Design and Pricing of Equity-Linked Life Insurance under Stochastic Interest Rates

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Abstract

A valuation model for equity-linked life insurance contracts incorporating stochastic interest rates is presented. Our model generalizes some previous pricing results based on deterministic interest rates. Moreover, a design of a new equity-linked product with some appealing features is proposed and compared with the periodical premium contract of Brennan and Schwartz (1976). Our new product is very simple to price and may easily be hedged either by long positions in the mutual fund of linkage or by European call options on the same fund.

1 Introduction

Equity-linked or unit-linked insurance contracts link the amount of benefit to a financial asset. This asset could be a certain stock, a stock index, a foreign currency, etc. For simplicity, we assume it is a mutual fund, commonly seen in practice. Such products seem to offer the insurance companies as well as the insurance customers advantages compared to traditional products. Customers may benefit from higher yields in financial markets and then again, the insurance industry may benefit from offering more competitive savings products. In addition, customers usually have some flexibility with respect to choosing, and subsequently changing, the mutual fund for linkage. Thereby they may influence the amount of financial risk of their policies.

Compared to classical insurance products, one distinguishing feature of equity-linked products is the random amount of benefit. The principle of equivalence, based on the philosophy that a company's income (premiums), and expenses (paid benefits) should balance in the long run, the traditional basis for pricing life insurance policies, does not deal with random benefits. Typically, financial valuation theories are used together with elements of actuarial theory to price such products.

The integration of the two types of theories is based on the assumptions of independence between financial and mortality factors and risk neutrality with respect to mortality. That is, the insurer does not receive any economic compensation for accepting mortality risk. This assumption is also implicit in the traditional principle of equivalence and is justified by the traditional pooling argument saying that the insurer can, at least in principle, eliminate mortality risk by adequately increasing the number of identical and independent contracts in his portfolio.

The focus of this paper is design and pricing of equity-linked contracts in a model with stochastic interest rates.

Our set-up includes a simple model of a financial market. In this market a mutual fund and default free bonds are traded. In order to keep the model simple, we restrict ourselves to two sources of uncertainty. The first reflects risk connected to the interest rate, the second risk connected to the mutual fund to which the policy is linked. For the stochastic interest rate we apply the term structure model by Heath, Jarrow, and Morton (1992) (henceforth referred to as HJM). This is a rather general framework which, e.g., includes the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) as special cases.

First, we calculate single premiums of two types of equity-linked policies which are similar to traditional pure endowment contracts, expiring upon survival at the term of the contract, and term insurances, expiring upon death before the term of the contract. These contracts include the characteristics of most interesting life insurance policies on single lives. Furthermore, under our set of assumptions the treatment of financial risk is independent of how complex the insurance contract is, and the results of this paper can easily be generalized to more complex life insurance policies.

As is the case for traditional life insurance, also equity-linked products are often paid by periodical premiums. For equity-linked insurance this periodical premium is typically designed as an investment plan, i.e., a certain proportion or a certain amount of the periodical premium is supposed to be invested in the mutual fund to which the contract is linked.

The second object of this paper is the design of an equity-linked policy without an explicit minimum guaranteed benefit. However, the insurer guarantees that the periodical premium would at least cover a given number of units of the mutual fund. Thus, this guarantee on the periodical premiums leads to a minimum guaranteed benefit expressed in number of units, and not as a fixed amount.

We claim that this contract has some appealing features compared to the periodical premium contract introduced by Brennan and Schwartz (1976) and extensively studied in the literature. First, it resembles contracts sold in real-world markets. Second, pricing and hedging of this contract is simple. Finally, we demonstrate how insurance companies may use their knowledge about mortality to level the periodical premiums, resulting in constant periodical premiums which may be desirable from insurance customers' point of view.

The paper is structured as follows: Section 2 categorizes parts of the existing literature on equity-linked policies. A description of the valuation framework follows in section 3. In section 4 single premiums of contracts similar to traditional pure endowment contracts and term insurances are priced. In section 5 a new type of contract, inspired by real-world contracts, is suggested and compared with the periodical premium contract of Brennan and Schwartz (1976). Section 6 contains some concluding remarks.

2 Literature on equity-linked contracts

The first treatments of equity-linked contracts based on financial theory of which we are aware are Brennan and Schwartz (1976), Brennan and Schwartz (1979a), Brennan and Schwartz (1979b) and Boyle and Schwartz (1977), subsequently (and somewhat ambiguously) referred to as BS. They recognized that the payoff of an equity-linked contract with guarantee is related to the payoffs of certain financial options, and applied the option pricing theory initiated by the results of Black and Scholes (1973) and Merton (1973). The more recent works are based on the martingale pricing theory, an extension of the Black-Scholes-Merton theory by Harrison and Kreps (1979) and Harrison and Pliska (1981).

The literature on unit-linked insurance is now rather abundant and may be classified along the following categories.

2.1 Structure of Benefit

The BS-contract is an endowment contract on a single life, i.e., the benefit is payable at the term of the contract or upon death, whatever comes first. A part of the single premium, in the case of a single premium contract, or otherwise a fixed amount of each periodical premium, is deemed to be invested in a mutual fund. Denoting by D_t the market value at time t of the accumulated investments in the mutual fund, the benefit of this contract is

$$\max[D_t, G_t],$$

where G_t is a possibly time-dependent deterministic minimum guarantee.

The relationship between this payoff and financial call and put options can be seen by writing

$$\max[D_t, G_t] = \max[G_t - D_t, 0] + D_t = \max[D_t - G_t, 0] + G_t.$$

Here $\max[G_t - D_t, 0]$ is the payoff of a put option and $\max[D_t - G_t, 0]$ is the payoff of a call option on the accumulated investments in the mutual fund, both with exercise price G_t and fixed expiration t.

More complicated benefits including caps, i.e. upper limits, are discussed in Ekern and Persson (1996) and Nonnenmacher and Russ (1997), as well as in the current paper. Hipp (1996) considers contracts with annual minimum guarantees in addition to a guarantee on expiration.

Persson (1993) analyzes a more general equity-linked insurance contract, e.g., including two or more lives and disability insurance.

2.2 Stochastic Interest Rate in Life Insurance

Traditionally the interest rate used for valuation of life insurance contracts is interpreted as the company's return on its investments. In a real world financial environment this rate will depend on the chosen investment strategy, which again depends on the company's attitude towards financial risk as well as legislation.

Whereas the majority of the literature so far assumes deterministic interest rates, empirical observations as well as current academic research stress the need for models incorporating stochastic interest rates.

The current article as well as Bacinello and Ortu (1993b), Bacinello and Ortu (1994), Nielsen and Sandmann (1995), Nielsen and Sandmann (1996), and Kurz (1996) apply stochastic models of interest rates.

2.3 Premium payment

In contrast to most financial products which are paid by a single amount at the initiation of the contract, life insurance products are usually paid by periodical premiums. The BS-study also includes the case of periodical premiums, and periodical payments have been further analyzed by Delbaen (1986), Bacinello and Ortu (1993a), Bacinello and Ortu (1994), Nielsen and Sandmann (1995), Nielsen and Sandmann (1996), and Kurz (1996).

Another view on periodical premiums has been taken by Aase and Persson (1994), where periodical premiums have been constructed in the more traditional way, i.e. by distributing the single premium over the period in which periodical premiums are supposed to be paid.

2.4 Hedging strategies

In addition to pricing issues for equity-linked products, financial theories may also suggest some hedging or replicating strategies that the insurance company may (or may not) use in order to reduce the financial risk often associated with such products. This problem has been studied by Brennan and Schwartz (1979a), Aase and Persson (1994), Hipp (1996), and Møller (1997). Aase and Persson (1994) and Møller (1997) use time continuous death probabilities and in Aase and Persson (1994) a connection between the celebrated

Thiele's differential equation of the actuarial sciences and the famous Black and Scholes equation is developed.

3 The valuation framework

The first two subsections contain a brief overview over the financial set-up. Details can be looked up in any advanced financial textbooks, such as Duffie (1996). The last subsection introduces the insurance factors.

3.1 The financial assets

A time horizon T is fixed and the financial uncertainty is generated by a 2-dimensional standard Brownian motion (W^1, W^2) defined on a probability space $(\Omega, \mathcal{F}, Q)$ together with the filtration $(\mathcal{F}_t, 0 \leq t \leq T)$, satisfying *the usual conditions* and representing the revelation of information. In particular, Q represents the equivalent martingale measure. All trade is assumed to take place in a frictionless market (no transaction costs or taxes, and short-sale allowed).

A unit discount bond is a default-free financial asset that entitles its owner to one unit of account at maturity without any intermediate coupon payments. We denote by $B_t(s)$ the market price at time t for a bond maturing at a fixed date $s \geq t$. By definition $B_s(s) = 1$.

We assume there is a continuum of such bonds maturing at all times $s, 0 \leq s \leq T$.

Furthermore, we assume that a mutual fund is traded and that its market price per unit S_t is given by the following stochastic differential equation under the equivalent martingale measure:

$$dS_t = r_t S_t dt + \sigma_1 S_t dW_t^1 + \sigma_2 S_t dW_t^2, \tag{1}$$

where σ_1 and σ_2 are constants and the initial value of the process S_0 is given. Here r_t represents the short term interest rate in the economy and $\sqrt{\sigma_1^2 + \sigma_2^2}$ may be interpreted as the instantaneous standard deviation of the rate of return on the mutual fund. As will soon be apparent, the Brownian motion W_t^2 is used to model mutual fund specific risk.

3.2 The Gaussian HJM model

The primitives of the HJM-model are the volatility structure and the initial instantaneous forward rates.

The volatility structure is given by the function $\sigma_t(u)$, for $t \leq u$. We assume it is deterministic, i.e., $\sigma_t(u)$ is a deterministic function of u and t.

Then we denote by $f_t(u), t \leq u \leq T$, the instantaneous forward rates prevailing at time $t \geq 0$. From the general relationship between the instantaneous forward rates and the bond price,

$$B_t(s) = \exp\left(-\int_t^s f_t(u)du\right),$$

and since $f_0(u)$ for $0 \le u \le T$ is given, the bond prices at time zero are known as well.

All relevant quantities are determined by the volatility structure and the initial instantaneous forward rates. For example, the short term interest rate under the equivalent martingale measure is given by

$$r_t = f_0(t) + \int_0^t \sigma_v(t) \int_v^t \sigma_v(s) ds dv + \int_0^t \sigma_v(t) dW_v^1.$$

The assumption of deterministic volatility structure implies that r_t is Gaussian, hence negative values of r_t have positive probability. This fact is a theoretical drawback of Gaussian term structure models, but does not seem to present a problem for reasonable parameter values.

The market price of the bond satisfies the following stochastic differential equation under the equivalent martingale measure:

$$dB_t(s) = r_t B_t(s) dt + a(t,s) B_t(s) dW_t^1,$$

where

$$a(t,s) = -\int_t^s \sigma_t(u) du,$$

and the initial value $B_0(t)$ is determined by the initial instantaneous forward rates.

The quantity

$$v(t) = \exp\left(-\int_0^t r_u du\right)$$

is sometimes called the discount function and represents the stochastic present value at time zero of one unit of account at time t.

In this model market prices of financial assets may be calculated as expectations of discounted cashflows under the equivalent martingale measure. In particular, the market price at time t of a unit discount bond expiring at time s may be calculated as

$$B_t(s) = E^Q\left[\frac{v(s)}{v(t)} | \mathcal{F}_t\right],$$

where $E^Q[\cdot|\mathcal{F}_t]$ denotes the conditional expectation under the probability measure Q.

As a second example consider a European call option on one unit of the mutual fund. The payoff of this option is $\max[S_t - G, 0]$, where the constant G represents the exercise price. Denote the market price at time zero of the described option with expiration at time t by $\pi_t(G)$. Given our model of the financial market it follows that

$$\pi_t(G) = S_0\Phi(d_t^1(G)) - GB_0(t)\Phi(d_t^2(G)), \tag{2}$$

where

$$d_t^1(G) = \frac{1}{\Theta_t}\left(\frac{1}{2}\Theta_t^2 + \ln\left(\frac{S_0}{B_0(t)G}\right)\right),$$

$$d_t^2(G) = d_t^1(G) - \Theta_t,$$

$$\Theta_t = \sqrt{\int_0^t a(s,t)^2 ds + (\sigma_1^2 + \sigma_2^2)t - 2\sigma_1 \int_0^t a(s,t)ds},$$

and $\Phi(\cdot)$ denotes the cumulative standard normal distribution function. See Amin and Jarrow (1992). The volatility parameter Θ_t depends on the volatility structure $(a(s,t))$ as well as the volatility parameters of the mutual fund (σ_1 and σ_2), in addition to time to expiration (t).

3.3 Insurance factors

Let $C(t)$ denote an arbitrary insurance benefit payable at time t, possibly dependent on the market value at time t of the mutual fund (formally, $C(t)$ is *adapted* to the filtration $(\mathcal{F}_t, 0 \leq t \leq T)$).

Let the random variable T_x, defined on another probability space $(\hat{\Omega}, \hat{\mathcal{F}}, P)$, denote the remaining life time of an x-year old person. We assume that the probability density function for T_x exists and denote it by $f_x(\cdot)$. Let ${}_tp_x = P(T_x > t)$ denote the survival probability of an x-year old policy buyer. By construction T_x is independent of W_t^1 and W_t^2, hence it is independent of all processes reflecting financial quantities. Finally, as indicated and explained in the introduction, we assume risk neutrality with respect to mortality.

4 Single premiums of insurance contracts

In this section we derive single premiums at time 0, for a life aged x, of some life insurance contracts of the pure endowment and term insurance types.

4.1 Pure endowment and term insurance types of contracts

From the assumed risk neutrality with respect to mortality and independence between mortality and financial risk, it follows that the market price at time 0 of a pure endowment insurance contract with benefit $C(T)$ payable at time T if the insured is alive is

$$\pi = {}_T p_x E^Q \left[v(T) C(T) \right]. \tag{3}$$

Similarly, the market price at time 0 of a term insurance with benefit $C(t)$ payable upon death at time $t \leq T$ is

$$\pi^1 = \int_0^T E^Q \left[v(t) C(t) \right] f_x(t) dt. \tag{4}$$

In the remainder of this section we consider three different kinds of benefits: $C_{(1)}(t) = 1$, $C_{(2)}(t) = \max[S_t, G_t]$, and $C_{(3)}(t) = \max[\min[S_t, K_t], G_t]$. In the first example the benefit is deterministic, as in traditional life insurance contracts. This example is included to isolate the effect of the stochastic interest rate. In the second example the benefit is the maximum of the value of one unit of the fund and a guaranteed amount G_t. In principle the amount the insurance company is obliged to pay under this contract has no upper bound. Therefore in the last contract a maximum amount, a cap, is included.

4.2 Deterministic benefits

We will calculate the single premiums of the policies at time zero. First we turn to the first example and calculate the market premiums of the pure endowment insurance and the term insurance. From the formulas of the previous subsection we obtain

$$\pi_{(1)} = {}_T p_x B_0(T)$$

and

$$\pi^1_{(1)} = \int_0^T B_0(t) f_x(t) dt.$$

For the pure endowment insurance the single premium is the market price at time zero of one unit of account payable at time T multiplied by the probability of payment.

For the term insurance $B_0(t) f_x(t) dt$ can, similarly, be interpreted as the market value at time zero of the expected payoff in the time interval $(t, t + dt)$. The single premium is then the sum of these expected payoffs over the whole term of the contract.

These formulas resemble the corresponding classical formulas. However, note carefully an important difference: The usual *present values* of future payoffs are replaced with *market values* of future payoffs.

Persson (1998) derives similar results for traditional life insurance based on the Vasicek (1977) model of the term structure.

4.3 Equity-Linked Policies with Guarantees

For the second example let

$$U^1(t) = E^Q[v(t)\max[S_t, G_t]].$$

Here $U^1(t)$ can be interpreted as the market value at time zero of a benefit which expires at time t with probability 1. We observe that the payoff $\max[S_t, G_t]$ resembles the structure of the benefit in the BS-model.

The calculation of the above expectation is presented in the following proposition.

Proposition 1 *The market value at time zero of the benefit* $\max[S_t, G_t]$ *payable at time* t *is*

$$U^1(t) = S_0\Phi(d_t^1(G_t)) + G_t B_0(t)\Phi(-d_t^2(G_t)).$$

Proof 1 *From equation (2) we know the time zero value of the claim* $\max[S_t - G_t, 0]$*. Observe that* $\max[S_t, G_t] = \max[S_t - G_t, 0] + G_t$*. The market price at time zero of the last term is* $B_0(t)G_t$*. The formula of Proposition 1 is then the sum of the two time zero market values.*

The resulting formula depends on the initial forward rates ($B_0(t)$) and five parameters: the parameters of the mutual fund price process (S_0, σ_1, σ_2), the guarantee (G_t), and time to expiration (t).

Incorporating the insurance aspects, the single premiums of the two contracts can now be expressed by exploiting relations (3) and (4) as

$$\pi_{(2)} = {}_Tp_x[S_0\Phi(d_T^1(G_T)) + G_T B_0(T)\Phi(-d_T^2(G_T))]$$

and

$$\pi_{(2)}^1 = \int_0^T [S_0\Phi(d_t^1(G_t)) + G_t B_0(t)\Phi(-d_t^2(G_t))] f_x(t)dt.$$

For constant interest rate these formulas reduce to the results in Theorem 1 and 2 of Aase and Persson (1994).

4.4 Equity-linked policies with minimum guarantees and capped benefits

We now turn to the third case. Define

$$U^2(t) = E^Q[v(t)\max[\min[S_t, K_t], G_t]],$$

where $G_t < K_t$. This expression can similarly be interpreted as the market value at time zero of the benefit $\max[\min[S_t, K_t], G_t]$ payable at time t.

Proposition 2 *The market value at time zero of the benefit* $\max[\min[S_t, K_t], G_t]$ *payable at time t is*

$$U^2(t) = S_0[\Phi(d_t^1(G_t)) - \Phi(d_t^1(K_t))] + G_t B_0(t)\Phi(-d_t^2(G_t)) + K_t B_0(t)\Phi(d_t^2(K_t)).$$

Proof 2 *Observe that* $\max[\min[S_t, K_t], G_t] = G_t + \max[S_t - G_t, 0] - \max[S_t - K_t, 0]$. *Thus, its time zero market value is* $G_t B_0(t) + \pi_t(G_t) - \pi_t(K_t)$*, where* $\pi_t(\cdot)$ *is given in expression (2).*

By comparing this formula to the formula for $U^1(t)$, we note that it depends on one more parameter, namely K_t, the cap. Observe that $U^2(t) < U^1(t)$ since high values ($> K_t$) of the mutual fund do not lead to higher benefit because of the cap.

As for the previous benefit, by incorporating the insurance aspects, the single premiums of the two contracts can now be expressed by equations (3) and (4) as $\pi_{(3)} = {}_T p_x U^2(T)$ and $\pi^1_{(3)} = \int_0^T U^2(t) f_x(t)dt$, respectively.

These formulas generalize the similar results under deterministic interest rate in Proposition 2.1 and Lemma 3.4 of Ekern and Persson (1996) to the case of stochastic interest rate.

5 A new equity-linked contract based on periodical premiums

In this section we describe an equity-linked life insurance policy of the endowment type, whose benefit is linked to the market value of the mutual fund in a specific way. There is no explicit guaranteed benefit. Currently equity-linked products without guarantees are sold, e.g., in Norway. However, for the contract we analyze there is a guarantee expressed in number of units connected to the periodical premium, which again leads to a time dependent minimum guaranteed benefit in number of units.

5.1 The periodical premium guarantee

Let $P_t, t = 0, 1, ..., T-1$, be the periodical premium paid at the beginning of each year, if the insured is alive. Assume that the contract specifies a fixed amount of the premium, denoted by d_t, deemed to be invested in the mutual fund. Without guarantees, the number of units acquired at time t should therefore be equal to d_t/S_t, but at this point we introduce the *minimum guarantee provision*, expressed by a minimum *number of units* guaranteed at time t. Let g_t represent this guarantee, and n_t denote the actual number of units deemed to be invested in the mutual fund at time t. Thus,

$$n_t = \max\left[g_t, \frac{d_t}{S_t}\right], t = 0, 1, ..., T-1.$$

The market value at time t of the periodical premium P_t must be equal to the value of n_t units at time t, i.e.,

$$P_t = n_t S_t = d_t + g_t \max\left[S_t - k_t, 0\right], \tag{5}$$

with $k_t = d_t/g_t$.

The time t payoff of the minimum guarantee provision, $P_t - d_t$, is then equal to the payoff of g_t call options on (units of) the mutual fund with exercise price k_t and maturity t.

Observe that the amount of periodical premium depends on the time t value of the mutual fund and, thus, is stochastic.

5.2 The benefit

If death occurs at time τ between t and $t+1$, with $t = 0, 1, ..., T-1$, the benefit $C(\tau)$ is simply the market value at time τ of the accumulated investments in the mutual fund, i.e.,

$$C(\tau) = S_\tau \sum_{j=0}^{t} n_j,$$

whereas the benefit at maturity T, due if the insured is alive, is

$$C(T) = S_T \sum_{j=0}^{T-1} n_j.$$

This contract thus merely represents a way of saving, and does not include any additional coverage against unfavorable events such as death or disability.

5.3 Constant periodical premium

The life insurance policy just described is a pure financial instrument, in which the mortality risk is completely absent from the insurance company's point of view. The mortality risk, indeed, determines only the time to expiration of the policy.

If the insured wishes to pay a fixed periodical premium determined at the inception of the contract, which is common in traditional life insurance, then this premium will be affected by the mortality factors.

To see this, denote by P the constant periodical premium and observe that the market value at time 0 of the stream of constant periodical premiums P, paid at the beginning of each year if the insured is alive, should equal the market value of the stream of time dependent periodical premiums P_t, i.e.,

$$P\sum_{t=0}^{T-1} B_0(t)\,{}_tp_x = \sum_{t=0}^{T-1}\left[d_t B_0(t) + g_t \pi_t(k_t)\right]{}_tp_x,$$

where $\pi_t(\cdot)$ for $t > 0$ is given in expression (2), and $\pi_0(k_0) = \max[S_0 - k_0, 0]$. The right hand side represents the market value at time zero of the periodical premium payments given by expression (5) paid until death or the term of the contract, whatever comes first. The left hand side is simply the similar market value at time zero of the constant periodical premiums P. From this equation P is determined as

$$P = \frac{\sum_{t=0}^{T-1}\left[d_t B_0(t) + g_t \pi_t(k_t)\right]{}_tp_x}{\sum_{t=0}^{T-1} B_0(t)\,{}_tp_x}.$$

Notice that P depends on the survival probabilities.

If, in particular, the amounts to be periodically invested in the mutual fund d_t and the minimum guaranteed numbers of units g_t are constants, i.e., if $d_t = d$ and $g_t = g$ for all t, then

$$P = d + g\frac{\sum_{t=0}^{T-1} \pi_t(k)\,{}_tp_x}{\sum_{t=0}^{T-1} B_0(t)\,{}_tp_x}, \tag{6}$$

with $k = d/g$. The periodical premium for the minimum guarantee provision, $P - d$, is proportional to the ratio between the time 0 value of a portfolio of European call options on the mutual fund, all with the same exercise price but different maturities, and the time 0 value of a portfolio of unit discount bonds with the same maturities of the options and held in the same proportions.

5.4 Financial risk and hedging

By this contract the financial risk exposed to the insured includes:

- the *payment of high premiums*, if the unit price of the mutual fund is "high" at the premium payment dates,

- the *collection of a low benefit*, if the unit price of the mutual fund is "low" when the contract expires.

One of the ideas behind unit-linked insurance is that the fund of linkage may represent a well diversified portfolio in the economy. An example of a well diversified portfolio is the market portfolio whose market value reflects the condition of the economy as a whole. Hence in a situation where the value of the benefit is low, the value of the whole market is low, implying that the amount of benefit is protected in "real terms". Another property of our suggested contract is that in situations when the market value of the fund is high you still receive a minimum number of units. Both this arguments indicate that that the second point above is not as severe as it may seem at first glance.

The first point could be undesirable from the insured's point of view, but we demonstrated in the previous subsection how the contract can be sold with fixed periodical premium payments.

The major financial risk facing the issuer of this contract is the risk of a future high market price of the fund. In this situation the guarantee becomes effective. This risk may be substantially reduced, or even totally eliminated, by the use of one or several hedging strategies explained below.

In the case of time dependent periodical premiums P_t if the insurer acquires n_t units of the mutual fund at each time t during the life of the contract, the benefit is replicated. By this dynamic strategy financial risk is not eliminated since the insurer faces a loss in each period the guarantee becomes effective.

A similar dynamic strategy could be implemented also in the case of constant periodical premiums. The replicating strategy described above would require to invest in the mutual fund, at each time t, an amount exactly equal to P_t, but in the case of constant premium the insurer receives P instead of P_t, and this amount may not be enough for buying the specified number n_t of units of the fund.

To hedge this risk at the inception of the contract the insurer could buy, for each identical and independent policy in his portfolio, a fraction ${}_tp_x$ of contingent-claims with payoff at time t equal to $\max[P_t - P, 0]$, for any t between 0 and $T-1$. Letting

$$h = \frac{\sum_{t=0}^{T-1} \pi_t(k)_t p_x}{\sum_{t=0}^{T-1} B_0(t)_t p_x}$$

and recalling that $P_t = d + g\max[S_t - k, 0]$ and $P = d + gh$, then

$$\max[P_t - P, 0] = g\max[S_t - (k+h), 0],$$

which corresponds to the payoff of g European call options on the mutual fund with exercise price $k+h$ and maturity t. Hence we have identified the relevant contingent claim for hedging as a European call option with exercise price $k+h$. This hedging strategy ensures that the insurer has the amount P_t available at each time t. Note that all financial risk is not eliminated since the amount P_t is not enough to buy the appropriate number of units in periods where the guarantee becomes effective.

Below we discuss how all financial risk may be eliminated from the insurer's point of view already at the inception of the contract. If the insured dies before maturity T, the contract expires and the number of periodical premium guarantees issued by the insurer is reduced. Here we only discuss the case where the insured survives. From a financial risk perspective this case represents in fact the 'worst case'.

There are two immediate ways the insurer may reduce the risk of a future high market price of the fund at the time of initiation of the contract. He can either buy units of the mutual fund or buy call options on the same fund with appropriate exercise prices.

Assuming the insured will survive the term of the contract, the minimum number of units he will be entitled to at time T is $G = \sum_{i=0}^{T-1} g_t$. If the insurer buys G units of the mutual fund at time zero, he will be protected against future high market values of the fund.

The same protection can be obtained by buying g_t call options on the mutual fund with expiration t and exercise price k_t for all t between 0 and $T-1$.

By following the dynamic strategies indicated above for both constant and time dependent periodical premium the insurer will at each time have sufficient investments in the fund to cover the benefit. This strategy is not riskless since the insurer suffers losses in periods where the guarantees are effective.

By appropriate investments in the fund of linkage or European call options all financial risk may be eliminated already at the inception of the contract. Many details of these hedging strategies are left out here, in particular comparisons of capital requirements and associated costs. Our main point here is just to demonstrate that the suggested insurance contract may easily be hedged by standard financial instruments.

5.5 Comparison with the BS-contract

The structure of the constant periodical premium contract presents some analogies, but also a fundamental difference, with respect to the celebrated BS-contract, in which there is no closed form solution for the periodical premium neither under the assumption of deterministic interest rates, see also Delbaen (1986), nor under the Vasicek (1977) model for the short term rate, see Bacinello and Ortu (1994). Also in the BS-contract a fixed part, d, of the periodical premium is deemed to be invested in a mutual fund, but the minimum amount guaranteed, at death or maturity, is *not stochastic*. In our model, instead, this guarantee is expressed in units of the mutual fund, and therefore its monetary value is unknown a priori. This fact, however, may constitute an appealing feature from the insured's point of view and, at the same time, allow him to hedge against alternative sources of economic risk such as inflation, currency devaluation, etc. Observe, indeed, that the reference fund with unit price S_t could be composed of equities, as well as of units of a foreign currency, gold, silver, and so on.

In order to compare our benefit $C(t)$ with the corresponding one in the BS-model, assume now that in case of death during the time interval $(t-1, t]$ this benefit is paid at the end of the year, i.e. at time t instead of at the time of death. The relations of subsection 5.2 are modified in the following way:

$$C(t) = \sum_{j=0}^{t-1} \max\left[gS_t, \frac{d \cdot S_t}{S_j}\right], t = 1, \ldots, T.$$

Observe that

$$C(t) \geq \max\left[gtS_t, \sum_{j=0}^{t-1} \frac{d \cdot S_t}{S_j}\right],$$

so that there is an implicit minimum benefit guaranteed at time t, given by the market value of $g \cdot t$ units of the mutual fund. We recall that in the BS-contract the benefit, that we denote by $C^*(t)$, is instead given by

$$C^*(t) = \max\left[G_t, \sum_{j=0}^{t-1} \frac{d \cdot S_t}{S_j}\right],$$

where G_t represents the minimum amount guaranteed at time t, expressed in the usual unit of account.

It is also interesting to compare the periodical premium for the minimum guarantee provision in both models. As already said, in our model this market price is proportional to the time 0 value of a portfolio of European call options on one unit of the mutual fund. We recall that for the BS-contract the periodical premium, denoted by P^*, is instead given by

$$P^* = d + \frac{E^Q\left[\sum_{t=1}^{T} \alpha_t v(t) \max\left[G_t - \sum_{j=0}^{t-1} d \cdot S_t / S_j, 0\right]\right]}{\sum_{t=0}^{T-1} B_0(t)_t p_x}, \tag{7}$$

where

$$\alpha_t = \begin{cases} {}_{t-1}p_x\left(1 - {}_1p_{x+t-1}\right), & t = 1, ..., T-1 \\ {}_{T-1}p_x, & t = T \end{cases}$$

represents the probability that the policy expires at time t. The periodical premium for the guarantee, $P^* - d$, is then proportional to the value at time 0 of a portfolio of European put options on the accumulated investments in the mutual fund, each one with maturity t and exercise price G_t, see Bacinello and Ortu (1994).

We observe, however, that for the BS-contract the minimum guarantee G_t can be fixed in such a way to supply the insured with an adequate coverage against early death. In our model, instead, the minimum guarantee could reach an adequate level only in the long run so that, even in the case of constant premiums, our policy may represent, from the insured's point of view, mainly an appealing way of investing money, but not a suitable coverage in the case of death during the first years of contract (since the mortality component has the only function of levelling the premium). Anyway, the goal of getting, in the same time, an interesting *financial* and *insurance* product can be easily attained if the insured buys, in addition to the policy here described, a standard term insurance contract.

5.6 Numerical results

In this subsection we present some numerical results for the constant premiums P and P^* defined in expressions (6) and (7) respectively, all obtained under the assumption of a constant volatility structure, i.e. $\sigma_t(u) \equiv \sigma$ for all t, u.

It is easy to check analytically the behavior of P with respect to the parameters on which it depends by the sign of its partial derivatives, all in closed form. In particular this premium is increasing with respect to the initial unit value of the mutual fund S_0, the minimum number of units guaranteed at each premium payment date g, the amount d deemed to be periodically invested in the fund, the instantaneous forward rates $f_0(t)$ prevailing at time 0, the volatility parameters $\sigma, \sigma_1, \sigma_2$ (at least when they are positive), while it is decreasing with respect to the time 0 prices of unit discount bonds $B_0(t)$. It is not a priori clear the behavior of P with respect to the maturity T and to the survival probabilities ${}_tp_x$ (or, alternatively, to the age x of the insured at the inception of the contract).

To study this behavior, to get a numerical intuition for the price of the minimum guarantee provision in our contract, $P - d$, and to compare it with the corresponding price in the BS-model, $P^* - d$ (not in closed form), some numerical examples are presented below.

For comparison, we have fixed the BS-parameter $G_t = gtS_0/B_0(t)$. This quantity can be interpreted as the *riskless* return at time t of the amount gtS_0 invested at time 0 in unit discount bonds with maturity t. If the same amount were invested in the mutual fund, its *stochastic* return at time t, gtS_t, would give exactly the implicit minimum guaranteed benefit in our model, as shown in the previous subsection.

To evaluate the expectation in expression (7) Monte Carlo simulations are employed. To this end we have simulated $1,000,000$ trajectories for the standard Brownian motions W_t^1 and W_t^2 in the time interval $(0, T]$ and used them for building corresponding trajectories of

$$r_t = f_0(t) + \frac{\sigma^2 t^2}{2} + \sigma W_t^1,$$

$$v(t) = \exp\left(-\int_0^t r_u du\right),$$

and

$$S_t = \frac{S_0}{v(t)} \exp\left(-\frac{t}{2}(\sigma_1^2 + \sigma_2^2) + \sigma_1 W_t^1 + \sigma_2 W_t^2\right).$$

As for the term structure of interest rates at time 0, in all our numerical examples we have set $f_0(t) = r_0 + qt$ (so that $B_0(t) = \exp\left(-r_0 t - qt^2/2\right)$), and although in most cases we have fixed $q = 0$ (*flat* term structure), we have also considered *increasing* ($q > 0$) and *decreasing* ($q < 0$) term structures. Moreover, we have fixed $d = g = S_0 = 1$. Finally, we have constructed the probabilities ${}_t p_x$ and α_t from the Italian Statistics for Males Mortality in 1991.

Table 1 reports some results obtained when the maturity T varies between 5 and 15 while the other parameters are fixed.

TABLE 1

$x = 40, f_0(t) = r_0 = 0.04, \sigma = 0.06, \sigma_1 = 0.03, \sigma_2 = 0.2$

T	P	P^*	$P^* - P$
5	1.1630	1.2025	0.0395
6	1.1984	1.2405	0.0421
7	1.2343	1.2801	0.0458
8	1.2711	1.3214	0.0503
9	1.3088	1.3628	0.0540
10	1.3473	1.4060	0.0587
11	1.3865	1.4467	0.0602
12	1.4263	1.4880	0.0617
13	1.4666	1.5246	0.0580
14	1.5073	1.5593	0.0520
15	1.5481	1.5915	0.0434

From Table 1 one can see that both P and P^* are increasing with respect to the maturity T. The price for the minimum guarantee provision is never negligible, and for the BS-contract it is on average 514 basis points (bp) higher than for our contract.

In Table 2 we show the behavior of the premiums with respect to an age of the insured at time 0 between 30 and 50.

TABLE 2

$T = 10, f_0(t) = r_0 = 0.04, \sigma = 0.06, \sigma_1 = 0.03, \sigma_2 = 0.2$

x	P	P^*	$P^* - P$
30	1.3480	1.4067	0.0587
31	1.3480	1.4067	0.0587
32	1.3480	1.4067	0.0587
33	1.3479	1.4067	0.0588
34	1.3479	1.4066	0.0587
35	1.3479	1.4066	0.0587
36	1.3478	1.4065	0.0587
37	1.3477	1.4064	0.0587
38	1.3476	1.4063	0.0587
39	1.3475	1.4061	0.0586
40	1.3473	1.4060	0.0587
41	1.3471	1.4058	0.0587
42	1.3469	1.4055	0.0586
43	1.3467	1.4053	0.0586
44	1.3464	1.4050	0.0586
45	1.3462	1.4047	0.0585
46	1.3459	1.4044	0.0585
47	1.3455	1.4040	0.0585
48	1.3451	1.4036	0.0585
49	1.3447	1.4031	0.0584
50	1.3442	1.4026	0.0584

From Table 2 one can observe that both P and P^* are decreasing with respect to x. However, the absolute differences are small and we are tempted to conclude that the insurer's age has only an imperceptible influence on the premiums. In this connection, we point out that also the use of different mortality tables proved to be almost irrelevant in the premium calculation. To interpret this fact recall that, at least in our model, the mortality component has the only function of levelling the premium. Also in these examples the premium P^* is higher than P, 586 bp on average.

The results reported in Tables 3 to 7 show the behavior of P and P^* with respect to the initial term structure. More precisely, in Table 3 we consider the case of flat term structures, and report the premiums corresponding to different values of the initial spot rate r_0. In Tables 4 and 5 we consider linearly increasing term structures corresponding to two different slopes and to various levels of the initial spot rate, while in Tables 6 and 7 we show similar results obtained when the initial forward rates $f_0(t)$ linearly decrease with respect to their time to maturity t.

TABLE 3

$x = 40, T = 10, f_0(t) = r_0, \sigma = 0.06, \sigma_1 = 0.03, \sigma_2 = 0.2$

r_0	P	P^*	$P^* - P$
0.02	1.2840	1.3364	0.0524
0.03	1.3146	1.3704	0.0558
0.04	1.3473	1.4060	0.0587
0.05	1.3819	1.4430	0.0611
0.06	1.4186	1.4813	0.0627
0.07	1.4572	1.5211	0.0639
0.08	1.4978	1.5620	0.0642
0.09	1.5403	1.6042	0.0639
0.10	1.5846	1.6476	0.0630

TABLE 4

$x = 40, T = 10, f_0(t) = r_0 + 0.001t, \sigma = 0.06, \sigma_1 = 0.03, \sigma_2 = 0.2$

r_0	P	P^*	$P^* - P$
0.02	1.2931	1.3473	0.0542
0.03	1.3242	1.3817	0.0575
0.04	1.3574	1.4176	0.0602
0.05	1.3926	1.4549	0.0623
0.06	1.4298	1.4936	0.0638
0.07	1.4689	1.5336	0.0647
0.08	1.5099	1.5748	0.0649
0.09	1.5529	1.6172	0.0643
0.10	1.5976	1.6608	0.0632

TABLE 5

$x = 40, T = 10, f_0(t) = r_0 + 0.002t, \sigma = 0.06, \sigma_1 = 0.03, \sigma_2 = 0.2$

r_0	P	P^*	$P^* - P$
0.02	1.3022	1.3582	0.0560
0.03	1.3340	1.3930	0.0590
0.04	1.3677	1.4293	0.0616
0.05	1.4034	1.4669	0.0635
0.06	1.4410	1.5059	0.0649
0.07	1.4807	1.5461	0.0654
0.08	1.5222	1.5876	0.0654
0.09	1.5655	1.6302	0.0647
0.10	1.6106	1.6739	0.0633

TABLE 6

$x = 40, T = 10, f_0(t) = r_0 - 0.001t, \sigma = 0.06, \sigma_1 = 0.03, \sigma_2 = 0.2$

r_0	P	P^*	$P^* - P$
0.02	1.2751	1.3256	0.0505
0.03	1.3052	1.3592	0.0540
0.04	1.3373	1.3944	0.0571
0.05	1.3714	1.4310	0.0596
0.06	1.4076	1.4691	0.0615
0.07	1.4457	1.5085	0.0628
0.08	1.4858	1.5493	0.0635
0.09	1.5278	1.5912	0.0634
0.10	1.5717	1.6344	0.0627

TABLE 7

$x = 40, T = 10, f_0(t) = r_0 - 0.002t, \sigma = 0.06, \sigma_1 = 0.03, \sigma_2 = 0.2$

r_0	P	P^*	$P^* - P$
0.02	1.2664	1.3148	0.0484
0.03	1.2959	1.3481	0.0522
0.04	1.3274	1.3829	0.0555
0.05	1.3610	1.4192	0.0582
0.06	1.3966	1.4569	0.0603
0.07	1.4343	1.4961	0.0618
0.08	1.4739	1.5365	0.0626
0.09	1.5154	1.5783	0.0629
0.10	1.5588	1.6212	0.0624

As expected, from Tables 3 to 7 one can notice that the initial spot rate r_0 has a strong influence on the premiums, no matter if the term structure is flat, increasing, or decreasing. Moreover, as analytically established under the assumption $B_0(t) = \exp\left(-r_0 t - qt^2/2\right)$, our premium is an increasing function both with respect to r_0 and with respect to q, and the same behavior can also be numerically verified for the corresponding premium for the BS-contract P^*, which is on the average 605 bp higher than P.

Table 8 reports some results obtained when the parameter σ, characterizing the volatility structure in the Gaussian HJM model, varies between 0 and 0.2 with step 0.01.

TABLE 8

$x = 40, T = 10, f_0(t) = r_0 = 0.04, \sigma_1 = 0.03, \sigma_2 = 0.2$

σ	P	P^*	$P^* - P$
0.00	1.2757	1.2895	0.0138
0.01	1.2806	1.2987	0.0181
0.02	1.2892	1.3162	0.0270
0.03	1.3008	1.3385	0.0377
0.04	1.3148	1.3625	0.0477
0.05	1.3304	1.3855	0.0551
0.06	1.3473	1.4060	0.0587
0.07	1.3649	1.4227	0.0578
0.08	1.3829	1.4350	0.0521
0.09	1.4012	1.4425	0.0413
0.10	1.4195	1.4448	0.0253
0.11	1.4376	1.4421	0.0045
0.12	1.4554	1.4343	-0.0211
0.13	1.4729	1.4218	-0.0511
0.14	1.4899	1.4047	-0.0852
0.15	1.5065	1.3836	-0.1229
0.16	1.5226	1.3588	-0.1638
0.17	1.5380	1.3311	-0.2069
0.18	1.5530	1.3011	-0.2519
0.19	1.5673	1.2695	-0.2978
0.20	1.5811	1.2372	-0.3439

The results displayed in Table 8 are as expected, as far as the constant premium P is concerned: the premium is increasing with respect to (and indeed sensitive to) the volatility parameter σ. The premium P^*, instead, increases only for relatively low values of the volatility parameter, reaches a peak corresponding to the value of $\sigma = 0.1$, before it decreases. Moreover, it is interesting to observe that the difference $P^* - P$ is maximum, equal to 587 bp, when $\sigma = 0.06$, which is just the value for the volatility parameter that we have fixed in all the numerical examples reported in Tables 1 to 7 and 9 to 10.

The difference $P^* - P$ increases rapidly for σ between 0 and 0.06, then it decreases roughly and becomes negative for $\sigma = 0.12$. When $\sigma = 0.2$, $P - P^* = -3439$ bp. This shows that for high values of σ the BS-contract could be cheaper than our contract. In our example a value of σ between 0.11 and 0.12 equates P and P^*.

In Table 9 we show the behavior of P and P^* with respect to the volatility parameter σ_1, which determines (together with σ) the instantaneous correlation between changes in the unit price S_t and changes in the term structure under the martingale measure. More precisely, the results here reported are obtained for σ_1 between -0.2 and 0.2 with step size 0.02.

TABLE 9

$x = 40, T = 10, f_0(t) = r_0 = 0.04, \sigma = 0.06, \sigma_2 = 0.2$

σ_1	P	P^*	$P^* - P$
-0.20	1.3024	1.3504	0.0480
-0.18	1.3008	1.3514	0.0506
-0.16	1.3003	1.3532	0.0529
-0.14	1.3009	1.3558	0.0549
-0.12	1.3026	1.3593	0.0567
-0.10	1.3055	1.3635	0.0580
-0.08	1.3094	1.3684	0.0590
-0.06	1.3143	1.3740	0.0597
-0.04	1.3202	1.3802	0.0600
-0.02	1.3270	1.3870	0.0600
0.00	1.3346	1.3942	0.0596
0.02	1.3429	1.4019	0.0590
0.04	1.3518	1.4101	0.0583
0.06	1.3613	1.4185	0.0572
0.08	1.3713	1.4273	0.0560
0.10	1.3817	1.4364	0.0547
0.12	1.3924	1.4457	0.0533
0.14	1.4034	1.4551	0.0517
0.16	1.4147	1.4647	0.0500
0.18	1.4261	1.4745	0.0484
0.20	1.4376	1.4843	0.0467

From Table 9 one can see that both P and P^* are more sensitive to changes in σ_1 when this parameter is positive. Moreover, while P is slightly decreasing for values of σ_1 between -0.2 and -0.16 and then increasing, the premium P^* is always increasing in our example. As for the difference $P^* - P$, it is always positive, 550 bp on the average.

The last table presented here, Table 10, displays the behavior of P and P^* with respect to the volatility parameter σ_2 characterizing the evolution of S_t, which varies between 0 and 0.5 with step 0.05.

TABLE 10

$x = 40, T = 10, f_0(t) = r_0 = 0.04, \sigma = 0.06, \sigma_1 = 0.03$

σ_2	P	P^*	$P^* - P$
0.00	1.2871	1.3696	0.0825
0.05	1.2918	1.3720	0.0802
0.10	1.3049	1.3792	0.0743
0.15	1.3240	1.3907	0.0667
0.20	1.3473	1.4060	0.0587
0.25	1.3735	1.4243	0.0508
0.30	1.4015	1.4451	0.0436
0.35	1.4307	1.4677	0.0370
0.40	1.4605	1.4914	0.0309
0.45	1.4904	1.5160	0.0256
0.50	1.5201	1.5409	0.0208

From Table 10 one can observe that, as expected, both premiums are very sensitive and increasing with respect to σ_2. Moreover, the BS-premium P^*, although being greater than P, tends to approach it as this volatility increases. The difference $P^* - P$, in fact, reaches 825 bp when $\sigma_2 = 0$ and goes down to 208 bp when $\sigma_2 = 0.5$.

To conclude this section we notice that, at least for the sets of parameters here considered and, in particular, when the volatility σ is sufficiently low, a riskless minimum amount guaranteed G_t is worth more than a minimum guarantee, with the same price at time 0, expressed in number of units of the mutual fund.

6 Concluding remarks

This paper demonstrates how formulas for equity-linked life insurance contracts based on deterministic interest rates may be generalized to stochastic interest rates following the HJM-model. This framework is also suitable for pricing insurance policies with different kinds of benefits and more general contracts involving e.g., more than one life or disability coverage etc. A new product is suggested which is simple both to price and to hedge. The new product is compared with the contract introduced by Brennan and Schwartz (1976).

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