The Flow of Quasiconformal Mappings on S^3 with Contact structure and a Family of Surfaces on the Heisenberg Group

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Contents

Introduction

Recent years have witnessed an increasing interest towards analysis and geometry in metric spaces, in the perspective of generalizing classical methods and results in Euclidean space to metric structures. Numerous topics of research, such as potential theory, Sobolev spaces, quasiconformal maps, and typical subjects of geometric measure theory, such as currents and rectifiable sets, have been reconsidered, adapted and generalized to the metric space setting.

Classical sub-Riemannian structure of the Heisenberg group $\mathbb{H}^1$ is a particular example of a metric space in which these investigations have been carried on with prosperous results. Having the Euclidean space $\mathbb{R}^3$ as the underlying manifold, the group $\mathbb{H}^1$ can be endowed with different, but equivalent, metrics. One metric arises from a homogeneous norm connected to the fundamental solution of Kohn's sub-Laplacian, given by G.B. Folland [Fol73], and the group structure of $\mathbb{H}^1$. Another one is the Carnot-Carathéodory metric, based on the length of horizontal curves. Endowed with any of these two metrics, that are homogeneous with respect to a special dilation $\delta_s : \mathbb{H}^1 \to \mathbb{H}^1$, $\delta_s(x, y, t) = (sx, sy, s^2t)$, $\mathbb{H}^1$ exhibits a behaviour, drastically different from Euclidean space. The difference is that in these metrics the horizontal coordinate axes x and y have Hausdorff dimension one while the vertical t-axis has Hausdorff dimension two. This yields to the fact that the Hausdorff dimension of $\mathbb{H}^1$ is four, which is strictly bigger than the topological dimension of the underlying manifold $\mathbb{R}^3$. By this reason many of the classical approaches do not work. The development of new, intrinsic methods was required.

A. Korányi and H.M. Reimann were among the pioneers adapting the classical quasiconformal mappings theory to the metric and group structures of the Heisenberg group. They conducted a deep study of such questions as quasiconformal deformations [KR85], moduli of families of curves [KR87], differentiability, Beltrami equation, Gehring L^p-integrability and capacities [KR95]. In parallel, G.B. Folland, E.M. Stein, L.P. Rothschild have developed the theory of subelliptic equations and singular integrals on nilpotent Lie groups [Fol73], [Fol75], [RS76] for which $\mathbb{H}^1$ is a typical and rich example. Particularly, Folland and Stein were first who introduced the polar decomposition of the Haar measure on homogeneous groups [FS82, Prop.1.15]. In [KR87] Korányi and Reimann obtained the exact value of the conformal modulus of a family of curves in a ring in $\mathbb{H}^1$ by making use of polar coordinates. Later on L. Capogna, D. Danielli and N. Garofalo in [CDG96, Th2.2] calculated the value of the capacity of the ring in Heisenberg-type groups, using the polar decomposition method of Folland and Stein. The next step to further generalization was done by Z. Balogh and J. Tyson, who introduced the class of groups in which the polar decomposition is possible. The use of this method allowed them to obtain the sharp constant in the Moser-Trudinger inequality [BT02, Cor.5.15].

This research unavoidably attracted the attention of the Finnish, Russian and Polish schools in quasiconformal mappings. An influential contribution to quasiconformal theory and Sobolev classes on metric spaces was done among others by Yu.G. Reshetnyak [Res78], [GR90], B.Bojarski and T. Iwaniec [BI87]. J. Heinonen together with P. Koskela founded the theory of quasiconformal maps in metric spaces with controlled geometry [HK98]. Koskela collaborated with B. Franchi and P. Hajłasz [FHK99], [HK95] on definitions of Sobolev classes on metric spaces and on regularity properties of solutions to a degenerate equation, and with J. Maly [KM03] on the theory of mappings of finite distortion. It is worth to mention joint work of J. Maly and W.P. Ziemmer [MZ97] on regularity theory of quasilinear second-order partial differential equations of elliptic type. See also the references in aforementioned literature.

The first attempt to develop geometric measure theory on the Heisenberg group traces back to the proof of the isoperimetric inequality in $\mathbb{H}^1$ [Pan82]. The theory of minimal surfaces (N. Garofalo, E. Giusti), differentiability (P. Pansu), area and co-area formulae (B. Franchi, R. Serapioni, F. Serra Cassano, V. Magnani) provided prosperous research topics.

Another subject which has been deeply analyzed is the possibility of giving appropriate definitions of rectifiability and currents, since the classical definition of Federer does not suit the geometry of $\mathbb{H}^1$. This problem, among others, occupied attention of L. Ambrosio, B. Franchi, R. Serapioni, F. Serra Cassano, L. Capogna, J. Cheeger [AK00], [FSSC01], [Che02].

The main goal of the thesis is to study quasiconformal mappings on the three-dimensional unit sphere $S^3 \subset \mathbb{C}^2$, endowed with the contact structure. This contact structure, furnished with a fiber bundle metric, gives an interesting example of a metric space, where the metric is not bi-Lipschitz equivalent to any Riemannian metric on the sphere. The corresponding quasiconformal maps have to preserve this contact structure, that restricts the class of admissible mappings and introduces a series of technical difficulties.

Another aim of the thesis is to generalize the notion of the family of surfaces for the Heisenberg group $\mathbb{H}^1$ and find a class of surfaces, on which the modulus is not degenerate and provides a fruitful theory. We intend to extend the classical result about the relation between the module of family of curves connecting two boundary components of the spherical domain and the module of a family of surfaces separating these components. This is the model example that reveals the main difficulties, possible approaches to the problem and opens a wide field for future research.

The structure of the thesis is the following. In Chapter 1 we set the context for the work and state the main features of the group $\mathbb{H}^1$. We consider in detail the connection between $\mathbb{H}^1$ and the three-dimensional unit sphere S^3 considered as contact, CR and sub-Riemannian manifolds.

In Chapter 2, using the results of Korányi and Reimann obtained on $\mathbb{H}^1$, we calculate the flow of quasiconformal maps on S^3. Korányi and Reimann in [KR85] presented the exact formulae for the vector field V that generates a flow of quasiconformal maps on $\mathbb{H}^1$ by making use of the Liebermann theorem [Lib59] stated for an arbitrary contact manifold. They described also a contact map between $\mathbb{H}^1$ and S^3, analogous to a stereographic projection. Given this map we calculate the push forward of the vector field V and present the exact formula for the vector field on sphere generating the flow of quasiconformal maps on S^3. The obtained formulae are quite complicated and it seems technically easier to study directly the properties of quasiconformal maps on the Heisenberg groups and then project them to the sphere. Further study of this question might be enlighting in future research.

Chapter 3 is devoted to the calculation of the p-modulus M_p of a family of curves in an annulus in $\mathbb{H}^1$. This is a natural continuation of the work of Korányi and Reimann, who found the value of M_p only for the conformal case $p = 4$. In this work we develop the calculations for an arbitrary exponent p of M_p. We extend this result to a wider class of Carnot groups called polarizable, following [BT02]. These groups admit an analogue of Euclidean spherical coordinates, which naturally give a set of radial curves for the annulus, that are rectifiable with respect of the Heisenberg metric. Our result is in a accordance with earlier results obtained by Capogna, Danielli and Garofalo in [CDG96, Th.2.2] via a more general approach.

Chapter 4 contains investigations inspired by some classical results of F. W. Gehring and W. P. Ziemer on the connection between the conformal capacity of the annulus and the extremal length of a family of surfaces separating the boundary components of the annulus in $\mathbb{R}^n$. We obtain an analogous relations in the setting of $\mathbb{H}^1$. The main difficulty is that in $\mathbb{H}^1$ the notion of admissible surfaces, that are essentially Lipschitz surfaces in the Euclidean space, has to be changed to a different one, that is compatible with the metric structure of the Heisenberg group. Due to this we develop some basic definitions and facts from geometric measure theory.

Basic notation

$A - B$	set-theoretic difference
A^c, $\bar{A}$	complement, closure
$\mathbb{R}^n$, $\mathbb{C}^n$	Euclidean space, complex n-space
S^n	the unit sphere in $\mathbb{R}^{n+1}$
$\mathbb{H}^1$	one-dimensional Heisenberg group
$\mathfrak{h}$	Lie algebra of $\mathbb{H}^1$
TM, T_xM	tangent bundle to a manifold M, tangent space at x
HM, H_xM	horizontal subbundle to M, horizontal subspace at x
$\|x\|$	Euclidean norm of $x \in \mathbb{R}^n$
$\|\xi\|_H$	Heisenberg norm of $\xi \in \mathbb{H}^1$
$\|\xi\|_{cc}$	Carnot-Carathéodory(CC) distance from $\xi \in \mathbb{H}^1$ to the origin
$\\|v\\|_0$	horizontal norm of a vector $v \in H\mathbb{H}^1$
$\langle v, w\rangle_0$	the scalar product in $H\mathbb{H}^1$
$\nabla_0 f$	horizontal gradient of f
$d\mathcal{L}^n$	Lebesgue measure in $\mathbb{R}^n$
dg	Haar measure in $\mathbb{H}^1$
$\mathcal{H}^k_d$	k-dimensional Hausdorff measure induced by a metric d
$\mathcal{H}^k_{cc}$	k-dimensional Hausdorff measure induced by CC metric d_{cc}
$\mathcal{H}^k_H$	k-dimensional Hausdorff measure induced by Heisenberg metric d_H
$C^k(U)$	continuous k-differentiable real-valued functions in U
$C^\infty(U)$	smooth real-valued functions in U
$C^\infty_0(U)$	functions in $C^\infty(U)$ with compact support in U
$HW^{1,p}$	horizontal Sobolev space
$ACL(U)$	functions, absolutely continuous on lines in U

Chapter 1

Contact manifolds.

1.1 Prerequisites

1.1.1 Smooth manifolds

Let us set the context for this work and recall some basic definitions of differential geometry.

Definition 1.1. *A C^∞-differentiable (or smooth) manifold M is a second countable Hausdorff space with a globally defined differential structure. By differential structure we mean a set of bijections $\varphi_i : M \supset U_i \to V_i \subset \mathbb{R}^m$ between a collection of open subsets of M (whose union covers M), and a set of open subsets of $\mathbb{R}^m$, which are C^∞-compatible in the following sense. Having two charts, i.e. two pairs (U_i, φ_i) and (U_j, φ_j), the transition map $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ between them is smooth.*

Let M be m-dimensional connected smooth manifold. Given a point $q \in M$, we define a tangent vector v_q to M at q as a first-order differential operator $v_q : C^\infty(M) \to C^\infty(M)$, where $C^\infty(M)$ is a set of all real-valued smooth functions on M. Since any two tangent vectors v_q, w_q to M at $q \in M$ satisfy

$$
\begin{aligned}
(bv)_q f &= b \cdot v_q f, \quad b \in \mathbb{R}, \\
(v + w)_q f &= v_q f + w_q f,
\end{aligned}
$$

we get that the set of all tangent vectors to M at a point q form a vector space. We call it the tangect space to M at q and denote by T_qM.

The union of all tangent spaces to M, endowed with a manifold structure, is a tangent bundle over M, denoted by

$$TM := \bigcup_{q \in M} T_qM.$$

The section $V : M \to TM$ of TM is called a vector field on M. We denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M. The dual space of T_qM is called a cotangent space and denoted by T_q^*M. Thus, the cotangent bundle of M is a manifold

$$TM^* := \bigcup_{q \in M} T_q^*M.$$

The section $\omega : M \to TM^*$ of TM^* is called a one-form on M.

Given connected smooth manifolds M^m and N^n of dimension m and n, correspondingly, we say that the map $F : M^m \to N^n$ is smooth if for any $q \in M$ there are charts: (U_i, φ_i) on M^m containing q and (W_j, ψ_j) on N^n with $W_j \supset F(U_i)$, such that the composition $\psi_j \circ F \circ \varphi_i^{-1} : \mathbb{R}^m \supset \varphi_i(U_i) \to \psi_j(W_j) \subset \mathbb{R}^n$ is a smooth function between $\varphi_i(U_i)$ and $\psi_j(W_j)$.

The map $F : M^m \to N^n$ between manifolds induces a corresponding mapping on tangent and cotangent spaces. Namely, given a point $q \in M$, we have maps $F_* : T_q M^m \to T_{F(q)} N^n$ and $F^* : T^*_{F(q)} N^n \to T^*_q M^m$ defined by

$$
\begin{aligned}
&(F_* V_q) f := V_q[f \circ F], \quad V_q \in T_q M^m,\ f \in C^\infty(N^n);\\
&(F^* \theta_{F(q)}) V_q := \theta_{F(q)}[F_* V_q] \quad \theta_{F(q)} \in T^*_{F(q)} N^n,\ V_q \in T_q M^m.
\end{aligned}
$$

With a choice of local basis, i.e. in one chart $(U, \varphi = (x_1, \cdots, x_m))$, we get the corresponding local coordinates $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})$ and $(dx_1, \ldots, dx_m)$ in tangent and cotangent spaces, satisfying the following relation:

$$dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij},$$

where δ_{ij} is a Kronecker symbol.

1.1.2 Lie groups

Definition 1.2. *A Lie group G is a smooth manifold, endowed with a group structure such that the map $G \times G \to G$ defined by $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ is smooth.*

Definition 1.3. *A Lie algebra $\mathfrak{g}$ is a real vector space $\mathfrak{g}$ together with bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for all $x, y, z \in \mathfrak{g}$:*

a. $[x, y] = -[y, x]$,

b. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Given a Lie group G, we define the left translation by an element $\sigma \in G$ as the automorphism $L_\sigma : G \to G$:

$$L_\sigma(\tau) = \sigma\tau.$$

A vector field V on G is said to be left-invariant if for each $\sigma, \tau \in G$

$$(L_\sigma)_* X(\tau) = X(L_\sigma(\tau)),$$

where $X(\tau) \in T_\tau G$. It turns out that every left-invariant vector field on G is smooth, see, for example, [War83, Prop.3.7]. Thus, the set of all left-invariant vector fields on G forms a Lie algebra, which we associate to a Lie group G and denote by $\mathfrak{g}$. For any left-invariant vector field $X \in \mathfrak{g}$, we have

$$X(\sigma) = (L_\sigma)_* X(e), \qquad \sigma \in G.$$

Therefore we can identify the Lie algebra $\mathfrak{g}$ of G with the tangent space to G at the identity via isomorphism $\alpha : \mathfrak{g} \to T_e G$ given by $\alpha(X) = X(e)$.

1.1.3 Contact structures

Definition 1.4. *Let M be a smooth manifold of dimension m, and let $1 \leq c \leq m$. A c-dimensional distribution $\mathcal{D}$ on M is a choice of a c-dimensional subspace $\mathcal{D}_q$ of $T_q(M)$ for each $q \in M$.*

A distribution $\mathcal{D}$ is smooth if for each $q \in M$ there is a neighbourhood N_q of q and there are c smooth vector fields $X_1, \dots, X_c$ on N_q which span $\mathcal{D}$ at each point of N_q. A vector field V on M is said to belong (or lie in) the distribution $\mathcal{D}$ if $V_q \in \mathcal{D}_q$ for each $q \in M$. A smooth distribution $\mathcal{D}$ is called involutive if $[X, Y] \in \mathcal{D}$ whenever X and Y are smooth vector fields lying in $\mathcal{D}$.

Definition 1.5. *A manifold M of dimension $2n + 1$ is said to be contact, if there exists a one-form ω, such that $\omega \wedge (d\omega)^n$ never vanishes. The form ω is called contact form.*

If M is a contact manifold, then the contact form ω defines a distribution $\mathcal{D} \subset TM$ on M, such that for each fiber $\mathcal{D}_q$:

$$\mathcal{D}_q = \{V \in T_qM : \omega(V) = 0\}.$$

We call the pair $(M, \mathcal{D})$ a contact structure on M, where $\mathcal{D} = \ker \omega \subset TM$ is a non-integrable distribution on M as was defined above.

Observe that if $\omega \wedge (d\omega)^n$ never vanishes, then the same is true for any differential form $\omega' = \lambda\omega$, where λ is a non-vanishing scalar function. Such a differential form $\omega' = \lambda\omega$ will be said to be equivalent to ω and we will extend this equivalence relation to measurable functions λ.

The equivalence class $[\omega]$ of differential forms can be taken as an alternative definition of contact structure, equivalent to the definition we used before and therefore we also use the notation (M, ω) for contact manifold.

Definition 1.6. *A diffeomorphism $F : M \to N$ between contact manifolds (M, ω) and (N, σ) is said to be a contact transformation, if $F_*\mathcal{D}_M = \mathcal{D}_N$, where $\mathcal{D}_M = \ker \omega$ and $\mathcal{D}_N = \ker \sigma$. In other words, there exists a nowhere vanishing function $\lambda : M \to \mathbb{R}$ such that $F^*\sigma = \lambda\omega$.*

1.1.4 CR structures

Definition 1.7. *Suppose V is a real vector space. The complexification of V is the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$ (or, for brevity, $V \otimes \mathbb{C}$).*

As an example, let M be a smooth manifold of real dimension m. For $q \in M$, $T_q(M) \otimes \mathbb{C}$ is called the complexified tangent space. The complexified tangent bundle $T^{\mathbb{C}}M$ is defined analogously to the real tangent bundle whose fiber at each point $q \in M$ is $T_q(M) \otimes \mathbb{C}$.

Definition 1.8. *Let V be a real vector space. A linear map $J : V \to V$ is called a complex structure map if $J \circ J = -\operatorname{Id}$, where $\operatorname{Id} : V \to V$ is the identity map.*

The complex structure map is extended to the complexified space $V \otimes \mathbb{C}$. Therefore, $J : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ has eigenvalues i and $-i$ with corresponding eigenspaces denoted by $V^{1,0}$ and $V^{0,1}$. From linear algebra, we have

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

Definition 1.9. *Let M be a smooth manifold and suppose $\mathbb{L}$ is a subbundle of $T^{\mathbb{C}}M$. The pair $(M, \mathbb{L})$ is called (an abstract) CR manifold (or CR structure) if:*

1. $\mathbb{L}_q \cap \bar{\mathbb{L}}_q = \{0\}$ *for each* $q \in M$,
2. $\mathbb{L}$ *is involutive.*

If M is a CR submanifold of $\mathbb{C}^n$, then $\mathbb{L} = T^{1,0}(M)$ and, accordingly

$$T^{\mathbb{C}}M = T^{1,0}(M) \oplus T^{0,1}(M).$$

The definitions and notation above are standard, see [Bog91, Chap.7].

In order to obtain a more geometric insight we recall the notion of an embedded CR manifold, in particular, the CR manifold of hypersurface type. We reduce our consideration to 3-dimensional case and refer to [CDPT07, Sec.3.3] for the following facts.

Let $\varphi : \mathbb{C}^2 \to \mathbb{R}$ be of class $C^2(\mathbb{C}^2)$. Then the set

$$\Omega = \{z = (z_1, z_2) \in \mathbb{C}^2 : \varphi(z) < 0, \nabla\varphi \neq 0\}$$

is a smooth subset of $\mathbb{C}^2$. The function φ is called defining function for Ω.

The tangent space to $\partial\Omega$ at $p \in \partial\Omega$ is given by

$$T_p\partial\Omega = \{Z \in \mathbb{C}^2 : \mathrm{Re}\langle\bar{\partial}\varphi(p), Z\rangle = 0\},$$

where

$$\bar{\partial}\varphi = \left(\frac{\partial}{\partial\bar{z}_1}\varphi, \frac{\partial}{\partial\bar{z}_2}\varphi\right),$$

and for $Z, W \in \mathbb{C}^2$, $\langle Z, W\rangle = Z_1\bar{W}_1 + Z_2\bar{W}_2$, denotes the Hermitian inner product. The maximal complex plane at p is given by

$$H_p\partial\Omega = \{Z \in \mathbb{C}^2 : \langle\bar{\partial}\varphi(p), Z\rangle = 0\}.$$

We also call $H_p\partial\Omega$ the horizontal plane.

Combining the conditions defining tangential complex lines ($\mathrm{Re}\langle\bar{\partial}\varphi(p), Z\rangle = 0$) and horizontal complex lines ($\langle\bar{\partial}\varphi(p), Z\rangle = 0$) we see that the horizontal lines tangential to $\partial\Omega$ are given by

$$\mathrm{Im}\langle\bar{\partial}\varphi(p), Z\rangle\langle(\partial - \bar{\partial})\varphi(p), Z\rangle = 0.$$

Equivalently, the horizontal distribution on $\partial\Omega$ is given by the tangential vector fields which are in the kernel of the form

$$\sigma = \frac{\partial}{\partial\bar{z}_1}\varphi d\bar{z}_1 + \frac{\partial}{\partial\bar{z}_2}\varphi d\bar{z}_2 - \frac{\partial}{\partial z_1}\varphi dz_1 - \frac{\partial}{\partial z_2}\varphi dz_2.$$

If Ω is strictly pseudoconvex, i.e., the Levi form

$$Z \mapsto L(p, Z) = \sum_{i,j=1}^{2} \frac{\partial^2 \varphi(p)}{\partial z_i \partial \bar{z}_j} Z_i \bar{Z}_j$$

is positive definite on $H_p\partial\Omega$ for all $p \in \partial\Omega$, then $H_p\partial\Omega$ is a contact distribution on $\partial\Omega$. In this case a defining contact form is given by σ, see [CDPT07, Sec.3.3].

1.2 Heisenberg group $\mathbb{H}^1$.

Heisenberg group is an important object that reveals itself related to many different topics. Originating from quantum mechanics, it has also wide application in theory of partial differential equations, harmonic analysis, representation theory of nilpotent groups, moduli of abelian varieties, structure theory of finite groups, homological algebra, ergodic theory etc. Heisenberg group is a very fruitful and interesting topic on its own. Some of the main results related to the Heisenberg group and its applications can be found, for example, in [Ste93]. We describe briefly the geometry and metric structure of the Heisenberg group in order to set the context for our work.

Definition 1.10. *The (first) Heisenberg group $\mathbb{H}^1$ is the analytic, nilpotent Lie group whose underlying manifold is $\mathbb{R}^3$ and whose Lie algebra $\mathfrak{h}$ is graded*

- $\mathfrak{h} = V_1 \oplus V_2$, *where V_1 has dimension 2 and V_2 has dimension 1, and*
- *has the following commutator relations:* $[V_1, V_1] = V_2$, $[V_1, V_2] = [V_2, V_2] = 0$.

Observe that since $\mathfrak{h}$ is nilpotent and graded the exponential map $\exp : \mathfrak{h} \to \mathbb{H}^1$ is a (global) diffeomorphism, see [FS82, Prop.1.2]. Fix an arbitrary basis X, Y of V_1 and let $T = [X, Y] \in V_2$. Then the group law reads off from the Baker-Campbell-Hausdorff formula:

$$\begin{aligned}\exp(v_1)\exp(v_2) &= \exp\left(v_1 + v_2 + \frac{1}{2}[v_1, v_2]\right) \\ &= \exp\left((x_1 + x_2)X + (y_1 + y_2)Y + (t_1 + t_2)T + \frac{1}{2}(x_1 y_2 - y_1 x_2)[X, Y]\right).\end{aligned}$$

Here we have denoted by $v_i = x_i X + y_i Y + t_i T$, $i = 1, 2$ a generic vector in $\mathfrak{h}$. By making use the normal coordinates on $\mathbb{H}^1$

$$(x, y, t) = \exp(xX + yY + tT),$$

we get the group multiplication law

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = \left(x_1 + x_2, y_1 + y_2, t_1 + t_2 + \frac{1}{2}(x_1 y_2 - y_1 x_2)\right).$$

The group identity is $e = (0,0,0)$, the inverse element of $\xi = (x,y,t)$ is $\xi^{-1} = (-x,-y,-t)$.

The natural group of automorphisms on $\mathbb{H}^1$ is given by left translations $L_\eta : \mathbb{H}^1 \to \mathbb{H}^1$, defined by $L_\eta(\xi) = \eta\xi$. The one-parametric subgroup of $\mathbb{H}^1$ is non-isotropic dilations $\delta_s(\xi) = (sx, sy, s^2 t)$. The bi-invariant Haar measure in $\mathbb{H}^1$ is denoted by dg and it is given by push-forward of the Lebesgue measure from $\mathbb{R}^3$.

In the remainder of the thesis we will almost invariably work with this model of the Heisenberg group, but with a group law adapted to our calculations:

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + 2(x_1 y_2 - y_1 x_2)) \,. \tag{1.1}$$

We shall also consider $\mathbb{H}^1$ as $\mathbb{C} \times \mathbb{R}$ by canonically indentifying (x,y,t) with (z,t) by $z = x + iy$. The group law reads

$$(z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}(\bar{z}_1 z_2)) \,.$$

By moving in a left invariant fashion the basis $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial t}$ of $T_e\mathbb{H}^1$ we obtain

$$\hat{X} = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \qquad \hat{Y} = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \qquad \hat{T} = \frac{\partial}{\partial t}. \tag{1.2}$$

The vector fields $\hat{X}, \hat{Y}$ and $\hat{T}$ are left invariant, first-order differential operators. By employing the canonical identification of the Lie algebra $\mathfrak{h}$ with the set of left-invariant vector fields on $\mathbb{H}^1$ we simply write X, Y, T for the basis (1.2). Observe that the vector fields X, Y and T are homogeneous of order 1 and 2 correspondingly with respect to the dilations $(\delta_s)_*$ induced on algebra $\mathfrak{h}$ by δ_s.

1.2.1 Heisenberg gauge and metric.

The Heisenberg group $\mathbb{H}^1$ can be endowed with the norm

$$|\xi|_H = \left((x^2 + y^2)^2 + t^2\right)^{1/4}, \tag{1.3}$$

also called Heisenberg (or Korányi) gauge. With the induced metric

$$d_H(\xi, \eta) = |\eta^{-1}\xi|_H, \tag{1.4}$$

$(\mathbb{H}^1, d_H)$ is a metric space. Indeed, the triangle inequality

$$d_H(\xi, \eta) \leqslant d_H(\xi, \nu) + d_H(\nu, \eta)$$

is a consequence of the norm inequality, see [CDPT07, p.18] for the proof:

$$|\xi\eta|_H \leqslant |\xi|_H + |\eta|_H.$$

Clearly, norm (1.3) and consequently the distance (1.4) are homogeneous of order 1 with respect to the dilations δ_s:

$$|\delta_s(\xi)|_H = s|\xi|_H.$$

In complex coordinates the Heisenberg gauge is written as

$$N(z,t) = |(z,t)|_H = (|z|^4 + t^2)^{1/4}. \tag{1.5}$$

We denote by $|\xi| = (x^2 + y^2 + t^2)^{1/2}$ the canonical Euclidean norm and by d_E the corresponding metric. Observe that the Heisenberg distance behaves like the Euclidean distance in horizontal directions X and Y, and like the square root of d_E in the missing direction T. By this reason the d_H-metric on $\mathbb{H}^1$ is not bi-Lipschitz equivalent to the Euclidean metric d_E on $\mathbb{R}^3$.

1.2.2 Carnot-Carathéodory distance.

There is another metric on $\mathbb{H}^1$ that comes from sub-Riemannian structure.

Definition 1.11. *A sub-Riemannian structure on a manifold M consists of a distribution $\mathcal{D}$, which is to say a vector subbundle of the tangent bundle of M, together with a fiber inner-product on this subbundle.*

A horizontal subbundle $H\mathbb{H}^1$ is the smooth subbundle of the tangent bundle $T\mathbb{H}^1$ with fibers $H_\xi\mathbb{H}^1 = \operatorname{span}\{X(\xi), Y(\xi)\}$, $\xi \in \mathbb{H}^1$. We endow $H\mathbb{H}^1$ with an inner product $\langle \cdot, \cdot \rangle_0$ with respect to which the vectors $X(\xi)$ and $Y(\xi)$ form an orthonormal basis in each fiber $H_\xi\mathbb{H}^1$, $\xi \in \mathbb{H}^1$. For a horizontal norm of a vector field $v \in H\mathbb{H}^1$, we write $\|v\|_0 = \langle v, v\rangle_0^{1/2}$.

Definition 1.12. *Let $S = \{a = r_0 \leqslant r_1 \leqslant \ldots \leqslant r_{n_S} = b\}$ be a subdivision of the interval $I = [a,b]$. A curve $\gamma : I \to \mathbb{H}^1$ is said to be rectifiable if*

$$\sup_S \sum_{k=1}^{n_S} |\gamma(r_k) - \gamma(r_{k-1})|_H < \infty.$$

Definition 1.13. *A curve $\gamma : I \to \mathbb{H}^1$ is said to be absolutely continuous on I if for given $\epsilon > 0$, there exists a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals (r_k, r_{k-1}) of I satisfies $\sum_k |r_k - r_{k-1}| < \delta$, then*

$$\sum_k |\gamma(r_k) - \gamma(r_{k-1})|_H < \epsilon.$$

A left-invariant sub-Riemannian metric on $\mathbb{H}^1$ is next defined by using horizontal curves. These are absolutely continuous curves $\gamma : [a,b] \to \mathbb{H}^1$ with tangents lying almost everywhere in the horizontal bundle, i.e. $\dot\gamma(t) \in H_{\gamma(t)}\mathbb{H}^1$ for a.e. $t \in [a,b]$. Equivalently, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is horizontal if it satisfies the condition

$$\dot\gamma_3(t) = 2(\dot\gamma_1(t)\gamma_2(t) - \gamma_1(t)\dot\gamma_2(t)) \qquad \text{for a.e. } t \in [a,b]. \tag{1.6}$$

To see this we write γ in the left-invariant basis

$$\dot{\gamma} = aX + bY + cT = a\left(\frac{\partial}{\partial x} + 2\gamma_2\frac{\partial}{\partial t}\right) + b\left(\frac{\partial}{\partial y} - 2\gamma_1\frac{\partial}{\partial t}\right) + c\frac{\partial}{\partial t}$$
$$= a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + (a\cdot 2\gamma_2 - b\cdot 2\gamma_1 + c)\frac{\partial}{\partial t} = \dot{\gamma}_1\frac{\partial}{\partial x} + \dot{\gamma}_2\frac{\partial}{\partial y} + \dot{\gamma}_3\frac{\partial}{\partial t},$$

Comparing the corresponding coordinates, we find $a = \dot{\gamma}_1(t)$, $b = \dot{\gamma}_2(t)$. In order to eliminate the movement in the direction T we set $c = 0$, which gives us the desired conditions(1.6) for horizontality.

For two given points in $\mathbb{H}^1$ there exists, by Chow's connectivity theorem ([Cho39]), a horizontal curve joining these points. We define the length of an absolutely continuous horizontal curve $\gamma : [a,b] \to \mathbb{H}^1$ to be

$$L_h(\gamma) := \int_a^b \left(\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle_0\right)^{\frac{1}{2}} dt = \int_a^b \|\gamma(t)\|_0\, dt = \int_a^b \left(\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2\right)^{\frac{1}{2}} dt.$$

Definition 1.14. *The Carnot-Carathéodory distance is defined as*

$$d_{cc}(\xi,\eta) := \inf_{\gamma} L_h(\gamma), \tag{1.7}$$

where the infimum is taken over all horizontal absolutely continuous curves γ joining ξ to η.

The Carnot-Carathéodory metric is left invariant:

$$d_{cc}(L_\eta(\xi), L_\eta(\nu)) = d_{cc}(\xi,\nu) \quad \text{for all } \xi,\eta,\nu \in \mathbb{H}^1,$$

and homogeneous with respect to dilation:

$$d_{cc}(\delta_s(\xi), \delta_s(\eta)) = s d_{cc}(\xi,\eta) \quad \text{for all } \xi,\eta \in \mathbb{H}^1 \text{ and } s > 0.$$

For the corresponding norm we write $|\xi|_{cc} = d_{cc}(\xi, 0)$. Observe that

$$|\xi|_E \leqslant |\xi|_{cc} \qquad \text{for all } \xi \in \mathbb{H}^1.$$

Since any two homogeneous functions are equivalent, the Carnot-Carathéodory metric (1.7) and the Heisenberg metric (1.4) are equivalent in the sense that there exist positive constants C_1, C_2 such that

$$C_1\, d_H(\xi,\eta) \leqslant d_{cc}(\xi,\eta) \leqslant C_2\, d_H(\xi,\eta), \qquad \xi,\eta \in \mathbb{H}^1.$$

The next lemma shows that the Heisenberg and CC-metrics generate the same infinitesimal structure, see [Str86].

Lemma 1.15. *If $\gamma : [0,1] \to \mathbb{R}$ is a C^1-curve and $t_i = i/n$, $i = 1, \ldots, n$ is a partition of $[0,1]$, then*

$$\lim_{n\to\infty} \sup \sum_{i=1}^{n} d_H(\gamma(t_i), \gamma(t_{i-1})) = \begin{cases} L_h(\gamma) & \textit{if } \gamma \textit{ is horizontal} \\ \infty & \textit{otherwise.} \end{cases}$$

The topology induced by the Carnot-Carathéodory metric coincides with the usual Euclidean topology on the underlying space.

1.2.3 Contact and CR structure of $\mathbb{H}^1$

The Heisenberg group $\mathbb{H}^1$ carries a CR structure. To see this we let

$$T^{1,0}\mathbb{H}^1 = \mathrm{span}(Z), \qquad T^{0,1}\mathbb{H}^1 = \mathrm{span}(\bar{Z}),$$

where

$$Z = \frac{1}{2}(X - iY) = \frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial t}.$$

The comutator relation is $[Z, \bar{Z}] = \dfrac{i}{2}\dfrac{\partial}{\partial t}$. Here we have used the standard notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Observe that $\mathbb{H}^1$ acts on $\mathbb{C}^2$ from the right by holomorphic affine transformations:

$$(z_1, z_2)(z, t) = \left(z_1 + z, z_2 + 4t + i|z|^2 + 2iz_1\bar{z}\right),$$

for $(z, t) \in \mathbb{H}^1$ and $(z_1, z_2) \in \mathbb{C}^2$. This action preserves the Siegel domain

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}\, z_2 - |z_1|^2 > 0\}$$

and its boundary ∂D, since

$$\begin{aligned} &\mathrm{Im}(z_2 + 4t + i|z|^2 + 2iz_1\bar{z}) - |z_1 + z|^2 \\ &\qquad = \mathrm{Im}(z_2) + |z|^2 + 2\,\mathrm{Im}(iz_1\bar{z}) - (|z|^2 + |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z})) = \mathrm{Im}(z_2) - |z_1|^2. \end{aligned}$$

Since this action is simply transitive on ∂D with a single fixed point at infinity, we may identify $\mathbb{H}^1$ with ∂D by the correspondence

$$(z, t) \mapsto (0, 0) \cdot (z, t) = (z, 4t + i|z|^2).$$

Under this identification the CR structure of $\mathbb{H}^1$ defined above coincides with the CR structure induced by the Euclidean metric in $\mathbb{C}^2$, i.e.,

$$H_{(z,4t+i|z|^2)}\partial D = T^{1,0}\mathbb{H}^1. \tag{1.8}$$

In order to prove this we observe that the holomorphic subspaces at the origin coincide as ∂D is tangent to the hyperplane $z_0 = 0$ there. Next, we remark that the CR structure on $\mathbb{H}^1$ is left invariant, and the action of $\mathbb{H}^1$ on ∂D is holomorphic, hence preserves the CR structure on ∂D.

Let us introduce the form

$$\omega = -2ydx + 2xdy + dt = -i\bar{z}dz + izd\bar{z} + dt. \tag{1.9}$$

At every point $\xi = (x, y, t) \in \mathbb{H}^1$ the forms dx, dy and ω represent a basis for the space of differential one-forms. This basis is a canonical dual to the left-invariant basis X, Y, T of the tangent space $T_\xi \mathbb{H}^1$ in the sense that $dx(X) = dy(Y) = \omega(T) = 1$ are the only non-trivial actions of one-forms on X, Y, T. Then any vector $v \in T_\xi \mathbb{H}^1$ can be expressed as

$$v = dx(v)\, X + dy(v)\, Y + \omega(v)\, T,$$

and, every differential one-form α can be represented by the following

$$\alpha = \alpha(X)\, dx + \alpha(Y)\, dy + \alpha(T)\, \omega.$$

The form $d\omega \wedge \omega$ is proportional to the euclidean volume form

$$d\omega \wedge \omega = (-2dy \wedge dx + 2dx \wedge dy) \wedge \omega = 4dx \wedge dy \wedge dt.$$

The tangent mapping f_* of a transformation $f : \mathbb{H}^1 \to \mathbb{H}^1$ can be expressed in terms of the basis given at $\xi = (x, y, t)$ as matrix

$$\begin{pmatrix} dx(f_*X) & dx(f_*Y) & dx(f_*T) \\ dy(f_*X) & dy(f_*Y) & dy(f_*T) \\ \omega(f_*X) & \omega(f_*Y) & \omega(f_*T) \end{pmatrix}.$$

Writing $f^*\omega$ in terms of the basis dx, dy, ω one sees that f is a contact transformation if and only if

$$f^*\omega(X) = \omega(f_*X) = 0,$$
$$f^*\omega(Y) = \omega(f_*Y) = 0$$

and

$$\lambda = f^*\omega(T) = \omega(f_*T) \neq 0.$$

Example 1.16. Left translation by a group element η clearly is a contact transformation. Let us fix an element $\eta = (x_0, y_0, t_0)$. Then for any $\xi = (x, y, t) \in \mathbb{H}^1$ the left translation $L_\eta(\xi) = (x_0 + x, y_0 + y, t_0 + t + 2(x_0 y - y_0 x))$ induces the corresponding map on cotangent space $L_\eta^* : T^*_{L_\eta(\xi)} \to T^*_\xi$ acting on the form ω as follows

$$L^*_{(x,y,t)} \omega_{L_\eta \xi} = \begin{pmatrix} 1 & 0 & 2y_0 \\ 0 & 1 & -2x_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2(y_0 + y) \\ 2(x_0 + x) \\ 1 \end{pmatrix} = \begin{pmatrix} -2y \\ 2x \\ 1 \end{pmatrix} = \omega_\xi.$$

1.3 Sphere S^3.

1.3.1 Group structure of S^3

Another example of a contact manifold is the sphere $S^3 \subset \mathbb{C}^2$. The sphere S^3 is the smooth manifold $\{w = (w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 = 1\}$ according to the regular point theorem ([dC92, Ex.4.3]), since $S^3 = f^{-1}(1)$, where $f : \mathbb{C}^2 \to \mathbb{R}$, $f(w_1, w_2) = |w_1|^2 + |w_2|^2$ is a smooth map and $q = 1$ is a regular value of f. The Lie group structure of S^3 can be seen by identifying it with the unitary group $SU(2)$:

$$SU(2) = \left\{ \begin{pmatrix} w_1 & w_2 \\ -\bar{w}_2 & \bar{w}_1 \end{pmatrix}, |w_1|^2 + |w_2|^2 = 1, w_1, w_2 \in \mathbb{C} \right\}.$$

The group operation on S^3 is induced by matrix multiplication in $SU(2)$:

$$(w_1, w_2) * (u_1, u_2) = (w_1u_1 - w_2\bar{u}_2, w_1u_2 + w_2\bar{u}_1), \quad \text{for } (w_1, w_2), (u_1, u_2) \in S^3.$$

To calculate left-invariant basis for the algebra of S^3 we rewrite the group law in real coordinates. For two points (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) of $\mathbb{R}^4$:

$$\begin{aligned}
& w_1u_1 - w_2\bar{u}_2 = (x_0 + ix_1)(y_0 + iy_1) - (x_2 + ix_3)(y_2 - iy_3) \\
& \ = (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + i(x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3) \\
& w_1u_2 + w_2\bar{u}_1 = (x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3) + i(x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3).
\end{aligned}$$

This rule induces a left translation $L_x(y)$ of an element $y = (y_0, y_1, y_2, y_3)$ by an element $x = (x_0, x_1, x_2, x_3)$. The matrix corresponding the tangent map $(L_x)_*$ is given by

$$(L_x)_* = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & -x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}.$$

We find left-invariant vector fields by left translating the basis vectors at the unity of the group, namely $X(x) = (L_x)_*(y)X(0)$:

$$\begin{aligned}
N(x) &= x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}, \\
X_1(x) &= -x_1\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}, \\
X_2(x) &= -x_2\frac{\partial}{\partial x_0} - x_3\frac{\partial}{\partial x_1} - x_0\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}, \\
X_3(x) &= -x_3\frac{\partial}{\partial x_0} + x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + x_0\frac{\partial}{\partial x_3}.
\end{aligned}$$

We observe that the vector $N(x)$ is the unit normal to S^3 at $x \in S^3$ with respect to the usual inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^4$. Moreover, for any $x \in S^3$

$$\langle N(x), X_1(x) \rangle = \langle N(x), X_2(x) \rangle = \langle N(x), X_3(x) \rangle = 0.$$

and

$$\langle N(x), N(x) \rangle = \langle X_1(x), X_1(x) \rangle = \langle X_2(x), X_2(x) \rangle = \langle X_3(x), X_3(x) \rangle = 1.$$

Since the matrix

$$\begin{pmatrix} x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & -x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}$$

has rank three, we conclude that the vector fields $\{X_1(x), X_2(x), X_3(x)\}$ form an orthonormal basis for T_xS^3 with respect to $\langle \cdot, \cdot \rangle$ at each point $x \in S^3$. The basic vector fields $\{X_1(x), X_2(x), X_3(x)\}$ possess the following commutator relations

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2.$$

By letting $\mathcal{D} = \text{span}\{X_2, X_3\}$ be the horisontal distribution generated by X_2 and X_3, we see that $\mathcal{D}$ is bracket-generating and it provides the sub-Riemannian structure for S^3.

1.3.2 Contact and CR structure of S^3

To see that S^3 is a CR manifold, we let the defining function introduced in Section 1.1.4 be $\varphi(w) = |w|^2 - 1$. Then Ω is the unit ball

$$\Omega = \{w = (w_1, w_2) \in \mathbb{C}^2 : |w|^2 - 1 < 0\},$$

and $\partial\Omega = S^3$. The Levi form is a constant multiple of the identity (and hence positive definite), and the horizontal distribution is given by the kernel of the contact form

$$\sigma = \bar{w}_1 dw_1 - w_1 d\bar{w}_1 + \bar{w}_2 dw_2 - w_2 d\bar{w}_2. \tag{1.10}$$

The subbundle $H_wS^3 = T_w^{1,0}S^3 = \text{span}(W_1)$, where

$$W_1 = i\frac{(1+w_2)^2}{1+\bar{w}_2}\left(\bar{w}_2\frac{\partial}{\partial w_1} - \bar{w}_1\frac{\partial}{\partial w_2}\right) = i\frac{(1+w_2)^2}{1+\bar{w}_2}W \tag{1.11}$$

and $T_w^{0,1}S^3 = \text{span}(\bar{W}_1)$

$$\bar{W}_1 = -i\frac{(1+\bar{w}_2)^2}{1+w_2}\left(w_2\frac{\partial}{\partial \bar{w}_1} - w_1\frac{\partial}{\partial \bar{w}_2}\right) = -i\frac{(1+\bar{w}_2)^2}{1+w_2}\bar{W}.$$

1.4 $\mathbb{H}^1$ vs. S^3

In this section we show the correspondence between $\mathbb{H}^1$ and S^3 via a contact map. Consequently, under this identification the CR structure on S^3 can be viewed as the CR structure of the one-point compactification of $\mathbb{H}^1$, namely, HS^3 corresponds to the horizontal distribution $H\mathbb{H}^1$, and σ corresponds to the contact form ω given in (1.9).

In order to write the exact correspondence between S^3 and $\mathbb{H}^1$, we require a special stereographic projection π based on the Cayley transform, which we define below.

First, we recall the definition of the Siegel domain

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 - |z_1|^2 > 0\}. \tag{1.12}$$

A defining function for D is

$$\varphi(z_1, z_2) = z_1\bar{z}_1 + \frac{i}{2}(z_2 - \bar{z}_2).$$

The horizontal structure $H\partial D$ is given by the kernel of the form

$$\tau = -i\bar{z}_1 dz_1 + iz_1 d\bar{z}_1 + \frac{1}{2}(dz_2 + d\bar{z}_2), \tag{1.13}$$

which is contact since the wedge

$$d\tau \wedge \tau = 4i dz_1 \wedge d\bar{z}_1 \wedge (dz_2 + d\bar{z}_2)$$

is the volume form on ∂D. To see this, we switch to the real coordinates by $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Hence,

$$\partial D = \{(x_1, y_1, x_2, x_1^2 + y_1^2)\},$$

and the form

$$d\tau \wedge \tau = 4i(dx_1 + idy_1) \wedge (dx_1 - idy_1) \wedge (2dx_2) = 16\, dx_1 \wedge dy_1 \wedge dx_2$$

is a multiple of a volume form in coordinates (x_1, y_1, x_2).

The vector fields

$$Z_1 = \frac{\partial}{\partial z_1} + 2i\bar{z}_1\frac{\partial}{\partial z_2} \quad \text{and} \quad \bar{Z}_1 = \frac{\partial}{\partial \bar{z}_1} - 2iz_1\frac{\partial}{\partial \bar{z}_2} \tag{1.14}$$

form a basis for $H\partial D$.

The Cayley transform

$$C(w_1, w_2) = \left(\frac{iw_1}{1 + w_2}, i\,\frac{1 - w_2}{1 + w_2}\right) \tag{1.15}$$

maps the unit ball $B \subset \mathbb{C}^2$ biholomorphically onto the Siegel domain D. With the help of the Cayley transform we can define a CR generalization of stereographic projection

$$\pi : c\backslash\{-e_2\} \to \mathbb{R}^3 \cong \mathbb{H}^1, \qquad e_2 = (0, 1) \tag{1.16}$$

as the composition of $C|_{\partial B}$ and the projection

$$\Pi : (z_1, z_2) \to (z_1, \operatorname{Re} z_2). \tag{1.17}$$

The stereographic projection π can be extended to a map from ∂B to the one-point compactification of $\mathbb{R}^3$, and the inverse map is given by

$$\pi^{-1}(z, t) = \left(\frac{-2iz}{1 + |z|^2 - 4it}, \frac{1 - |z|^2 + 4it}{1 + |z|^2 - 4it}\right), \tag{1.18}$$

with $t = \mathrm{Re}(z_2)$ and $z = z_1$.

The Cayley transform (1.15) is a holomorphic mapping. Its Jacobian is given by

$$\begin{pmatrix} \dfrac{\partial z_1}{\partial w_1} & \dfrac{\partial z_1}{\partial w_2} \\ \dfrac{\partial z_2}{\partial w_1} & \dfrac{\partial z_2}{\partial w_2} \end{pmatrix} = \frac{i}{|1+w_2|^2} \begin{pmatrix} 1+w_2 & -w_1 \\ 0 & -2 \end{pmatrix}.$$

The tangent mapping C_*^{-1} transforms the frame Z_1, $\bar{Z}_1$ for $H\partial D$ into the frame $W_1, \bar{W}_1$ generating HS^3 in the following way. For any $f \in C^\infty(B)$, we have

$$\begin{aligned}
(C_*^{-1} Z_1) f(w_1, w_2) &= Z_1(f \circ C^{-1}) = Z_1\left(f\left(\frac{-2iz_1}{1-iz_2}, \frac{1+iz_2}{1-iz_2} \right)\right) \\
&= \frac{\partial}{\partial z_1} f\left(\frac{-2iz_1}{1-iz_2}, \frac{1+iz_2}{1-iz_2} \right) + 2i\bar{z}_1 \frac{\partial}{\partial z_2} f\left(\frac{-2iz_1}{1-iz_2}, \frac{1+iz_2}{1-iz_2} \right) \\
&= -\frac{\partial}{\partial w_1} \frac{2i}{1-iz_2} f + 2i\bar{z}_1 \left(\frac{\partial}{\partial w_1} \frac{2z_1}{(1-iz_2)^2} + \frac{\partial}{\partial w_2} \frac{2i}{(1-iz_2)^2} \right) f \\
&= -\frac{\partial}{\partial w_1}\left(-i(1+w_2) + \frac{i|w_1|^2(1+w_2)^2}{1+\bar{w}_2} \right) f + \frac{\partial}{\partial w_2} \left(\frac{2\bar{w}_1}{1+\bar{w}_2} \frac{2i(1+w_2)^2}{4} \right) f \\
&= -i \frac{(1+w_2)^2}{1+\bar{w}_2} \left(\bar{w}_2 \frac{\partial}{\partial w_1} - \bar{w}_1 \frac{\partial}{\partial w_2} \right) f = W_1 f. \qquad (1.19)
\end{aligned}$$

The pull-back of the contact form τ in 1.13 is the contact form σ in 1.10 on the sphere S^3:

$$\begin{aligned}
C^*\tau|_{S^3} &= \frac{i}{|1+w_2|^2}(-\bar{w}_1 dw_1 + w_1 d\bar{w}_1 - \bar{w}_2 dw_2 + w_2 d\bar{w}_2) \\
&\quad + \frac{i}{|1+w_2|^4}(1 - |w_1|^2 - |w_2|^2)(-dw_2 + d\bar{w}_2) = \frac{i}{|1+w_2|^2}\, \sigma|_{S^3}.
\end{aligned}$$

This yields that the generalized stereographic projection π is a contact transformation, since

$$\pi^*\omega = \frac{i}{|1+w_2|^2}\, \sigma =: \lambda\, \sigma. \qquad (1.20)$$

Moreover, the generalized stereographic projection π preserves the metric structure as we shall see in the next subsection.

1.4.1 Metric structure of S^3

Let us consider S^3 as a metric space. We can employ a homogeneous distance function, that has been used by Mostow (see [Mos73]).

$$d_S^2(u, w) = |1 - (u, w)| = ||u - w|^2 - \overline{(u, w)}|, \qquad (1.21)$$

where $(u, w) = u_1\bar{w}_1 + u_2\bar{w}_2$ is a complex scalar product.

This distance function is equivalent via generalized stereographic projection π to the Heisenberg distance (1.4) in the following sense. Take points $[0,0], [z,t] \in \mathbb{H}^1$ and consider the distance d_S between their images $e_2 = (0,1)$ and $w = (w_1, w_2) \in S^3$ under generalized stereographic projection $\pi : S^3 \setminus \{-e_2\} \to \mathbb{R}^3$ (1.18):

$$d_S^2(e_2, w) = |1 - (e_2, w)| = |1 - \bar{w_2}| = \left|1 - \frac{1 - |z|^2 + it}{1 + |z|^2 - it}\right|$$
$$= 2\frac{||z|^2 + it|}{|1 + |z|^2 + it|} = (|z|^4 + t^2)^{\frac{1}{2}} \frac{2}{|1 + |z|^2 + it|}.$$

Plagging in

$$|1 + w_2| = \left|1 + \frac{1 - |z|^2 + it}{1 + |z|^2 - it}\right| = \frac{2}{|1 + |z|^2 - it|} = \frac{2}{|1 + |z|^2 + it|},$$

we get

$$d_S^2(e_2, w) = |1 + w_2| \cdot |[z,t]|^2 = |1 + w_2| \cdot d_H^2(\pi^{-1}(e_2), \pi^{-1}(w)).$$

Vice versa, one can take points $e_2 = (0,1), w = (w_1, w_2)$ on the sphere S^3 and check the distance between their images $\xi = [0,0], \eta = [z,t]$ on $\mathbb{H}^1$ under the inverse map:

$$\pi(w_1, w_2) = \left(\frac{iw_1}{1 + w_2}, \operatorname{Re}\left[i\frac{1 - w_2}{1 + w_2}\right]\right).$$

With some arrangements

$$t = \operatorname{Re}\left[i\frac{1 - w_2}{1 + w_2}\right] = \frac{i}{2}\left[\frac{1 - w_2}{1 + w_2} - \frac{1 - \bar{w_2}}{1 + \bar{w_2}}\right] = \frac{w_2 - \bar{w_2}}{2i}\frac{2}{|1 + w_2|^2} = \frac{2\operatorname{Im} w_2}{|1 + w_2|^2},$$
$$t^2 = 4\frac{\operatorname{Im} {w_2}^2}{|1 + w_2|^4} = -\frac{(w_2 - \bar{w_2})^2}{|1 + w_2|^4},$$
$$|z|^4 = \frac{|w_1|^4}{|1 + w_2|^4} = \frac{(1 - |w_2|^2)^2}{|1 + w_2|^4},$$

we have

$$d_H^4(\xi, \eta) = |z|^4 + t^2 = \frac{(1 - |w_2|^2)^2}{|1 + w_2|^4} - \frac{(w_2 - \bar{w_2})^2}{|1 + w_2|^4} = \frac{1 + |w_2|^4 - w_2^2 - \bar{w_2}^2}{|1 + w_2|^4}$$
$$= \frac{1 + (w_2\bar{w_2})^2 - w_2^2 - \bar{w_2}^2}{|1 + w_2|^4} = \frac{(1 - w_2^2)(1 - \bar{w_2}^2)}{|1 + w_2|^4} = \frac{|1 + w_2|^2|1 - w_2|^2}{|1 + w_2|^4}$$
$$= \frac{|1 - w_2|^2}{|1 + w_2|^2} = \frac{d_S^4(\pi(p), \pi(q))}{|1 + w_2|^2}.$$

And we get the following correspondence between two distance functions:

$$d_H(\xi, \eta) = |\lambda|^{1/4}\, d_S(\pi(\xi), \pi(\eta)) \qquad \text{for every } \xi, \eta \in \mathbb{H}^1,$$

where $|\lambda| = |\lambda(w)| = |1 + w_2|^{-1}$ is the same coefficient that arises in (1.20).

Chapter 2

Flow of quasiconformal mappings

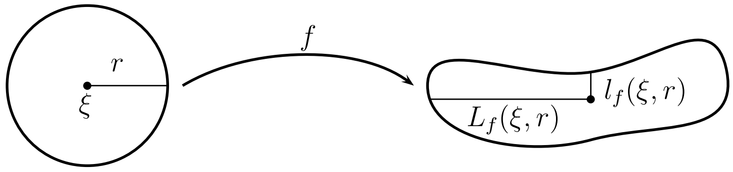
2.1 Quasiconformal mappings on $\mathbb{H}^1$

Quasiconformal (qc) mappings arise in complex function theory, for example in the study of multiply connected domains and in the Teichmüller problem. They are encountered also in the theory of partial differential equations as the univalent solutions of Beltrami systems. Finally, the study of such mappings is interesting in its own right, for though the theory usually parallels that of conformal mapping, there are striking instances where the analogy breaks down. Moreover, this study sometimes casts new light on the theory of conformal mapping, since often one must employ different methods when dealing with this more general class of mappings.

Classically, qc maps may be defined in metrical, geometrical or analytical way. With any homeomorphism $f : U \to \mathbb{H}^1$, where U is a domain in $\mathbb{H}^1$, we associate the functions

$$L_f(\xi, r) = \sup_{d_H(\xi,\eta)=r} d_H(f(\xi), f(\eta)),$$
$$l_f(\xi, r) = \inf_{d_H(\xi,\eta)=r} d_H(f(\xi), f(\eta)).$$

The functions are well defined if $d_H(\xi, \partial U) > r$.



In addition we set for every $\xi \in U$

$$H_f(\xi) = \limsup_{r\to 0} \frac{L_f(\xi, r)}{l_f(\xi, r)},$$

Definition 2.1. *A homeomorphism $f : U \to \mathbb{H}^1$ is a qc mapping, if H_f is uniformly bounded in the domain U. If in addition*

$$\operatorname*{ess\,sup}_{\xi \in U} |H_f(\xi)| \leqslant K$$

then f is called a K-qc mapping.

The following theorem can be found with detailed proof in [KR85, Th.8]:

Theorem 2.2. *Any smooth conformal map in $\mathbb{H}^1$ is a composition of maps of the following types:*

- *left translations,*
- *dilations,*
- *rotations about the t-axis*

$$\mathcal{R}(x, y, t) := (R(x, y), t)\,,\ R \in SO(2),$$

- *the Heisenberg inversion*

$$j_H(z, t) := \left(\frac{-z}{|z|^2 + 4it}, \frac{-t}{|z|^4 + 16t^2} \right).$$

We see that smooth conformal mappings on $\mathbb{H}^1$ are necessarily group actions.

2.2 Contact transformations vs. qc mappings.

Korányi and Reimann also demonstrated the existence of an extensive supply of nontrivial (e.g., nonconformal) quasiconformal mappings of $\mathbb{H}^1$ by characterizing the infinitesimal generators of one-parameter flows of smooth quasiconformal maps. We will sketch the construction in the following theorems, for the details of proofs reader is referred to [KR85].

Theorem 2.3. *A differentiable quasiconformal mapping with a non-singular derivative is a contact transformation.*

The idea of the proof is to show that the tangent mapping $f_* : T_\xi \mathbb{H}^1 \to T_{f(\xi)}\mathbb{H}^1$ maps the horizontal tangent plane

$$P_\xi = \{V \in T_\xi \mathbb{H}^1 : \omega(V) = 0\}$$

into the horizontal tangent plane $P_{f(\xi)} \subset T_{f(\xi)}\mathbb{H}^1$ for any $\xi \in \mathbb{H}^1$. In other words, differentiable qc mappings respect the geometry of $\mathbb{H}^1$. Moreover, enough differentiability and bounded growth give a converse of the last theorem.

Let

$$\lambda_1(\xi) = \sup_{\substack{V\in H_\xi\mathbb{H}^1\\ \|V\|_0=1}} \|f_* V\|_0,$$
$$\lambda_2(\xi) = \inf_{\substack{V\in H_\xi\mathbb{H}^1\\ \|V\|_0=1}} \|f_* V\|_0.$$

Theorem 2.4. *A contact transformation which is twice differentiable and satisfies*

$$\frac{\lambda_1}{\lambda_2}(\xi) \leqslant K$$

is a K-quasiconformal mapping.

Corollary 2.5. *The stereographic projection is a conformal mapping.*

The Corollary follows from the fact that the stereographic projection is a contact mapping with

$$\lambda_1 = \lambda_2 = \left|\frac{(1+w_2)^2}{1+\bar{w}_2}\right|^{-1} = |1+w_2|^{-1}.$$

The following theorem is a special case of a theorem due to P. Liebermann [Lib59]. The presented version is a specially adapted by Korányi and Reimann for the contact structure on the Heisenberg group.

Theorem 2.6. *For any $p \in C^1(\mathbb{H}^1)$, C^2-vector fields of the form*

$$V = -\frac{1}{4}(Yp)X + \frac{1}{4}(Xp)Y + pT, \tag{2.1}$$

generate local one-parameter group of contact transformations.

Conversely, every C^2-vector field V which generates a local one-parameter group of contact transformations is necessarily of this form with $p = \omega(V)$.

As a next step they exhibit an explicit estimate for the constant of quasiconformality of a one-parameter group of quasiconformal mappings generated by a C^2-vector field.

Theorem 2.7. *Assume that V is a C^2-vector field of the form* (2.1)*, which generates a one-parameter group f_s of contact tranformations. If*

$$|ZZp| \leqslant k \quad \textit{for} \quad Z = \frac{1}{2}(X - iY),$$

then f_s is K-quasiconformal with

$$\frac{1}{2}\left(K + \frac{1}{K}\right) \leqslant e^{\sqrt{2}k|s|}.$$

2.3 Liebermann theorem on S^3

Using the results of Korányi and Reimann sketched in the previous section we can describe the flow of qc mappings on the sphere S^3. Let us start with the C^2-vector field $v \in \mathfrak{X}(\mathbb{H}^1)$, which generates a local one-parameter group of contact transformations on $\mathbb{H}^1$ as in the Theorem 2.6, i.e.

$$v = -\frac{1}{4}(Yp)X + \frac{1}{4}(Xp)Y + pT,$$

where p is at least once differentiable real valued function on $\mathbb{H}^1$.

Using the fact that the generalized stereographic projection $\pi : S^3 \setminus \{-e_2\} \to \mathbb{H}^1$ is a contact transformation, we can pull back given vector field v to the sphere, so that it generates contact transformations as well. In order to do that, we will find first the images of the vector fields X, Y, T under the corresponding map between tangent spaces of $\mathbb{H}^1$ and S^3. We find $\pi_*^{-1}v$ as a linear combination of those images.

The map π was defined in (1.16) as the composition of the Cayley transform (1.15) $C : S^3 \setminus \{-e_2\} \to D$ restricted to $S^3 \setminus \{-e_2\}$ followed by the projection (1.17) $\Pi : D \to \partial D \cong \mathbb{H}^1$.

Thus the tangent map $(\pi^{-1})_* : T\mathbb{H}^1 \to TS^3$ is a composition of two linear transformations $(\pi^{-1})_* = (C^{-1})_* \circ (\Pi^{-1})_*$. Let us consider the tangent map $(\Pi^{-1})_* : T\mathbb{H}^1 \to T\partial D$. In the standars basis of $\mathbb{R}^4$ it is given by the matrix:

$$(\Pi^{-1})_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 2y & 1 \end{pmatrix}.$$

It pulls back the left-invariant basis X, Y, T as follows:

$$\tilde{X} = d\Pi^{-1}(X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 2y & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2y \\ 2x \end{pmatrix}$$

$$\tilde{Y} = d\Pi^{-1}(Y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 2y & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2x \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2x \\ 2y \end{pmatrix}$$

$$\tilde{T} = d\Pi^{-1}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 2y & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Recalling the complex notation

$$X = Z + \bar{Z}, \qquad Y = i(Z - \bar{Z}), \qquad T = \frac{i}{2}[\bar{Z}, Z],$$

we get the following vector fields on $T\partial D$:

$$\begin{aligned} Z_1 &= \frac{1}{2}(\tilde{X} - i\tilde{Y}) = \frac{\partial}{\partial z_1} + 2i\bar{z}_1\frac{\partial}{\partial z_2} \\ \bar{Z}_1 &= \frac{1}{2}(\tilde{X} + i\tilde{Y}) = \frac{\partial}{\partial \bar{z}_1} - 2iz_1\frac{\partial}{\partial \bar{z}_2} \\ [\bar{Z}_1, Z_1] &= 2i\tilde{T} = 2i(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial \bar{z}_2}). \end{aligned}$$

The vector field v is mapped onto the vector field $\tilde{v} \in T\partial D$

$$\begin{aligned} \tilde{v} = (\Pi^{-1})_*(v) = -\frac{1}{4}(\tilde{Yp}) \left\{ \frac{\partial}{\partial x_1} + 2y_1\frac{\partial}{\partial x_2} + 2x_1\frac{\partial}{\partial y_2} \right\} \\ + \frac{1}{4}(\tilde{Xp}) \left\{ \frac{\partial}{\partial y_1} - 2x_1\frac{\partial}{\partial x_2} + 2y_1\frac{\partial}{\partial y_2} \right\} + \tilde{p}\frac{\partial}{\partial x_2}. \end{aligned}$$

where the coefficients $\tilde{Yp}, \tilde{Xp}, \tilde{p} : D \to \mathbb{R}$ are extensions of $Yp, Xp, p : \partial D \to \mathbb{R}$.

We already saw that under the tangent map $(C^{-1})_*$ the vector fields Z_1 and $\bar{Z}_1$ on $T\partial D$ are mapped onto the vector fields W_1 and $\bar{W}_1$, restricted to S^3. By linearity we find $(C^{-1})_*(\tilde{X})$, $(C^{-1})_*(\tilde{Y})$ as

$$\begin{aligned} \hat{X} &= (C^{-1})_*(\tilde{X}) = (C^{-1})_*(Z_1 + \bar{Z}_1) = W_1 + \bar{W}_1 \\ \hat{Y} &= (C^{-1})_*(\tilde{Y}) = i(C^{-1})_*(Z_1 - \bar{Z}_1) = i(W_1 - \bar{W}_1) \end{aligned}$$

However, since the map $(C^{-1})_*$ is not a Lie algebra homeomorphism, it does not preserve the brackets and

$$(C^{-1})_*[Z_1, \bar{Z}_1] \neq [(C^{-1})_*(Z_1), (C^{-1})_*(\bar{Z}_1)].$$

By this reason we have to calculate the matrix of $(C^{-1})_*$ explicitely.

For the generic points $(z_1, z_2) = (x_1, y_1, x_2, y_2) \in D$ and $(w_1, w_2) = (u_1, v_1, u_2, v_2) \in B$, the inverse of the Cayley transform (1.15) in real coordinates reads

$$C^{-1}(x_1, y_1, x_2, y_2) = r(2(y_1(1 + y_2) + x_1x_2), 2(y_1x_2 - x_1(1 + y_2)), 1 - x_2^2 - y_2^2, 2x_2),$$

where $r = r(x_2, y_2) = x_2^2 + (1 + y_2)^2$

The differential map $(C^{-1})_*$ is given by the following matrix:

$$\frac{2}{r^2} \begin{pmatrix} x_2 r & (1+y_2)r & -2y_1x_2(1+y_2) + x_1\left((1+y_2)^2 - x_2^2\right) & -2x_1x_2(1+y_2) - y_1\left((1+y_2)^2 - x_2^2\right) \\ -(1+y_2)r & x_2 r & 2x_1x_2(1+y_2) + y_1\left((1+y_2)^2 - x_2^2\right) & -2y_1x_2(1+y_2) + x_1\left((1+y_2)^2 - x_2^2\right) \\ 0 & 0 & -2x_2(1+y_2) & -\left((1+y_2)^2 - x_2^2\right) \\ 0 & 0 & (1+y_2)^2 - x_2^2 & -2x_2(1+y_2) \end{pmatrix}.$$

Now we can find

$$\hat{T} = (C^{-1})_*(\tilde{T}) = (C^{-1})_* \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{2}{r} \begin{pmatrix} -2y_1x_2(1+y_2) + x_1\left((1+y_2)^2 - x_2^2\right) \\ 2x_1x_2(1+y_2) + y_1\left((1+y_2)^2 - x_2^2\right) \\ -2x_2(1+y_2) \\ (1+y_2)^2 - x_2^2 \end{pmatrix}$$

Finally, we can write

$$\hat{v} = (C^{-1})_*(\tilde{v}) = \frac{1}{4}(-\hat{Y}p + i\hat{X}p)W_1 - \frac{1}{4}(\hat{Y}p + i\hat{X}p)\bar{W}_1 + \hat{p}\hat{T}$$

where $\hat{X}p = \tilde{X}p \circ C^{-1}$, $\hat{Y}p = \tilde{Y}p \circ C^{-1}$, $\hat{p} = \tilde{p} \circ C^{-1}$ are real-valued functions. The resulting vector field $\hat{v} \in TS^3$ generates a local one-parametric flow of contact maps on S^3.

Chapter 3

Modulus of a family of curves on polarizable groups

3.1 Modulus of a family of curves on $\mathbb{H}^1$.

The notion of extremal length was originally introduced in early 1930's by A. Beuring (and published in a joint paper with L. Ahlfors in 1950) as a conformal invariant, in order to investigate function theoretic properties of domains of the complex plane. Later this notion has been generalized and utilized in various problems in function theory.

In [KR87] A. Korányi and H. M. Reimann showed in a simple and elegant way how to prove some of known explicit formulas of potential theory in the Heisenberg group. Particularly they calculated the conformal modulus of a spherical ring in $\mathbb{H}^1$. We will employ their method and generalize this result to the p-modulus of spherical ring in $\mathbb{H}^1$.

We recall that if $\gamma : [a, b] \to \mathbb{H}^1$ is an absolutely continuous curve in $\mathbb{H}^1$ that is not horizontal for some open subinterval $U \subset [a, b]$, then it is nonrectifiable. Thus, when computing the p-modulus of a family of curves we can restrict ourselves to horizontal curves, since a p-modulus of a family of non-rectifiable curves vanishes. If γ is parametrized by arc length then $\|\gamma'(u)\|_0 = 1$ a.e. in $[a, b]$.

Definition 3.1. *The line integral of a non-negative, Borel measurable function f can be defined for continuous rectifiable curves $\gamma : [a, b] \to \mathbb{H}^1$ by*

$$\int_\gamma f = \int_a^b f(\gamma^0(s))\, ds,$$

where γ^0 is the reparametrization of γ in terms of the arc length parameter. If γ is a C^1-curve we also have

$$\int_\gamma f = \int_a^b f(\gamma(u))\, \|\gamma'(u)\|_0\, du.$$

Given a family of horizontal curves Γ, we denote by $F(\Gamma)$ the class of non-negative

Borel-measurable functions σ such that for all $\gamma \in \Gamma$

$$\int_\gamma \sigma \geqslant 1. \tag{3.1}$$

The functions $\sigma \in F(\Gamma)$ are usually called admissible functions.

Definition 3.2. *Given $p > 1$, we set the p-modulus of Γ to be the value*

$$M_p(\Gamma) = \inf_{\sigma \in F(\Gamma)} \int_{\mathbb{H}^1} \sigma^p dg.$$

Given positive real numbers a and b, where $a < b$, we denote by R_{ab} the ring given by

$$R_{ab} := \{\xi \in \mathbb{H}^1 : a \leqslant N(\xi) \leqslant b\},$$

where $N(\xi) = N(z,t) = (|z|^4 + t^2)^{1/4}$ is the homogeneous norm on $\mathbb{H}^1$. In what follows, Γ_{ab} denotes the family of all horizontal curves joining the sphere $S_a = \{N(\xi) = a\}$ to the sphere S_b in R_{ab}.

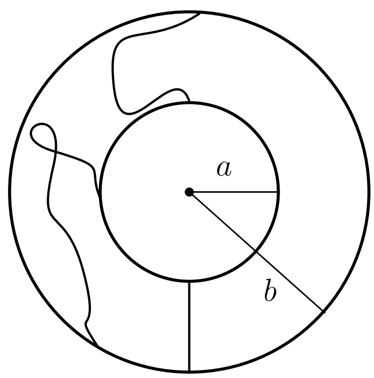


Figure 3.1: The family of curves Γ_{ab}

Our goal is to compute the exact value of the p-modulus $M_p(\Gamma_{ab})$.

Theorem 3.3. *The value of the p-modulus of the family of curves Γ_{ab} is given by*

$$M_p(\Gamma_{ab}) = \begin{cases} C(p)\left(\dfrac{p-4}{p-1}\right)^{p-1}\left(b^{\frac{p-4}{p-1}} - a^{\frac{p-4}{p-1}}\right)^{1-p}, & p \neq 4 \\ \pi^2\left(\log \dfrac{b}{a}\right)^{-3}, & p = 4, \end{cases} \tag{3.2}$$

where the coefficient $C(p)$ is given by

$$C(p) := \frac{2\pi\sqrt{\pi}\,\Gamma\left(\frac{p}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{p}{4} + 1\right)}.$$

In order to calculate the integral over the ring R_{ab} of an arbitrary admissible function, we need to find convenient coordinates. In their joint work [FS82] Folland and Stein developed 'polar coordinates' formula valid for any homogeneous group G, which is a connected and simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ is endowed by dilations $\{\delta_s\}$. In application to $\mathbb{H}^1$ this formula reads as follows:

Proposition 3.4. *Let $S = \{\eta \in \mathbb{H}^1 : N(\eta) = 1\}$ be a unit sphere in $\mathbb{H}^1$. Then there is a unique Radon measure v on S such that for all $u \in L^1(\mathbb{H}^1)$,*

$$\int_{\mathbb{H}^1} u\, dg = \int_0^\infty \int_S u(\delta_s(\eta)) s^3\, dv(\eta) ds.$$

However this formula is of no use for our calculations since the curves $\delta_s : \mathbb{R} \supset I \to \mathbb{H}^1$ are not horizontal by (1.6), and consequently, not rectifiable. It turns out that the extremal family of curves for a p-modulus is a family of horizontal curves normal to each sphere $S_r\,(r > 0)$. Such curves are integral curves for the vector field $\nabla_0 N$. The construction of such a family is given by the following lemma:

Lemma 3.5. *For every continuous function f, we have the decomposition*

$$\int_{R_{ab}} f\, dg = \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\alpha \int_a^b f(\gamma_\xi(r))\, r^3\, dr, \tag{3.3}$$

where γ_ξ are horizontal curves normal to each sphere $S_r\,(r > 0)$.

Proof of Lemma 3.5. Elementary calculations show that

$$\nabla_0 N = \left(\left(\frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t} \right) N, \left(\frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t} \right) N \right) = N^{-3}(|z|^2 x + yt, |z|^2 y - xt),$$

and

$$\|\nabla_0 N\|_0^2 = \langle \nabla_0 N, \nabla_0 N \rangle = \frac{|z|^2}{N^2}.$$

Therefore, the normalized vector field

$$V = \frac{N^2}{|z|^2} \nabla_0 N$$

is normal to each S_r and satisfies $\langle V(\eta), \nabla_0 N(\eta) \rangle = 1$, whenever

$$\eta \notin \mathcal{Z} = \{\xi \in \mathbb{H}^1 : \nabla_0 N(\xi) = 0\}.$$

The set $\mathcal{Z}$ is called the characteristic set of N. We are looking for the family of horizontal curves normal to each sphere $S_r\,(r > 0)$ as a solution $\gamma_\xi\colon [0,h] \to \mathbb{H}^1$ of the following Cauchy problem:

$$\begin{cases} \dot{\gamma}_\xi(s) = V(\gamma_\xi(s)) \\ \gamma_\xi(h) = \xi. \end{cases}$$

We introduce the polar change of coordinates (ρ, θ, t) on $\mathbb{H}^1$ by $z = \rho e^{i\theta}$ in the z-plane, which is a diffeomorphism in $\mathbb{H}^1 \backslash \mathcal{Z}$. Thus we can rewrite the vector field V as

$$V = \frac{1}{N}\left\{\rho\frac{\partial}{\partial\rho} + 2t\frac{\partial}{\partial t} - \frac{t}{\rho^2}\frac{\partial}{\partial\theta}\right\}.$$

We parametrize the sphere $S_h = \{N(\xi) = h\}$ by

$$S_h = \left\{[he^{i\phi}\cos^{\frac{1}{2}}\alpha, h^2\sin\alpha], -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}, 0 \leqslant \phi \leqslant 2\pi\right\}.$$

Let γ_ξ denote the integral curve of V such that $\gamma_\xi(h) = \xi$, where $\xi = \xi(h, \alpha, \phi) \in \mathbb{H}^1$. Observe that if γ is an integral curve of V and if $N(\gamma(h)) = h$ for some h, then $N(\gamma(r)) = r$ for all $r > 0$. This property holds because

$$\frac{\partial}{\partial r}N(\gamma(r)) = \langle\nabla_0 N(\gamma(r)), \dot\gamma(r)\rangle = \langle\nabla_0 N, V\rangle = 1.$$

Consequently, along an integral curve of V, the term $N(\gamma(r))$ equals r. For future considerations, it is convenient to have an explicit expression for γ_ξ, which can be found by solving the system of differential equations:

$$\dot\gamma_\xi = \dot\rho\frac{\partial}{\partial\rho} + \dot t\frac{\partial}{\partial t} + \dot\theta\frac{\partial}{\partial\theta} = \frac{1}{r}\left\{\rho\frac{\partial}{\partial\rho} + 2t\frac{\partial}{\partial t} - \frac{t}{\rho^2}\frac{\partial}{\partial\theta}\right\} = V(\gamma_\xi).$$

Namely,

$$\begin{cases} \dot\rho = \dfrac{\rho}{r} \\ \dot t = \dfrac{2t}{r} \\ \dot\theta = -\dfrac{t}{r\rho^2} \end{cases} \implies \begin{cases} \rho = k_1 r \\ t = k_2 r^2 \\ \theta = -\tan\alpha\log r + k_3, \end{cases}$$

where the integration constants k_1, k_2, k_3 can be found from the initial data $\gamma_\xi(h) = \xi$. Finally, the curve γ_ξ is given by

$$\begin{cases} \rho(r) = r\cos^{\frac{1}{2}}\alpha, \\ t(r) = r^2\sin\alpha, \\ \theta(r) = \phi - \tan\alpha\log\dfrac{r}{h}. \end{cases} \tag{3.4}$$

The variables (r, α, ϕ) can be regarded as spherical coordinates of the point $\gamma_\xi(r)$. Since the curves γ_ξ are normal to the spheres S_h, this is the natural analogue in $\mathbb{H}^1$ of the Euclidean spherical coordinates.

We find the expression for the volume element dg in terms of the new coordinate system. Since $\frac{\partial\rho}{\partial\theta} = \frac{\partial t}{\partial\theta} = 0$ and $\frac{\partial\theta}{\partial\phi} = 1$, the Jacobian is given by

$$\frac{\partial(\rho,t,\theta)}{\partial(r,\alpha,\phi)}=\frac{\partial(\rho,t)}{\partial(r,\alpha)}=\left|\begin{pmatrix}\cos^{\frac12}\alpha & -\frac{r}{2}\cos^{-\frac12}\alpha\sin\alpha\\ 2r\sin\alpha & r^2\cos\alpha\end{pmatrix}\right|=r^2\cos^{-\frac12}\alpha=\frac{r^3}{\rho}.$$

The equality $dg=\rho\,d\rho\,dt\,d\theta$, yields $dg=r^3\,dr\,d\alpha\,d\phi$. Thus, for every continuous function f, we have

$$\int_{R_{ab}} f\,dg=\int_0^{2\pi}d\phi\int_{-\pi/2}^{\pi/2}d\alpha\int_a^b f(\gamma_\xi(r))\,r^3dr \tag{3.5}$$

and Lemma 3.5 is proved. □

The horizontal norm of $\dot\gamma$ for a horizontal C^1-curve γ in terms of the spherical coordinate system is given by

$$\begin{aligned}\|\dot\gamma\|_0&=\left\{\left(\frac{d}{dr}(\rho(r)\cos\theta(r))\right)^2+\left(\frac{d}{dr}(\rho(r)\sin\theta(r))\right)^2\right\}^{\frac12}\\ &=\Big((\dot\rho\cos\theta-\dot\theta\rho\sin\theta)^2+(\dot\rho\sin\theta+\dot\theta\rho\cos\theta)^2\Big)^{\frac12}\\ &=(\dot\rho^2+\dot\theta^2\rho^2)^{\frac12}=\cos^{\frac12}\alpha(1+\tan^2\alpha)^{\frac12}=\cos^{-\frac12}\alpha.\end{aligned}$$

Proof of Theorem 3.3. Let us estimate the value of $M_p(\Gamma_{ab})$. For all admissible functions $\sigma\in F(\Gamma_{ab})$ we have,

$$1\leqslant\left(\int_{\gamma_\xi}\sigma\right)^p=\left(\int_a^b\sigma(\gamma_\xi(r))\cos^{-\frac12}\alpha\,dr\right)^p. \tag{3.6}$$

We estimate (3.6), using Hölder's inequality:

$$\begin{aligned}\cos^{\frac{p}{2}}\alpha\leqslant\left(\int_a^b\left(\sigma r^{\frac3p}\right)r^{-\frac3p}dr\right)^p&\leqslant\left[\left(\int_a^b\sigma^pr^3dr\right)^{\frac1p}\left(\int_a^b r^{-\frac{3}{p-1}}dr\right)^{\frac{p-1}{p}}\right]^p\\ &=\left(\int_a^b\sigma^pr^3dr\right)\left(\int_a^b r^{-\frac{3}{p-1}}dr\right)^{p-1}.\end{aligned}$$

The conformal case $p=4$ has been considered in details in [KR87]. We separate this case from the general one, since the calculations are different.

Let $C_{ab}(p)$ denote the value of the integral below:

$$C_{ab}(p):=\begin{cases}\displaystyle\int_a^b r^{-\frac{3}{p-1}}\,dr=\frac{p-1}{p-4}\left(b^{\frac{p-4}{p-1}}-a^{\frac{p-4}{p-1}}\right), & p\neq4\\ \displaystyle\int_a^b\frac1r\,dr=\log\frac{b}{a}, & p=4,\end{cases}$$

therefore

$$\int_a^b \sigma^p(\gamma_\xi(r))\, r^3 dr \geqslant C_{ab}(p)^{1-p} \cos^{\frac{p}{2}} \alpha,$$

and for all $p > 1$

$$\int_{R_{ab}} \sigma^p \, dg = \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\alpha \int_a^b \sigma^p(\gamma_\xi(r)) r^3 dr \geqslant$$

$$C_{ab}(p)^{1-p} \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} \cos^{\frac{p}{2}} \alpha \, d\alpha = 2\pi \, C_{ab}(p)^{1-p} \int_{-\pi/2}^{\pi/2} \cos^{\frac{p}{2}} \alpha \, d\alpha.$$

Let $C(p)$ denote the value of the integral on the unit sphere

$$C(p) := \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} \cos^{\frac{p}{2}} \alpha \, d\alpha = \frac{2\pi\sqrt{\pi}\, \Gamma\left(\frac{p}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{p}{4} + 1\right)}.$$

It follows that

$$M_p(\Gamma_{ab}) \geqslant C_{ab}(p)^{1-p} C(p). \tag{3.7}$$

Let us exhibit an admissible function on which the value of the estimate above is attained. In non-conformal case $p \neq 4$ let

$$\sigma_0 = \left(b^{\tau+1} - a^{\tau+1}\right)^{-1} \|\nabla_0 \left(N^{\tau+1}\right)\|_0,$$

where $\tau + 1 = \frac{p-4}{p-1}$. For $p = 4$ let

$$\sigma_0 = \left(\log \frac{b}{a}\right)^{-1} \|\nabla_0 \, \log N\|_0 = C_{ab}(p)^{-1} N^{-1} \|\nabla_0 N\|_0.$$

Calculating $\|\nabla_0 \left(N^{\tau+1}\right)\|_0$ explicitly, we see that

$$\left(b^{\tau+1} - a^{\tau+1}\right)^{-1} \|\nabla_0 N^{\tau+1}\|_0 = \left(b^{\tau+1} - a^{\tau+1}\right)^{-1} (\tau+1) N^\tau \|\nabla_0 N\|_0 = C_{ab}(p)^{-1} N^\tau \|\nabla_0 N\|_0.$$

By using the parametrization of $\gamma_\xi(r)$ by spherical coordinates given in (3.4), we have for all $p > 1$ the equality

$$\sigma_0(\gamma_\xi(r)) = C_{ab}(p)^{-1}\, r^\tau \left(\frac{r \cos^{\frac{1}{2}} \alpha}{r}\right) = \cos^{\frac{1}{2}} \alpha \, C_{ab}(p)^{-1} r^{-\frac{3}{p-1}},$$

Using Lemma 3.5, we can calculate the integral

$$\begin{aligned}
\int_{R_{ab}} \sigma_0^p \, dg &= \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\alpha \int_a^b \sigma_0^p(\gamma_\xi(r)) r^3 \, dr \\
&= \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\alpha \int_a^b \cos^{\frac{p}{2}} \alpha \, C_{ab}(p)^{-p} r^{-\frac{3}{p-1}} \, dr \\
&= C_{ab}(p)^{-p} \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} \cos^{\frac{p}{2}} \alpha \, d\alpha \int_a^b r^{-\frac{3}{p-1}} \, dr = C_{ab}(p)^{1-p} C(p).
\end{aligned}$$

Thus, if σ_0 is an admissible function, we have immediately that

$$M_p(\Gamma_{ab}) \leqslant \int_{R_{ab}} \sigma_0^p \, dg = C_{ab}(p)^{1-p} C(p). \tag{3.8}$$

Combining the estimates (3.7) and (3.8), we get the exact value of the p-modulus of the family of curves Γ_{ab}.

$$M_p(\Gamma_{ab}) = C_{ab}(p)^{1-p} C(p) = \begin{cases} C(p) \left(\dfrac{p-4}{p-1}\right)^{p-1} \left(b^{\frac{p-4}{p-1}} - a^{\frac{p-4}{p-1}}\right)^{1-p}, & p \neq 4 \\ C(4) \left(\log \dfrac{b}{a}\right)^{-3} = \pi^2 \left(\log \dfrac{b}{a}\right)^{-3}, & p = 4. \end{cases}$$

The function σ_0 is admissible by Lemma 3.6. This finishes the proof. □

Lemma 3.6. *For all $p > 1$ the function σ_0 is admissible for Γ_{ab}.*

Proof. Let $\gamma(s)$, $s \in [0, l]$, be any curve in Γ_{ab} parametrized by arc length. Recall that $a = N(\gamma(0))$ and $b = N(\gamma(l))$ since curves in Γ_{ab} connect the spheres S_a and S_b. Then, by Schwarz inequality

$$\langle \nabla_0 N, \dot{\gamma}(s) \rangle_0 \leqslant \|\nabla_0 N\|_0$$

for a.a. $s \in [0, l]$. It follows that, for $p \neq 4$

$$\begin{aligned} \int_\gamma \sigma_0 = \int_0^l \sigma_0(\gamma(s)) ds &= \left(b^{\tau+1} - a^{\tau+1}\right)^{-1} \int_0^l (\tau+1) N^\tau(\gamma(s)) \|\nabla_0 N(\gamma(s))\|_0 \, ds \\ &\geqslant \left(b^{\tau+1} - a^{\tau+1}\right)^{-1} \int_0^l (\tau+1) N^\tau(\gamma(s)) \langle \nabla_0 N(\gamma(s)), \dot{\gamma}(s) \rangle ds \\ &= \left(b^{\tau+1} - a^{\tau+1}\right)^{-1} \int_0^l \frac{d}{ds} N^{\tau+1}(\gamma(s)) ds. \end{aligned}$$

Since $N^{\tau+1} \circ \gamma$ is absolutely continuous, by Fundamental Theorem of Calculus

$$\int_\gamma \sigma_0 \geqslant \left(b^{\tau+1} - a^{\tau+1}\right)^{-1} \left(N^{\tau+1}(\gamma(l)) - N^{\tau+1}(\gamma(0))\right) = 1.$$

The same arguments are valid if $p = 4$, see [KR87] for more details. □

3.2 Modulus of a family of curves on H-type groups

In ([CDG96]) Capogna, Danielli and Garofalo obtained the result, generalizing Theorem 3.3 in the setting of H-type groups. They calculated the value of p-capacity of the spherical ring on H-type group. It is known that in this particular case its value coincides with the value of the p-modulus of a family of curves contained in the ring, see [Mar03].

Our goal is to extend this result for more general setting of so-called polarizable groups by applying techniques developed by Korányi and Reimann in [KR87] and improved later by Balogh in [BT02].

Definition 3.7. *A Carnot group G is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ nilpotent and has a stratification*

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_l,$$

where $[V_1, V_j] = V_{j+1}$ for all $j \in \mathbb{N}$ with $V_j = \{0\}$ whenever $j > l$. The positive integer l is called the step of the group.

Throughout this section we assume the following notation. Let G be a Carnot group of step l with Lie algebra $\mathfrak{g}$. Let $X_1, \cdots, X_k$ be an orthonormal basis of V_1 with respect to the inner product $\langle\,,\rangle_0$ in V_1. Then the horizontal tangent bundle relative to this basis has fibers $H_xG = \operatorname{span}\{X_1(x), \cdots, X_k(x)\}$.

As a simply connected nilpotent group, admitting dilations δ_s, G is globally diffeomorphic to $\mathfrak{g} \cong \mathbb{R}^m$, $m = \sum_{i=1}^{l} i \dim V_i$, via the exponential map. We identify an element $g \in G$ with $(x_1, \ldots, x_k, t_{k+1}, \ldots, t_m) \in \mathbb{R}^m$ by the formula

$$g = \exp\left(\sum_{i=1}^{k} x_i X_i + \sum_{i=k+1}^{m} t_i T_i\right),$$

where $T_{k+1}, \cdots, T_m$ denotes a set of nonhorizontal vectors, extending $X_1, \cdots, X_k$ to a basis of $\mathfrak{g}$. The Haar measure on G is induced by the exponential mapping from the Lebesgue measure on $\mathfrak{g} \cong \mathbb{R}^m$ and we denote it by $d\mathbf{g}$.

Recall that given a domain $U \subset G$, the function $f \in C^2(U)$ is called p-harmonic if it satisfies p-sub-Laplace equation in U:

$$\Delta_{0,p} := \sum_{i=1}^{k} X_i \left(\|\nabla_0 f\|_0^{p-2} X_i f\right) = 0,$$

or, correspondingly, ∞-harmonic if it satisfies ∞-sub-Laplace equation in U:

$$\Delta_{0,\infty} := \frac{1}{2}\langle \nabla_0 \|\nabla_0 f\|_0^2, \nabla_0 f\rangle_0 = 0.$$

By a result of Folland [Fol75, Th.2.1] in any Carnot group G there exists a unique fundamental solution u_2 to the Kohn sub-Laplacian $\Delta_{0,2}$ which is smooth away from zero and homogeneous of degree $2-Q:\ u_2 \circ \delta_s = s^{2-Q} u_2$.

Definition 3.8. *We say that a Carnot group G is polarizable if the fundamental solution u_2 of the Kohn sub-Laplacian $\Delta_{0,2}$ has the property that the homogeneous norm $N_G = u_2^{\frac{1}{2-Q}}$ associated to u_2 is ∞-harmonic away from zero in G.*

The examples of polarizable groups are $\mathbb{R}^m$ and n-th Heisenberg group $\mathbb{H}^n$. The main result of Balogh in [BT02] is that in any polarazible group it is possible to carry out the construction of 'spherical coordinates' in the same manner as it has been done by Korányi and Reimann in $\mathbb{H}^1$.

We denote by $R_{ab}(G)$ the ring defined by $\{g \in G : 0 < a \leqslant N_G(g) \leqslant b < \infty\}$, where N_G is the homogeneous norm associated to Folland's solution u_2. In the following $\Gamma_{ab}(G)$ will denote the family of all curves joining the sphere $S_a = \{N_G(g) = a\}$ to the sphere S_b and $S = S_1$ will stand for the unit sphere.

As before, the idea is to construct family of horizontal curves $\phi(\cdot, \xi) : (0, \infty) \to G$, where $\xi \in S$, and a positive Radon measure dv on S so that the integration formula

$$\int_G f(g) d\mathbf{g} = \int_{S \setminus \mathcal{Z}} \int_0^\infty f(\phi(s, \xi)) s^{Q-1} ds\, dv(\xi) \tag{3.9}$$

is valid for all $f \in L^1(G)$, where $d\mathbf{g}$ denotes the Haar measure on G and

$$\mathcal{Z} = \{0\} \cup \{g \in G \setminus \{0\} : \nabla_0 N(g) = 0\}$$

is a characteristic set of N_G.

Theorem 3.9. *Let G be a polarazible Carnot group of the Hausdorff dimension Q with the homogeneous norm N_G associated to Folland's solution of the Kohn sub-Laplacian. Let $p > 1$, then*

$$M_p(\Gamma_{ab}(G)) = \begin{cases} C_p(G) \Big(\dfrac{|p-Q|}{p-1} \Big)^{p-1} \Big| (b^{\frac{p-Q}{p-1}} - a^{\frac{p-Q}{p-1}}) \Big|^{1-p}, & p \neq Q \\ C_p(G) \Big(\log \dfrac{b}{a} \Big)^{1-Q}, & p = Q, \end{cases}$$

where coefficient $C_p(G)$ is given by an integral

$$C_p(G) = \int_{S \setminus \mathcal{Z}} \left(\frac{\|\nabla_0 N_G(\phi(s,\xi))\|_0}{N_G(\xi)} \right)^p dv(\xi).$$

The radial flow we are looking for is the solution $\phi(\cdot, \xi) \colon [0, 1] \to \mathbb{G}$ of the following Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial s} \phi(s, \xi) = \dfrac{N_G(\phi(s,\xi))}{s} \cdot \dfrac{\nabla_0 N_G(\phi(s,\xi))}{\|\nabla_0 N_G(\phi(s,\xi))\|_0^2}, \\ \phi(1, \xi) = \xi, \end{cases} \tag{3.10}$$

which is the generalization of the Cauchy problem that we have solved on $\mathbb{H}^1$. We construct the flow ϕ so that it has the required properties:

Lemma 3.10. *The flow ϕ satisfies the following properties:*

i. $N_G(\phi(s, \xi)) = s N_G(\xi)$ *for* $s > 0$ *and* $\xi \in G \setminus \mathcal{Z}$;

ii. $\|(\partial\phi/\partial s)\|_0$ *is independent of* s, *i.e.*

$$\|(\partial\phi/\partial s)\|_0 = \frac{N_G(\xi)}{\|\nabla_0 N_G(\phi(s,\xi))\|_0} =: \lambda(\xi)^{-1},$$

for nonzero real-valued function λ *on* $G \setminus \mathcal{Z}$;

iii. $\det D_x\phi(s,\xi) = s^Q$ *for* $s>0$ *and* $\xi \in G\backslash\mathcal{Z}$ *where* $D_\xi\phi$ *denotes the differential of the map* $\phi(s,\cdot): G\backslash\mathcal{Z} \to G\backslash\mathcal{Z}$*, so that* $\phi(s,\cdot)$ *understood as a mapping from* $\mathbb{R}^m$ *to* $\mathbb{R}^m$*.*

Proof of Theorem 3.9. Let us use the spherical decomposition (3.9) in order to calculate the p-modulus of $\Gamma_{ab}(G)$. For all admissible functions $\sigma \in F(\Gamma_{ab}(G))$ we have

$$1 \leqslant \left(\int_{\phi_\xi} \sigma\right)^p = \left(\int_a^b \sigma(\phi_\xi(s))\lambda(\xi)^{-1} ds\right)^p.$$

Hölder's inequality implies

$$\lambda(\xi)^p \leqslant \left(\int_a^b \left(\sigma s^{\frac{Q-1}{p}}\right) s^{-\frac{Q-1}{p}} ds\right)^p \leqslant \left[\left(\int_a^b \sigma^p s^{Q-1} ds\right)^{\frac{1}{p}} \left(\int_a^b s^{\frac{1-Q}{p-1}} ds\right)^{\frac{p-1}{p}}\right]^p$$
$$= \left(\int_a^b \sigma^p s^{Q-1} ds\right) \left(\int_a^b s^{\frac{1-Q}{p-1}} ds\right)^{p-1}.$$

Just as in case of the first Heisenberg group, the calculations split into conformal case $p=Q$ and non-conformal case $p \neq Q$. The logic is the same as before, that is why we will consider in detail non-conformal case only.

Let $C_{ab}(G)$ denote the value of the integral on the right

$$C_{ab}(G) := \int_a^b s^{\frac{1-Q}{p-1}}\, ds = \frac{p-1}{p-Q}\, s^{\frac{p-Q}{p-1}}\Big|_a^b = \frac{p-1}{p-Q}\left(b^{\frac{p-Q}{p-1}} - a^{\frac{p-Q}{p-1}}\right),$$

Therefore

$$\int_a^b \sigma^p(\phi_\xi(s)) s^{Q-1} ds \geqslant C_{ab}(G)^{1-p}\lambda(\xi)^p,$$

and,

$$\int_{R_{ab}(G)} \sigma^p(g)\, d\mathbf{g} = \int_{S\backslash\mathcal{Z}}\int_a^b \sigma^p(\phi_\xi(s)) s^{Q-1} ds\, dv(\xi) \geqslant C_{ab}(G)^{1-p}\int_{S\backslash\mathcal{Z}} \lambda(\xi)^p\, dv(\xi).$$

Let $C_p(G)$ denote the value of the integral

$$C_p(G) := \int_{S\backslash\mathcal{Z}} \lambda(\xi)^p\, dv(\xi).$$

It follows that

$$M_p(\Gamma_{ab}(G)) \geqslant C_{ab}(G)^{1-p} C_p(G). \tag{3.11}$$

The extremal function on which this estimate is attained is given by

$$\sigma_0 = \left(b^{\tau+1} - a^{\tau+1}\right)^{-1} \|\nabla_0\left(N_G^{\tau+1}\right)\|_0,$$

with $\tau+1=\frac{p-Q}{p-1}$.

Considering σ_0 along the radial curves flow $\phi(s,\xi)=\phi_\xi(s)$ yields:

$$\sigma_0(\phi_\xi(s))=C_{ab}(G)^{-1}\,s^\tau\lambda(\xi).$$

Using the spherical decomposition (3.9) we can calculate the integral

$$\begin{aligned}\int_{R_{ab}(G)}\sigma_0^p\,d\mathbf{g}&=\int_{S\setminus\mathcal{Z}}\int_a^b\sigma_0^p(\phi_\xi(s))s^{Q-1}ds\,dv(\xi)\\&=C_{ab}(G)^{-p}\int_{S\setminus\mathcal{Z}}\int_a^b s^{\frac{p(1-Q)}{p-1}}\lambda(\xi)^p s^{Q-1}ds\,dv(\xi)\\&=C_{ab}(G)^{-p}\int_{S\setminus\mathcal{Z}}\lambda(\xi)^p dv(\xi)\int_a^b s^{\frac{1-Q}{p-1}}ds=C_{ab}(G)^{1-p}C_p(G).\end{aligned}$$

The function σ_0 is admissible by the same argument as in Lemma 3.6, and we have

$$M_p(\Gamma_{ab}(G))\leqslant\int_{R_{ab}(G)}\sigma_0^p\,d\mathbf{g}=C_{ab}(G)^{1-p}C_p(G).$$

Finally, the value of the p-modulus for $\Gamma_{ab}(G)$ in non-conformal case is given by

$$M_p(\Gamma_{ab}(G))=C_{ab}(G)^{1-p}C_p(G)=C_p(G)\Big(\frac{|p-Q|}{p-1}\Big)^{p-1}\Big|(b^{\frac{p-Q}{p-1}}-a^{\frac{p-Q}{p-1}})\Big|^{1-p}\tag{3.12}$$

This finishes the proof. $\square$

Observe that the only information one needs to carry on the construction of spherical coordinates on a Carnot group is existence of ∞-harmonic homogeneous norm N_G. We know that the fundamental solution of 2-sub-Laplacian always exists, but it is not necessarily ∞-harmonic. This is the case for the Heisenberg group. There is a larger class of Carnot groups for which this is true, these are so-called H-type groups.

Definition 3.11. *We say that a Carnot group G is of H-type if its the Lie algebra*

$$\mathfrak{g}=V_1\oplus V_2$$

is two-step and if the inner product $\langle\,,\rangle_0$ in V_1 can be extended to an inner product $\langle\,,\rangle$ in all of $\mathfrak{g}$ so that the linear map $J:V_2\to\mathrm{End}(V_1)$ defined by

$$\langle J_ZU,V\rangle=\langle Z,[U,V]\rangle$$

has the property $J_Z^2=-\|Z\|^2\,\mathrm{Id}$ for all $Z\in V_2$, where $\|Z\|^2=\langle Z,Z\rangle$.

Let G be a group of H-type. Since the exponential map of G is an analytic diffeomorphism, we can define analytic mappings $U : G \to V_1$ and $Z : G \to V_2$ by

$$x = \exp\left(U(x) + Z(x)\right), \qquad x \in G.$$

Let

$$N_G(x) = \left(\|U(x)\|^4 + 16\|Z(x)\|^2\right)^{1/4}$$

be the homogeneous norm on G, then it is well known that N_G is smooth on $G\setminus\{0\}$, see [Kap80, Th.2].

It is known that H-type groups are polarizable, see the proof in [BT02, p. 719]. We extend Teorem 3.3 by calculating the flow $\phi(s,\xi)$ and the measure dv on the unit N_G-sphere in an H-type group. As a consequence we are able to compute the constant $C_p(G)$ in the expression for p-modulus of $\Gamma_{ab}(G)$. We state the following results from [BT02]:

Lemma 3.12. *In an H-type group G with norm N_G as above, the flow defined by* (3.10) *is given by*

$$\phi(s,\xi) = \exp(u(s) + z(s)), \qquad \xi \in G \setminus \mathcal{Z},$$

where

$$u(s) \;:=\; U(\phi(s,\xi)) \;=\; s\cos\left(\frac{4\|Z(\xi)\|}{\|U(\xi)\|^2}\log s\right)U(\xi) \;+\; s\sin\left(\frac{4\|Z(\xi)\|}{\|U(\xi)\|^2}\log s\right)\frac{J_{Z(\xi)}U(\xi)}{\|Z(\xi)\|}$$

and

$$z(s) := Z(\phi(s,\xi)) = s^2 Z(\xi).$$

Lemma 3.13. *Let G be of H-type with $k = \dim V_1$ and $l = \dim V_2 \geqslant 1$. Then the set*

$$\begin{aligned} S \setminus \mathcal{Z} &= \{\xi \in G : N_G(\xi) = 1, \nabla_0 N_G(\xi) \neq 0\} \\ &= \exp\{U(\xi) + Z(\xi) \in V_1 \oplus V_2 : \|U(\xi)\|^4 + 16\|Z(\xi)\|^2 = 1, U(\xi) \neq 0\} \end{aligned}$$

can be parametrized by coordinates $(\alpha, x, y) \in [0, \frac{\pi}{2}) \times S^{k-1} \times S^{l-1}$ as follows

$$U(\xi) = \sqrt{\cos\alpha}\cdot x,$$
$$Z(\xi) = \frac{1}{4}\sin\alpha \cdot y,$$

and the measure dv on $S\setminus\mathcal{Z}$ is given by

$$d\sigma = \frac{1}{4^l}\cos^{\frac{k}{2}-1}\alpha\,\sin^{l-1}\alpha\,d\alpha\,dx\,dy, \tag{3.13}$$

where dx and dy are the usual surface area measures on S^{k-1} and S^{l-1} respectively.

Corollary 3.14. *In an H-type group G, the constant $C_p(G)$ of Theorem 3.3 is given by*

$$C_p(G) = \frac{2\pi^{k+l/2}\,\Gamma\left(\frac{k+p}{4}\right)}{4^l\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+2l+p}{4}\right)}.$$

Proof of Corollary 3.14. By definition

$$C_p(G) := \int_{S \setminus \mathcal{Z}} \lambda(\xi)^p \, dv(\xi).$$

We defined the function λ in Lemma 3.10 as

$$\lambda(\xi) = \frac{\|\nabla_0 N_G(\phi(s,\xi))\|_0}{N_G(\xi)}$$

Emlpoying the formula from [CDG96, (2.5)]:

$$\|\nabla_0 N\|_0^2 = \|\nabla_0 N\|^2 = \frac{\|U\|}{N^2},$$

we have that

$$\lambda(\xi) = \frac{\|\nabla_0 N_G(\xi)\|}{N_G(\xi)} = \frac{\|U(\xi)\|}{N_G^2(\xi)} = \|U(\xi)\| \qquad \text{for all } \xi \in S \setminus \mathcal{Z}.$$

Using Lemma 3.12 and Lemma 3.13 we see that

$$\begin{aligned} C_p(G) &:= \int_{S \setminus \mathcal{Z}} \lambda(\xi)^p \, dv(\xi) \\ &= \frac{1}{4^l} \int_{S^{l-1}} \int_{S^{k-1}} \int_0^{\frac{\pi}{2}} \cos^{\frac{p}{2}} \alpha \, \cos^{\frac{k}{2}-1} \alpha \, \sin^{l-1} \alpha \, d\alpha dx dy \\ &= \frac{2\pi^{k+l/2} \, \Gamma\left(\frac{k+p}{4}\right)}{4^l \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{k+2l+p}{4}\right)}. \end{aligned}$$

□

Chapter 4

Geometric Measure Theory

4.1 Modulus of a system of measures.

In a celebrated paper [Fug57] B. Fuglede introduced the notion of the modulus of a system of measures as a generalization of the concept of extremal length. Let $(X, \mathfrak{M})$ be an abstract measurable space with a fixed basic measure $m : \mathfrak{M} \to [0, +\infty]$ defined on a sigma-algebra $\mathfrak{M}$ of open subsets of X.

We denote by $\mathcal{M}$ the system of all measures μ in X, whose domains of definition contain $\mathfrak{M}$. For $\mu \in \mathcal{M}$, $\bar{\mu}$ denotes the completion of μ.

With an arbitrary system of measures $E \subset \mathcal{M}$ we associate a class of non-negative m-measurable functions f defined in X and satisfying the condition

$$\int_X f \, d\mu \geqslant 1, \qquad \mu \in E \tag{4.1}$$

We write $f \wedge \mu$ if (4.1) holds and $f \wedge E$ if (4.1) holds for every $\mu \in E$.

Definition 4.1. *For $0 < p < \infty$, the modulus $M_p(E)$ of E is defined as*

$$M_p(E) = \inf_{f \wedge E} \int_X f^p \, dm,$$

interpreted as $+\infty$ if the set $\{f : f \wedge E\}$ is empty.

A statement concerning measures $\mu \in \mathcal{M}$ is said to hold for M_p-a.a. μ if it fails to hold for a system E_0 of measures of zero p-modulus. A system E_0 in this case is called p-exceptional.

The proofs of the following statements can be found in [Fug57, Ch.1]:

Theorem 4.2. *Let $(X, \mathfrak{M}, m)$ be an abstract measure space where m is a fixed basic measure defined on a σ-sub-algebra of $\mathfrak{M}$. Then the following properties hold:*

i. $M_p(E) \leqslant M_p(E')$ *if* $E \subset E'$, *ii.* $M_p(E) \leqslant \sum_{i=1}^{\infty} M_p(E_i)$ *if* $E = \bigcup_{i=1}^{\infty} E_i$.

iii. *If* $A \subset X$ *and* $\bar{m}(A) = 0$, *then* $\bar{\mu}(A) = 0$ *for* M_p*-a.e.* $\mu \in \mathcal{M}$.

iv. *If* $f \in L^p(X, \bar{m})$, *then* f *is* $\bar{\mu}$*-integrable for* M_p*-a.e.* $\mu \in \mathcal{M}$.

v. *If* $\|f_i - f\|_{L^p(X,\bar{m})} \to 0$ *then there is a subsequence* f_{i_j} *such that*

$$\int_X |f_{i_j} - f| \, d\bar{\mu} \to 0 \qquad \text{for } M_p\text{-a.e. } \mu \in \mathcal{M}.$$

vi. *If* $E \subset \mathcal{M}$, *then* E *is* p*-exceptional if and only if, there exists a non-negative function* $f \in L^p(X, m)$ *such that*

$$\int_X f \, d\mu = +\infty \qquad \text{for every } \mu \in E.$$

vii. *If* $p > 1$ *and* $E \subset \mathcal{M} \setminus \{\mu \equiv 0\}$ *then there exists a non-negative function* f *such that*

$$\int_X f^p \, dm = M_p(E) \quad \text{and} \int_X f \, d\mu \geqslant 1 \text{ for } M_p\text{-a.e. } \mu \in E.$$

This is indeed a natural generalization of the concept of modulus of the family of curves. For if, having the family Γ of rectifiable curves in $\mathbb{R}^n$, one can regard the arc length of a curve as a linear measure defined on the Borel class of sets in $\mathbb{R}^n$ and, thus, get a corresponding system of measures. This construction was carefully developed in detail in [Oht03, Chap.2]. We have already considered this 'linear' case in setting of $\mathbb{H}^1$ group.

The next step is to study this question for systems of measures in $\mathbb{H}^1$. The rigorous study of extremal length for vector measures in $\mathbb{R}^n$ was done by the H. Aikawa and M. Ohtsuka in [AO99]. Their main result is a reciprocal relation between so-called extremal distance and extremal width. This is a result that we are keen to obtain in our setting of $\mathbb{H}^1$ (easily generalizable to $\mathbb{H}^n$, or maybe even to Carnot groups). However instead of following Aikawa and Ohtsuka in their functional approach, we go for a classic geometric way of Gehring, Ziemer, Fuglede and others.

We shall begin with more concrete type of measures, namely, measures on the hypersurfaces in $\mathbb{H}^1$. Our first task will be to understand what kind of surfaces it makes sense to consider. In the case of the system of curves, we saw that the class of curves, compatible with the definition of the modulus (admissible curves) coincided with the class of rectifiable curves. Now we want to obtain a similar notion for the sets in $\mathbb{H}^1$. So that we could define the modulus of the family of admissible (or rectifiable) sets in such a way that the modulus of a family of unrectifiable sets would be trivial. This is quite interesting and intensively studied problem in Geometric Measure Theory. We shall briefly review some core results in this area and use them to build our construction.

Let us first consider $\mathbb{R}^n$ with Lebesgue measure $\mathcal{L}^n$ defined on Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$. Recall the classic definition of the modulus of the system of surfaces by Fuglede ([Fug57, p.187]):

Definition 4.3. *Let $1 \leqslant k \leqslant n-1$. A non-empty subset $S \subset \mathbb{R}^n$ will be called a k-dimensional Lipschitz surface (or manifold) in $\mathbb{R}^n$ if there corresponds to every point $x^* \in S$ an open set $U \in \mathbb{R}^n$ such that $x^* \in U$, and $S \cap U$ is a Lipschitz image of some open set $T \in \mathbb{R}^k$.*

From Lindelöf's covering theorem follows that S is a Borel subset of $\mathbb{R}^n$. In fact, S is a countable union of compact sets. Moreover, it is known that there exists one and only one measure μ_S defined on $\mathcal{B}(S)$ which agrees with the surface measure μ_X on every Lipschitz image $X = S \cap U$.

Definition 4.4. *Let $\mathcal{S}^k$ be the system of all k-dimensional Lipschitz surfaces in $\mathbb{R}^n$. For any sub-system $E \subset \mathcal{S}^k$ we define its modulus $M_p(E)$ as the p-modulus M_p of the system of measures μ_σ, $\sigma \in E$. Thus, for any $0 < p < \infty$,*

$$M_p(E) = \inf_{f \wedge E} \int\limits_{\mathbb{R}^n} f(x)^p \, d\mathcal{L}^n(x),$$

where $f \wedge E$ means that f is a non-negative Borel function such that

$$\int\limits_{\sigma} f \, d\sigma \geqslant 1 \qquad \textit{for every } \sigma \in E.$$

The results listed above for the system of measures may be carried over. In particular, $E \subset \mathcal{S}^k$ is exceptional of order p, i.e. $M_p(E) = 0$ if and only if, there is a non-negative Borel function $f \in L^p$, such that

$$\int_{\sigma} f \, d\sigma = +\infty \qquad \text{for every } \sigma \in E.$$

We see that in this definition admissible surfaces are Lipschitz surfaces. More generally, this is a particular case of so-called rectifiable subsets of $(\mathbb{R}^n, d_E)$. In the following section we shall discuss this notion more in details.

4.2 Rectifiability of sets in $\mathbb{H}^1$.

We shall limit ourselves to the case of one-dimensional Heisenberg group $\mathbb{H}^1$ and compare the notion of 2-Euclidean rectifiability with the notion of H-rectifiability introduced in [FSSC01]. We start from the classical definition of m-rectifiability in a general metric space due to Federer (see [Fed69]).

Definition 4.5. *Recall that the k-dimensional spherical Hausdorff measure $\mathcal{H}^k_d$ of a set S on a metric space (X, d) is a measure defined as*

$$\mathcal{H}^k_d(S) = \lim_{\delta \to 0} \inf_{\mathcal{B}} \sum_i (\operatorname{diam} B_i)^k, \tag{4.2}$$

where the infimum is taken over all coverings $\mathcal{B}$ of the set S by balls B_i with diameter with respect to the metric d, $\operatorname{diam} B_i < \delta$.

The Hausdorff dimension of S can be defined as

$$\dim_{Haus}(S) = \inf\{k \geqslant 0 : \mathcal{H}_d^k(S) = 0\}.$$

.

The left invariance and scaling properties of the Carnot-Carathéodory metric d_{cc} (or, equivalently, Heisenberg metric d_H) yield the corresponding properties for the Hausdorff measures $\mathcal{H}_{cc}^k$ in $(\mathbb{H}^1, d_{cc})$, see, for example [FSSC01, Prop.2.4]:

$$\mathcal{H}_{cc}^k(L_\eta(E)) = \mathcal{H}_{cc}^k(E),$$
$$\mathcal{H}_{cc}^k(\delta_s(E)) = s^k \mathcal{H}_{cc}^k(E),$$

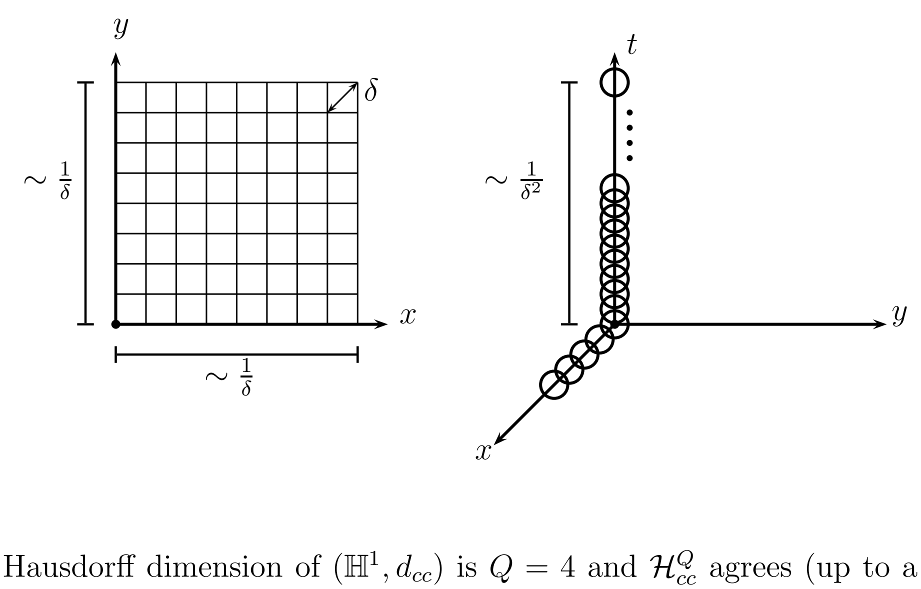
for all $s, k > 0, \eta \in \mathbb{H}^1$, and $E \subset \mathbb{H}^1$.

In particular, for each k there exists $c(k) \in [0, \infty]$ so that

$$\mathcal{H}_{cc}^k(B(\xi, r)) = c(k) r^k$$

for all $\xi \in \mathbb{H}^1$ and $r > 0$, where $B(\xi, r)$ denotes the metric ball in $(\mathbb{H}^1, d_{cc})$.

We want to discuss the important property, that the Hausdorff dimension of $\mathbb{H}^1$, $Q = 4$ is strictly bigger than the topological dimension, which is three. Intuitively, it can be seen as follows. Recall that the Heisenberg metric is homogeneous of order 1 in directions x and y and of order 2 in direction of t. Thus, when covering, for example, a unit cube by balls of diameter $\delta > 0$, we need $\frac{1}{\delta}$ balls in directions x and y and $\frac{1}{\delta^2}$ balls in direction t. This makes the resulting number of balls $\frac{1}{\delta^4}$. Therefore we get the finite Hausdorff measure, when the exponent k in the definition of $\mathcal{H}_d^k$ is 4, consequently the Hausdorff measure of the cube is 4. Essentially the same argument is valid for any set in $\mathbb{H}^1$.



Thus the Hausdorff dimension of $(\mathbb{H}^1, d_{cc})$ is $Q = 4$ and $\mathcal{H}_{cc}^Q$ agrees (up to a constant multiplicative factor) with the Haar measure on $\mathbb{H}^1$.

Definition 4.6 (Federer). *Given a metric space (X, d) and a positive integer m, we say that a Borel set $E \subseteq X$ is m-rectifiable if there exists a countable collection of Lipschitz maps $f_i : A_i \subseteq (\mathbb{R}^m, d_E) \to (X, d)$ such that*

$$\mathcal{H}^m_d\Big(E \backslash \bigcup_{i=1}^{\infty} f_i(A_i)\Big) = 0.$$

It turned out however that this notion of rectifiability is not appropriate in the setting of the Heisenberg group endowed with the Heisenberg metric. Indeed, Ambrosio and Kirchheim (see [AK00]) proved that the Heisenberg group $(\mathbb{H}^1, d_H)$ is purely m-unrectifiable for $m = 2, 3, 4$, i.e. that for any Lipschitz map

$$f : A \subseteq (\mathbb{R}^m, d_E) \to (\mathbb{H}^1, d_H)$$

one has $\mathcal{H}^m_H(f(A)) = 0$.

This lack of rectifiable sets in the classical sense suggests that more intrinsic definitions of rectifiability could be useful instead. To this aim, in [FSSC01], an intrinsic definition of rectifiability in the Heisenberg group was introduced in the codimension one case. The idea was to replace the images of Lipschitz mappings in Federer's definition by surfaces given as level sets of C^1-functions with non-vanishing gradient. This led to the following definitions.

Definition 4.7. *We shall say that $S \subset \mathbb{H}^1$ is a H-regular hypersurface if for every $\xi \in S$ there exists $r > 0$ and a function $f \in C^1(B(\xi, r))$ such that*

i. $S \cap B(\xi, r) = \{\eta \in B(\xi, r) : f(\eta) = 0\}$*;*

ii. $\nabla_0 f(\xi) \neq 0$.

Remark 4.8. We point out that the class of H-regular hypersurfaces is different from the class of Euclidean regular hypersurfaces, in the sense that there are H-regular hypersurfaces that are not Euclidean continuously differentiable submanifolds, and conversely there are continuously differentiable 2-submanifolds in $\mathbb{R}^3$ that are not H-regular hypersurfaces, see [FSSC01, Remarks 6.2 and 6].

Definition 4.9. *We shall say that $\Gamma \subset \mathbb{H}^1$ is m-dimensional H-rectifiable if there exists a sequence of H-regular hypersurfaces $(S_j)_{j\in N}$ such that*

$$\mathcal{H}^m_H\Big(\Gamma \backslash \bigcup_{j\in N} S_j\Big) = 0. \tag{4.3}$$

Example 4.10. An immediate example of 3-dimensional H-rectifiable set in $\mathbb{H}^1$ is a Heisenberg sphere $S = \{\xi \in H^1 : N(\xi) = 1\}$. The characteristic set $\mathcal{Z}$ of N is a set of points

$$\mathcal{Z} = \{\xi \in \mathbb{H}^1 : \nabla_0 N(\xi) = 0\} = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 = 1,\, t = 0\} \cup \{(0, 0, 1)\}.$$

Thus the sphere S can be covered by three H-regular hypersurfaces, namely

$$\begin{aligned} S_x^+ &= \{(x,y,t) \in \mathbb{R}^3 : N(\xi) - 1 = 0, t > 0, x > 0\}, \\ S_x^- &= \{(x,y,t) \in \mathbb{R}^3 : N(\xi) - 1 = 0, t > 0, x < 0\}, \\ S^- &= \{(x,y,t) \in \mathbb{R}^3 : N(\xi) - 1 = 0, t < 0\} \end{aligned}$$

So that

$$\begin{aligned} &S \setminus \left(S_x^+ \cup S_x^- \cup S^-\right) \\ &\qquad = \{(x,y,t) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ t = 0\} \cup \{(x,y,t) \in \mathbb{R}^3 : t^2 + y^4 = 1,\ t > 0\} \end{aligned}$$

is a union of $\mathcal{H}_H^1$-dimensional and $\mathcal{H}_H^2$-dimensional sets correspondingly, and consequently

$$\mathcal{H}_H^3\Big(S \setminus \left(S_x^+ \cup S_x^- \cup S^-\right)\Big) = 0.$$

4.3 Modulus of a family of separating sets in $\mathbb{H}^1$.

In this section we shall define the modulus of the system of surfaces in $\mathbb{H}^1$, using the intrinsic notion of rectifiability in $\mathbb{H}^1$. To begin with, we shall consider a special system of sets, namely a family of separating sets. Let $\hat{\mathbb{H}}^1 = \mathbb{H}^1 \cup \{\infty\}$ denote the one-point compactification of $\mathbb{H}^1$.

Definition 4.11. *Given two disjoint non-empty compact sets $C_0, C_1 \subset \hat{\mathbb{H}}^1$, the (C_0, C_1)-condenser is the open subset $\Omega \subset \hat{\mathbb{H}}^1$ such that $\hat{\mathbb{H}}^1 \setminus \Omega = C_0 \cup C_1$.*

We will use Ω throughout this chapter for a general (C_0, C_1)-condenser. Without loss of generality we assume that C_0 is a component containing point at ∞ (if not, make a conformal transformation of variables).

Definition 4.12. *We will say that a set $\sigma \subset \mathbb{H}^1$ separates C_0 from C_1 in Ω if $\sigma \cap \Omega$ is closed in Ω and if there are disjoint open sets $A, B \subset \hat{\mathbb{H}}^1$, such that $\Omega - \sigma = (A \cup B) \cap \Omega$, $C_0 \subset A$ and $C_1 \subset B$.*

Let Σ denote the class of all $(Q-1)$-dimensional H-rectifiable sets that separate C_0 from C_1 in Ω. With every $\sigma \in \Sigma$, associate the complete measure μ in the following way. For every $\mathcal{H}_H^{Q-1}$-measurable set $A \subset \mathbb{H}^1$, define

$$\mu(A) = \mathcal{H}_H^{Q-1}(A \cap \sigma \cap \Omega).$$

From the properties of Hausdorff measure, it is clear that the Borel sets of $\mathbb{H}^1$ are μ-measurable and therefore the modulus of Σ can be defined analogously to the modulus of set of measures, namely

Definition 4.13. *Given* $1 < p < \infty$,

$$M_p(\Sigma) = \inf_{f \wedge \Sigma} \int_{\mathbb{H}^1} f^p \, dg,$$

where dg *is the Haar measure on* $\mathbb{H}^1$ *and* $f \wedge \Sigma$ *means that* f *is non negative Borel function on* $\mathbb{H}^1$ *such that*

$$\int_{\sigma \cap \Omega} f \, d\mathcal{H}_H^{Q-1} \geqslant 1 \qquad \text{for every } \sigma \in \Sigma.$$

Here, as before Q is the Hausdorff dimension of $\mathbb{H}^1$. We prefer to use Q rather then its concrete value $Q = 4$, baring in mind the generalization to higher dimensional cases.

4.4 A p-capacity of a condenser.

A p-capacity of a general condenser Ω can be defined as follows.

Definition 4.14. *Given* $1 < p < \infty$, *a p-capacity of a condenser* Ω *in* $\mathbb{H}^1$ *is the quantity*

$$\operatorname{Cap}_p(\Omega) = \inf_{u \in F(\Omega)} \int_{\mathbb{H}^1} \|\nabla u\|_0^p \, dg, \tag{4.4}$$

where infimum is taken over all admissible functions

$$F(\Omega) = \{u \in C_0^\infty(\hat{\mathbb{H}}^1), u|_{C_0} = 0, u|_{C_1} = 1\}.$$

In what follows we will work with conformal case $p = Q$. This is a model case and results can be generalazed in the future research for an arbitrary exponent $1 < p < \infty$. We also restrict ourselves to simplest condensers on $\mathbb{H}^1$, namely, non-degenerate rings. A ring R is a condenser with connected complementary components C_0 and C_1. As before we will use the notation R_{ab} for a spherical N-ring in $\mathbb{H}^1$.

Definition 4.15. *Let* U *be a domain in* $\mathbb{H}^1$. *For* $p \geqslant 1$ *we say that* $u : U \to \mathbb{R}$ *is in the horizontal Sobolev space* $HW^{1,p}(U)$ *if* $u \in L^p(U)$ *and its distributional partial derivatives* Xu *and* Yu *are in* $L^p(U)$, *i.e.*

$$HW^{1,p}(U) = \{u \in L^p(U) : Xu, \, Yu \in L^p(U)\},$$

where by distributional partial derivative we mean

$$\int_U Xu \cdot \phi \, dg = -\int_U u \cdot X\phi \, dg, \qquad \phi \in C_0^\infty(U).$$

It can be shown by a standard regularization process that the function $u \in HW^{1,p}(U)$ can be approximated by functions $u_n \in C^\infty(U)$ such that

$$\begin{aligned} &u_n \to u \qquad \text{in } L^p(U) \\ &Xu_n \to Xu, \; Yu_n \to Yu \qquad \text{in } L^p(U). \end{aligned}$$

Definition 4.16. *A continuous real valued function u defined on an open set $U \subset \mathbb{H}^1$ is said to be absolutely continuous on lines (or in sense of Tonelli), $u \in ACL(U)$ on U if for any domain U', $\bar{U}' \subset U$ and any foliation χ defined by left-invariant vector fields X, Y, the function u is absolutely continuous on $\gamma \cap U'$ with respect to the Hausdorff measure $\mathcal{H}^1$ for $d\gamma$-almost all curves $\gamma \in \chi$.*

For such a function u the derivatives Xu, Yu exist almost everywhere in U. If Xu, Yu belong to $L^p(U)$, then u is said to be in $ACL^p(U)$. We also use a notation

$$ACL^p(\bar{U}) = \{u \in C(\bar{U}) : u|_U \in ACL^p(U)\}.$$

Example 4.17. Let us give the example of a function of ACL^p class in $\mathbb{H}^1$. Let $U = R_{ab}$. Let us show that the norm $N : \mathbb{H}^1 \to \mathbb{R}$ belongs to $ACL^p(R_{ab})$.

$$XN(\xi) = \left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right) N = N^{-3}(|z|^2 x + yt), \quad \xi = (x, y, t) \in \mathbb{H}^1.$$

Making use of spherical coordinates representation in Theorem 3.4, we have

$$\begin{aligned}
&\left(\int_{R_{ab}} |N^{-3}(|z|^2 x + yt)|^p \, dg\right)^{1/p} \\
&\quad = \left(\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\alpha \int_a^b r^{-3p} \, |r^2 \cos\alpha \, r\cos\theta + r\sin\theta \, r^2 \sin\alpha|^p \, r^3 \, dr\right)^{1/p} \\
&\quad = \left(\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\alpha \int_a^b |\cos\alpha \, \cos\theta + \sin\theta \, \sin\alpha|^p \, r^3 \, dr\right)^{1/p} \\
&\quad \leqslant \left(\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\alpha \int_a^b r^3 \, dr\right)^{1/p} < \infty.
\end{aligned}$$

Essentially the same estimate holds for the derivative $YN(\xi)$, $\xi \in \mathbb{H}^1$, and consequently both XN and YN are in $L^p(R)$, which yields that $N \in ACL^p(R_{ab})$.

We shall make use of the following results of Korányi and Reimann, see [KR95, Prop.11, Prop.9] for the details of proof.

Proposition 4.18. *For an open subset U of $\mathbb{H}^1$, the following is true:*

$$ACL^p(U) = C(U) \cap HW^{1,p}(U).$$

Proposition 4.19. *Let*

$$F^*(R) = \{u \in ACL^Q(\bar{R}), u|_{\partial R \cap C_0} = 0, u|_{\partial R \cap C_1} = 1\}.$$

Then for a ring R the following relation holds

$$\mathrm{Cap}_Q(R) = \inf_{u \in F(R)} \int_{\mathbb{H}^1} \|\nabla_0 u\|_0^Q \, dg = \inf_{u \in F^*(R)} \int_R \|\nabla_0 u\|_0^Q \, dg.$$

Observe that the extremal function $u \in F^*(R)$ is a continuous function from $HW^{1,Q}(R)$, thus it is a weak solution for Q-sub-Laplacian and satisfies the variational condition

$$\int_R \|\nabla_0 u\|_0^{Q-2} \langle \nabla_0 u, \nabla_0 w \rangle_0 \, dg = 0 \tag{4.5}$$

for all test functions $w \in C_0^\infty(R)$. Since the function $w \in HW^{1,Q}(R)$ can be approximated by smooth functions, we can consider our test function w to be in $HW^{1,Q}(R)$.

4.5 Modulus vs. capacity

The application of the theory of extremal length in quasiconformal mapping theory begins essentially with a theorem proved by Gehring [Geh62], that the conformal capacity of a ring $R \subset \mathbb{R}^3$ is directly related to the extremal length of the family of curves that join the boundary components of R. He has also shown that the conformal capacity is related to the extremal length of a family of surfaces that separate the boundary components of R.

Gehring assumes that the separating surfaces are sufficiently smooth. Krivov [Kri64] establishes a similar result under the assumption that the extremal metric is well-behaved. Under similar assumptions, other authors have dealt with the extremal length of separating surfaces, cf. [Fug57, Th.9], J. Hersch, B. Shabat.

In [Zie67] the author eliminates the need for these assumptions. He considered the general case of two disjoint compact sets C_0, C_1 contained in the closure of a bounded, open, connected set $U \subset \mathbb{R}^n$. It is proved that the conformal capacity $\mathrm{Cap}_n(U; C_0, C_1)$ of C_0, C_1 relative to U is related to the $n/n-1$-dimensional modulus $M_{n/n-1}(\Sigma)$ of all closed sets that separate C_0 from C_1 in the closure of U by

$$\mathrm{Cap}_n(U; C_0, C_1)^{-\frac{1}{n-1}} = M_{n/n-1}(\Sigma).$$

Let Γ_R denote the family of rectifiable curves connecting C_0 and C_1 in a ring $R \subset \mathbb{R}^n$. The coincidence of the p-modulus of Γ_R and the p-capacity of R was studied by many autors, see for example, [Hes75], for general setting in $\mathbb{R}^n$. In setting of $\mathbb{H}^1$ and, more general, in Carnot groups, this problem is also well-studied, see, for example [Mar03]. In view of this, for our particular case of the spherical N-ring in $\mathbb{H}^1$, we will take for granted the coincidence of these two quantities.

Thus our goal is to adapt the result of Ziemer [Zie67, Th.3.13] to $\mathbb{H}^1$, which we present in the following theorem:

Theorem 4.20. *Given a spherical N-ring $R_{ab} \subset \mathbb{H}^1$ with boundary components C_0 and C_1, let Σ denote the class of all $(Q-1)$-dimensional H-rectifiable sets that separate C_0 from C_1 in R_{ab}. Then*

$$M_{Q'}(\Sigma) = \mathrm{Cap}_Q(R_{ab})^{-\frac{1}{Q-1}}$$

where $\frac{1}{Q} + \frac{1}{Q'} = 1$.

In order to obtain the desirable result, we shall prove two opposite inequalities in Lemma 4.22 and Lemma 4.23. The first inequality easily follows from a well-known co-area formula, see, for example, [CDPT07].

Lemma 4.21 (The co-area formula in $\mathbb{H}^1$)**.** *The formula*

$$\int\limits_{\mathbb{H}^1} u(\xi)\|\nabla_0 f(\xi)\|_0\, dg = \int_0^\infty \int\limits_{f=t} u(\eta)\, d\mathcal{H}^{Q-1}(\eta)dt \qquad [1] \tag{4.6}$$

holds for all smooth f and non-negative measurable u.

Lemma 4.22. *In the setting of the Theorem 4.20, the following relation holds*

$$M_{Q'}(\Sigma) \geqslant \operatorname{Cap}_Q(R_{ab})^{-\frac{1}{Q-1}}.$$

Proof. Choose $\epsilon > 0$ and let f be a function, such that $f \wedge \Sigma$. Let $u \in \tilde{F}(R_{ab})$, where

$$\tilde{F}(R_{ab}) = F(R_{ab}) \cap \{u : \nabla_0 u|_{R_{ab}} \neq 0\}$$

is a subset of a set of admissible functions for $C := \operatorname{Cap}_Q(R_{ab})$, such that

$$\int_{\mathbb{H}^1} \|\nabla u\|_0^Q\, dg < C + \epsilon.$$

Observe that

$$u^{-1}(s) = \{\xi \in \mathbb{H}^1 : u(\xi) - s = 0\}$$

is H-regular hypersurface since $\tilde{u}(\xi) = u(\xi) - s$ belongs to $C_0^\infty(R_{ab})$ and $\nabla_0 \tilde{u} = \nabla_0 u \neq 0$ in R_{ab}. Thus, since R_{ab} is connected, the level sets $u^{-1}(s) \in \Sigma$ for all $0 < s < 1$. Hence, Hölder's inequality and co-area formula (4.6) yield

$$\begin{aligned}\left(\int_{\mathbb{H}^1} f^{Q'}\, dg\right)^{\frac{1}{Q'}} (C+\epsilon)^{\frac{1}{Q}} > \left(\int_{\mathbb{H}^1} f^{Q'}\, dg\right)^{\frac{1}{Q'}} \left(\int_{\mathbb{H}^1} \|\nabla u\|_0^Q\, dg\right)^{\frac{1}{Q}} \\ \geqslant \int_{R_{ab}} f\|\nabla u\|_0\, dg \geqslant \int_0^1 \int\limits_{u^{-1}(s)} f\, d\mathcal{H}^{Q-1} ds \geqslant 1.\end{aligned}$$

Since ϵ is arbitrary, we have

$$\int_{R_{ab}} f^{Q'}\, dg \geqslant \left(\inf_{u\in\tilde{F}(R_{ab})} \int_{R_{ab}} u^Q\, dg\right)^{-\frac{1}{Q-1}} \geqslant C^{-\frac{1}{Q-1}}.$$

Taking infimum over all $f \wedge \Sigma$, we get the desired result.

□

[1] In this section we will omit the subscript $\mathcal{H}_H^{Q-1}$, emphasizing the Heisenberg metric structure of Hausdorff measure, and write simply $\mathcal{H}^{Q-1}$.

The opposite inequality requires some work. For the full picture we exhibit the result right away and place the proofs of all intermediate steps afterwards.

Lemma 4.23. *In the setting of the Theorem 4.20, the following relation holds*

$$M_{Q'}(\Sigma) \leqslant \mathrm{Cap}_Q(R_{ab})^{-\frac{1}{Q-1}}.$$

Proof. Let $u \in F(R_{ab})$ be the extremal function for $C := \mathrm{Cap}_Q(R_{ab})$. Then, by definition of $F(R_{ab})$, u is Q-harmonic in R_{ab}. By argument of Balogh and Tyson in [BT02, Th.2.15] such a function allways exists and, moreover, there is an explicit formula for p-harmonic functions on $\mathbb{H}^1 \setminus \{0\}$, $1 < p < \infty$.

Let $f = \|\nabla_0 u\|_0^{Q-1} C^{-1}$. Then by Lemma 4.24

$$C^{-1} \int_{\sigma \cap R_{ab}} \|\nabla_0 u\|_0^{Q-1} d\mathcal{H}^{Q-1} \geqslant C^{-1} C = 1, \tag{4.7}$$

which means that $f \wedge \Sigma$. Thus we can write

$$M_{Q'}(\Sigma) \leqslant \int_{R_{ab}} f^{Q'} dg = C \cdot C^{-\frac{Q}{Q-1}} = C^{-\frac{1}{Q-1}}$$

□

Now we shall prove all the intermediate results we have used.

Lemma 4.24. *Let $u \in F(R_{ab})$ be an extremal function for R_{ab}, and let $\sigma \in \Sigma$. Then the following is true:*

$$\int_{\sigma \cap R_{ab}} \|\nabla_0 u\|_0^{Q-1} \, d\mathcal{H}^{Q-1} \geqslant \mathrm{Cap}_Q(R_{ab})$$

where $\frac{1}{Q} + \frac{1}{Q'} = 1$.

Lemma 4.25. *Let R_{ab} be an N-ring. Let $u \in F^*(R_{ab})$ be the extremal function for $\mathrm{Cap}_Q R_{ab}$, and let $\sigma \in \Sigma$. In addition, let $\sigma(b) = \{\xi \in \mathbb{H}^1 : d_H(\xi, \sigma) < b\}$ and $\delta(\xi) = d_H(\xi, \sigma)$. Then*

$$\int_{\sigma(b)} \|\nabla_0 u\|_0^{Q-1} \, \|\nabla_0 \delta\|_0 \, dg \geqslant 2b \; \mathrm{Cap}_Q(R_{ab})$$

whenever $0 < b < d_H(\sigma, \partial R_{ab})$.

Proof. Fix $0 < b < d_H(\sigma, \partial R_{ab})$.

For $i = 0$, 1 let D_i be the component of σ^c that contains C_i.

Let $E_i = \{\xi \in H^1 : 0 < d_H(\xi, D_i^c) < b\}$. Then $E_i \subset D_i$, $E_0 \cup E_1 \subset \sigma(b)$. Hence it is sufficient to show that

$$\int_{E_i} \|\nabla_0 u\|_0^{Q-1} \, \|\nabla_0 \delta\|_0 \, dg \geqslant b \; \mathrm{Cap}_Q(R_{ab}) \qquad i = 0,\, 1.$$

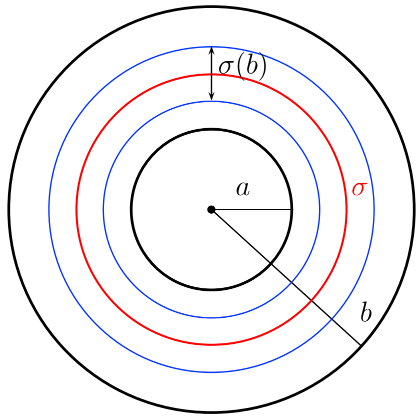


Figure 4.1: The tubular neighborhood $\sigma(b)$

CASE i=1 (For the case i=0 the considerations are analogous).

Set $w = v - bu$, where

$$v(\xi) = \min\{b, d_H(\xi, D_1^c)\} = \begin{cases} 0 & \text{if } \xi \in D_1^c = D_0 \\ \inf_{\eta \in D_1^c} d_H(\xi,\eta) & \text{if } \xi \in E_1 \\ b & \text{if } \xi \in D_1 - E_1. \end{cases}.$$

By Example 4.17 $v \in HW^{1,Q}(R_{ab})$. Consequently, by Proposition 4.18 $w \in HW^{1,Q}(R_{ab})$.

Observe that, $0 < \|\nabla_0 v\| = \|\nabla_0 \delta\|$ a.e. in E_1, and $\|\nabla_0 v\| = 0$ a.e. in $R_{ab} - E_1$. The variational condition for u with test function $w \in HW^{1,Q}(R_{ab})$ reads

$$\int_{R_{ab}} \|\nabla_0 u\|_0^{Q-2} \langle \nabla_0 u, \nabla_0 w\rangle \, dg = \int_{R_{ab}} \|\nabla_0 u\|_0^{Q-2} \langle \nabla_0 u, (\nabla_0 v - b\nabla_0 u)\rangle \, dg = 0,$$

which yields, together with Cauchy inequality,

$$\int_{E_i} \|\nabla_0 u\|_0^{Q-1} \|\nabla_0 \delta\|_0 \, dg \geqslant \int_{R_{ab}} \|\nabla_0 u\|_0^{Q-1} \|\nabla_0 v\|_0 \, dg$$
$$\geqslant \int_{R_{ab}} \|\nabla_0 u\|_0^{Q-2} \langle \nabla_0 u, \nabla_0 v\rangle \, dg = b \int_{R_{ab}} \|\nabla_0 u\|_0^{Q} \, dg.$$

$\square$

Lemma 4.26. *Let $\sigma \in \Sigma$ and let*

$$f_r(\xi) = g(B(\xi,r))^{-1} \int_{B(\xi,r)} \|\nabla_0 u\|_0^{Q-1} dg(\eta)$$

be an integral average of u. Then

$$\int_\sigma f_r \, d\mathcal{H}^{Q-1} \geqslant \mathrm{Cap}_Q(R_{ab})$$

whenever $r < d_H(\sigma, \partial R_{ab})$.

Proof. Choose $b > 0, r > 0$ so that $b + r < d_H(\sigma, \partial R_{ab})$. If σ_η denotes the translation of σ by the vector η, then Fubini's Theorem and Lemma 4.25 imply

$$\int_{\sigma(b)} f_r(\xi)\, \|\nabla_0 \delta\|_0 \, dg(\xi) = g(B(\xi, r))^{-1} \int_{B(0,r)} \int_{\sigma(b)} \|\nabla_0 u(\xi + \eta)\|_0^{Q-1} \, \|\nabla_0 \delta\|_0 \, dg(\xi) dg(\eta)$$
$$= g(B(\xi, r))^{-1} \int_{B(0,r)} \int_{\sigma_\eta(b)} \|\nabla_0 u(\xi)\|_0^{Q-1} \, \|\nabla_0 \delta\|_0 \, dg(\xi) dg(\eta) \geqslant 2b \, \mathrm{Cap}_Q(R_{ab}),$$

since the Heisenberg distance d_H is translation invariant.

In addition to this, if as before $\delta(\xi) = d_H(\xi, \sigma)$ then the co-area formula (4.6) yields

$$\int_{\sigma(b)} f_r(\xi)\, \|\nabla_0 \delta\|_0 \, dg(\xi) = \int_0^b \int_{\delta^{-1}(s)} f_r(\eta) \, d\mathcal{H}^{Q-1}(\eta) ds.$$

Let $F(s)$ denote the inner integral on the right

$$F(s) := \int_{\delta^{-1}(s)} f_r(\eta) \, d\mathcal{H}^{Q-1}.$$

Since f_r is continuous on $\mathbb{H}^1$, it is clear that

$$\lim_{s \to 0} F(s) = 2 \int_\sigma f_r(\eta) \, d\mathcal{H}^{Q-1}.$$

Hence

$$\mathrm{Cap}_Q(R_{ab}) \leqslant \lim_{b \to 0} (2b)^{-1} \int_0^b F(s) ds = \int_\sigma f_r(\eta) d\mathcal{H}^{Q-1}.$$

□

Proof of Lemma 4.24. Select $\sigma \in \Sigma$ and let D_1 be that part of σ^c that contains C_1. Since $\|\nabla_0 u\|_0 = 0$ outside R_{ab}, we can choose r_0 so small that the support of f_{r_0} is contained in R_{ab} (and therefore for all $r \leqslant r_0$) and $r_0 < d_H(\partial D_1, \partial R_{ab})$, where f_r is the integral average of $\|\nabla_0 u\|_0^{Q-1}$ and given by

$$f_r(\xi) = g(B(\xi, r))^{-1} \int\limits_{B(\xi, r)} \|\nabla_0 u\|_0^{Q-1} dg.$$

We refer to [Ste93, Cor.1, p.13] for the fact that f_r is continuous for $r > 0$, and that

$$f_r \to \|\nabla_0 u\|_0^{Q-1} \quad g\text{-a.e. as } r \to 0.$$

Consequently, for $\frac{1}{Q} + \frac{1}{Q'} = 1$

$$|f_r|^{Q'} \to \|\nabla_0 u\|_0^{Q} \quad g\text{-a.e. as } r \to 0.$$

Thus, by Lebesgue's dominated convergence theorem,

$$\int_{R_{ab}} |f_r|^{Q'}\, dg \to \int_{R_{ab}} |\|\nabla_0 u\|_0^{Q-1}|^{Q'}\, dg \quad \text{as } r \to 0.$$

and

$$|f_r - \|\nabla_0 u\|_0^{Q-1}|_{L^{Q'}} \to 0 \quad \text{as } r \to 0.$$

By (iv) and (v) of Theorem 4.2

$$\int_{R_{ab}} |f_r - \|\nabla_0 u\|_0^{Q-1}|\, d\mathcal{H}^{Q-1} \to 0 \quad \text{as } r \to 0. \tag{4.8}$$

Therefore Lemma 4.26, implying

$$\int_{\sigma} f_r\, d\mathcal{H}^{Q-1} \geqslant \mathrm{Cap}_Q(R_{ab}) \quad \text{for all } r \leqslant r_0. \tag{4.9}$$

together with (4.8) yield the desired result. □

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