ : \ \ 1 : 0] & \text{if } a = 0; \\ [t^2 : -t : 1] & \text{if } a = 1 \end{cases}$$

induces a group structure on C_{ns} that satisfies Property ♣ [14, 16]. Consider the points

$$\begin{aligned} P_H &:= \phi(\ \ 0.2, 0) = [0.20 : \ \ 1.0 : 0], \\ P_{K_1} &:= \phi(-0.8, 1) = [0.64 : \ \ 0.8 : 1], \\ P_{K_2} &:= \phi(\ \ 0.4, 1) = [0.16 : -0.4 : 1] \end{aligned}$$

on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) for $m = 4$, one finds a generalized principal lattice.



4.3.6 C reducible, the union of a conic and a secant

Suppose that C is the union of a conic C_1 and a line L passing through two distinct points P, Q of C_1. Such a curve C is sometimes called a *secant cubic*. In dual space, the line L corresponds to a vertex v, the point P to a line L_P, and the point Q to a line L_Q. The dual to C_1 is a conic $\widehat{C}_1$. Since C_1 has degree two, it intersects L only in the points P, Q. The lines L_P, L_Q are therefore the only tangents of $\widehat{C}_1$ that pass through the vertex v. This is only possible when v is a point *outside of* $\widehat{C}_1$. Picking an integer $m \geq 0$ and three lines $H, K_1, K_2 \neq L_P, L_Q$ as in Theorem 13.(2), one finds a generalized principal lattice.

For instance, let $C = C_1 \cup L$ be the union of the conic $C_1 : F(w,x,y) = 4xy - w^2$ and the line $L : w = 0$ at infinity. The tangent line to C_1 at the point $R = [w_0 : x_0 : y_0]$

is given by

$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y = -2w_0 w + 4y_0 x + 4x_0 y = 0.$$

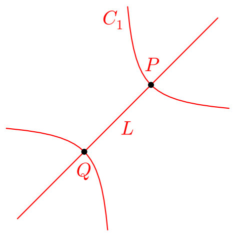
It follows that the triples $[a:b:c] = [-2w_0 : 4y_0 : 4x_0]$ form the conic $\widehat{C}_1 : bc - a^2 = 0$ in the dual plane. The line L corresponds to the vertex $v = [1:0:0]$. The bijection $\phi : \mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow C_{\text{ns}}$ given by

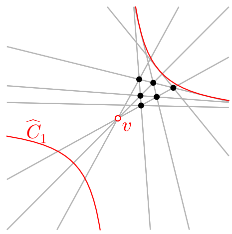
$$(t,a,b) \longmapsto \begin{cases} [\ \ 0 : -(-1)^a \mathrm{e}^t : 1\ ] & \text{if } b = 0; \\ [\ -2 : (-1)^a \mathrm{e}^{-t} : (-1)^a \mathrm{e}^t\ ] & \text{if } b = 1 \end{cases}$$

induces a group structure on C_{ns} that satisfies Property ♣ [14,16]. Consider the points

$$\begin{aligned} P_H &:= \phi(\ \ 0.6, 0, 0) = [\ \ 0 : -\mathrm{e}^{+0.6} : \ 1\ ], \\ P_{K_1} &:= \phi(-1.3, 0, 1) = [-2 : \ \mathrm{e}^{+1.3} : \mathrm{e}^{-1.3}], \\ P_{K_2} &:= \phi(\ \ 0.7, 0, 1) = [-2 : \ \mathrm{e}^{-0.7} : \mathrm{e}^{+0.7}] \end{aligned}$$

on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) for $m = 2$, one finds a generalized principal lattice.





4.3.7 C irreducible, with an isolated singular point

Suppose that C is an irreducible cubic with an isolated singular point P. Such a point is also called an *acnode*, as it corresponds to an ordinary node on the complexification $C_{\mathbb{C}} := C \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ of C. By [23, Theorem 6], such a curve has precisely three *real* inflection points. Using Equations 4.6–4.8, we find that the dual curve $\widehat{C}$ is a quartic of genus zero without nodes and with three real cusps. It follows that $\widehat{C}$ is a *tricuspidal quartic with three real cusps*.

For instance, let $C : F(w,x,y) = (x^2 + y^2)w + 2xy^2 = 0$ (cf. [15, §6]). Its Hessian curve is given by

$$H_C : \det \begin{bmatrix} 0 & 2x & 2y \\ 2x & 2w & 4y \\ 2y & 4y & 2w + 4x \end{bmatrix} = -16x^3 - 8wx^2 + 32xy^2 - 8wy^2 = 0.$$

Besides the singularity $P = [1:0:0]$, the intersection $C \cap H_C$ contains the real points of inflection $Q_1 = \left[1 : -2 : \sqrt{4/3}\right], Q_2 = \left[1 : -2 : -\sqrt{4/3}\right]$, and $Q_3 = [0:0:1]$. These points correspond to the tangents of three cusps on $\widehat{C}$.

Let us now explicitly describe $\widehat{C}$. The tangent line to C at the point $R = [w_0 : x_0 : y_0]$ is given by

$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y =$$
$$(x_0^2 + y_0^2)w + 2(w_0x_0 + y_0^2)x + 2y_0(w_0 + 2x_0)y = 0.$$

Using Gröbner bases to eliminate the variables w_0, x_0, y_0, for instance with the program *Singular* [43], it follows that the triples $[a : b : c] = \big[x_0^2+y_0^2 : 2(w_0x_0+y_0^2) : 2y_0(w_0+2x_0)\big]$ form the quartic

$$-8a^3b + 12a^2b^2 - a^2c^2 - 6ab^3 + 10abc^2 + b^4 + 2b^2c^2 + c^4 = 0$$

in the dual plane. The bijection $\phi : \mathbb{S}^1 \longrightarrow C_{\text{ns}}$ given by

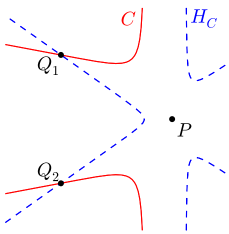
$$t \longmapsto \begin{cases} [\ \sin(t) : -\tan(t/2) : 1\ ] & \text{if } t \neq \pi; \\ [\ 0 : 1 : 0\ ] & \text{if } t = \pi \end{cases}$$

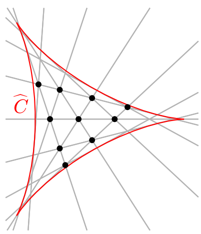
induces a group structure on C_{ns} that satisfies Property ♣ [15, 16]. Consider the points

$$P_H := \phi\left(\frac{1}{2}\right) = \left[\sin\left(\frac{1}{2}\right) : -\tan\left(\frac{1}{4}\right) : 1\right],$$
$$P_{K_1} := \phi\left(-\frac{5}{2}\right) = \left[\sin\left(-\frac{5}{2}\right) : \tan\left(\frac{5}{4}\right) : 1\right],$$
$$P_{K_2} := \phi\left(\frac{3}{2}\right) = \left[\sin\left(\frac{3}{2}\right) : -\tan\left(\frac{3}{4}\right) : 1\right]$$

on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) one finds a generalized principal lattice.

Below in the left figure, the solid curve represents C in the finite plane, while the dashed curve represents H_C. Both curves have an isolated singularity at P and pass through the three inflection points Q_1, Q_2, Q_3 of C (two of which are shown). The figure to the right shows the generalized principal lattice for H, K_1, K_2 as above and $m = 3$, formed by tangents to the tricuspidal quartic $\widehat{C}$.





4.3.8 C irreducible, with a nodal singularity

Suppose that C is an irreducible cubic with an ordinary node P (more precisely, a *crunode*). By [23, Theorem 6], such a curve has precisely one *real* inflection point.

Using Equations 4.6–4.8, we find that the dual curve $\widehat{C}$ is a quartic of genus zero with three cusps, one of which is real, and no other singularities. It follows that $\widehat{C}$ is a *tricuspidal quartic with one real cusp.*

For instance, let $C : F(w,x,y) = wy^2 - x^2(x+w) = 0$. Its Hessian curve is given by

$$H_C : \det \begin{bmatrix} 0 & -2x & 2y \\ -2x & -2w-6x & 0 \\ 2y & 0 & 2w \end{bmatrix} = -8wx^2 + 8(w+3x)y^2 = 0$$

Besides the singularity $P = [1:0:0]$, the intersection $C \cap H_C$ contains the real point of inflection $Q = [0:0:1]$. This point corresponds to the tangent of a cusp on $\widehat{C}$.

Let us now explicitly describe $\widehat{C}$. The tangent line to C at the point $R = [w_0 : x_0 : y_0]$ is given by

$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y = (y_0^2 - x_0^2)w - (2w_0 + 3x_0)x_0 x + 2w_0 y_0 y = 0.$$

Using Gröbner bases to eliminate the variables w_0, x_0, y_0, it follows that the triples $[a:b:c] = [y_0^2 - x_0^2 : -(2w_0+3x_0)x_0 : 2w_0y_0]$ form the quartic

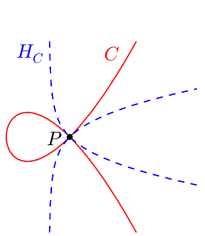
$$\widehat{C} : 4ab^3 - 4b^4 + 27a^2c^2 - 36abc^2 + 8b^2c^2 - 4c^4 = 0$$

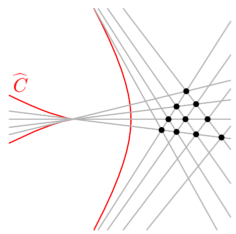
in the dual plane. The bijection $\phi : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow C_{\mathrm{ns}}$ that sends

$$(t,a) \longmapsto \left[\left(1-(-1)^a e^t\right)^3 : 4(-1)^a e^t\left(1-(-1)^a e^t\right) : 4(-1)^a e^t\left(1+(-1)^a e^t\right)\right]$$

induces a group structure on C_{ns} that satisfies Property ♣.

Consider the points $P_H := \phi(\frac{1}{4}, 0), P_{K_1} := \phi(-2, 1), P_{K_2} := \phi(\frac{3}{2}, 1)$ on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) one finds a generalized principal lattice. Below in the left figure, the solid curve represents C in the chart $w = 1$, while the dashed curve represents H_C. Both curves pass through the singular point P and the inflection point Q of C at infinity. The figure to the right shows the generalized principal lattice for H, K_1, K_2 as above and $m = 3$, formed by tangents to the curve $\widehat{C}$.





4.3.9 C irreducible, with a cuspidal singularity

Suppose that C is an irreducible cubic with a cusp P. By [23, Theorem 6], such a curve has precisely one *real* inflection point. Using Equations 4.6–4.8, we find that the dual curve $\widehat{C}$ is a cubic of genus zero without nodes and with one real cusp. Such a curve $\widehat{C}$ is called a *semicubical parabola*.

For instance, let $C : F(w, x, y) = wy^2 + x^3 = 0$. Its Hessian curve is given by

$$H_C : \det \begin{bmatrix} 0 & 0 & 2y \\ 0 & 6x & 0 \\ 2y & 0 & 2z \end{bmatrix} = -24xy^2 = 0.$$

Besides the singularity $P = [1 : 0 : 0]$, the intersection $C \cap H_C$ contains the inflection point $Q = [0 : 0 : 1]$ at infinity. This inflection point corresponds to the tangent of the cusp of $\widehat{C}$.

Let us now explicitly describe $\widehat{C}$. The tangent line to C at the point $R = [w_0 : x_0 : y_0]$ is given by

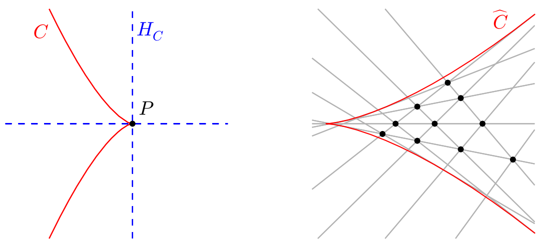
$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y = y_0^2 w + 3x_0^2 x + 2w_0 y_0 y = 0.$$

It follows that the triples $[a : b : c] = [y_0^2 : 3x_0^2 : 2w_0y_0]$ form the cubic $\widehat{C} : 27ac^2 - 4b^3 = 0$ in the dual plane. The bijection $\phi : \mathbb{R} \longrightarrow C_{\text{ns}}$ given by $t \longmapsto [t^3 : -t : 1]$ induces a group structure on C_{ns} that satisfies Property ♣ (compare [16]). Consider the points

$$\begin{aligned} P_H &:= \phi\left(\frac{1}{4}\right) = \left[\frac{1}{64} : -\frac{1}{4} : 1\right], \\ P_{K_1} &:= \phi\left(-\frac{3}{2}\right) = \left[-\frac{27}{8} : \frac{3}{2} : 1\right], \\ P_{K_2} &:= \phi(1) = [1 : -1 : 1] \end{aligned}$$

on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) one finds a generalized principal lattice.

Below in the left figure, the solid curve represents C in the finite plane, while the dashed curve represents H_C. Both curves pass through the singular point $P = [1 : 0 : 0]$ and the inflection point $Q = [0 : 0 : 1]$ of C. The figure to the right shows the generalized principal lattice for H, K_1, K_2 as above and $m = 3$, formed by tangents to the semicubical parabola $\widehat{C}$.



4.3.10 C irreducible, smooth

If C is an irreducible and smooth cubic curve, then C has genus 1. Such a curve is also called an *elliptic curve*. For this reason, we write $E = C$ in this section.

Let us briefly recall some facts about elliptic curves. The group law $\oplus$ that satisfies Property ♣ turns E into an Abelian variety, as both $\oplus : E \times E \longrightarrow E$ and $\ominus : E \longrightarrow E$ are regular maps [75, Theorem III.3.6]. Up to an isomorphism of Abelian varieties, one distinguishes an infinite number of elliptic curves [74, Theorem V.2.3]. As E is smooth and as any regular map is built up from polynomial functions, E is a real Lie group as well. Since there are far more real analytic isomorphisms than there are biregular maps, one can expect fewer Lie group isomorphism classes of elliptic curves. In fact, there are only two such classes, and there is an isomorphism of real Lie groups

$$E \simeq \begin{cases} \mathbb{S}^1 & \text{if } \Delta(E) < 0; \\ \mathbb{S}^1 \times \mathbb{Z}_2 & \text{if } \Delta(E) > 0, \end{cases} \tag{4.9}$$

where the *modular discriminant* $\Delta(E)$ is the discriminant of some Weierstrass equation of E [74, Corollary V.2.3.1].

After a change of coordinates, any elliptic curve $E \subset \mathbb{P}^2$ can be assumed to be of the form

$$E : wy^2 - 4x^3 + g_2w^2x + g_3w^3 = wy^2 - (x - e_1w)(x - e_2w)(x - e_3w) = 0.$$

Since E is nonsingular, the discriminant $\Delta(E) = g_2^3 - 27g_3^2$ is necessarily nonzero. Note that the sign of $\Delta(E)$, and therefore the number of components of E, depends on the number of real roots of the polynomial $h(x) = 4x^3 - g_2x - g_3 = 0$. More precisely, E has two components whenever h has three real roots $e_1 > e_2 > e_3$, and E has one component whenever h has one real root e_2 and two conjugated nonreal roots e_1, e_3.

As E has genus 1, it cannot be parameterized by rational functions. However, an isomorphism as in Equation 4.9 can be given explicitly in terms of elliptic functions. Let us denote the complexification of any real curve C by attaching the symbol $\mathbb{C}$ as a subindex. That is, $C_{\mathbb{C}} := C \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$. Let $\mathbb{P}_{\mathbb{C}}$ denote the *Riemann sphere*, and let $u : \mathbb{P}_{\mathbb{C}} \longrightarrow \mathbb{C}$ be the multivalued function defined by the elliptic integral

$$u(y) = \int_y^\infty \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2s - g_3}}.$$

The function $u(y)$ is multivalued because the value of the integral depends on the choice of the path. More precisely, every loop around e_i will contribute a certain number ω_i, called a *period*, to the integral (see [46, §9.6] for explicit expressions of $\omega_1, \omega_2, \omega_3$ in terms of the equation of $E_{\mathbb{C}}$). By the topology of the Riemann sphere, these periods satisfy the linear relation $\omega_1 + \omega_2 + \omega_3 = 0$. Defining a lattice $\Lambda := \{m\omega_1 + n\omega_2 \ : \ m, n \in \mathbb{Z}\} \subset \mathbb{C}$, one finds that $u(y) : \mathbb{P}_{\mathbb{C}} \longrightarrow \mathbb{C}/\Lambda$ is a single-valued function.

One defines the *Weierstrass $\wp$ function*, $\wp : \mathbb{C}/\Lambda \longrightarrow \mathbb{P}_{\mathbb{C}}$ as the inverse function $\wp(u)$ of $u(y)$. An explicit formula for $\wp$ is

$$\wp(z) = \wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right). \tag{4.10}$$

As an application of Liouville's theorem, one can show that $\wp$ satisfies the differential equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, where moreover the *elliptic invariants* g_2, g_3 satisfy

$$g_2 = g_2(\Lambda) := 60 \sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^4}, \qquad g_3 = g_3(\Lambda) := 140 \sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^6}.$$

We thus obtain a map $\phi : \mathbb{C}/\Lambda \longrightarrow E_{\mathbb{C}}$ given by $z \longmapsto [1 : \wp(z) : \wp'(z)]$. Because of the *addition formula* for Weierstrass $\wp$ functions,

$$\det \begin{bmatrix} 1 & \wp(z_0) & \wp'(z_0) \\ 1 & \wp(z_1) & \wp'(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) \end{bmatrix} = 0 \qquad \Longleftrightarrow \qquad z_0 + z_1 + z_2 \in \Lambda,$$

this parameterization is in fact an analytic group isomorphism [46, Theorem 9.4.3] [75, Corollary VI.5.1.1] [55, §7D].

In case E is defined over $\mathbb{R}$, then one can take the lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_3$ to be rectangular (that is, $\omega_1 \in \mathbb{R}$ and $\omega_3 \in i\mathbb{R}$) [75, Ex. VI.6.7]. Restricting ϕ to real numbers and identifying $\mathbb{S}^1 \simeq \mathbb{R}/\omega_1\mathbb{Z}$ will yield a parameterization $\phi_1 : \mathbb{S}^1 \longrightarrow E$ of one connected component of the real elliptic curve E. Note that ϕ_1 is a *topological group isomorphism*, in the sense that it is both a homeomorphism and a group isomorphism.

C irreducible, smooth with one real connected component

Suppose that E is an elliptic curve with only one real component. By Equations 4.6–4.8 the dual curve $\widehat{E}$ is a sextic of geometric genus 1 whose only singularities are nine cusps, three of which are real.

For instance, let $E : F(w, x, y) = wy^2 - x^3 - w^3 = 0$. Its Hessian curve is given by

$$H_E : \det \begin{bmatrix} -6w & 0 & 2y \\ 0 & -6x & 0 \\ 2y & 0 & 2w \end{bmatrix} = 24(3w^2 + y^2)x = 0.$$

The intersection $E \cap H_E$ consists of the inflection points $Q_1 = [1 : 0 : 1], Q_2 = [1 : 0 : -1], Q_3 = [0 : 0 : 1]$ of E, each of which corresponds to the tangent of a cusp of $\widehat{E}$.

Let us now explicitly describe $\widehat{E}$. The tangent line to E at the point $R = [w_0 : x_0 : y_0]$ is given by

$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y = \left(y_0^2 - 3w_0^2\right)w - 3x_0^2 x + 2w_0 y_0 y = 0.$$

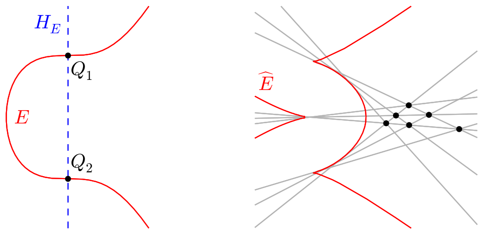
Using Gröbner bases to eliminate the variables w_0, x_0, y_0, it follows that the triples $[a : b : c] = [y_0^2 - 3w_0^2 : -3x_0^2 : 2w_0 y_0]$ form the sextic

$$\widehat{E} : 4a^3b^3 - 4b^6 + 27a^4c^2 - 36ab^3c^2 - 54a^2c^4 + 27c^6 = 0$$

in the dual plane.

If E has only one connected component, then the map ϕ_1 will be a topological group isomorphism that, by the addition formula for Weierstrass $\wp$ functions, satisfies Property ♣. Consider the points $P_H = \phi_1(0.1), P_{K_1} = \phi_1(-1.75), P_{K_2} = \phi_1(1.63)$ on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) one finds a generalized principal lattice.

Below in the left figure, the solid curve represents E in the finite plane, while the dashed curve represents H_E. Both curves pass through the inflection points Q_1, Q_2, Q_3 of E. The figure to the right shows the generalized principal lattice for H, K_1, K_2 as above and $m = 2$, formed by tangents to the curve $\widehat{E}$.



C irreducible, smooth with two real connected components

Suppose that E is an elliptic curve with two real components. By Equations 4.6–4.8 the dual curve $\widehat{E}$ is a sextic of geometric genus 1 with two components whose only singularities are nine cusps, three of which are real.

For instance, let $E : F(w,x,y) = wy^2 - x^3 + w^2x = 0$. Its Hessian curve is given by

$$H_E : \det \begin{bmatrix} 2x & 2w & 2y \\ 2w & -6x & 0 \\ 2y & 0 & 2w \end{bmatrix} = -8w^3 - 24(xw - y^2)x = 0.$$

The intersection $E \cap H_E$ comprises the inflection points

$$Q_\pm = \left[1 : \sqrt{1 + \frac{2}{\sqrt{3}}} : \pm\sqrt{2}\frac{\sqrt[4]{3 + 2\sqrt{3}}}{\sqrt{3}}\right], \qquad Q = [0:0:1].$$

Each of these inflection points corresponds to the tangent of a cusp of $\widehat{E}$.

Let us now explicitly describe $\widehat{E}$. The tangent line to E at the point $R = [w_0 : x_0 : y_0]$ is given by

$$\frac{\partial F}{\partial w}(R)w + \frac{\partial F}{\partial x}(R)x + \frac{\partial F}{\partial y}(R)y = \left(y_0^2 + 2w_0x_0\right)w + \left(w_0^2 - 3x_0^2\right)x + 2w_0y_0y = 0.$$

Using Gröbner bases to eliminate the variables w_0, x_0, y_0, it follows that the triples $[a:b:c] = \left[y_0^2 + 2w_0x_0 : w_0^2 - 3x_0^2 : 2w_0y_0\right]$ form the sextic

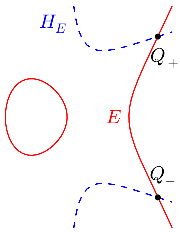
$$\widehat{E} : 4a^3b^3 - 4ab^5 + 27a^4c^2 - 30a^2b^2c^2 - b^4c^2 - 24abc^4 - 4c^6 = 0$$

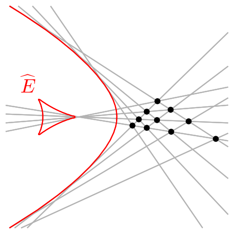
in the dual plane. The map ϕ_1 from the previous section yields a parameterization of one component of E. The other component can be parameterized by the map $\phi_2 : \mathbb{R}/\omega_1\mathbb{Z} \longrightarrow E$ given by $t \longmapsto \phi(t + \frac{1}{2}\omega_3)$, as Equation 4.10 implies that $\wp(t + \frac{1}{2}\omega_3)$ and $\wp'(t + \frac{1}{2}\omega_3)$ are real for all real t. Identifying $\mathbb{R}/\omega_1\mathbb{Z}$ with $\mathbb{S}^1$, one finds a topological group isomorphism $\psi : \mathbb{S}^1 \times \mathbb{Z}_2 \longrightarrow E$ defined by the rule

$$(t,a) \longmapsto \begin{cases} \phi_1(t) & \text{if } a = 0; \\ \phi_2(t) & \text{if } a = 1 \end{cases},$$

which shows that E satisfies Property ♣. Consider the points $P_H := \phi(0.12, 0), P_{K_1} := \phi(-0.7, 1), P_{K_2} := \phi(0.5, 1)$ on C_{ns}, and let H, K_1, K_2 be the associated lines in dual space. Applying Theorem 13.(2) one finds a generalized principal lattice.

Below in the left figure, the solid curve represents E in the finite plane, while the dashed curve represents H_E. Both curves pass through the inflection points Q_+, Q_-, Q of E. The figure to the right shows the generalized principal lattice for H, K_1, K_2 as above and $m = 3$, formed by tangents to the curve $\widehat{E}$.





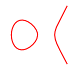
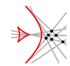
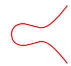
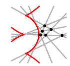
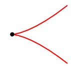
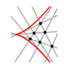
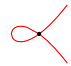
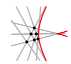
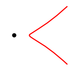
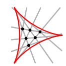
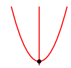
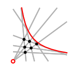
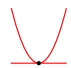
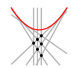
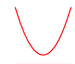
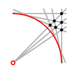
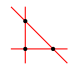
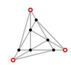
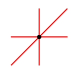
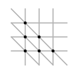
	type of cubic	an equation	a curve	type of mesh	a GPL	group
(a)	elliptic with two components	$0 = y^2 - x(x-1)(x-2)$		tangents to a sextic with three real cusps and two components		$\mathbb{S}^1 \times \mathbb{Z}_2$
(b)	elliptic with one component	$0 = y^2 - x^3 + x^2 + 1$		tangents to a sextic with three real cusps and one component		$\mathbb{S}^1$
(c)	irreducible with a cuspidal singularity	$0 = y^2 - x^3$		tangents to a semicubical parabola		$\mathbb{R}$
(d)	irreducible with a nodal singularity	$0 = y^2 - x^2(x+1)$		tangents to a tricuspidal quartic with one real cusp		$\mathbb{R} \times \mathbb{Z}_2$
(e)	irreducible with an isolated singularity	$0 = y^2 - x^2(x-1)$		tangents to a tricuspidal quartic with three real cusps		$\mathbb{S}^1$
(f)	reducible union of a conic and a secant	$0 = (y - x^2)x$		tangents to a conic and lines through a vertex outside this conic		$\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
(g)	reducible union of a conic and a tangent	$0 = (y - x^2)y$		tangents to a conic and lines through a vertex on this conic		$\mathbb{R} \times \mathbb{Z}_2$
(h)	reducible disjoint union of a conic and a line	$0 = (y - x^2 - 1)y$		tangents to a conic and lines through a vertex inside a conic		$\mathbb{S}^1 \times \mathbb{Z}_2$
(i)	reducible union of three nonconcurrent lines	$0 = xy(x + y - 1)$		3-pencil lattice		$\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3$
(j)	reducible union of three concurrent lines	$0 = xy(x - y)$		principal lattice		$\mathbb{R} \times \mathbb{Z}_3$

Table 4.1: *A classification of the generalized principal lattices in the plane.*

4.4 Generalized principal lattices in space

In this section we construct generalized principal lattices in higher-dimensional projective space. Our discussion will mainly focus on generalized principal lattices in 3-dimensional projective space, but we remark that many of the techniques we apply can be seen to work in any higher-dimensional projective space.

In Section 4.4.1, we introduce some basic notation, define generalized principal lattices in n-dimensional projective space for arbitrary n, and give two examples. In the next section, we remark that all known examples of generalized principal lattices in n-dimensional projective space correspond to a curve of degree $n+1$ and arithmetic genus 1. Moreover, in 3-dimensional projective space, the curves are always the complete intersection of two quadrics. We subsequently ask the question whether, conversely, any such curve define generalized principal lattices of arbitrary high degree. Section 4.4.3 recalls two properties **P1** and **P2** for families of parameterized curves from [17], which can be used as a technical tool for showing that these curves defines generalized principal lattices of arbitrary degree. In Section 4.4.4, we recall the classification of complete intersections of quadrics in complex 3-dimensional project space, and single out the types that have the potential to define generalized principal lattices. In Sections 4.4.5 – 4.4.9, we transform each of these curve types into a normal form by means of a real projective change of coordinates and find a parameterization satisfying Properties **P1**, **P2**. The resulting classification is summarized in Table 4.4.

4.4.1 Definition, notation, and examples

In this section, we define the notion of a generalized principal lattice in higher-dimensional projective space and introduce some notation needed for the remainder of the chapter. We end the section with two simple examples.

For any $m \geq 0$ and $n \geq 2$, the set

$$S_m^n := \big\{(i_0, i_1, \ldots, i_n) \,:\, i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, m\},\; i_0 + i_1 + \cdots + i_n = m\big\}$$

will function as an index set for the points of a generalized principal lattice. For any $n \geq 1$, the symbol $\mathbb{P}^n$ represents the real projective space of dimension n, and the symbol $\widehat{\mathbb{P}}^n$ represents its dual space. Similarly, the symbol $\mathbb{P}^n_{\mathbb{C}}$ represents the complex projective space of dimension n and the symbol $\widehat{\mathbb{P}}^n_{\mathbb{C}}$ its dual space. Both spaces come equipped with projective coordinates $[x_0 : x_1 : \cdots : x_n]$, which we denote by $[w : x : y : z]$ for the case $n = 3$. Usually, we think of $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) as lying inside of $\mathbb{P}^n$ (respectively $\mathbb{P}^n_{\mathbb{C}}$) by means of the injection $(x_1, \ldots, x_n) \longrightarrow [1 : x_1 : \cdots : x_n]$. The remaining points in $\mathbb{P}^n$ (respectively $\mathbb{P}^n_{\mathbb{C}}$) form the hyperplane $x_0 = 0$, which is called the *hyperplane at infinity*. By contrast, the points in $\mathbb{R}^n$ (respectively $\mathbb{C}^n$) are called the *finite points* in $\mathbb{P}^n$ (respectively $\mathbb{P}^n_{\mathbb{C}}$). Moreover, we think of $\mathbb{P}^n$ as lying inside $\mathbb{P}^n_{\mathbb{C}}$, via the trivial injection

$$\mathbb{P}^n \longrightarrow \mathbb{P}^n_{\mathbb{C}}, \qquad [x_0 : x_1 : \cdots : x_n] \longmapsto [x_0 : x_1 : \cdots : x_n].$$

Now the formalities are out of the way, let us get to business. The following definition is the straightforward generalization of Definition 11 taken from [17, Definition 1].

Definition 15 (generalized principal lattice in $\mathbb{P}^n$)**.** Let $m \geq 0$, $n \geq 2$ and consider $n+1$ families

$$H_0^0, H_1^0, \ldots, H_m^0, \qquad H_0^1, H_1^1, \ldots, H_m^1, \qquad \ldots, \qquad H_0^n, H_1^n, \ldots, H_m^n$$

of $m+1$ hyperplanes in $\mathbb{P}^n$, for which any two of the $(n+1)(m+1)$ hyperplanes are distinct. Suppose that

GPL$'_1$ Any intersection $H_{i_1}^{r_1} \cap H_{i_2}^{r_2} \cap \cdots \cap H_{i_n}^{r_n}$, corresponding to distinct indices $0 \leq r_1 < r_2 < \cdots < r_n \leq n$, consists of exactly one point.

GPL$'_2$ The intersection $H_{i_0}^0 \cap H_{i_1}^1 \cap \cdots \cap H_{i_n}^n \neq \emptyset$ whenever $(i_0, i_1, \ldots, i_n) \in S_m^n$.

Under these assumptions, the set of points

$$X := \big\{x_{\mathbf{i}} \,:\, H_{i_0}^0 \cap H_{i_1}^1 \cap \cdots \cap H_{i_n}^n = \{x_{\mathbf{i}}\},\ \mathbf{i} = (i_0, i_1, \ldots, i_n) \in S_m^n\big\} \tag{4.11}$$

is a *generalized principal lattice of degree m in* $\mathbb{P}^n$ if, additionally,

GPL$'_3$ For any $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, m\}$,

$$H_{i_0}^0 \cap H_{i_1}^1 \cap \cdots \cap H_{i_n}^n \cap X \neq \emptyset \implies (i_0, i_1, \ldots, i_n) \in S_m^n.$$

In this case, we say that the $n+1$ families $\mathcal{H}^r = \{H_0^r, H_1^r, \ldots, H_m^r\}$ of $m+1$ hyperplanes *define a generalized principal lattice of degree m in* $\mathbb{P}^n$.

We proceed with constructing two important examples of generalized principal lattices in $\mathbb{P}^n$.

Example 8 (triangular meshes in $\mathbb{R}^n$)**.** Let us reconstruct the motivating example of a generalized principal lattice in $\mathbb{P}^n$, namely that of a *triangular mesh of degree m in* $\mathbb{R}^n$. Following our convention, we identify $\mathbb{R}^n$ with the chart $x_0 = 1$ of $\mathbb{P}^n$. We verify that the hyperplanes

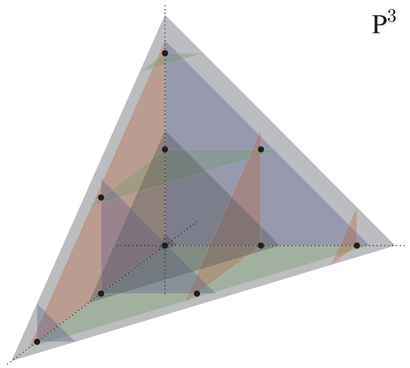
$$H_i^0 : x_1 + \cdots + x_n - (m-i)x_0 = 0,$$

$$H_i^r : x_r - i x_0 = 0, \qquad\qquad r = 1, \ldots, n, \qquad i = 0, \ldots, m,$$

satisfy the conditions of Definition 15.

Consider any intersection $I := H_{i_1}^{r_1} \cap H_{i_2}^{r_2} \cap \cdots \cap H_{i_n}^{r_n}$ corresponding to distinct indices $0 \leq r_1 < r_2 < \cdots < r_n \leq n$. For any point $[1 : x_1 : \cdots : x_n] \in I$, one has $x_{r_j} = i_j$ for $j = 2, \ldots, n$. The remaining coordinate is fixed by the equation of $H_{i_1}^{r_1}$. We conclude that the hyperplanes $\{H_i^r\}$ satisfy Property **GPL$'_1$**. Moreover, the intersection $H_{i_0}^0 \cap H_{i_1}^1 \cap \cdots \cap H_{i_n}^n$ is nonempty if and only if

$$0 = \det \begin{bmatrix} i_0 - m & 1 & 1 & \cdots & 1 \\ -i_1 & 1 & 0 & \cdots & 0 \\ -i_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -i_n & 0 & 0 & \cdots & 1 \end{bmatrix} = i_0 + i_1 + \cdots + i_n - m,$$



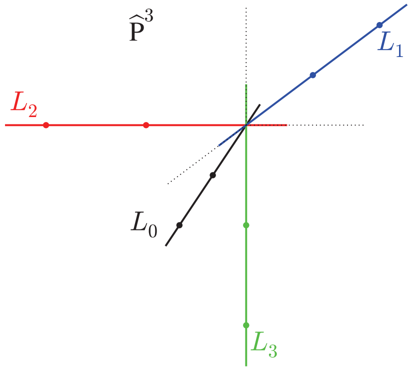


Figure 4.4: *Figure (a) shows the triangular mesh of degree* 2 *in* $\mathbb{R}^3$ *together with its* $3+1$ *families of* $2+1$ *hyperplanes. Figure (b) shows the lines* L_0, L_1, L_2, L_3 *in dual space together with the points* p_i^r *(except for the points* p_m^0, p_0^1, p_0^2*, and* p_0^3*, which lie at infinity).*

implying Properties $\mathbf{GPL}_2'$ and $\mathbf{GPL}_3'$. We conclude that any triangular mesh of degree m in $\mathbb{R}^n$ is a generalized principal lattice of degree m in $\mathbb{P}^n$.

In a triangular mesh in $\mathbb{R}^n$, each family $\mathcal{H}^r = \{H_0^r, H_1^r, \ldots, H_m^r\}$ is contained in a pencil $\mathcal{P}_r$ of hyperplanes. For $r \neq 0$, the pencil $\mathcal{P}_r$ is spanned by the hyperplanes $x_0 = 0$ and $x_r = 0$, while for $r = 0$ it is spanned by the hyperplanes $x_0 = 0$ and $x_1 + x_2 + \cdots + x_n = 0$. Via the dual correspondence, each hyperplane H_i^r in $\mathbb{P}^n$ corresponds to a point p_i^r in $\widehat{\mathbb{P}}^n$. Moreover, the base locus of the pencil $\mathcal{P}_r$ corresponds to a line L_r in $\widehat{\mathbb{P}}^n$, with

$$L_0 = \big\{[\alpha : \beta : \beta : \cdots : \beta] \in \widehat{\mathbb{P}}^n \;:\; [\alpha : \beta] \in \mathbb{P}^1\big\},$$

$$L_1 = \big\{[\alpha : \beta : 0 : \cdots : 0] \;\in \widehat{\mathbb{P}}^n \;:\; [\alpha : \beta] \in \mathbb{P}^1\big\},$$

$$\vdots$$

$$L_n = \big\{[\alpha : 0 : 0 : \cdots : \beta] \;\in \widehat{\mathbb{P}}^n \;:\; [\alpha : \beta] \in \mathbb{P}^1\big\}.$$

As each pencil contains the hyperplane $x_0 = 0$ at infinity, the lines $L_0, L_1, \ldots, L_n$ meet in the point $[1 : 0 : \cdots : 0] \in \widehat{\mathbb{P}}^n$.

Figure 4.4a shows the triangular mesh in $\mathbb{R}^3$ with $m = 2$. Planes of the same color form a family $\mathcal{H}_r$ that is part of a pencil $\mathcal{P}_r$ with base locus at infinity. For each pencil $\mathcal{P}_r$, the corresponding line L_r is drawn with the same color in 4.4b, together with the points p_i^r visible in the finite plane.

Example 9 ($(n+1)$-pencil lattices)**.** Analogously to the planar case, one can consider more general configurations of pencils of hyperplanes. The planar principal lattices discussed in Section 4.1, for instance, can without difficulty be generalized to higher-dimensional projective space. More generally, Lee and Phillips introduced the notion of an $(n+1)$-pencil lattice [53]. Although principal lattices are strictly speaking not $(n+1)$-pencil lattices, it is shown in [53, Section 4] that the principal lattices are recovered from the $(n+1)$-pencil lattices as a degenerate case. We proceed to show Lee and Phillips' construction in a special case, from which any other $(n+1)$-pencil lattice can be obtained by means of a real projectivity of $\mathbb{P}^n$.

Let $y_0 := [1:0:\cdots:0], y_1 := [0:1:\cdots:0], \ldots, y_n := [0:0:\cdots:1]$ be the vertices of the simplex of reference in $\mathbb{P}^n$. Here the indices are considered cyclically, so that for instance $y_{-1} = y_n$. Let $\mathcal{P}_r$ be the pencil of hyperplanes passing through $\{y_0, y_1, \ldots, y_n\} \backslash \{y_{r-1}, y_r\}$. In other words, any hyperplane in $\mathcal{P}_r$ is the zeroset of a linear form $\alpha x_{r-1} + \beta x_r$ for some $[\alpha : \beta] \in \mathbb{P}^1$. Let m be any nonnegative integer and $\mu \neq 0, 1$ a real number. Consider the $n+1$ families $\mathcal{H}^r = \{H_0^r, H_1^r, \ldots, H_m^r\}$, $r = 0, 1, \ldots, n$, of hyperplanes defined by

$$\begin{aligned} &H_i^0 : x_0 - \mu^{m-i} x_n = 0, \\ &H_i^r : x_{r-1} - \mu^i x_r \ = 0, \qquad\qquad i = 0, 1, \ldots, m, \qquad r = 1, \ldots, n. \end{aligned}$$

Let us show that these hyperplanes satisfy the conditions of Definition 15. The intersection of any $n+1$ hyperplanes $H_{i_0}^0, H_{i_1}^1, \ldots, H_{i_n}^n$ is nonempty if and only if

$$0 = \det \begin{bmatrix} 1 & 0 & 0 & \cdots & -\mu^{m-i_0} \\ 1 & -\mu^{i_1} & 0 & \cdots & 0 \\ 0 & 1 & -\mu^{i_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -\mu^{i_n} \end{bmatrix} = (-1)^n \big(\mu^{i_1+\cdots+i_n} - \mu^{m-i_0}\big),$$

which happens precisely when $i_0 + i_1 + \cdots + i_n = m$, implying Properties $\mathbf{GPL}_2'$ and $\mathbf{GPL}_3'$. Since any n rows in the above matrix are linearly independent, the hyperplanes $\{H_i^r\}$ satisfy Property $\mathbf{GPL}_1'$. The $(n+1)$-*pencil lattice* associated with $\{H_i^r\}$ is the set X from (4.11) explicitly given by

$$X = \left\{ \left[1 : \mu^{-\alpha_1} : \mu^{-\alpha_1-\alpha_2} : \cdots : \mu^{-\alpha_1-\cdots-\alpha_n}\right] \,:\, (\alpha_0, \alpha_1, \ldots, \alpha_n) \in S_m^n \right\}.$$

4.4.2 What curves give rise to a generalized principal lattice?

Both for the triangular mesh in $\mathbb{R}^n$ and the $(n+1)$-pencil lattice in $\mathbb{P}^n$, the points dual to the hyperplanes $\{H_i^r\}$ lie on the union of $n+1$ lines. This leads to the following question. What if we consider other projective curves $C \subset \mathbb{P}^n$ of degree $n+1$, can we do the same as in the planar case? More precisely, we wish to investigate when such a curve C has the following property.

★ For any nonnegative integer m, there exist $n+1$ families of $m+1$ points on C such that the corresponding hyperplanes in dual space define a generalized principal lattice of degree m in $\widehat{\mathbb{P}}^n$ in the sense of Definition 15.

Clearly, Property ★ does not hold for every curve $C \subset \mathbb{P}^n$ of degree $n+1$. In the examples below, suppose $C \subset \mathbb{P}^3$ contains points p_i^r, with $i = 0, 1, \ldots, m$ and $r = 0, 1, \ldots, n$, such that the corresponding planes H_i^r in dual space define a generalized principal lattice X of degree m in $\widehat{\mathbb{P}}^3$.

Example 10. Suppose that C is *degenerate*, in the sense that it is contained in a plane. Then all of its points will obviously be coplanar, and the planes H_i^r will all meet in the same point, which is a generalized principal lattice of degree 0 in $\widehat{\mathbb{P}}^3$. In other words, Property ★ does not hold for $m \geq 1$ for such a curve.

Example 11. Suppose that C is not reduced, in the sense that it is set-theoretically equivalent to a reduced curve C' of degree lower than 4. By definition of the *degree* of a projective curve, a plane will either intersect C' in fewer than 4 points, or contain a whole component of C'. In the former case, there will be no four independent hyperplanes that meet in a point of X, while the latter case contradicts Condition $\mathbf{GPL}'_3$ for large enough m.

So which curves *do* have Property ★? A partial answer to this question was given in [17]. In this paper, Carnicer, Gasca, and Sauer find many examples of generalized principal lattices in $\mathbb{P}^n$ (of the types (c), (d), (e), (i), (j), (k) in Table 4.4), by applying the theory of Haar systems to find compatible 1-parameter families of hyperplanes. These families are parameterized by the topological groups $\mathbb{R}$, $\mathbb{R} \times \mathbb{Z}_2$, and $\mathbb{S}^1$. They explicitly state that they do not claim to cover all possibilities, and there are indeed many types missing already in the case $n = 3$.

The examples in [17] provide us with a clue where to look for generalized principal lattices in $\mathbb{P}^3$. In each example, the 1-parameter families of hyperplanes form a curve in dual space that is the complete intersection of two quadric surfaces. It is well known that such a curve is of degree 4 and arithmetic genus 1 [45, Exercise I.7.2d]. This is comparable to the classification of the planar generalized principal lattices from Table 4.1, where each curve is a of degree 3 and arithmetic genus 1.

The goal of the remainder of this chapter is to answer the following question.

Question 16. Which curves $C \subset \mathbb{P}^3$ that are the complete intersection of two quadrics satisfy Property ★, and how can we explicitly construct generalized principal lattices from these?

In the next section, we show how one can equip such curves with additional structure to keep track of coplanarity of its points, similar to how the group law for cubic curves exhibited in Theorem 12 keeps track of collinearity.

4.4.3 Encoding cohyperplanarity

In this section, we recall from [17] a sufficient criterion for a curve $C \subset \mathbb{P}^n$ of degree $n+1$ to satisfy Property ★. Let C_{ns} denote the nonsingular part of C.

Let us consider an example to show that the situation for space quartics differs from that of planar cubics. Let C be the union of the twisted cubic C' and the tangent line L to some point $P \in C'$. Then C_{ns} will be homeomorphic to the disjoint union of two copies of the real line $\mathbb{R}$. By Proposition 18 in [54], any topological Abelian group with this topology is isomorphic to $\mathbb{R} \times \mathbb{Z}_2$. Suppose we are given four points on C_{ns}, three of which lie on C' and one on L. Whether these four points are coplanar or not, the corresponding points in $\mathbb{R} \times \mathbb{Z}_2$ can never sum to $(0, 0)$. Yet in Section 4.4.7 it is shown that the curve C satisfies Property ★. We cannot, therefore, in general expect the existence of a group law on C_{ns} that encodes coplanarity the way Property ♣ encodes collinearity for planar curves.

In [17], a different method for finding generalized principal lattices in $\mathbb{P}^n$ is proposed. For any Abelian group G and any $n+1$ families of hyperplanes $\mathcal{H}^r = \big\{H^r(\omega) : \omega \in G\big\}, r = 0, 1, \ldots, n$, satisfying two compatibility conditions, they present a way to pick $(m+1)$ hyperplanes $H_0^r, H_1^r, \ldots, H_m^r$ from each family $\mathcal{H}^r$ so that the set X defined by Equation 4.11 is a generalized principal lattice of degree m in $\mathbb{P}^n$. Using the theory of Haar systems, they find many examples of such compatible families of hyperplanes and of corresponding generalized principal lattices.

Let us present the projective dual of this construction. For any Abelian group G and $n+1$ parameterized curves

$$\chi_r : G \longrightarrow C_r \subset \widehat{\mathbb{P}}^n, \qquad r = 0, 1, \ldots, n,$$

some of which may coincide, we introduce the following two properties.

P1 Any distinct points $\chi_{r_1}(g_1), \chi_{r_2}(g_2), \ldots, \chi_{r_n}(g_n)$ corresponding to distinct indices $0 \leq r_1 < r_2 < \cdots < r_n \leq n$ span a hyperplane.

P2 Any distinct points $\chi_0(g_0), \chi_1(g_1), \ldots, \chi_n(g_n)$ lie in a hyperplane if and only if $g_0 + g_1 + \cdots + g_n = 0$.

The following proposition is the dual version of Proposition 5 in [17], and it shows how one can construct generalized principal lattices of any degree from such a system of curves.

Proposition 17. *Let be given an Abelian group G and $n+1$ parameterized curves $\chi_r : G \longrightarrow C_r \subset \widehat{\mathbb{P}}^n, r = 0, 1, \ldots, n$, satisfying properties **P1** and **P2**. Let $g_0, g_1, \ldots, g_n, \delta \in G$ be such that $g_0 + g_1 + \cdots + g_n + m\delta = 0$. If the $(n+1)(m+1)$ points*

$$p_i^r := \chi_r(g_r + i\delta), \qquad i = 0, 1, \ldots, m, \qquad r = 0, 1, \ldots, n,$$

are distinct, then the corresponding hyperplanes in dual space define a generalized principal lattice of degree m in $\mathbb{P}^n$.

As an example, let us try to reconstruct the triangular mesh in $\mathbb{R}^3$ in this setting. Let $G = \mathbb{R}$ and let be given the parameterized curves

$$\chi_0 : \mathbb{R} \longrightarrow C_0 \subset \widehat{\mathbb{P}}^3, \qquad t \longmapsto [+t : 1 : 1 : 1],$$

$$\chi_1 : \mathbb{R} \longrightarrow C_1 \subset \widehat{\mathbb{P}}^3, \qquad t \longmapsto [-t : 1 : 0 : 0],$$

$$\chi_2 : \mathbb{R} \longrightarrow C_2 \subset \widehat{\mathbb{P}}^3, \qquad t \longmapsto [-t : 0 : 1 : 0],$$

$$\chi_3 : \mathbb{R} \longrightarrow C_3 \subset \widehat{\mathbb{P}}^3, \qquad t \longmapsto [-t : 0 : 0 : 1].$$

The Zariski closures of these curves are lines in $\widehat{\mathbb{P}}^3$ that meet in the point $[1 : 0 : 0 : 0]$, the only point not in the image of the parameterizations. Clearly any distinct points $\chi_{r_1}(t_1), \chi_{r_2}(t_2), \chi_{r_3}(t_3)$ corresponding to distinct indices $0 \leq r_1 < r_2 < r_3 \leq 3$ are linearly independent, implying that Property **P1** holds for these parameterized curves.

Moreover, any distinct points $\chi_0(t_0), \chi_1(t_1), \chi_2(t_2), \chi_3(t_3)$ lie in a hyperplane if and only if

$$0 = \det \begin{bmatrix} +t_0 & 1 & 1 & 1 \\ -t_1 & 1 & 0 & 0 \\ -t_2 & 0 & 1 & 0 \\ -t_3 & 0 & 0 & 1 \end{bmatrix} = t_0 + t_1 + t_2 + t_3,$$

implying that Property **P2** holds. Let $t_0 = -m$, $t_1 = t_2 = t_3 = 0$, and $\delta = 1$, so that $t_0 + t_1 + t_2 + t_3 + m\delta = 0$. Define, for $i = 0, 1, \ldots, m$, the points

$$p_i^0 := \chi_0(t_0 + i\delta) = \chi_0(i - m) = [i - m : 1 : 1 : 1],$$

$$p_i^1 := \chi_1(t_1 + i\delta) = \chi_1(i) = [-i : 1 : 0 : 0],$$
$$p_i^2 := \chi_2(t_2 + i\delta) = \chi_2(i) = [-i : 0 : 1 : 0],$$
$$p_i^3 := \chi_3(t_3 + i\delta) = \chi_3(i) = [-i : 0 : 0 : 1].$$

Then these points p_i^r are distinct, and the associated planes

$$H_i^0 : x_1 + x_2 + x_3 + (i - m)x_0 = 0,$$

$$H_i^1 : x_1 - ix_0 = 0, \qquad H_i^2 : x_2 - ix_0 = 0, \qquad H_i^3 : x_3 - ix_0 = 0$$

in dual space define the triangular mesh of degree m in $\mathbb{R}^3$ (compare Figures 4.4a and 4.4b).

4.4.4 Pencils of quadrics in space

In order to answer Question 16, we need to know a bit more about the intersection of two quadrics in $\mathbb{P}^3$. In this section, we briefly recall some definitions and facts from the theory of quadratic forms and pencils of quadrics in $\mathbb{P}^n$, for $n \geq 2$.

Any quadric hypersurface $Q \subset \mathbb{P}^n$ can be *represented* by an $(n+1) \times (n+1)$ symmetric matrix M, unique up to scalar multiplication, as the zeroset $Q = \{x \in \mathbb{P}^n : x^{\mathrm{t}} M x = 0\}$. The *rank* of the quadric Q is the rank of any representing matrix M. One verifies directly that Q is singular if and only if M is singular. In fact, the singular locus of Q coincides with the kernel of M.

How is such a representation affected by a projective change of coordinates? Any projectivity $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ can be represented by an $(n+1) \times (n+1)$ invertible matrix A, unique up to scalar multiplication, as a map $x \longmapsto y$ with $Ay = x$. One finds that, in terms of the new coordinates $y = A^{-1}x$, the quadric $Q : x^{\mathrm{t}} M x = y^{\mathrm{t}} A^{\mathrm{t}} M A y = 0$ is represented by the matrix $A^{\mathrm{t}} M A$. Since M is symmetric, there always exists a real projectivity with orthogonal matrix A for which $A^{\mathrm{t}} M A$ is a real diagonal matrix (see [68, Theorem 10.19]).

Let $X \subset \mathbb{P}^n$ be the complete intersection of two quadric hypersurfaces $Q_1, Q_2 \subset \mathbb{P}^n$. For any nonsingular matrix

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{bmatrix}, \tag{4.12}$$

the variety X can equally well be represented as the complete intersection of the quadrics $\lambda_1 Q_1 + \lambda_2 Q_2$ and $\mu_1 Q_1 + \mu_2 Q_2$, where we denote a quadric hypersurface and its quadratic form by the same symbol. To remove this ambiguity, we consider the *pencil of quadrics* $\mathcal{P} = \{\lambda_1 Q_1 + \lambda_2 Q_2 \,:\, [\lambda_1 : \lambda_2] \in \mathbb{P}^1\}$ and define X as the *base locus* $\bigcap_{Q\in\mathcal{P}} Q$. In this case, we say that $\mathcal{P}$ is *generated by* Q_1 *and* Q_2.

To examine all possible base loci obtained in this way, we sometimes need to consider the *complexification of* $\mathcal{P}$, which is simply the set $\mathcal{P}_{\mathbb{C}}$ of all complex linear combinations of the equations of the quadrics in $\mathcal{P}$. That is, for $\mathcal{P}$ generated by Q_1 and Q_2,

$$\mathcal{P}_{\mathbb{C}} = \big\{\lambda_1 Q_1 + \lambda_2 Q_2 \,:\, [\lambda_1 : \lambda_2] \in \mathbb{P}^1_{\mathbb{C}}\big\}.$$

Consider a pencil $\mathcal{P}$ generated by two quadrics Q_1, Q_2 whose general member is a nonsingular quadric. One defines the *discriminant* of the pair (Q_1, Q_2) as $\Delta := \det(sQ_1 + tQ_2)$, which is a homogeneous form in the coordinates $[s : t]$ on $\mathbb{P}^1$ of degree 4. Whenever Q_2 is nonsingular, Δ is, up to multiplication by a scalar, the characteristic polynomial of $Q_1 Q_2^{-1}$ in the variable $\lambda := -t/s$. Every singular quadric in the pencil $\mathcal{P}$ corresponds to a factor of Δ, and we can define the *multiplicity* of each singular quadric in $\mathcal{P}$ as the multiplicity of this factor in Δ. Note that this definition is independent of the choice of the generators Q_1, Q_2 of $\mathcal{P}$. The multiplicity will be useful in distinguishing projectively inequivalent pencils in Table 4.4.

The pencils $\mathcal{P}_{\mathbb{C}}$ of quadric hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$ with nonsingular general member are classically classified in terms of the Segre characteristic [8, Section III.18] [81, Section 6.5], which is defined as follows. Let $\mathcal{P}_{\mathbb{C}}$ be any pencil of quadrics in $\mathbb{P}^n_{\mathbb{C}}$ generated by two linearly independent quadrics $Q_1 : x^{\mathrm{t}} M_1 x = 0$ and $Q_2 : x^{\mathrm{t}} M_2 x = 0$, of which Q_2 is nonsingular. Suppose that the matrix $Q_1 Q_2^{-1}$ has eigenvalues $\lambda_1, \ldots, \lambda_k$ with algebraic multiplicities $m_1, \ldots, m_k$. Additionally, suppose that the Jordan normal form of $Q_1 Q_2^{-1}$ has, corresponding to each of its eigenvalues λ_i, l_i Jordan blocks

$$\begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ 0 & 0 & \lambda_i & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix}$$

with sizes $s_i^1 \geq s_i^2 \geq \cdots \geq s_i^{l_i}$. Then the *Segre characteristic* of $\mathcal{P}_{\mathbb{C}}$, or *characteristic* for short, is defined as the symbol

$$\big[\big(s_1^1 s_1^2 \cdots s_1^{l_1}\big)\big(s_2^1 s_2^2 \cdots s_2^{l_2}\big) \cdots \big(s_k^1 s_k^2 \cdots s_k^{l_k}\big)\big],$$

where we leave out the parentheses for eigenvalues with only one Jordan block (c.f. [81, Section 5.5]). This definition can be shown to be independent of the choice of the generators Q_1 and Q_2 of a given pencil. For instance, if $n = 3$ and $Q_1 Q_2^{-1}$ has Jordan normal form

$$\left[\begin{array}{cc|c|c} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 \end{array}\right], \qquad \lambda_1 \neq \lambda_2,$$

characteristic	description of the pencil and its *base locus*
[1111]	*an elliptic normal curve* four distinct cones in the pencil with vertices in general position
[211]	*a nodal quartic* all quadrics touch at the node
[31]	*a cuspidal quartic* the quadrics have stationary contact at the cusp
[22]	*a twisted cubic and a secant* the quadrics touch at the points of intersection
[4]	*a twisted cubic and a tangent*
[(111)1]	*a double conic* all quadrics touch along the conic (*ring contact*)
[(11)11]	*two conics in different planes and meeting in two distinct points* all quadrics touch at these points
[(21)1]	*two conics in different planes and meeting tangentially* all quadrics have stationary contact at the point of contact
[2(11)]	*two lines and a conic intersecting at three points,* at which the quadrics touch
[(31)]	*two lines and a conic touching the plane of the lines at the intersection*
[(211)]	*two intersecting lines counted twice* all quadrics touch along these lines
[(22)]	*a double line meeting two (single) lines at distinct points*
[(11)(11)]	*four lines arranged as a skew quadrilateral*

Table 4.2: *Complex equivalence classes of complex pencils of quadric surfaces in $\mathbb{P}^3$ with nonsingular general member by Segre characteristic. (The Segre characteristic [(1111)] is omitted, as it does not correspond to a proper pencil.)*

then the corresponding Segre characteristic is [(21)1].

Consider pencils $\mathcal{P}_{\mathbb{C}}$ generated by quadrics $Q_1, Q_2 \subset \mathbb{P}^n_{\mathbb{C}}$ and $\mathcal{P}'_{\mathbb{C}}$ generated by quadrics $Q'_1, Q'_2 \subset \mathbb{P}^n_{\mathbb{C}}$, where

$$Q_i : x^{\mathrm{t}} M_i x = 0, \qquad Q'_i : x^{\mathrm{t}} M'_i x = 0, \qquad M_i, M'_i \in \mathbb{C}^{(n+1)\times(n+1)}, \qquad i = 1, 2.$$

The pencils $\mathcal{P}_{\mathbb{C}}$ and $\mathcal{P}'_{\mathbb{C}}$ are said to be *complex equivalent*, if there exists a complex projectivity of $\mathbb{P}^n_{\mathbb{C}}$ with matrix $A \in \mathbb{C}^{(n+1)\times(n+1)}$ such that $M'_1 = A^{\mathrm{t}} M_1 A$ and $M'_2 = A^{\mathrm{t}} M_2 A$. Similarly, real pencils are said to be *real equivalent* whenever they can be transformed into each other by a real projective change of coordinates. From [81, Section 6.4] we have the following theorem.

Theorem 18. *Two pencils $\mathcal{P}_{\mathbb{C}}$ and $\mathcal{P}'_{\mathbb{C}}$ of quadrics in $\mathbb{P}^n_{\mathbb{C}}$ with nonsingular general member are complex equivalent if and only if their Segre characteristics coincide.*

Let us apply this result to the case $n = 3$. From [81, Sections 6.5] and [8, Section III.18], it is known that the pencils of quadric surfaces in $\mathbb{P}^3_{\mathbb{C}}$ with nonsingular general

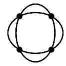
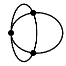
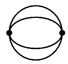
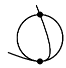
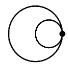
characteristic	description of the pencil and its *base locus*	figure
[111]	*four points of intersection*, no three of which are collinear	
[21]	*three points of intersection*, all quadrics have simple contact at one of them	
[(11)1]	*two points of intersection*, all quadrics have simple contact at both of them	
[3]	*two points of intersection*, at one of which all quadrics have contact of the second order	
[(21)]	*one point of intersection*, at which all quadrics have contact of the third order	

Table 4.3: *Complex equivalence classes of complex pencils of conics in $\mathbb{P}^2$ with nonsingular general member by Segre characteristic. (The Segre characteristic [(111)] is omitted, as it does not correspond to a proper pencil.)*

member are as in Table 4.2. What about the remaining case of pencils whose general member is singular? Whenever a pencil $\mathcal{P}_\mathbb{C}$ contains a nonsingular quadric Q_1, then continuity of the determinant function ensures that $\mathcal{P}_\mathbb{C}$ contains a continuous family of nonsingular quadrics $\{Q_1 + tQ_2\}_{t\in(0,\varepsilon)}$ for some other quadric $Q_2 \in \mathcal{P}_\mathbb{C}$ and for some $\varepsilon > 0$. It follows that the discriminant of Q_1 and Q_2 is well defined and that $\mathcal{P}_\mathbb{C}$ contains at most four singular quadrics. If $\mathcal{P}_\mathbb{C}$ is a pencil whose general member is singular, therefore, all of its members must be singular. From [8, Section III.17.5] we have the following theorem.

Theorem 19. *Let $\mathcal{P}_\mathbb{C}$ be a pencil of singular quadrics, no two of which have a singular point in common. Then these vertices lie along a straight line L that is contained in each of the cones, and the cones have a common tangent plane along L. The base locus of $\mathcal{P}_\mathbb{C}$ is the union of the double line L and a conic.*

If, on the other hand, $\mathcal{P}_\mathbb{C}$ is a pencil of singular quadrics in which two quadrics share a singular point, then this point must be a singularity of the other quadrics in $\mathcal{P}_\mathbb{C}$ as well. After a projective change of coordinates, we can assume this point to be $[0:0:0:1]$. Then each quadric in $\mathcal{P}_\mathbb{C}$ is represented by a matrix of the form

$$\begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the study of pencils of this type reduces to the study of pencils of conics in $\mathbb{P}^2$. By [81, Section 6.3] and [8, Section III.15], it follows that the base locus of $\mathcal{P}_\mathbb{C}$ is either the cone over one of the configurations of points of Table 4.3, or the cone over the base locus of a pencil of conics in $\mathbb{P}^2_{\mathbb{C}}$ with singular general member.

The situation is slightly more complicated over the real numbers, as two pencils with the same Segre characteristic might not be related by a real projective change of coordinates. Every complex equivalence class of pencils is partitioned into real equivalence classes of pencils. This gives rise to real curves of different type but with equal Segre characteristic.

Which of the base loci in Tables 4.2 and 4.3 do not have the potential to satisfy property ★? Any double conic (characteristic [(111)1]) or intersection of two double lines (characteristic [(211)]) is contained in a plane and can therefore not satisfy Property ★ by Example 10. Moreover, a double line meeting two single lines at distinct points (characteristic [(22)]), the singular case of Theorem 19, the cones over the configurations with at most three points in Table 4.3 (characteristics [21], [(11)1], [3], and [(21)]), and the cones over the base locus of a pencil of conics in $\mathbb{P}^2_{\mathbb{C}}$ with singular general member are not reduced. By Example 11, they can therefore not satisfy Property ★.

In Section 4.4.5, we exhibit two continuous families of parameterized real elliptic normal curves, each of which satisfies Properties **P1** and **P2**. In Sections 4.4.6 – 4.4.9, we obtain for each of the remaining base loci C a convenient coordinate system, in which we can find parameterizations of the components of C satisfying Properties **P1** and **P2**. In each case, Proposition 17 implies that C satisfies Property ★. These results are summarized in Table 4.4.

4.4.5 Elliptic normal curves

Just like one can equip any real plane elliptic curve with a group structure that encodes collinearity of its points, one can embed any real elliptic curve into $\mathbb{P}^n$ in such a way that its group structure encodes cohyperplanarity of its points. From [75, Ex. 3.11], we have the following proposition in the complex case.

Proposition 20. *Let $E_{\mathbb{C}}$ be an elliptic curve with distinguished point O, and choose a basis $f_0, \ldots, f_n$ for the Riemann-Roch space $\mathcal{L}\big((n+1)O\big)$. For $n \geq 2$, the map*

$$\chi : E_{\mathbb{C}} \longrightarrow \mathbb{P}^n, \qquad t \longmapsto [f_0(t) : \cdots : f_n(t)]$$

is an isomorphism of $E_{\mathbb{C}}$ onto its image that embeds $E_{\mathbb{C}}$ as an elliptic normal curve of degree $n+1$. Moreover, identifying $E_{\mathbb{C}}$ with its image, any $n+1$ distinct points $P_0, \ldots, P_n \in E_{\mathbb{C}}$ lie in a hyperplane if and only if $P_0 \oplus \cdots \oplus P_n = O$.

Let $E_{\mathbb{C}}$ be an elliptic curve isomorphic to the torus $\mathbb{C}/\Lambda$, where $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. The space $\mathcal{L}\big((n+1)O\big)$ is the vector space of meromorphic functions on $E_{\mathbb{C}}$ with a pole of order at most $n+1$ at O and no other poles. Let $\wp(z)$ be the Weierstrass function associated to Λ. As the meromorphic functions on $E_{\mathbb{C}}$ form a field isomorphic to $\mathbb{C}\big(\wp(z), \wp'(z)\big)$ [75, Theorem VI.3.2], one can write down a basis for $\mathcal{L}\big((n+1)O\big)$ in terms of $\wp(z)$ and $\wp'(z)$. Keeping in mind that $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ and that $\wp^{(i)}$ has a pole of order $i+2$ at zero, one finds a basis $\{f_0, \ldots, f_n\} = \{1, \wp, \wp', \wp^2, \wp'\wp, \wp^3, \ldots\}$.

Alternatively, we can choose the basis $\{f_0, \ldots, f_n\} = \big\{1, \wp, \wp', \ldots, \wp^{(n-1)}\big\}$ of $\mathcal{L}\big((n+1)O\big)$. Distinct points $\chi(z_0), \chi(z_1), \ldots, \chi(z_n)$ of $\chi(E_{\mathbb{C}})$ lie in a hyperplane if and only they are linearly dependent. Multiplying the coordinates of $\chi(z)$ by z^{n+1}, we remove the poles without introducing additional zeros. The following proposition is a generalization of the *General Addition Theorem* for Weierstrass functions.

Proposition 21. *Let $\wp(z) = \wp(z;\Lambda)$ be the Weierstrass elliptic function with respect to some lattice $\Lambda \subset \mathbb{C}$. Then, for any complex numbers $z_0, z_1, \ldots, z_n$ not differing from each other by an element of Λ,*

$$\Delta := \det \begin{bmatrix} z_0^{n+1} & z_0^{n+1}\wp(z_1) & z_0^{n+1}\wp'(z_0) & \cdots & z_0^{n+1}\wp^{(n-1)}(z_0) \\ z_1^{n+1} & z_1^{n+1}\wp(z_2) & z_1^{n+1}\wp'(z_1) & \cdots & z_1^{n+1}\wp^{(n-1)}(z_1) \\ z_2^{n+1} & z_2^{n+1}\wp(z_3) & z_2^{n+1}\wp'(z_2) & \cdots & z_2^{n+1}\wp^{(n-1)}(z_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n^{n+1} & z_n^{n+1}\wp(z_m) & z_n^{n+1}\wp'(z_n) & \cdots & z_n^{n+1}\wp^{(n-1)}(z_n) \end{bmatrix} = 0 \qquad (4.13)$$

if and only if $z_0 + z_1 + \cdots + z_n \equiv 0 \mod \Lambda$.

Before we proceed with the proof, we introduce, following [86, p. 447], the *Weierstrass sigma function* as the Weierstrass product

$$\sigma(z) = \sigma(z;\Lambda) := z \prod_{0\neq\omega\in\Lambda} \left[\left(1 - \frac{z}{\omega}\right)\exp\left(\frac{z}{\omega} + \frac{z^2}{\omega^2}\right)\right],$$

which can be shown to converge absolutely and uniformly in any bounded domain of values of z. Then $\wp(z) = -\frac{d^2 \log \sigma(z)}{dz^2}$. One can show that $\sigma(z)$ is an entire function whose zeros are located at the lattice points $\omega \in \Lambda$, each of which has order one.

Any elliptic function can be factorized into a product and quotient of σ functions [50, p. 406]. The idea of the proof below is to use such a factorization for the elliptic function $\Delta/\left(z_0^{n+1} z_1^{n+1} \cdots z_n^{n+1}\right)$ to get ahold of its zeros.

Proof. From [86, p. 458] and [37, p. 179], we have the *Frobenius-Stickelberger addition formula*

$$\det \begin{bmatrix} 1 & \wp(z_0) & \wp'(z_0) & \cdots & \wp^{(n-1)}(z_0) \\ 1 & \wp(z_1) & \wp'(z_1) & \cdots & \wp^{(n-1)}(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) & \cdots & \wp^{(n-1)}(z_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(z_n) & \wp'(z_n) & \cdots & \wp^{(n-1)}(z_n) \end{bmatrix}$$

$$= (-1)^{n(n-1)/2} 1!2!\cdots n! \frac{\sigma(z_0 + z_1 + \cdots + z_n)}{\left(\sigma(z_0)\sigma(z_1)\cdots\sigma(z_n)\right)^{n+1}} \prod_{0\leq i<j\leq n} \sigma(z_i - z_j)$$

for any $n \geq 0$. In particular for $z_0, z_1, \ldots, z_n$ as in the proposition, the determinant Δ from Equation 4.13 vanishes if and only if

$$\left(\frac{z_0 z_1 \cdots z_n}{\sigma(z_0)\sigma(z_1)\cdots\sigma(z_n)}\right)^{n+1} \sigma(z_0 + z_1 + \cdots + z_n) \prod_{0\leq i<j\leq n} \sigma(z_i - z_j) = 0.$$

Since $\sigma(z)$ is an entire function with only zeros *of order one* at $z \equiv 0 \mod \Lambda$, the quotient $\frac{z_0 z_1 \cdots z_n}{\sigma(z_0)\sigma(z_1)\cdots\sigma(z_n)}$ and factors $\sigma(z_i - z_j)$ are nonzero complex numbers. We conclude that Δ vanishes if and only if $z_0 + z_1 + \cdots + z_n \equiv 0 \mod \Lambda$. □

Let us apply these results to find real elliptic normal curves $E \subset \mathbb{P}^3$ that satisfy Property ★. By [28, p. 456], E is the base locus of some pencil $\mathcal{P}$ of quadrics spanned by two real quadrics $Q, Q' \subset \mathbb{P}^3$. By [81, Section 6.5], there are precisely four singular quadrics Q_0, Q_1, Q_2, Q_3 in the complexified pencil $\mathcal{P}_{\mathbb{C}}$, each a rank 3 cone of multiplicity 1. These singular quadrics correspond to the linear factors of the discriminant of the pair (Q, Q'), which is a homogeneous form $P := \det(sQ + tQ')$ of degree 4. There are therefore three possibilities for the four singular quadrics Q_0, Q_1, Q_2, Q_3. Either

(1) all four belong to $\mathcal{P}$, or

(2) two of them belong $\mathcal{P}$ and two of them form a conjugate pair in $\mathcal{P}_{\mathbb{C}} \backslash \mathcal{P}$, or

(3) none of them belongs to $\mathcal{P}$ and they form two conjugate pairs in $\mathcal{P}_{\mathbb{C}} \backslash \mathcal{P}$.

Clearly these three categories are left invariant under changing coordinates by a real projectivity of $\mathbb{P}^3$. The examples given in the next two sections fall into the categories (1) and (2), and we are not quite sure what role the third category plays. In the second case the discriminant of P is negative, while it is positive in the first and third cases, suggesting that the categories (1) and (3) somehow belong together. These categories are therefore grouped together into one category in Table 4.4a.

Elliptic normal curve with one real connected component

Let $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \subset \mathbb{C}$ be a lattice with $\omega_1, i\omega_2 \in \mathbb{R}$, and consider the complex elliptic curve $E_{\mathbb{C}} = \mathbb{C}/\Lambda$ with distinguished point $O = 0 + \Lambda$. Applying Proposition 20 to the basis $\{1, \wp(z), \wp'(z), \wp^2\}$ of $\mathcal{L}(4O)$, we obtain a map

$$\chi' : E_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}, \qquad z \longmapsto \big[1 : \wp(z) : \wp'(z) : \wp^2\big] \tag{4.14}$$

that embeds $E_{\mathbb{C}}$ as the elliptic normal curve in $\mathbb{P}^3_{\mathbb{C}}$ that is the base locus of the pencil $\mathcal{P}_{\mathbb{C}}$ generated by the quadrics $Q_1 : wz - x^2 = 0$ and $Q_2 : y^2 - 4xz + g_2wx + g_3w^2 = 0$, with $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ the Weierstrass invariants of $E_{\mathbb{C}}$. The singular quadrics in this pencil correspond to the factors of

$$\det\left(s\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} + t\begin{bmatrix} g_3 & \frac{1}{2}g_2 & 0 & 0 \\ \frac{1}{2}g_2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \right) = 4t(4s^3 - g_2st^2 - g_3t^3). \tag{4.15}$$

By Proposition 20 the curve $E_{\mathbb{C}}$ is embedded normally, implying that it cannot be contained in the union of two planes. All singular quadrics in $\mathcal{P}_{\mathbb{C}}$ must therefore be rank 3 cones. Since $E_{\mathbb{C}}$ is nonsingular, the *modular discriminant* $\Delta(\Lambda) := g_2^3 - 27g_3^2$ is nonzero. If $\Delta(\Lambda) < 0$, then two of the cones in $\mathcal{P}_{\mathbb{C}}$ are elements of $\mathcal{P}$, while the other two cones are not. If $\Delta(\Lambda) > 0$, on the other hand, then all four cones belong to $\mathcal{P}$.

Alternatively, the basis $\{1, \wp(z), \wp'(z), \wp''(z)\}$ of $\mathcal{L}(4O)$ yields the embedding

$$\chi'' : E_{\mathbb{C}} \longrightarrow \mathbb{P}^3_{\mathbb{C}}, \qquad z \longmapsto \big[1 : \wp(z) : \wp'(z) : \wp''(z)\big]. \tag{4.16}$$

How is this embedding related to the embedding χ'? Differentiating the identity $\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 0$, one finds that $\wp$ satisfies the differential equation $\wp'' = 6\wp^2 - \frac{1}{2}g_2 = 0$.

It follows that the curves $\chi'(E_{\mathbb{C}})$ and $\chi''(E_{\mathbb{C}})$ are related by the real projectivity of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}g_2 & 0 & 0 & 6 \end{bmatrix}.$$

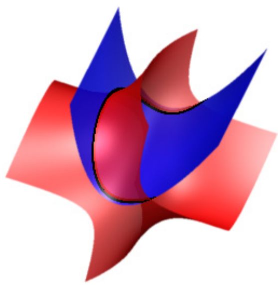
Let $\pi : \mathbb{P}^3_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ be the projection from the point $[0 : 0 : 0 : 1] \in E_{\mathbb{C}}$ onto the plane $z = 0$, and let $\psi : E_{\mathbb{C}} \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ be the embedding defined by $z \longmapsto [1 : \wp(z) : \wp'(z)]$. Then there is a commutative diagram

$$\begin{array}{ccc} E_{\mathbb{C}} & \xrightarrow{\chi'} & \mathbb{P}^3 \\ & \underset{\psi}{\searrow} & \downarrow \pi \\ & & \mathbb{P}^2 \end{array} \tag{4.17}$$

and the restriction of π to $\chi'(E_{\mathbb{C}})$ is an isomorphism $\pi' : \chi'(E_{\mathbb{C}}) \longrightarrow \psi(E_{\mathbb{C}})$. The latter curve is given by the Weierstrass equation $wy^2 = 4x^3 - g_2w^2x - g_3w^3$. It follows that the real elliptic normal curve $E \subset \mathbb{P}^3$ defined as the base locus of the real pencil of quadrics generated by Q_1 and Q_2 has one real component if $\Delta(\Lambda) < 0$ and two real components if $\Delta(\Lambda) > 0$.

Suppose $\Delta(\Lambda) < 0$. Identifying $\mathbb{S}^1 \simeq \mathbb{R}/\omega_1\mathbb{Z}$, the restriction $\chi_1 = \chi'|_{\mathbb{R}/\omega_1\mathbb{Z}} : \mathbb{S}^1 \longrightarrow E$ is a parameterization of the only real connected component of $E \subset \mathbb{P}^3$. It follows directly from Proposition 20 that the four coinciding parameterized curves $\chi_1 : \mathbb{S}^1 \longrightarrow E$ have Properties **P1** and **P2**. We conclude that E is a curve that satisfies Property $\bigstar$.

As an example, let ω_1 and ω_2 be such that $g_2 = 0$ and $g_3 = -4$. Then $\Delta(\Lambda) < 0$, and the discriminant of $Q_1 : wz - x^2 = 0$ and $Q_2 : y^2 - 4xz - 4w^2 = 0$ from Equation 4.15 factors as $16t(s+t)(s^2 - st + t^2)$. We find two rank 3 cones $Q_1, Q_1 - Q_2$ in $\mathcal{P}$ and a pair of complex conjugated rank 3 cones in $\mathcal{P}_{\mathbb{C}} \backslash \mathcal{P}$. In the chart $w = 1$, the quadrics Q_1, Q_2, and the base locus E of $\mathcal{P}$ can be visualized as follows.



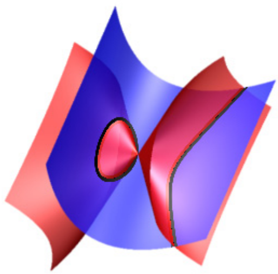
Elliptic normal curve with two real connected components

Suppose $\Delta(\Lambda) > 0$ and identify $\mathbb{S}^1 \simeq \mathbb{R}/\omega_1\mathbb{Z}$. The curve E has two real connected components, one of which is parameterized by $\chi_1 : \mathbb{S}^1 \longrightarrow E$. By the commutative diagram from Equation 4.17 and by Section 4.3.10, the other component is parameterized by the map $\chi_2 : \mathbb{S}^1 \longrightarrow \mathbb{P}^3$ defined by $t \longmapsto \chi'(t + \frac{1}{2}\omega_2)$. These two parameterizations can be combined into a single parameterization

$$\chi : \mathbb{S}^1 \times \mathbb{Z}_2 \longrightarrow E, \qquad (t,a) \longmapsto \begin{cases} \chi_1(t) = \chi'(t) & \text{if } a = 0 \\ \chi_2(t) = \chi'(t + \frac{1}{2}\omega_2) & \text{if } a = 1 \end{cases}$$

By Proposition 20, any four points $\chi(t_0, a_0), \chi(t_1, a_1), \chi(t_2, a_2), \chi(t_3, a_3)$ are coplanar if and only if $t_0 + t_1 + t_2 + t_3 + (a_0 + a_1 + a_2 + a_3)\frac{1}{2}\omega_2 \equiv 0 \mod \Lambda$, which happens if and only if $t_0 + t_1 + t_2 + t_3 \equiv 0 \mod \omega_1$ and $a_0 + a_1 + a_2 + a_3 \equiv 0 \mod 2$. We conclude that the four coinciding parameterized curves $\chi : \mathbb{S}^1 \times \mathbb{Z}_2 \longrightarrow E$ satisfy Properties **P1**, **P2** and that E is a curve that satisfies Property ★.

As an example, let ω_1 and ω_2 be such that $g_2 = 1$ and $g_3 = 0$. This is sometimes called the *lemniscatic case* [1, Section 18.14]. Then $\Delta(\Lambda) > 0$, and the discriminant of $Q_1 : wz - x^2 = 0$ and $Q_2 : y^2 - 4xz + wx = 0$ from Equation 4.15 factors as $4st(2s - t)(2s + t)$. We find the four rank 3 cones $Q_1, Q_2, 2Q_1 + Q_2, 2Q_1 - Q_2$ in $\mathcal{P}$. In the chart $w = 1$, the quadrics Q_1, Q_2, and the base locus E of $\mathcal{P}$ can be visualized as follows.



4.4.6 Singular irreducible quartics

There are two types of singular irreducible curves $C_{\mathbb{C}} \subset \mathbb{P}^3_{\mathbb{C}}$ of degree 4 and arithmetic genus 1, namely the cuspidal quartic (corresponding to characteristic [31] in Table 4.2) and the nodal quartic (corresponding to characteristic [211] in Table 4.2). Similarly to the situation in the projective plane, there are two types of real curves $C \subset \mathbb{P}^3$ whose complexification $C_{\mathbb{C}} \subset \mathbb{P}^3_{\mathbb{C}}$ is a nodal quartic. The real curve C is said to have a *crunode* if the tangent cone at the singularity of $C_{\mathbb{C}}$ comprises two real lines, and C is said to have an *acnode* (or *isolated point*) if the tangent cone at the singularity of $C_{\mathbb{C}}$ comprises a pair of complex conjugated lines.

Irreducible quartic with a cusp

Suppose $C \subset \mathbb{P}^3$ is an irreducible curve of degree 4 with a cusp at Q. Then C is rational and has only one real component. There exist a rational normal curve $C' \subset \mathbb{P}^4$ of degree 4 and a linear projection $\pi : \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ whose projection center P lies on a tangent line $L_{Q'}$ of some point Q' of C' for which $\pi(C') = C$ and $\pi(Q') = Q$. We can assume C' is the standard rational normal curve of degree 4 parameterized by

$$\phi : \mathbb{P}^1 \longrightarrow C' \subset \mathbb{P}^4, \qquad [s : t] \longmapsto \left[s^4 : s^3t : s^2t^2 : st^3 : t^4\right], \tag{4.18}$$

The following lemma states that we can move around any three points on such a curve by changing coordinates by a projectivity.

Lemma 22. *Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree $n \geq 1$. For any six points $P_1, P_2, P_3, P_1', P_2', P_3' \in C$, there exists a real projectivity $\psi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ that restricts to an automorphism $\psi|_C : C \longrightarrow C$ and maps P_i to P_i' for $i = 1, 2, 3$.*

Proof. After applying a projective change of coordinates, we can assume that C is the standard rational normal curve of degree n parameterized by

$$\phi : \mathbb{P}^1 \longrightarrow C \subset \mathbb{P}^n, \qquad [s : t] \longmapsto \big[s^n : s^{n-1}t : \cdots : st^{n-1} : t^n\big]. \tag{4.19}$$

By the fundamental theorem for projectivities, there exists a projectivity $\psi' : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ that maps $\phi^{-1}(P_i)$ to $\phi^{-1}(P_i')$ for $i = 1, 2, 3$. Let $\psi : C \longrightarrow C$ be the associated automorphism that makes the diagram

$$\begin{array}{ccccc} \mathbb{P}^1 & \xrightarrow{\phi} & C & \hookrightarrow & \mathbb{P}^n \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \\ \mathbb{P}^1 & \xrightarrow[\phi]{} & C & \hookrightarrow & \mathbb{P}^n \end{array}$$

commute. Let us show that this automorphism ψ can be lifted to a projectivity of $\mathbb{P}^n$. If $[s' : t'] := \psi'([s : t])$ is given by the matrix multiplication

$$\begin{bmatrix} s' \\ t' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix},$$

then $s'^{n-k}t'^k = (as + bt)^{n-k}(cs + dt)^k$ for $k = 0, \ldots, n$ and ψ is given by the matrix multiplication

$$\begin{bmatrix} s'^n \\ s'^{n-1}t' \\ \vdots \\ t'^n \end{bmatrix} = \begin{bmatrix} a^n & na^{n-1}b & \cdots & b^n \\ a^{n-1}c & a^{n-1}d + (n-1)a^{n-2}bc & \cdots & b^{n-1}d \\ \vdots & \vdots & \ddots & \vdots \\ c^n & nc^{n-1}d & \cdots & d^n \end{bmatrix} \begin{bmatrix} s^n \\ s^{n-1}t \\ \vdots \\ t^n \end{bmatrix}.$$

The above matrix is invertible, and its inverse is found by substituting

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We conclude that ψ can be lifted to a projectivity $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ that satisfies the conditions of the lemma. □

As a consequence of this lemma, we can choose $Q' = [0 : 0 : 0 : 0 : 1]$. The point P will have moved along to some point $[0 : 0 : 0 : a : b]$ on the tangent line to C' at Q'. The projectivity of $\mathbb{P}^1$ with matrix

$$\begin{bmatrix} 4 & 0 \\ -b & 4a \end{bmatrix}$$

induces a projectivity of $\mathbb{P}^4$ with matrix

$$\begin{bmatrix} 256b^4 & 0 & 0 & 0 & 0 \\ -64b^4 & 256ab^3 & 0 & 0 & 0 \\ 16b^4 & -128ab^3 & 256a^2b^2 & 0 & 0 \\ -4b^4 & 48ab^3 & -192a^2b^2 & 256a^3b & 0 \\ b^4 & -16ab^3 & 96a^2b^2 & -256a^3b & 256a^4 \end{bmatrix},$$

which restricts to an automorphism of C' that maps Q' to itself and P to $[0:0:0:1:0]$. We may thus assume that

$$\pi : \mathbb{P}^4 \longrightarrow \mathbb{P}^3, \qquad \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_4 \end{bmatrix}.$$

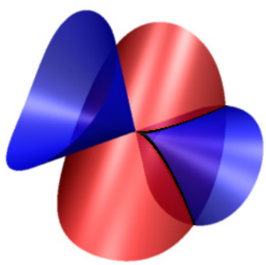
Restricting the composition $\pi \circ \phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ to the affine chart defined by $s = 1$, one finds a parameterization

$$\chi : \mathbb{R} \longrightarrow C_{\mathrm{ns}}, \qquad t \longmapsto [1 : t : t^2 : t^4].$$

It follows that C is the base locus of the pencil of quadrics spanned by the rank 3 cones $Q_1 : wy - x^2 = 0$ and $Q_2 : wz - y^2 = 0$. As

$$\det\left(s \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \right) = -\frac{1}{4} st^3,$$

Q_1 appears with multiplicity 3 and Q_2 with multiplicity 1 in this pencil, and there are no other singular fibers. In the chart $z = 1$, the objects Q_1, Q_2, and C can be visualized as follows.



Any four distinct points $\chi(t_0), \chi(t_1), \chi(t_2), \chi(t_3)$ on C_{ns} are coplanar if and only if

$$0 = \det \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^4 \\ 1 & t_1 & t_1^2 & t_1^4 \\ 1 & t_2 & t_2^2 & t_2^4 \\ 1 & t_3 & t_3^2 & t_3^4 \end{bmatrix} = (t_0 + t_1 + t_2 + t_3) \prod_{0 \le i < j \le 3} (t_i - t_j),$$

which happens precisely when $t_0 + t_1 + t_2 + t_3 = 0$. It follows that χ induces a topological group structure on C_{ns} and that the four coinciding parameterized curves $\chi : \mathbb{R} \longrightarrow C$ have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

Irreducible quartic with a crunode

Suppose $C \subset \mathbb{P}^3$ is an irreducible curve of degree 4 with a crunode at Q. Then there exist a rational normal curve $C' \subset \mathbb{P}^4$ of degree 4 and a linear projection $\pi : \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ whose projection center P lies on a line L that meets C in two points Q'_1, Q'_2 for which $\pi(C') = C$ and $\pi(Q'_1) = \pi(Q'_2) = Q$. By Lemma 22, we can assume that C' is the standard rational normal curve parameterized as in Equation 4.18, $Q_1 = [1:0:0:0:0]$, $Q_2 = [0:0:0:0:1]$, and $P = [a:0:0:0:b]$ for some $a, b \neq 0$.

The projectivity of $\mathbb{P}^1$ with matrix

$$\begin{bmatrix} |a|^{-1/4} & 0 \\ 0 & |b|^{-1/4} \end{bmatrix}$$

induces a projectivity of $\mathbb{P}^4$ with matrix

$$\begin{bmatrix} |a|^{-1} & 0 & 0 & 0 & 0 \\ 0 & |a|^{-3/4}|b|^{-1/4} & 0 & 0 & 0 \\ 0 & 0 & |a|^{-2/4}|b|^{-2/4} & 0 & 0 \\ 0 & 0 & 0 & |a|^{-1/4}|b|^{-3/4} & 0 \\ 0 & 0 & 0 & 0 & |b|^{-1/4} \end{bmatrix}$$

which maps C' to itself, leaves Q'_1 and Q'_2 invariant, and maps P to $[1:0:0:0:\pm 1]$. The projection π then becomes

$$\pi : \mathbb{P}^4 \longrightarrow \mathbb{P}^3, \qquad \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 & \mp 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_0 \mp x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

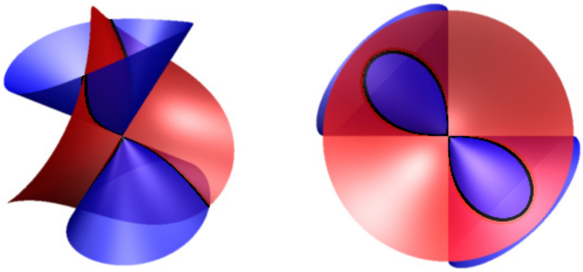
Let $\iota : \mathbb{R}^* \longrightarrow \mathbb{P}^1$ be the inclusion defined by $t \longmapsto [1 : t]$. The composition

$$\chi' := \pi \circ \phi \circ \iota : \mathbb{R}^* \longrightarrow C_{\text{ns}}, \qquad t \longmapsto \left[1 \mp t^4 : t : t^2 : t^3\right]$$

parameterizes the nonsingular points of C. It follows that C is the base locus of the pencil $\mathcal{P}$ of quadrics spanned by $Q_1 : xz - y^2$ and $Q_2 : x^2 - wy \mp z^2 = 0$. As

$$\det\left(s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \end{bmatrix} \right) = \frac{1}{4} t^2 \left(\frac{1}{4} s^2 \pm t^2 \right),$$

the rank 3 cone Q_1 appears with multiplicity 2. In case of the "lower sign", there are two additional singular fibers in $\mathcal{P}$, each a rank 3 cone of multiplicity 1. In case of the "upper sign", there are no other singular fibers in $\mathcal{P}$ (but there are two additional rank 3 cones of multiplicity 1 in $\mathcal{P}_{\mathbb{C}}$). In the chart $w = 1$, the objects Q_1, Q_2, and C can be visualized as follows (the left figure corresponding to the upper sign and the right figure to the lower sign).



Any four distinct points $\chi'(t_i)$, $i = 0, 1, 2, 3$, are coplanar if and only if

$$0 = \det \begin{bmatrix} 1 \pm t_0^4 & t_0 & t_0^2 & t_0^3 \\ 1 \pm t_1^4 & t_1 & t_1^2 & t_1^3 \\ 1 \pm t_2^4 & t_2 & t_2^2 & t_2^3 \\ 1 \pm t_3^4 & t_3 & t_3^2 & t_3^3 \end{bmatrix} = (1 \mp t_1 t_2 t_3 t_4) \prod_{0 \le i < j \le 3} (t_j - t_i).$$

Precomposing χ' by the homeomorphism $\psi : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow \mathbb{R}^*$ defined by $(u, a) \longmapsto (-1)^a \mathrm{e}^u$, we find a parameterization $\chi := \chi' \circ \psi : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow C_{\mathrm{ns}}$ given by

$$(u, a) \longmapsto \left[1 \pm \mathrm{e}^{4u} : (-1)^a \mathrm{e}^u : \mathrm{e}^{2u} : (-1)^a \mathrm{e}^{3u}\right]$$

that turns C_{ns} into a topological group. Any four distinct points $\chi(u_0, a_0), \chi(u_1, a_1), \chi(u_2, a_2), \chi(u_3, a_3)$ on C_{ns} are coplanar if and only if

$$(-1)^{a_0 + a_1 + a_2 + a_3} \mathrm{e}^{u_0 + u_1 + u_2 + u_3} = \pm 1.$$

In case of the upper sign, one finds that the four coinciding parameterized curves $\chi : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow C$ have Properties **P1** and **P2**. In case of the lower sign, the three coinciding parameterized curves

$$\mathbb{R} \longrightarrow C, \qquad u \longmapsto \chi(u, 0)$$

and the parameterized curve

$$\mathbb{R} \longrightarrow C, \qquad u \longmapsto \chi(u, 1)$$

together have Properties **P1** and **P2**. We conclude that in each case C is a curve that satisfies Property $\bigstar$.

Irreducible quartic with an acnode

Suppose $C \subset \mathbb{P}^3$ is an irreducible curve of degree 4 with an acnode at Q. Then there exist a rational normal curve $C' \subset \mathbb{P}^4$ of degree 4 and a linear projection $\pi : \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ whose projection center P lies on a line L that meets $C'_{\mathbb{C}}$ in two complex conjugated points $Q', \overline{Q}'$ for which $\pi(C') = C$ and $\pi(Q') = \pi(\overline{Q}') = Q$.

Similar to Lemma 22, the following lemma states that we can move around any pair of complex conjugated points on $C'_{\mathbb{C}}$ by a real projectivity.

Lemma 23. *Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree $n \geq 1$ and $C_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$ its complexification. For any two pairs $(P_1, \overline{P}_1), (P_2, \overline{P}_2)$ of complex conjugated points in $C_{\mathbb{C}}$, there exists a real projectivity $\psi : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ that restricts to an automorphism $\psi|_C : C \longrightarrow C$ and maps P_1 to P_2 and $\overline{P}_1$ to $\overline{P}_2$.*

Proof. After changing coordinates by a real projectivity, we can assume that C is the standard rational normal curve of degree n parameterized by the map ϕ from Equation 4.19. For $i = 1, 2$, the point P_i is nonreal and corresponds to a point $[\alpha_i + \beta_i i : 1] := \phi^{-1}(P_i) \in \mathbb{P}^1_{\mathbb{C}}$ with $\beta_i \neq 0$. The real projectivity $\psi' : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ with matrix

$$\begin{bmatrix} \beta_2/\beta_1 & \alpha_2 - \alpha_1\beta_2/\beta_1 \\ 0 & 1 \end{bmatrix}$$

maps $[\alpha_1 \pm \beta_1 i : 1]$ to $[\alpha_2 \pm \beta_2 i : 1]$ and induces an automorphism $\psi : C_{\mathbb{C}} \longrightarrow C_{\mathbb{C}}$ that maps $P_1 \longmapsto P_2$ and $\overline{P}_1 \longmapsto \overline{P}_2$ for which the diagram

$$\begin{array}{ccccc} \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\ \phi\ } & C_{\mathbb{C}} & \hookrightarrow & \mathbb{P}^n_{\mathbb{C}} \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} & & \\ \mathbb{P}^1_{\mathbb{C}} & \xrightarrow[\ \phi\ ]{} & C_{\mathbb{C}} & \hookrightarrow & \mathbb{P}^n_{\mathbb{C}} \end{array}$$

commutes. As in the proof of Lemma 22, this automorphism ψ can be lifted to a real projectivity $\mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$ satisfying the conditions of the lemma. □

As a consequence of this lemma, we can assume that C' is the standard rational normal curve parameterized as in Equation 4.18,

$$Q' = \phi([i : 1]) = [1 : -i : -1 : i : 1], \qquad \overline{Q}' = \phi([-i : 1]) = [1 : i : -1 : -i : 1],$$

and $P = [a : b : -a : -b : a]$. Suppose $a = 1$ and let α be one of the (real) roots of the polynomial

$$1 - 4bt - 6t^2 + 4bt^3 + t^4 = \left(t^2 + 2(\sqrt{b^2+1} + b)t - 1\right)\left(t^2 + 2(\sqrt{b^2+1} - b)t - 1\right).$$

The projectivity of $\mathbb{P}^1$ with matrix

$$\begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}$$

induces a real projectivity of $\mathbb{P}^4_{\mathbb{C}}$ with matrix

$$\begin{bmatrix} 1 & -4\alpha & 6\,\alpha^2 & -4\,\alpha^3 & \alpha^4 \\ \alpha & -3\alpha^2 + 1 & 3\,\alpha^3 - 3\,\alpha & -\alpha^4 + 3\,\alpha^2 & -\alpha^3 \\ \alpha^2 & -2\alpha^3 + 2\alpha & \alpha^4 - 4\,\alpha^2 + 1 & 2\,\alpha^3 - 2\,\alpha & \alpha^2 \\ \alpha^3 & -\alpha^4 + 3\alpha^2 & -3\,\alpha^3 + 3\,\alpha & -3\,\alpha^2 + 1 & -\alpha \\ \alpha^4 & 4\alpha^3 & 6\,\alpha^2 & 4\,\alpha & 1 \end{bmatrix}$$

that leaves the points $Q', \overline{Q}'$ invariant and moves P to $[0:1:0:-1:0]$. We may therefore assume that

$$\pi : \mathbb{P}^4 \longrightarrow \mathbb{P}^3, \qquad \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_0 \\ x_2 \\ x_4 \end{bmatrix}.$$

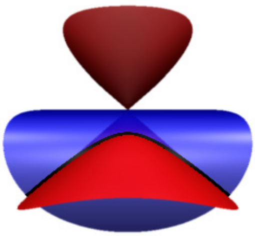
From the parameterization

$$\chi' = \pi \circ \phi : \mathbb{P}^1 \longrightarrow C_{\mathrm{ns}}, \qquad [s:t] \longmapsto [s^3 t + st^3 : s^4 : s^2t^2 : t^4]$$

one obtains that C is the base locus of the pencil $\mathcal{P}$ of quadrics spanned by $Q_1 : y^2 - xz = 0$, $Q_2 : w^2 - xy - 2y^2 - yz = 0$. As

$$\det \left(s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -2 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix} \right) = -\frac{1}{4}(s-t)^2 st,$$

the singular quadrics in $\mathcal{P}$ are Q_1, Q_2, both rank 3 cones of multiplicity 1, and a third rank 3 cone $Q_3 = Q_1 + Q_2$ of multiplicity 2. In the chart $z = 1$, the objects Q_1, Q_3, and C can be visualized as follows.



Four distinct points $\chi'([s_i : t_i]), i = 0,1,2,3$, are coplanar if and only if

$$0 = \det \begin{bmatrix} s_0^3 t_0 + s_0 t_0^3 & s_0^4 & s_0^2 t_0^2 & t_0^4 \\ s_1^3 t_1 + s_1 t_1^3 & s_1^4 & s_1^2 t_1^2 & t_1^4 \\ s_2^3 t_2 + s_2 t_2^3 & s_2^4 & s_2^2 t_2^2 & t_2^4 \\ s_3^3 t_3 + s_3 t_3^3 & s_3^4 & s_3^2 t_3^2 & t_3^4 \end{bmatrix} = - \prod_{0 \le i < j \le 3} (s_i t_j - s_j t_i)$$

$$\times \big((s_0 t_1 + s_1 t_0)(s_2 s_3 - t_2 t_3) + (s_2 t_3 + s_3 t_2)(s_0 s_1 - t_0 t_1) \big),$$

which happens if and only if the latter factor is zero. Identifying $\mathbb{S}^1$ with $\mathbb{P}^1$ by the homeomorphism $\psi : \mathbb{S}^1 \longrightarrow \mathbb{P}^1$ defined by $\theta \longmapsto [\cos(\theta/2) : \sin(\theta/2)]$, this factor is zero if and only if

$$\sin\left(\frac{\theta_0}{2} + \frac{\theta_1}{2}\right) \cos\left(\frac{\theta_2}{2} + \frac{\theta_3}{2}\right) + \sin\left(\frac{\theta_2}{2} + \frac{\theta_3}{2}\right) \cos\left(\frac{\theta_0}{2} + \frac{\theta_1}{2}\right)$$

$$= \sin\left(\frac{\theta_0}{2}+\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}\right)=0 \qquad \Longleftrightarrow \qquad \theta_0+\theta_1+\theta_2+\theta_3\equiv 0 \mod 2\pi,$$

where we made use of the angle-sum trigonometric identities. It follows that $\chi = \chi' \circ \psi$ induces a topological group structure on C_{ns} and that the four coinciding parameterized curves $\chi : \mathbb{R} \longrightarrow C$ together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

4.4.7 Union of a cubic and a line

Let $C \subset \mathbb{P}^3$ be the union of an irreducible cubic C' and a line L. If C is the complete intersection of two quadrics, then C' is a twisted cubic and L is either a secant (corresponding to characteristic [22] in Table 4.2) of C' or a tangent (corresponding to characteristic [4] in Table 4.2) to C'. After changing coordinates by a projectivity, we may assume that C' is the standard twisted cubic in $\mathbb{P}^3$ [29, Theorem 6.8]. Parametrically, C' is given as the image of the map

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \qquad [s:t] \longmapsto \left[s^3 : s^2t : st^2 : t^3\right], \tag{4.20}$$

and implicitly C' is the base locus of the net of quadrics spanned by

$$Q_1 : wy - x^2 = 0, \qquad Q_2 : xz - y^2 = 0, \qquad Q_3 : wz - xy = 0.$$

The singular quadrics $sQ_1 + tQ_2 + uQ_3$ in this net are those for which

$$\det\left(s\begin{bmatrix} 0 & 0 & \frac{1}{2} & 0\\ 0 & -1 & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} + t\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -1 & 0\\ 0 & \frac{1}{2} & 0 & 0\end{bmatrix} + u\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & 0\\ 0 & -\frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\end{bmatrix}\right)$$
$$= \frac{1}{16}(st-u^2)^2 = 0$$

and are therefore parameterized by the points of the double conic $C^0 : (st - u^2)^2 = 0$. Since the twisted cubic is not contained in a union of two planes, every such singular quadric is a cone of rank 3.

As the complete intersection of two quadrics is a curve of degree 4, the base locus of any pencil $\mathcal{P}$ containing C' is the union of C' and some line L. The pencil $\mathcal{P}$ corresponds to a line L^0 of points $[s:t:u] \in \mathbb{P}^2$. One can thus distinguish three types of pencils:

(1) L^0 cuts C^0 in two distinct points with multiplicity 2.

(2) L^0 touches C^0 in one point with multiplicity 4.

(3) L^0 is disjoint from C^0.

In case (1), the vertices P_1, P_2 of the cones in $\mathcal{P}$ are double points of $C' \cup L$, implying that L cuts C' (transversely) in P_1, P_2 and therefore that L is a secant of C'. This is the case of characteristic [22] in Table 4.2.

In case (2), the vertex P of the cone in $\mathcal{P}$ is the only singularity of $C' \cup L$, implying that L cuts C' (tangentially) in P and therefore that L is a tangent of C'. This is the case of characteristic [4] in Table 4.2.

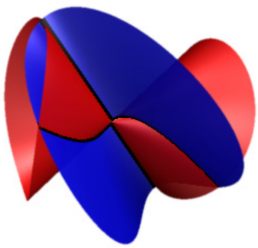
As before, let us denote the complexification of any real curve C by attaching the symbol $\mathbb{C}$ as a subindex. In case (3), the complex line $L^0_{\mathbb{C}}$ cuts $C^0_{\mathbb{C}}$ in two complex conjugated points with multiplicity 2. This is the case of characteristic [22] in Table 4.2. It follows that the complexication $\mathcal{P}_{\mathbb{C}}$ of $\mathcal{P}$ contains two singular fibers of multiplicity 2, rank 3, and with conjugated equations. These cones have conjugated vertices $P, \overline{P}$ that are double points of $C_{\mathbb{C}}$, implying that $L_{\mathbb{C}}$ cuts $C'_{\mathbb{C}}$ in $P, \overline{P}$, and therefore that $L_{\mathbb{C}}$ is a secant of $C'_{\mathbb{C}}$, whose real part L is disjoint from C'.

A twisted cubic and a real secant

Suppose that L is a real secant of C'. That is, C' and L meet in two real points P_1, P_2. By Lemma 4.19, we can assume that $P_1 = [1:0:0:0], P_2 = [0:0:0:1]$ and therefore that $L : x = y = 0$. Then C is the base locus of the pencil $\mathcal{P}$ of quadrics spanned by Q_1 and Q_2. The singular fibers in this pencil correspond to the factors of

$$\det\left(s \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \right) = \frac{1}{16} s^2 t^2.$$

It follows that the general fiber in this pencil is a smooth quadric and that there are two rank 3 cones in $\mathcal{P}$, each of multiplicity 2, with vertices at P_1 and P_2. The following image shows C as the intersection of these cones Q_1, Q_2 in the chart $w = 1$.



The two components of the open set $C' \backslash \{P_1, P_2\}$ can be parameterized by precomposing ϕ by the map $\mathbb{R} \times \mathbb{Z}_2 \longrightarrow \mathbb{P}^1$ defined by $(t, a) \longmapsto [1 : (-1)^a \mathrm{e}^t]$, resulting into a homeomorphism $\chi_1 : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow C' \backslash \{P_1, P_2\}$. We wish to find a homeomorphism $\chi_2 : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow L \backslash \{P_1, P_2\}$ sending $(t, a) \longmapsto [1 : 0 : 0 : f(t, a)]$, compatible with χ_1 in the sense that

$$\det \begin{bmatrix} 1 & 0 & 0 & f(t_0, a_0) \\ 1 & (-1)^{a_1} \mathrm{e}^{t_1} & \mathrm{e}^{2t_1} & (-1)^{a_1} \mathrm{e}^{3t_1} \\ 1 & (-1)^{a_2} \mathrm{e}^{t_2} & \mathrm{e}^{2t_2} & (-1)^{a_2} \mathrm{e}^{3t_2} \\ 1 & (-1)^{a_3} \mathrm{e}^{t_3} & \mathrm{e}^{2t_3} & (-1)^{a_3} \mathrm{e}^{3t_3} \end{bmatrix}$$

$$= \big((-1)^{a_1} \mathrm{e}^{t_1} - (-1)^{a_2} \mathrm{e}^{t_2}\big)\big((-1)^{a_2} \mathrm{e}^{t_2} - (-1)^{a_3} \mathrm{e}^{t_3}\big)$$

$$\times\big((-1)^{a_3}\mathrm{e}^{t_3}-(-1)^{a_1}\mathrm{e}^{t_1}\big)\big((-1)^{a_1+a_2+a_3}\mathrm{e}^{t_1+t_2+t_3}-f(t_0,a_0)\big)$$

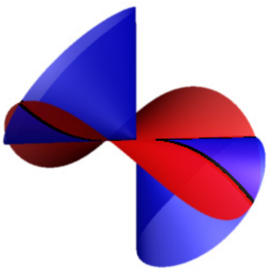
is zero if and only if either two of the pairs $(t_1,a_1),(t_2,a_2),(t_3,a_3)$ coincide, or $a_0+a_1+a_2+a_3=0$ and $t_0+t_1+t_2+t_3=0$. Clearly $f(t_0,a_0)=(-1)^{a_0}\mathrm{e}^{-t_0}$ satisfies this condition. Moreover, any three rows in the above matrix are independent (as long as they correspond to distinct points). It follows that the three coinciding parameterized curves $\chi_1:\mathbb{R}\times\mathbb{Z}_2\longrightarrow C'$ and the parameterized curve $\chi_2:\mathbb{R}\times\mathbb{Z}_2\longrightarrow L$ together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

A twisted cubic and a tangent

Suppose that L is a tangent of C'. By an argument similar to that of the previous section, we can assume that $L: y=z=0$ is the tangent line to the point $P=[1:0:0:0]$ of C'. Then C is the base locus of the pencil of quadrics spanned by $Q_2: xz-y^2=0$ and $Q_3: wz-xy=0$. The singular fibers correspond to the factors of

$$\det\left(s\begin{bmatrix}0&0&0&0\\0&0&0&\frac{1}{2}\\0&0&-1&0\\0&\frac{1}{2}&0&0\end{bmatrix}+t\begin{bmatrix}0&0&0&\frac{1}{2}\\0&0&-\frac{1}{2}&0\\0&-\frac{1}{2}&0&0\\\frac{1}{2}&0&0&0\end{bmatrix}\right)=\frac{1}{16}t^4.$$

It follows that the general fiber is a smooth quadric. There is one singular fiber of multiplicity 4, which is the rank 3 cone Q_2 with vertex at P. The following image shows C as the intersection of the cone Q_2 and the smooth quadric Q_3 in the chart $w=1$.



Precomposing the parameterization ϕ from Equation 4.20 with the embedding $\mathbb{R}\longrightarrow\mathbb{P}^1$ given by $t\longmapsto[t:1]$ yields a homeomorphism $\chi_1:\mathbb{R}\longrightarrow C'\backslash\{P\}$. We wish to find a homeomorphism $\chi_2:\mathbb{R}\longrightarrow L\backslash\{P\}$ sending $t\longmapsto[a(t):1:0:0]$, compatible with χ_1 in the sense that

$$\det\begin{bmatrix}a(t_0)&b(t_0)&0&0\\t_1^3&t_1^2&t_1&1\\t_2^3&t_2^2&t_2&1\\t_3^3&t_3^2&t_3&1\end{bmatrix}=(t_1-t_2)(t_2-t_3)(t_3-t_1)\big(t_1+t_2+t_3-a(t_0)\big)$$

is zero if and only if either two of the t_1,t_2,t_3 coincide, or $t_0+t_1+t_2+t_3=0$. Clearly $a(t)=-t$ satisfies this condition. Moreover, any three rows in the above matrix are independent (as long as they correspond to distinct points). It follows that the three coinciding parameterized curves $\chi_1 : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow C'$ and the parameterized curve $\chi_2 : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow L$ together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

A twisted cubic and a secant through complex conjugated points

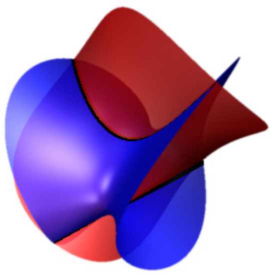
Suppose that C is the disjoint union of the standard twisted cubic C' and a line L such that $L_\mathbb{C}$ cuts $C'_\mathbb{C}$ in two complex conjugated points $P, \overline{P}$. By Lemma 23, we can assume that

$$P = \phi([i:1]) = [-i:-1:i:1], \quad \overline{P} = \phi([-i:1]) = [i:-1:-i:1],$$

and $L : w + y = x + z = 0$. It follows that C is the base locus of the pencil $\mathcal{P}$ of quadrics spanned by Q_3 and $Q_4 := Q_1 - Q_2 : wy - x^2 - xz + y^2 = 0$. The singular quadrics in this pencil correspond to the factors of

$$\det\left(s\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix}\right) = \frac{1}{16}(s^2+t^2)^2.$$

It follows that the general fiber in $\mathcal{P}$ is a smooth quadric, that there are no singular quadrics in $\mathcal{P}$, and that there are two rank 3 cones of multiplicity 2 in $\mathcal{P}_\mathbb{C}$. The following image shows Q_3, Q_4, and C in the chart $w = 1$.



As real topological spaces, both L and C' are homeomorphic to $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Precomposing ϕ by the homeomorphism $\mathbb{S}^1 \longrightarrow \mathbb{P}^1$ defined by $\theta \longmapsto [\cos(\theta/2) : \sin(\theta/2)]$ yields a homeomorphism

$$\chi_1 : \mathbb{S}^1 \longrightarrow C', \qquad \theta \longmapsto \left[\cos^3\frac{\theta}{2} : \cos^2\frac{\theta}{2}\sin\frac{\theta}{2} : \cos\frac{\theta}{2}\sin^2\frac{\theta}{2} : \sin^3\frac{\theta}{2}\right].$$

Let us write c_i for $\cos(\theta_i/2)$ and s_i for $\sin(\theta_i/2)$. We wish to find a parameterization

$$\chi_2 : \mathbb{S}^1 \longrightarrow L, \qquad \theta \longmapsto [a(\theta_0) : -b(\theta_0) : -a(\theta_0) : b(\theta_0)]$$

of the line L compatible with χ_1 in the sense that

$$\det\begin{bmatrix} a(\theta_0) & -b(\theta_0) & -a(\theta_0) & b(\theta_0) \\ c_1^3 & c_1^2 s_1 & c_1 s_1^2 & s_1^3 \\ c_2^3 & c_2^2 s_2 & c_2 s_2^2 & s_2^3 \\ c_3^3 & c_3^2 s_3 & c_3 s_3^2 & s_3^3 \end{bmatrix} = 0 \tag{4.21}$$

if and only if

$$\theta_0 + \theta_1 + \theta_2 + \theta_3 \equiv 0, \quad \theta_1 \equiv \theta_2, \quad \theta_1 \equiv \theta_3, \qquad \text{or} \qquad \theta_2 \equiv \theta_3 \mod 2\pi.$$

The determinant can be written as the product of the factor $-(c_1 s_2 - c_2 s_1)(c_1 s_3 - c_3 s_1)(c_2 s_3 - c_3 s_2)$ and the factor

$$\begin{aligned} &a(\theta_0)\big[s_3(c_1c_2 - s_1s_2) + c_3(c_1s_2 + c_2s_1)\big] \\ &-b(\theta_0)\big[c_1(s_2s_3 - c_2c_3) + s_1(c_2s_3 + c_3s_2)\big]. \end{aligned}$$

The former factor is zero if and only if two of the angles $\theta_1, \theta_2, \theta_3$ coincide. Via the angle-sum trigonometric identities, the latter factor can be rewritten to

$$\begin{aligned} &a(\theta_0)\left[+s_3 \cos\left(\frac{\theta_1}{2} + \frac{\theta_2}{2}\right) + c_3 \sin\left(\frac{\theta_1}{2} + \frac{\theta_2}{2}\right)\right] \\ &-b(\theta_0)\left[-c_1 \cos\left(\frac{\theta_2}{2} + \frac{\theta_3}{2}\right) + s_1 \sin\left(\frac{\theta_2}{2} + \frac{\theta_3}{2}\right)\right] \\ &= a(\theta_0) \sin\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2}\right) + b(\theta_0)\cos\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2}\right). \end{aligned}$$

Choosing $a(\theta_0) = \cos(\theta_0/2)$ and $b(\theta_0) = \sin(\theta_0/2)$ and using an angle-sum trigonometric identity, we find that the latter factor is zero if and only if

$$0 = \sin\left(\frac{\theta_0}{2} + \frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2}\right) \qquad \Longleftrightarrow \qquad \theta_0 + \theta_1 + \theta_2 + \theta_3 = 0 \mod 2\pi.$$

Note that any three rows in the matrix of Equation 4.21 are independent (as long as they correspond to distinct points). It follows that the three coinciding parameterized curves $\chi_1 : \mathbb{S}^1 \longrightarrow C'$ and the parameterized curve $\chi_2 : \mathbb{S}^1 \longrightarrow L$ together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property $\bigstar$.

4.4.8 Curves containing a nondegenerate conic

Suppose that C contains as an irreducible component a nondegenerate conic C_1. After changing coordinates by a projectivity, we may assume that C_1 is the rational normal curve of degree 2 in $\mathbb{P}^3$ given parametrically as the image of the map

$$\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \qquad [s:t] \longmapsto \big[s^2 : st : t^2 : 0\big]$$

and implicitly by the equations $C_1 : x^2 - wy = z = 0$. The quadric surfaces passing through C_1,

$$Q_{[a:b:c:d:e]} : 2a(x^2 - wy) + 2bwz + 4cxz + 2dyz + 2ez^2 = 0,$$

where $[a : b : c : d : e] \in \mathbb{P}^4$, form a linear system $\mathcal{L} \simeq \mathbb{P}^4$. The singular quadrics $Q_{[a:b:c:d:e]}$ in this linear system are those for which

$$0 = \det M_{[a:b:c:d:e]} = 4(c^2 - ae - bd)a^2, \quad \text{with } M_{[a:b:c:d:e]} := \begin{bmatrix} 0 & 0 & -a & b \\ 0 & 2a & 0 & 2c \\ -a & 0 & 0 & d \\ b & 2c & d & 2e \end{bmatrix}.$$

Of these, the quadrics $Q_{[a:b:c:d:e]}$ satisfying $c^2 - ae - bd = 0$ and $a \neq 0$ have rank 3, the quadrics $Q_{[0:b:c:d:e]}$ have rank 2, except for the quadric $Q_{[0:0:0:0:1]}$, which has rank 1.

Any pencil $\mathcal{P} \subset \mathcal{L}$ of quadrics through C_1 corresponds to a line $L \subset \mathbb{P}^4$. If we demand that the base locus C of $\mathcal{P}$ is a curve that defines a generalized principal lattice, then this leads to some restrictions on L. Clearly L cannot be contained in the plane $a = 0$, as this would imply that C contains the plane $z = 0$. Moreover, L cannot pass through $[0 : 0 : 0 : 0 : 1]$, as this would imply that C is contained in the plane $z = 0$. It follows that L intersects the hyperplane $a = 0$ in some point $[0 : b : c : d : e] \neq [0 : 0 : 0 : 0 : 1]$, which corresponds to the rank 2 quadric $2z(bw + 2cx + dy + ez) = 0$. If the line L were contained in the hypersurface defined by $c^2 - ae - bd = 0$, then C would not be reduced (C would be the union of C_1 and the double line formed by singularities of the cones in $\mathcal{P}$, which is the case of Theorem 19).

According to how L intersects the hypersurface $4(c^2-ae-bd)a^2$ of singular quadrics, the relevant cases from Section 4.4.4 for the complexification $C_{\mathbb{C}}$ are

(1) two conics meeting in two distinct points (characteristic [(11)11] in Table 4.2),

(2) two conics meeting tangentially in one point (characteristic [(21)1] in Table 4.2),

(3) two intersecting lines intersecting a conic at two points (characteristic [2(11)] in Table 4.2),

(4) two intersecting lines and a conic touching the plane of the lines at the intersection (characteristic [(31)] in Table 4.2).

There are two types of real curves $C = C_1 \cup C_2$ whose complexification is of type (1). Either C_1 and C_2 meet in two real points, or C_1 and C_2 are disjoint but their complexifications meet in a pair of complex conjugated points.

Two conics meeting in two real points

Suppose that C is the union of two nondegenerate conics C_1, C_2 that lie in different planes H_1, H_2 and meet in two points P_1, P_2. After changing coordinates by a projectivity of $\mathbb{P}^3$, we can assume that C_1 is the standard rational normal curve of degree two parameterized by

$$\phi : \mathbb{P}^1 \longrightarrow C_1 \subset H_1, \qquad [s : t] \longmapsto \left[s^2 : st : t^2 : 0\right], \qquad H_1 : z = 0, \tag{4.22}$$

and implicitly defined by $C_1 : x^2 - wy = z = 0$. By Lemma 22, we can assume that $P_1 = [1 : 0 : 0 : 0]$ and $P_2 = [0 : 0 : 1 : 0]$. As the secant of C_2 through these points is given by $x = z = 0$, the conic C_2 is contained in a plane of the form $H_2 : x + \alpha z = 0$ for some real number α. Changing coordinates by the projectivity of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we can assume that $H_2 : x = 0$ and $C_2 : Bwy + Dwz + Eyz + Fz^2 = x = 0$ for certain real numbers B, D, E, F. Since C_2 is nondegenerate, we can assume that $B = 1$ and

$$0 \neq \det \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2}D \\ \frac{1}{2} & 0 & \frac{1}{2}E \\ \frac{1}{2}D & \frac{1}{2}E & F \end{bmatrix} = \frac{1}{4}(DE - F).$$

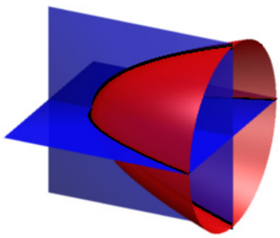
Let $\pm$ denote the sign of this determinant and $\mp := -\pm$. Changing coordinates by the projectivity of $\mathbb{P}^3$ with matrix

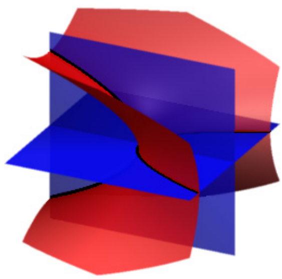
$$\begin{bmatrix} 1 & 0 & 0 & E \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & D \\ 0 & 0 & 0 & \sqrt{\pm DE \mp F} \end{bmatrix},$$

we can assume that $C_2 : \pm z^2 - wy = x = 0$ and that C is the base locus of the pencil $\mathcal{P}$ of quadrics generated by $Q_1 : xz = 0$ and $Q_2 : x^2 \pm z^2 - wy = 0$. The singular quadrics in this pencil correspond to the factors of

$$\det \left(s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix} \right) = \frac{1}{16}(s^2 \mp 4t^2)t^2.$$

The rank 2 cone Q_1 appears with multiplicity 2 in this pencil. In case of the "upper sign", there are two additional rank 3 cones $2Q_1 + Q_2$ and $-2Q_1 + Q_2$ in $\mathcal{P}$, each appearing with multiplicity 1. In case of the "lower sign", there are no additional singular quadrics in $\mathcal{P}$, but there are two additional rank 3 cones $2iQ_1 + Q_2$ and $-2iQ_1 + Q_2$ in $\mathcal{P}_{\mathbb{C}} \backslash \mathcal{P}$, each appearing with multiplicity 1. In the chart $w = 1$, the objects Q_1, Q_2, and C can be visualized as follows (the left figure corresponding to the upper sign and the right figure to the lower sign).





The homeomorphism $\chi : \mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow C_{\text{ns}}$ defined by

$$(t, a, b) \longmapsto \begin{cases} [\; 1 : (-1)^a e^t : e^{2t} : 0 \;] & \text{if } b = 0, \\ [\; e^{2t} : 0 : \pm 1 : (-1)^a e^t \;] & \text{if } b = 1. \end{cases}$$

induces a topological group structure on C_{ns}. Four distinct points $\chi(t_0, a_0, 0), \chi(t_1, a_1, 0)$, $\chi(t_2, a_2, 1), \chi(t_3, a_3, 1)$ on C_{ns} are coplanar if and only if

$$0 = \det \begin{bmatrix} 1 & (-1)^{a_0}\mathrm{e}^{t_0} & \mathrm{e}^{2t_0} & 0 \\ 1 & (-1)^{a_1}\mathrm{e}^{t_1} & \mathrm{e}^{2t_1} & 0 \\ \mathrm{e}^{2t_2} & 0 & \pm 1 & (-1)^{a_2}\mathrm{e}^{t_2} \\ \mathrm{e}^{2t_3} & 0 & \pm 1 & (-1)^{a_3}\mathrm{e}^{t_3} \end{bmatrix} = \big((-1)^{a_2}\mathrm{e}^{t_2} - (-1)^{a_3}\mathrm{e}^{t_3}\big)$$

$$\times \big((-1)^{a_0}\mathrm{e}^{t_0} - (-1)^{a_1}\mathrm{e}^{t_1}\big)\big(\pm 1 - (-1)^{a_0+a_1+a_2+a_3}\mathrm{e}^{t_0+t_1+t_2+t_3}\big).$$

In case of the upper sign, one finds that the parameterized curves

$$\mathbb{R} \times \mathbb{Z}_2 \longrightarrow C_1, \qquad (t, a) \longmapsto \chi(t, a, 0),$$

$$\mathbb{R} \times \mathbb{Z}_2 \longrightarrow C_2, \qquad (t, a) \longmapsto \chi(t, a, 1),$$

each counted twice, together have Properties **P1** and **P2**. In case of the lower sign, the parameterized curves

$$\mathbb{R} \longrightarrow C_1, \qquad t \longmapsto \chi(t, 0, 0) \qquad \text{(counted twice)},$$

$$\mathbb{R} \longrightarrow C_2, \qquad t \longmapsto \chi(t, 0, 1),$$

$$\mathbb{R} \longrightarrow C_2, \qquad t \longmapsto \chi(t, 1, 1),$$

together have Properties **P1** and **P2**. In each case, we conclude that C is a curve that satisfies Property ★.

Two conics meeting in two complex conjugated points

Suppose that C is the union of two nondegenerate disjoint conics C_1, C_2 that lie in different planes H_1, H_2 and whose complexifications $C_{1,\mathbb{C}}, C_{2,\mathbb{C}}$ meet in two complex conjugated points $P, \overline{P}$. After changing coordinates by a real projectivity of $\mathbb{P}^3$, we can assume that C_1 and H_1 are as in Equation 4.22. By Lemma 23, we can assume that $P = [-1 : i : 1 : 0]$ and $\overline{P} = [-1 : -i : 1 : 0]$. As the secant of $C_{2,\mathbb{C}}$ passing through these points is given by $x + y = z = 0$, the conic C_2 is contained in a plane of the form $H_2 : (w + y) + \alpha z = 0$ for some real number α. Changing coordinates by the projectivity of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we can assume that $H_2 : w+y = 0$ and $C_2 : C(x^2+y^2)+2Dxz+2Eyz+Fz^2 = w+y = 0$, for certain real numbers C, D, E, F. Furthermore, since C_2 is nondegenerate, we can assume that $C = 1$ and that the characteristic polynomial

$$\det \begin{bmatrix} 1-\lambda & 0 & D \\ 0 & 1-\lambda & E \\ D & E & F-\lambda \end{bmatrix} = (1-\lambda)\big(\lambda^2 - (F+1)\lambda + F - D^2 - E^2\big)$$

has nonzero constant term. Moreover, if $F - D^2 - E^2 > 0$, then Descartes' rule of signs implies that all eigenvalues are positive, in which case C_2 does not define a curve. It follows that $F - D^2 - E^2 < 0$.

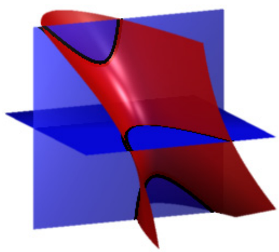
Changing coordinates by the projectivity of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} 1 & 0 & 0 & -E \\ 0 & -1 & 0 & -D - \sqrt{D^2 + E^2 - F} \\ 0 & 0 & 1 & E \\ 0 & 0 & 0 & -2\sqrt{D^2 + E^2 - F} \end{bmatrix},$$

we can assume that $C_2 : x^2 + y^2 - xz = w + y = 0$ and that C is the base locus of the pencil of quadrics spanned by $Q_1 : (w + y)z = 0$ and $Q_2 : x^2 - wy - xz = 0$. As,

$$\det\left(s \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \end{bmatrix} \right) = -\frac{1}{16}(2s - t)(2s + t)t^2,$$

one finds a rank 2 cone Q_1 of multiplicity 2, rank 3 cones $Q_1 \pm 2Q_2$ of multiplicity 1, and no other singular quadrics in this pencil. The following figure shows Q_1, Q_2, and C in the chart $w = 1$.



Consider the homeomorphism $\chi' : \mathbb{P}^1 \times \mathbb{Z}_2 \longrightarrow C_{\mathrm{ns}}$ defined by

$$\big([s : t], a\big) \longmapsto \begin{cases} [\, st : t^2 : -st : s^2 + t^2 \,] & \text{if } a = 0, \\ [\, s^2 : -st : t^2 : 0 \,] & \text{if } a = 1. \end{cases}$$

Four distinct points $\chi'\big([s_0 : t_0], 0\big), \chi'\big([s_1 : t_1], 0\big), \chi'\big([s_2 : t_2], 1\big), \chi'\big([s_3 : t_3], 1\big)$ are coplanar if and only if

$$0 = \det \begin{bmatrix} s_0t_0 & t_0^2 & -s_0t_0 & s_0^2 + t_0^2 \\ s_1t_1 & t_1^2 & -s_1t_1 & s_1^2 + t_1^2 \\ s_2^2 & -s_2t_2 & t_2^2 & 0 \\ s_3^2 & -s_3t_3 & t_3^2 & 0 \end{bmatrix} = -(s_2t_3 - s_3t_2)(s_0t_1 - s_1t_0)$$

$$\times \big((s_0s_1 - t_0t_1)(s_2s_3 - t_2t_3) - (s_0t_1 + s_1t_0)(s_2t_3 + s_3t_2)\big). \tag{4.23}$$

The latter factor can be simplified by substituting trigonometric functions for the s_i and t_i. More precisely, identifying $\mathbb{S}^1 \times \mathbb{Z}_2 \simeq \mathbb{P}^1 \times \mathbb{Z}_2$ by the homeomorphism

$\psi : (\theta, a) \longmapsto \left(\left[\cos\left(\frac{\theta}{2} - \frac{\pi}{8}\right) : \sin\left(\frac{\theta}{2} - \frac{\pi}{8}\right) \right], a \right)$, the angle-sum trigonometric identities imply that Equation 4.23 is zero if and only if

$$0 = \cos\left(\frac{\theta_0}{2} - \frac{\pi}{8} + \frac{\theta_1}{2} - \frac{\pi}{8} + \frac{\theta_2}{2} - \frac{\pi}{8} + \frac{\theta_3}{2} - \frac{\pi}{8}\right) = \sin\left(\frac{\theta_0 + \theta_1 + \theta_2 + \theta_3}{2}\right),$$

which happens precisely when $\theta_0 + \theta_1 + \theta_2 + \theta_3 \equiv 0 \mod 2\pi$. It follows that the parameterized curves

$$\mathbb{S}^1 \longrightarrow C_1, \qquad \theta \longmapsto \chi(\theta, 1),$$
$$\mathbb{S}^1 \longrightarrow C_2, \qquad \theta \longmapsto \chi(\theta, 0),$$

each counted twice, together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property $\bigstar$.

Two conics meeting tangentially in one point

Suppose that C is the union of two nondegenerate conics C_1 and C_2 that lie in different planes and meet tangentially in a point P. After changing coordinates by a projectivity of $\mathbb{P}^3$, we can assume that C_1 and H_1 are as in Equation 4.22. By Lemma 22, we can assume that $P = [1 : 0 : 0 : 0]$. The tangent line to C_1 at the point P is given by $T_P C_1 : y = z = 0$. As C_2 passes through P and shares the tangent to C_1 at this point, the conic C_2 is contained in a plane of the form $H_2 : y + \alpha z = 0$ for some real number α. Changing coordinates by the projectivity of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we can assume that $H_2 : y = 0$ and $C_2 : x^2 + Dwz + Exz + Fz^2 = y = 0$ for certain real numbers D, E, F with $D \neq 0$ (since C_2 is nondegenerate). Changing coordinates by the projectivity of $\mathbb{P}^3$ with matrix

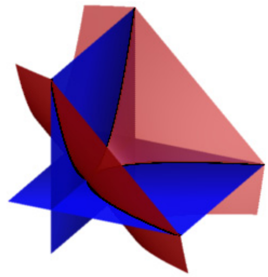
$$\begin{bmatrix} 1 & 0 & 0 & (F - \frac{1}{4}E^2)/D \\ 0 & 1 & 0 & \frac{1}{2}E \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -D \end{bmatrix},$$

one can assume that $C_2 : x^2 - wz = y = 0$.

The curve C is the base locus of the pencil of quadrics spanned by the rank 2 cone $Q_1 : yz = 0$ and the rank 3 cone $Q_2 : x^2 - wy - wz = 0$. As

$$\det\left(s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \right) = \frac{1}{4} s t^3,$$

Q_1 appears with multiplicity 3 and Q_2 with multiplicity 1, and there are no other singular quadrics in this pencil. The following figure shows Q_1, Q_2, and C in the chart $w = 1$.



The homeomorphism

$$\chi : \mathbb{R} \times \mathbb{Z}_2 \longrightarrow C_{\text{ns}}, \qquad (t,a) \longmapsto \begin{cases} [t^2 : +t : 1 : 0] & \text{if } a = 0, \\ [t^2 : -t : 0 : 1] & \text{if } a = 1 \end{cases}$$

turns C_{ns} into a topological group. Moreover, four distinct points $\chi(t_0,0), \chi(t_1,0), \chi(t_2,1), \chi(t_3,1)$ on C_{ns} are coplanar if and only if

$$0 = \det \begin{bmatrix} t_0^2 & +t_0 & 1 & 0 \\ t_1^2 & +t_1 & 1 & 0 \\ t_2^2 & -t_2 & 0 & 1 \\ t_3^2 & -t_3 & 0 & 1 \end{bmatrix} = (t_2 - t_3)(t_0 - t_1)(t_0 + t_1 + t_2 + t_3),$$

which happens precisely when $t_0 + t_1 + t_2 + t_3 = 0$. It follows that the parameterized curves

$$\mathbb{R} \longrightarrow C_1, \qquad t \longmapsto \chi(t,0),$$

$$\mathbb{R} \longrightarrow C_2, \qquad t \longmapsto \chi(t,1),$$

each counted twice, together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property $\bigstar$.

Two lines meeting a conic in distinct points

Suppose that C is the union of a nondegenerate conic C_1, contained in some hyperplane $H_1 \subset \mathbb{P}^3$, and two lines L_1, L_2 that meet C_1 in two distinct points P_1, P_2 and meet each other in a point $P_0 \notin H_1$. After applying a projective change of coordinates, we can assume that C_1 and H_1 are as in Equation 4.22. By Lemma 22, we can assume that $P_1 = [1:0:0:0]$ and $P_2 = [0:0:1:0]$. Writing $P_0 = [w_0 : x_0 : y_0 : 1]$ and changing coordinates by a projectivity of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} 1 & 0 & 0 & -w_0 \\ 0 & 1 & 0 & -x_0 \\ 0 & 0 & 1 & -y_0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we can assume that $P_0 = [0:0:0:1]$, $L_1 : x = y = 0$, and $L_2 : w = x = 0$.

The homeomorphism $\chi : \mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow C_{\mathrm{ns}}$ defined by

$$(t, a, b) \longmapsto \begin{cases} [\ 1 : (-1)^a \mathrm{e}^t : \mathrm{e}^{2t} : 0\ ] & \text{if } b = 0, \\ [\ (-1)^a \mathrm{e}^t : 0 : 0 : 1\ ] & \text{if } b = 1, \\ [\ 0 : 0 : 1 : (-1)^a \mathrm{e}^t\ ] & \text{if } b = 2 \end{cases}$$

turns C_{ns} into a topological group. Moreover, any four distinct points $\chi(t_0, a_0, 0)$, $\chi(t_1, a_1, 0), \chi(t_2, a_2, 1), \chi(t_3, a_3, 2)$ on C_{ns} are coplanar if and only if

$$0 = \det \begin{bmatrix} 1 & (-1)^{a_0} \mathrm{e}^{t_0} & \mathrm{e}^{2t_0} & 0 \\ 1 & (-1)^{a_1} \mathrm{e}^{t_1} & \mathrm{e}^{2t_1} & 0 \\ (-1)^{a_2} \mathrm{e}^{t_2} & 0 & 0 & 1 \\ 0 & 0 & 1 & (-1)^{a_3} \mathrm{e}^{t_3} \end{bmatrix}$$
$$= \big((-1)^{a_1} \mathrm{e}^{t_1} - (-1)^{a_0} \mathrm{e}^{t_0}\big) \cdot \big((-1)^{a_0+a_1+a_2+a_3} \mathrm{e}^{t_0+t_1+t_2+t_3} - 1\big),$$

which happens precisely when $a_0 + a_1 + a_2 + a_3 = 0$ and $t_0 + t_1 + t_2 + t_3 = 0$. It follows that the parameterized curves

$$\mathbb{R} \times \mathbb{Z}_2 \longrightarrow C_1, \qquad (t, a) \longmapsto \chi(t, a, 0), \qquad \text{(counted twice)}$$
$$\mathbb{R} \times \mathbb{Z}_2 \longrightarrow L_1, \qquad (t, a) \longmapsto \chi(t, a, 1),$$
$$\mathbb{R} \times \mathbb{Z}_2 \longrightarrow L_2, \qquad (t, a) \longmapsto \chi(t, a, 2),$$

together satisfy Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

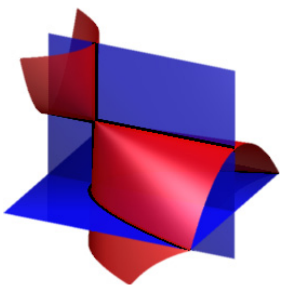
The curve C is difficult to draw, as none of the charts of $\mathbb{P}^3$ shows all components of C. Changing coordinates by the projectivity of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

one obtains a curve C that is the base locus of the pencil of quadrics spanned by the rank 2 cone $Q_1 : xz = 0$ and the rank 3 cone $Q_2 : x^2 - (w - z)y = 0$. As

$$\det \left(s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \right) = \frac{1}{16} s^2 t^2,$$

both of these quadrics appear with multiplicity 2, and there are no other singular quadrics in the pencil. The following figure shows Q_1, Q_2, and C in the chart $w = 1$.



Two lines meeting a conic in the same point

Suppose that C is the union of a conic C_1, contained in some plane $H_1 \subset \mathbb{P}^3$, and two lines L_+, L_- that meet C_1 transversely in a single point P, such that the plane H_2 spanned by L_+ and L_- contains the tangent line to C_1 at P. After applying a projective change of coordinates, we can assume that C_1 and H_1 are as in Equation 4.22. By Lemma 22, we can assume that $P = [1:0:0:0]$. As H_2 contains the tangent line $T_PC_1 : y = z = 0$, it must be of the form $H_2 : y - y_0 z = 0$ for some real number y_0. Since L_+, L_- meet C_1 transversely in P, it follows that there exist points of the form $P_\pm = [1 : x_1 \pm x_2 : y_0 : 1]$ in $L_\pm$. Changing coordinates by a projectivity of $\mathbb{P}^3$ with matrix

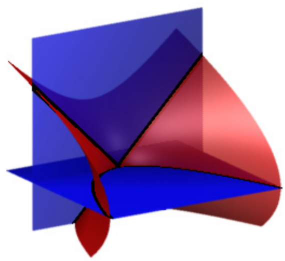
$$\begin{bmatrix} \frac{1}{x_2^2} & 0 & 0 & 1-\frac{1}{x_2^2} \\ 0 & \frac{1}{x_2} & 0 & -\frac{x_1}{x_2} \\ 0 & 0 & 1 & -y_0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we can assume that $P_\pm = [1 : \pm 1 : 0 : 1]$ and $L_\pm : y = x \mp z = 0$.

The curve C is the base locus of the pencil of quadrics spanned by $Q_1 : yz = 0$ and $Q_2 : x^2 - wy - z^2 = 0$. As

$$\det\left(s \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right) = \frac{1}{4}t^4,$$

the rank 2 cone Q_1 appears with multiplicity 4, and there are no other singular quadrics in the pencil. The following figure shows Q_1, Q_2, and C in the chart $w = 1$.



The homeomorphism

$$\chi : \mathbb{R} \times \mathbb{Z}_3 \longrightarrow C_{\mathrm{ns}}, \qquad (t,a) \longmapsto \begin{cases} [\ t^2 : t : 1 : 0\ ] & \text{if } a = 0, \\ [\ -2t : 1 : 0 : 1\ ] & \text{if } a = 1, \\ [\ -2t : 1 : 0 : -1\ ] & \text{if } a = 2 \end{cases}$$

turns C_{ns} into a topological group. Moreover, any four distinct points $\chi(t_0, 0), \chi(t_1, 0), \chi(t_2, 1), \chi(t_3, 2)$ on C_{ns} are coplanar if and only if

$$0 = \det \begin{bmatrix} t_0^2 & t_0 & 1 & 0 \\ t_1^2 & t_1 & 1 & 0 \\ -2t_2 & 1 & 0 & 1 \\ -2t_3 & 1 & 0 & -1 \end{bmatrix} = 2(t_0 - t_1)(t_0 + t_1 + t_2 + t_3),$$

which happens precisely when $t_0 + t_1 + t_2 + t_3 = 0$. It follows that the parameterized curves

$$\mathbb{R} \longrightarrow C_1, \qquad t \longmapsto \chi(t,0), \qquad \text{(counted twice)}$$

$$\mathbb{R} \longrightarrow L_+, \qquad t \longmapsto \chi(t,1),$$

$$\mathbb{R} \longrightarrow L_-, \qquad t \longmapsto \chi(t,2),$$

together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

4.4.9 Union of four lines

Suppose that C is the union of four lines L_0, L_1, L_2, L_3 and that C is the complete intersection of two quadric surfaces. By Section 4.4.4, C must either be a *skew quadrilateral* (characteristic [(11)(11)] in Table 4.2), or the union of four concurrent lines, no three of which span a plane (which is the cone over a configuration of points with characteristic [111] in Table 4.3).

A union of four lines that form a skew quadrilateral

We start with a characterization of these curves that will make it easy to give explicit examples. We say that four planes H_0, H_1, H_2, H_3 are *in general position* if $H_0 \cap H_1 \cap H_2 \cap H_3 = \emptyset$, or equivalently if

$$\det \begin{bmatrix} \text{---} & H_0 & \text{---} \\ \text{---} & H_1 & \text{---} \\ \text{---} & H_2 & \text{---} \\ \text{---} & H_3 & \text{---} \end{bmatrix} \neq 0, \tag{4.24}$$

where we denote the coordinate (row) vector of H_i by H_i as well.

Proposition 24. *Any skew quadrilateral $C \subset \mathbb{P}^3$ is the base locus of a pencil of quadrics spanned by two rank 2 quadrics $Q = H_0 \cup H_2$ and $Q' = H_1 \cup H_3$, for which the planes H_0, H_1, H_2, H_3 are in general position. Conversely, the base locus of any such pencil is a skew quadrilateral.*

Proof. Suppose C is the union of four lines $L_{01}, L_{12}, L_{23}, L_{30}$, no three of which are coplanar, for which any two consecutive lines $L_{i-1,i}$ and $L_{i,i+1}$ intersect. Here we think of the indices as elements of $\mathbb{Z}_4$, so that $L_{34} = L_{30}$. Let H_i denote the plane spanned by the lines $L_{i-1,i}$ and $L_{i,i+1}$. Then the planes H_0, H_1, H_2, H_3 are in general position. Writing $Q = H_0 \cup H_2$ and $Q' = H_1 \cup H_3$, it follows that

$$Q \cap Q' = (H_0 \cap H_1) \cup (H_1 \cap H_2) \cup (H_2 \cap H_3) \cup (H_3 \cap H_0) = L_{01} \cup L_{12} \cup L_{23} \cup L_{30}$$

and that C is the base locus of the pencil of quadrics spanned by Q and Q'.

Conversely, let C be the base locus of the pencil of quadrics spanned by two quadrics $Q = H_0 \cup H_2$ and $Q' = H_1 \cup H_3$ for which the planes H_i are in general position (as in Equation 4.24). Then $C = L_{01} \cup L_{12} \cup L_{23} \cup L_{30}$, where $L_{i,i+1} := H_i \cap H_{i+1}$. Any two consecutive lines $L_{i-1,i}, L_{i,i+1}$ span the plane H_i and meet in the point $P_i \in \mathbb{P}^3$ for which

$$\operatorname{span}\{P_i\} = \ker \begin{bmatrix} \text{——} & H_{i-1} & \text{——} \\ \text{——} & H_i & \text{——} \\ \text{——} & H_{i+1} & \text{——} \end{bmatrix}. \tag{4.25}$$

Moreover, the lines L_{01}, L_{23} (and similarly the lines L_{12}, L_{30}) do not lie in a plane H, as this would imply the existence of $[\alpha : \beta], [\alpha' : \beta'] \in \mathbb{P}^1$ for which

$$\alpha H_0 + \beta H_1 = H = \alpha' H_2 + \beta' H_3,$$

contradicting the requirement that H_0, H_1, H_2, H_3 are in general position. In particular no three of the four lines can lie in a plane. We conclude that C is a union of four lines, no three of which are coplanar, that intersect cyclically. That is, C is a skew quadrilateral. □

As an example, suppose C is the union of four lines $L_{01}, L_{12}, L_{23}, L_{30}$ such that $L_{i,i+1} = H_i \cap H_{i+1}$ with

$$H_0 : y = 0, \qquad H_1 : z = 0, \qquad H_2 : w = 0, \qquad H_3 : x = 0.$$

Then

$$P_0 = [1:0:0:0], \qquad P_1 = [0:1:0:0],$$

$$P_2 = [0:0:1:0], \qquad P_3 = [0:0:0:1]$$

are as in Equation 4.25. Consider the homeomorphism $\chi : (\mathbb{R} \times \mathbb{Z}_2) \times (\mathbb{Z}_2)^2 \longrightarrow C_{\mathrm{ns}}$ defined by

$$(t,a,b,c) \longmapsto \begin{cases} [-(-1)^a e^t : 0 : 0 : 1] & \text{if } (b,c) = (0,0), \\ [1 : -(-1)^a e^t : 0 : 0] & \text{if } (b,c) = (1,0), \\ [0 : 1 : -(-1)^a e^t : 0] & \text{if } (b,c) = (0,1), \\ [0 : 0 : 1 : -(-1)^a e^t] & \text{if } (b,c) = (1,1). \end{cases}$$

Any four distinct points

$$\chi(t_0, a_0, 0, 0), \quad \chi(t_1, a_1, 1, 0), \quad \chi(t_2, a_2, 0, 1), \quad \chi(t_3, a_3, 1, 1)$$

are coplanar if and only if

$$0 = \det \begin{bmatrix} -(-1)^{a_0} e^{t_0} & 0 & 0 & 1 \\ 1 & -(-1)^{a_1} e^{t_1} & 0 & 0 \\ 0 & 1 & -(-1)^{a_2} e^{t_2} & 0 \\ 0 & 0 & 1 & -(-1)^{a_3} e^{t_3} \end{bmatrix}$$
$$= -1 + (-1)^{a_0+a_1+a_2+a_3} e^{t_0+t_1+t_2+t_3},$$

which happens if and only if $a_0 + a_1 + a_2 + a_3 = 0$ and $t_0 + t_1 + t_2 + t_3 = 0$. It follows that the four parameterized curves

$$\mathbb{R} \times \mathbb{Z}_2 \longrightarrow L_{30}, \qquad (t,a) \longmapsto \chi(t,a,0,0),$$

$$\mathbb{R}\times\mathbb{Z}_2 \longrightarrow L_{01}, \qquad (t,a)\longmapsto \chi(t,a,1,0),$$
$$\mathbb{R}\times\mathbb{Z}_2 \longrightarrow L_{12}, \qquad (t,a)\longmapsto \chi(t,a,0,1),$$
$$\mathbb{R}\times\mathbb{Z}_2 \longrightarrow L_{23}, \qquad (t,a)\longmapsto \chi(t,a,1,1)$$

together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

Now let P_0', P_1', P_2', P_3' be four linearly independent points of $\mathbb{P}^3$, and consider the curve $C' = L_{01}' \cup L_{12}' \cup L_{23}' \cup L_{30}' \subset \mathbb{P}^3$, where $L_{i,i+1}'$ is the line passing through P_i' and P_{i+1}' for $i = 0,1,2,3 \bmod 4$. The projectivity ψ of $\mathbb{P}^3$ with matrix

$$\begin{bmatrix} | & | & | & | \\ P_0' & P_1' & P_2' & P_3' \\ | & | & | & | \end{bmatrix}$$

sends $L_{i,i+1}$ to $L_{i,i+1}'$ for $i = 0,1,2,3 \bmod 4$, and the homeomorphism $\chi' := \psi\circ\chi : (\mathbb{R}\times\mathbb{Z}_2)\times(\mathbb{Z}_2)^2 \longrightarrow C_{\mathrm{ns}}'$ turns C_{ns}' into a topological group. By construction, the curve C' satisfies Property ★.

In particular, for the points

$$P_0' = [1:0:0:0], \qquad P_1' = [1:1:0:0],$$
$$P_2' = [1:0:1:0], \qquad P_3' = [1:0:0:1],$$

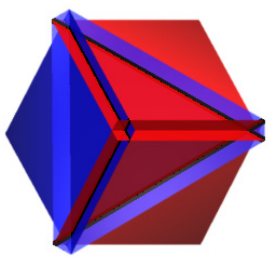
one finds a curve C' that is the union of the four lines

$$L_{01}' : y = z = 0, \qquad L_{12}' : z = x+y+z-w = 0,$$
$$L_{23}' : x+y+z-w = x = 0, \qquad L_{30}' : x = y = 0.$$

These lines form the base locus of the pencil of quadrics spanned by $Q : xz = 0$ and $Q' : y(x+y+z-w) = 0$. The singular quadrics in this pencil are the quadrics $sQ+tQ'$ for which

$$0 = \det\left(s\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}\right) = \frac{1}{16}s^2t^2.$$

It follows that the rank 2 quadrics Q and Q' appear in this pencil with multiplicity 2, and there are no other singular quadrics in this pencil. The following figure shows Q, Q', and C' in the chart $w = 1$.



Four concurrent lines, no three of which span a plane

As in the previous section, we start with a characterization of these curves that will make it easy to give explicit examples.

Proposition 25. *Any curve $C \subset \mathbb{P}^3$ that is the union of four concurrent lines, no three of which are coplanar, is the base locus of a pencil of quadrics spanned by two rank 2 quadrics $Q = H_{01} \cup H_{23}$ and $Q' = H_{02} \cup H_{13}$, for which the coordinate (row) vectors to the planes $H_{01}, H_{23}, H_{02}, H_{13}$ form a matrix*

$$M := \begin{bmatrix} \text{——} & H_{01} & \text{——} \\ \text{——} & H_{23} & \text{——} \\ \text{——} & H_{02} & \text{——} \\ \text{——} & H_{13} & \text{——} \end{bmatrix} \tag{4.26}$$

of rank 3 in which any three rows are linearly independent. Conversely, the base locus of any such pencil is the union of four concurrent lines, no three of which are coplanar.

Proof. Let $C \subset \mathbb{P}^3$ be the union of four lines L_0, L_1, L_2, L_3, no three of which are coplanar, that meet in a single point P. Write H_{ij} for the plane spanned by L_i and L_j, and let $Q = H_{01} \cup H_{23}$ and $Q' = H_{02} \cup H_{13}$. Since no three of the lines are coplanar, one finds

$$\begin{aligned} Q \cap Q' &= (H_{01} \cap H_{02}) \cup (H_{01} \cap H_{13}) \cup (H_{23} \cap H_{02}) \cup (H_{23} \cap H_{13}) \\ &= L_0 \cup L_1 \cup L_2 \cup L_3 \end{aligned}$$

and C is the base locus of the pencil of quadrics spanned by Q and Q'. Note that the matrix M from Equation 4.26 has rank 3, as all planes meet in a single point P. Moreover, no three of the planes share a line, as no three of the lines are coplanar. It follows that any three rows in M are linearly independent.

Conversely, suppose $C \subset \mathbb{P}^3$ is the intersection of two rank 2 quadrics $Q = H_{01} \cup H_{23}, Q' = H_{02} \cup H_{13}$ for which the matrix M in Equation 4.26 has rank 3 and any three of the four rows are linearly independent. Define lines

$$\begin{aligned} L_0 &:= H_{01} \cap H_{02}, \qquad & L_1 &:= H_{01} \cap H_{13}, \\ L_2 &:= H_{23} \cap H_{02}, \qquad & L_3 &:= H_{23} \cap H_{13}. \end{aligned}$$

As $\operatorname{rank}(M) = 3$, the lines L_0, L_1, L_2, L_3 meet in a single point P. No three of the lines, say L_0, L_1, L_2, lie in a plane H, as this would imply the existence of $[\alpha_0 : \beta_0], [\alpha_1 : \beta_1], [\alpha_2 : \beta_2] \in \mathbb{P}^1$ for which

$$H = \alpha_0 H_{01} + \beta_0 H_{02} = \alpha_1 H_{01} + \beta_1 H_{13} = \alpha_2 H_{23} + \beta_2 H_{02},$$

contradicting the assumption that any three of the planes are linearly independent. We conclude that C is the union of four concurrent lines, no three of which are coplanar. □

Suppose we are given a curve C that is the union of four concurrent lines L_0, L_1, L_2, L_3, no three of which are coplanar. Denote the point of intersection by P, and let H be any plane that intersects C in four distinct points R_0, R_1, R_2, R_3 on the lines

L_0, L_1, L_2, L_3. Since no three of the lines are coplanar, the points R_i are in general position inside the plane H (that is, no three of them are collinear). After applying a projectivity, we may assume that $P = [0:0:0:1]$ and H is the plane defined by $z = 0$. Since the points R_i are in general position inside the plane H, the fundamental theorem for projectivities implies that we can assume that

$$R_0 = [1:0:0:0], \qquad R_1 = [1:1:0:0],$$

$$R_2 = [1:0:1:0], \qquad R_3 = [1:1:1:0],$$

so that the components of C are given by

$$L_0 : x = y = 0, \qquad L_1 : x - w = y = 0,$$

$$L_2 : x = y - w = 0, \qquad L_3 : x - w = y - w = 0.$$

The curve C is the base locus of the pencil of quadrics spanned by $Q = H_{01} \cup H_{23}$ and $Q' = H_{02} \cup H_{13}$, where

$$H_{01} : y = 0, \qquad H_{23} : y - w = 0,$$

$$H_{02} : x = 0, \qquad H_{13} : x - w = 0.$$

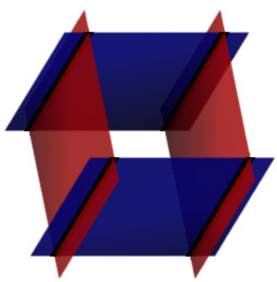
Any quadric $sQ + tQ'$ in the pencil has matrix

$$s \begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}t & -\frac{1}{2}s & 0 \\ -\frac{1}{2}t & t & 0 & 0 \\ -\frac{1}{2}s & 0 & s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

implying that a generic quadric in the pencil is a rank 3 cone. As

$$\det \begin{bmatrix} 0 & -\frac{t}{2} & -\frac{s}{2} \\ -\frac{t}{2} & t & 0 \\ -\frac{s}{2} & 0 & s \end{bmatrix} = -\frac{1}{4}(s+t)st,$$

the rank 2 quadrics in the pencil are Q, Q', and $Q - Q'$, each of multiplicity 1. The following figure shows Q, Q', and C in the chart $w = 1$.



Consider the homeomorphism

$$\chi : \mathbb{R} \times (\mathbb{Z}_2)^2 \longrightarrow C_{\mathrm{ns}}, \qquad (t,a,b) \longmapsto \begin{cases} [1:0:0:+t] & \text{if } (a,b)=(0,0), \\ [1:1:0:-t] & \text{if } (a,b)=(1,0), \\ [1:0:1:-t] & \text{if } (a,b)=(0,1), \\ [1:1:1:+t] & \text{if } (a,b)=(1,1). \end{cases}$$

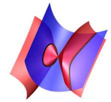
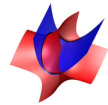
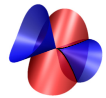
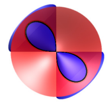
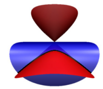
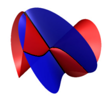
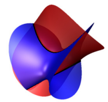
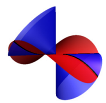
Any four distinct points $\chi(t_0,0,0), \chi(t_1,1,0), \chi(t_2,0,1), \chi(t_3,1,1)$ of C_{ns} are coplanar if and only if

$$0 = \det \begin{bmatrix} 1 & 0 & 0 & +t_0 \\ 1 & 1 & 0 & -t_1 \\ 1 & 0 & 1 & -t_2 \\ 1 & 1 & 1 & +t_3 \end{bmatrix} = t_0 + t_1 + t_2 + t_3.$$

It follows that the four parameterized curves

$$\begin{aligned} \mathbb{R} &\longrightarrow L_0, \qquad t \longmapsto \chi(t,0,0), \\ \mathbb{R} &\longrightarrow L_1, \qquad t \longmapsto \chi(t,1,0), \\ \mathbb{R} &\longrightarrow L_2, \qquad t \longmapsto \chi(t,0,1), \\ \mathbb{R} &\longrightarrow L_3, \qquad t \longmapsto \chi(t,1,1) \end{aligned}$$

together have Properties **P1** and **P2**. We conclude that C is a curve that satisfies Property ★.

	type of curve C	characteristic	quadrics in the pencil $\mathcal{P}_{\mathbb{C}}$	example pencil generators	picture
(a)	elliptic normal with two components	[1111]	smooth quadrics, either 4 rank 3 cones of mult. 1 in $\mathcal{P}$, or 4 rank 3 cones of mult. 1 in $\mathcal{P}_{\mathbb{C}}\backslash\mathcal{P}$	$0 = wz - x^2$ $0 = y^2 - 4xz + wx$	
(b)	elliptic normal with one component	[1111]	smooth quadrics, 2 rank 3 cones of mult. 1 in $\mathcal{P}$, 2 rank 3 cones of mult. 1 in $\mathcal{P}_{\mathbb{C}}\backslash\mathcal{P}$	$0 = wz - x^2$ $0 = y^2 - 4xz - 4w^2$	
(c)	irreducible, singular with a cusp	[31]	smooth quadrics, 1 rank 3 cone of mult. 3, 1 rank 3 cone of mult. 1	$0 = wy - x^2$ $0 = wz - y^2$	
(d)	irreducible, singular with a crunode	[211]	smooth quadrics, 1 rank 3 cone of mult. 2, 2 rank 3 cones of mult. 1 in $\mathcal{P}_{\mathbb{C}}$	$0 = xz - y^2$ $0 = x^2 - wy \mp z^2$	
(e)	irreducible, singular with an acnode	[211]	smooth quadrics, 1 rank 3 cone of mult. 2, 2 rank 3 cones of mult. 1	$0 = y^2 - xz$ $0 = w^2 - xy - xz - y^2 - yz$	
(f)	reducible, union of a twisted cubic and a real secant	[22]	smooth quadrics, 2 rank 3 cones of mult. 2	$0 = wy - x^2$ $0 = xz - y^2$	
(g)	reducible, union of a twisted cubic and a complex secant	[22]	smooth quadrics, 2 rank 3 cones of mult. 2 in $\mathcal{P}_{\mathbb{C}}\backslash\mathcal{P}$, with conjugated vertices	$0 = wz - xy$ $0 = wy - x^2 - xz + y^2$	
(h)	reducible, union of a twisted cubic and a tangent	[4]	smooth quadrics, 1 rank 3 cone of mult. 4	$0 = xz - y^2$ $0 = wz - xy$	

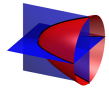
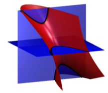
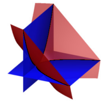
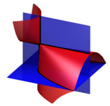
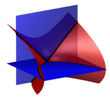
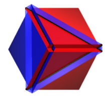
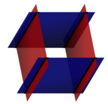
(i)	two conics, meeting in two real points	[(11)11]	smooth quadrics, rank 2 cone of mult. 2, 2 rank 3 cones of mult. 1 in $\mathcal{P}_{\mathbb{C}}$	$0 = x^2 \pm z^2 - wy$ $0 = xz$	
(j)	two conics, meeting in two nonreal points	[(11)11]	smooth quadrics, rank 2 cone of mult. 2, 2 rank 3 cones of mult. 1 in $\mathcal{P}$	$0 = x^2 - wy - xz$ $0 = (w + y)z$	
(k)	two conics, meeting in one point	[(21)1]	smooth quadrics, rank 2 cone of mult. 3, rank 3 cone of mult. 1	$0 = x^2 - wy - wz$ $0 = yz$	
(l)	a conic, meeting two lines in distinct points	[2(11)]	smooth quadrics, rank 2 cone of mult. 2, rank 3 cone of mult. 2	$0 = x^2 - (w - z)y$ $0 = xz$	
(m)	a conic, meeting two lines in one point	[(31)]	smooth quadrics, rank 2 cone of mult. 4	$0 = x^2 - wy - z^2$ $0 = yz$	
(n)	skew quadrilateral	[(11)(11)]	smooth quadrics, 2 rank 2 cones of mult. 2	$0 = y(x + y + z - w)$ $0 = xz$	
(o)	four concurrent lines, no three of which are coplanar	undefined	rank 3 cones, 3 rank 2 cones of mult. 1	$0 = x(x - w)$ $0 = y(y - w)$	

Table 4.4: *A real projective classification of the nondegenerate reduced real curves $C \subset \mathbb{P}^3$ that can be realized as the intersection of two quadrics. In Section 4.4.5, the existence was shown of continuous families of (projectively inequivalent) curves of types (a) and (b) satisfying Property ★. In Sections 4.4.6–4.4.9, it was shown that any curve of type (c) – (o) satisfies Property ★.*

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