A disaggregated gravity model

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Abstract
The gravity model is used to estimate trip distributions. The estimates are in form of trip frequencies. This study is devoted to an entropy problem where the solution decomposes trip frequencies to the underlying probability distribution.

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Introduction

According to Sen & Smith (1998), the analogy of gravity forces in social sciences can be dated back to Carey and his studies, published in 1858, on the sociological phenomena of group behavior. In modern times, the gravity model is mostly used in areas such as transportation, physical planning, environmental studies, regional economics and geographical analysis. In this paper, a disaggregated gravity model will be presented, and applied on data of commuting flows. This also means that the often-used words, such as commuting and traffic flows are specific for the empirical example. With respect to the model, these words might be substituted by a far more general interaction concept. Somewhat loosely, we might say that the gravity model is an attempt to model a rather common observation: Traffic flow, or a level of interaction, between any pair of nodes is decreasing with the distance between the nodes, and increasing with the population in the nodes. This phenomenon is an analogy to Newton's law of gravity, and thereby the name of this class of models. In the late sixties, A.G. Wilson introduced the concept of entropy maximization in a formal deviation of the gravity model (i.e. Wilson (1967, 1970)). This optimization approach is often referred as the *most probable state approach*, and is similar to the *classical maximum entropy principles* (Golan et al. 1996). Since the seminal work by Wilson, the gravity model has been developed in many directions and it is now possible to derive it from many different principles other than the entropy based most probable state approach. An overview of the class of gravity models is presented in Erlander & Stewart (1990) and Sen & Smith (1998).

The traditional gravity model enables planners to predict traffic flows between any pair of locations. The predictions are in form of positive real numbers. This kind of information may be very useful in many situations, i.e. in evaluating consequences by new or upgraded road links in a transportation network. In this paper, we will consider traffic flows as random numbers. We specify an entropy based gravity model that gives us, a generally unique, probability distribution of the traffic flow between any pair of locations. From the calibrated probability distributions, it is obviously possible to calculate expected traffic flows. In addition, the probabilities give information not contained in the classical gravity model. This disaggregated information might be useful in sensitivity analysis in traffic planning models. New road links or upgraded road links are often based on an expectation of future use. If the expected level of traffic is too low, this is an argument against such investments. On the other hand, a high predicted future usage would provide an argument for the investments. From a planner's point of view, knowledge of the probability for achieving a minimum level of traffic will give the planner information about the risk of the project. Capacity planning is somewhat similar. Here, the planners are concerned about the upper tail of the traffic level distribution.

In this article, we will use this disaggregated information to do some illustrative sensitivity analysis according to the consequences mentioned above. But first the classical gravity model will be presented, and then the classical model is extended to a more disaggregated level.

The Classical Problem

The statistical interpretation of the gravity model, as found in Wilson (1967), is derived from a trip distribution problem within a network with a specific number of zones, or nodes, where each zone potentially might appear both as a point of origin and a point of destination. The transport flows resulting from individual choices of residential and working zones can be specified by an origin-destination matrix (O-D matrix). If it is possible to identify each individual in such a description, these constitute a microstate of the system. On the other hand, the macrostate of the system contains only the total number of workers corresponding to each cell of the O-D matrix. Such a matrix is often labeled a trip matrix. The number of different microstates that results in the same macrostate is given by the multinomial coefficient:

$$W\{T_{ij}\} = \frac{T!}{\prod_{ij} T_{ij}!} \tag{1}$$

Here, $W\{T_{ij}\}$ is the number of microstates that are associated with the trip matrix $\{T_{ij}\}$. i=1,2,…I is an index of origins, and j=1,2,…,J is an index of destinations. T is the total number of commuters in the system, $T = \sum_{ij} T_{ij}$.

By taking the natural logarithms of eq. (1), using the Stirlings approximation and ignoring constant terms, the entropy becomes (the deviation is similar to Golan et al. (1997), p. 8-9):

$$H\{T_{ij}\} = -\sum_{ij} T_{ij} \ln T_{ij} \tag{2}$$

The doubly constrained gravity model results from maximization of the entropy (2) under the following three kinds of restrictions, and a non negativity constraint:

$$\sum_j T_{ij} = O_i \tag{3}$$

$$\sum_i T_{ij} = D_j \tag{4}$$

$$\sum_i \sum_j c_{ij} T_{ij} \le C \tag{5}$$

$$T_{ij} \ge 0$$

Here, O_i is the population of commuters living at origin i. D_j is the number of working places at destination j. c_{ij} is the unity cost of commuting from origin i to destination j. C is a cost restriction for the system.

The solution of the problem is given by (se Erlander and Stewart, 1990):

$$T_{ij} = \exp(\alpha_i + \beta_j - \lambda c_{ij}) \tag{6}$$

Here, α_i, β_j and λ are lagrangeian multipliers. The parameters α_i, β_j are often named balancing factors since they ensure the fulfillment of the marginal constraints (3) and (4) in the trip matrix. In order to use the gravity model (6) in numerical problems, the parameters α_i, β_j and λ need to be estimated. This is usually done by an iterative two-step procedure. First the balancing factors $\{\alpha_i\},\{\beta_j\}$ are calibrated against observed marginals $\{O_i\},\{D_j\}$. Secondly, the essential parameter λ is usually calibrated on information of the total transportation cost in the system C by Newton's method. These steps are repeated until convergence.

In the first step, dealing with the balancing factors, it is rather usual to use a method referred to as the (biproportional) growth-factor, or row-column balancing method. Bregman (1967) proved the convergence of the balancing method, and the method is often denoted as "Bregman method". Formal derivation of the method can for example be found in Lamond and Stewart (1981) or Erlander and Stewart (1990). With the structure of the gravity model as in eq. (6), the *i*-th marginal row restriction eq. (3), might be expressed as:

$$\sum_j \exp(\alpha_i + \beta_j - \lambda c_{ij}) = O_i$$

Solving for α_i, we get:

$$\alpha_i = \ln O_i - \ln \sum_j \exp(\beta_j - \lambda c_{ij}) \quad (7)$$

With similar operations on the column restrictions, the solutions for β_j become:

$$\beta_j = \ln D_j - \ln \sum_i \exp(\alpha_i - \lambda c_{ij}) \quad (8)$$

According to Bregman's method, an iteration step consists of first calculating the balancing factors for the rows, using eq. (7), then calculating the balancing factors for the columns, using eq. (8). The Bregman iterations continue until the marginal constraints (3) and (4) are sufficiently met. With respect to the balancing factors, the relation between Bregman's method and Newton's method is shown to be equivalente (Erlander and Stewart, 1990, pp. 168-170). The computation time used by calibrating the classical gravity model is usual insignificant.

If we are interested in more than just the expected number of commuters in a certain relation we could extend the gravity model in a two-stage fashion. First we solve the classical gravity model and get as a result the expected values for each cell in the trip matrix. Since we know the upper bound for each cell value (min$\{O_i,D_j\}$) , we then solve the I×J entropy problems, one entropy problem for each cell. This procedure gives a probability distribution for each cell. The inherent problem with such an extension is that the interdependence between the probability distributions on the cells are lost. However, this simple extension makes it possible to add cell specific information in the second stage if there exists such information. In the absence of such additional information for at least some of the cells, the second stage problem will result in the trivial solution of uniform within cell probability distribution.

The disaggregated gravity model

In the classical gravity model the prediction of commuters reciding in zone *i* and working in zone *j*, T_{ij}, is a non-stochastic number. It is very likely that such flows of commuters are varying over time. In larger populations, and in populations with high turnover rates, it is probable that the level of commuting is changing on a daily basis. In many circumstances, the randomness of the commuting flow might be a minor problem, that does not call for an implementation in the model. It is possible to consider the predicted flow, T_{ij}, as an expected value. In other cases, such as in capacity planning, the upper tails of the distributions of commuting flows are of major interest, and far more interesting than the expectations (see Galambos, 1987). One simple way to incorporate the randomness in the gravity model, is to add an entropy term for each cell of the trip matrix. That means, instead of using T_{ij} as argument in the objective function, we use an underlying probability distribution $\{p_{ijx_{ij}}\}$. Here, p_{ijx} is the probability that the commuting flow from origin *i* to destination *j* is exactly x_{ij}, and x_{ij} is a random variable, limited to the positive natural numbers. To ensure the marginal constraints, the upper limit for the variable x_{ij} has to be min$\{O_i,D_j\}$. This simple extension gives us following optimization problem:

$$\max H\{p_{ijx}\} = -\sum_i \sum_j \sum_{x_{ij}} p_{ijx_{ij}} \ln p_{ijx_{ij}} \tag{9}$$

$$st. \begin{cases} \sum_j \sum_{x_{ij}} x_{ij} p_{ijx_{ij}} = O_i \\ \sum_i \sum_{x_{ij}} x_{ij} p_{ijx_{ij}} = D_j \\ \sum_i \sum_j c_{ij} \sum_{x_{ij}} x_{ij} p_{ijx_{ij}} \leq C \\ \sum_{x_{ij}} p_{ijx_{ij}} = 1 \\ p_{ijx_{ij}} \geq 0 \end{cases} \tag{10}$$

The solution of this problem is derived from the first order conditions for the probabilities, and gives probabilities of the form:

$$p_{ijk} = \frac{\exp\{(\alpha_i' + \beta_j' - \lambda' c_{ij})k\}}{\sum_{x_{ij}} \exp\{(\alpha_i' + \beta_j' - \lambda' c_{ij})x_{ij}\}} \tag{11}$$

Here, the $\{\alpha_i'\}, \{\beta_j'\}$ and λ' are parameters, similar to the parameters in the traditional gravity model in eq. (6). After calibrating this kind of disaggregated gravity model, the probabilities in eq. (11) are the essential bricks that can be used in studying the probability of every possible trip matrix of interest.

As seen from the table 1 and 2, the classical and the disaggregated gravity model does not give the same expected cell frequencies. It is possible, however, to incorporate cell specific weight factors γ_{ij}, in the objective function in eq. (9):

$$\max H\{p_{ijx}\} = -\sum_i \sum_j \gamma_{ij} \sum_{x_{ij}} p_{ijx_{ij}} \ln p_{ijx_{ij}}$$

If it is important for, some reason, to ensure the similar expected cell frequencies from the disaggregated model as in the classical model, it is easy to find such a set of weight factors.

Calibration and numerical calculus

As mention above, the usual calibration of the classical gravity model is to choose the set of parameter values fulfilling the restrictions (3) to (5). Restriction (5) is commonly represented by an equality. (The calibration process is given a detailed discussion in Erlander and Stewart (1990)). In case of the disaggregated gravity models, it is possible to calibrate the model against the same marginal observations as in the classical case, with the exception that traffic flow is substituted by expected traffic flow. However, there are three main differences in the calibration process of the classical and the disaggregated gravity model. First, in the disaggregated model, each traffic level is decomposed to an expectation based on many probabilities. Hence, a considerable increase in calibration complexity follows. Second, the probabilities in the disaggregated model consist of factors of the form $\exp(u)^x$, where x might be very large. Therefore, even small changes in *u*, might give enormous changes in the probabilities and overflow problems might appear. Third, while the classical gravity model could use the efficient Bregman algorithm, this opportunity does not exist in the disaggregated model. It is not possible to isolate the balancing factors of (11) as done in the Bregman equations (7) and (8). This calls for either use of general methods for solving nonlinear equations, or a development of a specific algorithm for the calibration of the disaggregated gravity model. In the following numerical example, such general methods have been used. The application of gradient methods did not work well on the test problem. The Jacobian matrix was ill conditioned, and standard scaling procedures did not significantly improve the condition. We reformulated the standard calibration process to a minimization problem and minimized the sum of squared residuals from the calibrating equations (the first three equations of restriction (10)). Then we successfully applied an irregular-simplex iteration sequence (see Nelder an Mead, 1965). This method represents a practical opportunity when the dimension of the trip matrix is not too large.

Numerical estimates

In the numerical evaluation of the disaggregated gravity model (11), data from Karmøy, an island on the south-west coast of Norway, were applied. For a detailed presentation of the data see Thorsen and Gitlesen (1998). The island was divided into 11 postal delivery zones, representing 11 nodes in the network. The distance matrix is thereby symmetrical with rank 11. In the calibration we used the marginal observations O_i, D_j and the total distance traveled by commuters, as a cost measure.

We calibrated the classical model and the disaggregated model on the same data set. The estimated trip matrices are presented in tables 1 and 2. It is easy to see that the two different models give different aggregated information.

[Table 1 about here]

[Table 2 about here]

The trip frequencies are not equal. Although the estimates are different, they also seem to represent the same spatial interaction patterns. The correlation between the two trip matrices is 0.96. When the distance between the two matrices is measured by the root of squared cell differences, we get a distance of 507. That means an average distance per cell of 507/121=4.19, or an average distance per trip of 507/6547=0.08. The spread of the cell frequencies seems to be larger in the disaggregated model than in the classical model. The standard deviation of the frequencies in the disaggregated model is 148.42. The frequencies in the classical model have a standard deviation of 123.67. From the classical gravity model, the distance deterrence parameter λ was estimated to be –0.158537. In the disaggregated gravity model the estimate of the distance deterrence parameter $\lambda`$ was -0.00351798. The maximum entropy of the classical model was -3.58722, in the disaggregated model the maximum entropy was -3.48699. To get a similar reference in the entropy measure, we used here the same entropy measure for both models: $-\sum_{ij} \frac{T_{ij}}{T} \ln \frac{T_{ij}}{T}$, $T = \sum_{ij} T_{ij}$.

[Figure 1 about here]

Although it is possible to compare the models in some respects, the disaggregated information is not contained in the classical gravity model. From the disaggregated solution we will just present some examples of the additional information. Figure 1 presents the (cumulative) probability distribution for the number of people living and working in zone 1. From the classical model, we find the expected number of commuters to be 618.28. From the disaggregated model we find the expected number to be 625.58. From the disaggregated information we find the probability for less than 244 employees resides and working in zone 1 is 0.00995. This number is less than 483 with probability 0.01. With probability 0.10, the number is greater than 722. Standard deviation of the number of people living and working in zone 1 is 105.84.

[Figure 2 about here]

Figure 2 presents a somewhat different story. We are here looking at commuting flow from node 1 to node 7. From the classical model we find the estimated number of commuting from origin 1 to destination 7 to be 123.68. From the disaggregated model, the similar estimated expected number is 100.68. From the disaggregated information we find standard deviation to be 99.34. The probability of less than 463 commuters from node 1 to node 7 is 0.99. The probability of less than 232 commuters is 0.90, and it is a probability of 0.99 for at least 10 commuters.

Closing comments

In this paper we have presented an extension of the gravity model based on the entropy principle, and shown how it is possible to estimate the whole probability distribution for each travel frequency in the trip matrix. At an aggregated level, the numerical example suggests that the disaggregated model and the classical model are rather similar with respect to the predicted trip distribution, the differences between the two matrices are rather small. The additional distributional information in the disaggregated model is clearly useful, and it is hard to see how such information could be extracted from other well-accepted principles. Classical statistics is excluded, within this tradition we need observed frequencies. Without an observed trip matrix, classical statistics can’t even give an estimated trip matrix from the classical gravity model. Bayesian statistics might be an opportunity, but we don’t see how to realize such an approach. In the focus of this article, the strength of the entropy principle is clearly also a major weakness: We get a solution from a clearly underdetermined system, but we don’t know how well the solution describe the real word. To study the goodness of fit one will need a large sample of observed trip matrices and measure the distance between observed and predicted distribution of traffic levels. This is probably very costly. The computational difficulties experienced when calibrating the disaggregated model on the test problem show there is a need to develop efficient special purpose calibration algorithms. Secondly, it would be interesting to incorporate more flexible probability distributions.

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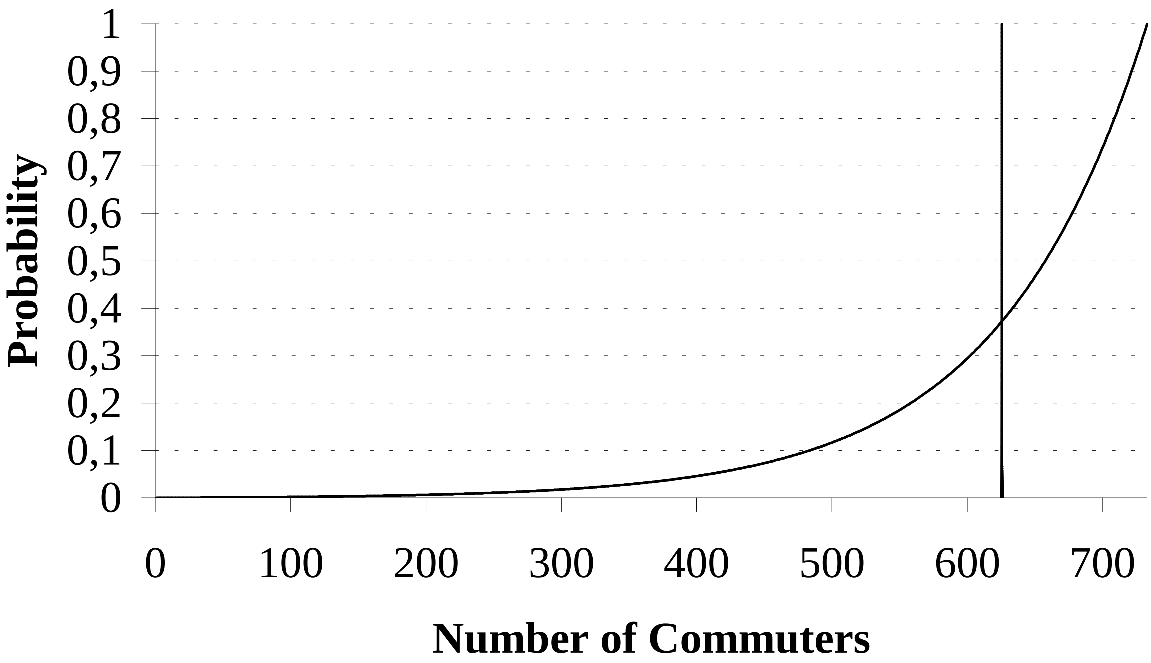

FIGURE 1: The cumulative probability distribution of workers living and working in zone 1. (Expected value is indicated with the vertical line).

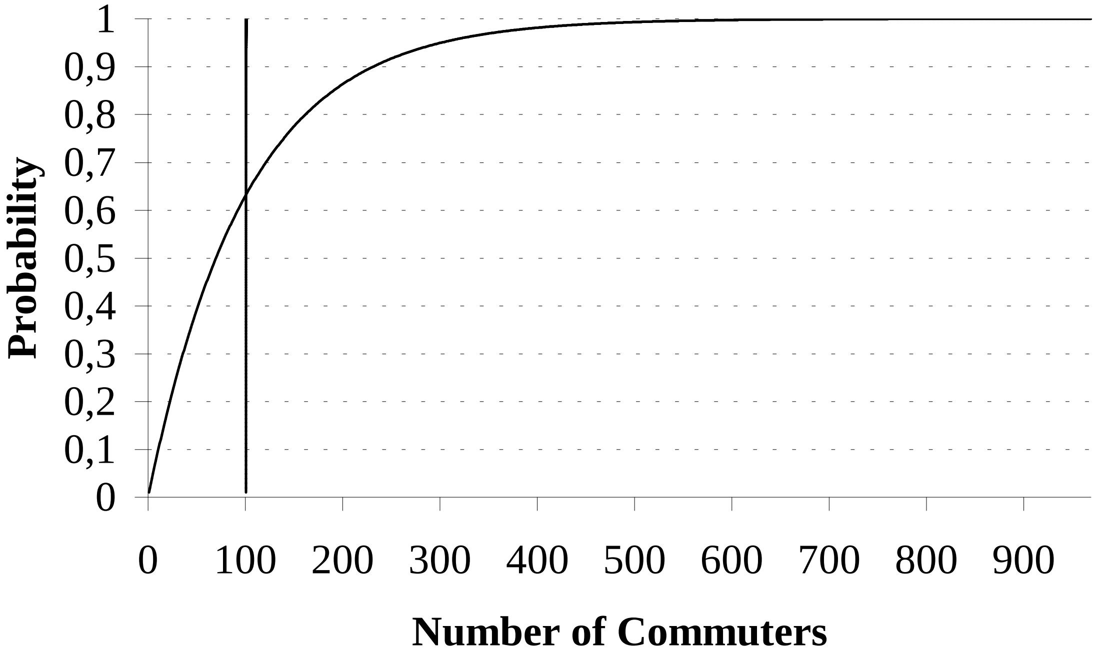

FIGURE 2: The cumulative probability distribution of workers living in zone 1 and working in zone 7. (Expected value is indicated with the vertical line).

TABLE 1: Estimated trip matrix by the classical gravity model

		Destination											
		1	2	3	4	5	6	7	8	9	10	11	Total
Origin	1	618.28	22.87	16.71	81.40	12.91	12.14	123.68	64.85	10.44	0.90	4.82	969
	2	56.23	10.15	7.42	36.14	5.73	5.39	39.99	28.79	4.63	0.40	2.14	197
	3	15.28	2.76	13.51	65.83	10.44	9.82	72.85	52.44	8.44	0.73	3.90	256
	4	19.26	3.48	17.02	294.84	46.75	43.98	326.26	234.87	37.80	3.27	17.47	1045
	5	5.10	0.92	4.51	78.10	17.00	16.00	118.66	85.42	13.75	1.19	6.36	347
	6	4.41	0.80	3.90	67.54	14.70	49.18	193.47	139.28	22.42	1.94	10.36	508
	7	11.76	1.55	7.57	131.16	28.55	50.65	972.68	509.96	82.07	7.11	37.94	1841
	8	0.74	0.13	0.65	11.28	2.46	4.36	60.93	155.92	25.09	0.84	11.60	274
	9	1.15	0.21	1.02	17.63	3.84	6.81	95.21	243.65	139.39	4.66	64.44	578
	10	0.21	0.04	0.19	3.21	0.70	1.24	17.34	17.14	9.81	1.60	4.53	56
	11	0.58	0.10	0.51	8.88	1.93	3.43	47.94	122.67	70.18	2.35	217.43	476
	Total	733	43	73	796	145	203	2069	1655	424	25	381	6547

TABLE 2: Estimated trip matrix by the disaggregated gravity model

		Destination											
		1	2	3	4	5	6	7	8	9	10	11	Total
Origin	1	625.60	5.10	8.72	85.35	16.24	19.97	100.93	69.24	21.48	2.28	14.10	969
	2	24.61	4.80	7.81	36.37	13.18	15.49	31.05	33.36	16.36	2.24	11.75	197
	3	14.38	4.30	8.47	59.65	15.34	18.61	47.20	52.47	19.90	2.27	13.42	256
	4	12.37	4.14	7.77	412.76	19.85	25.88	196.90	317.69	28.57	2.32	16.76	1045
	5	10.70	3.97	7.11	70.93	17.63	22.16	79.53	93.50	24.05	2.30	15.14	347
	6	9.94	3.88	6.79	47.17	15.62	38.15	148.79	190.64	28.11	2.31	16.61	508
	7	9.38	3.73	6.29	28.96	13.03	19.15	1365.50	347.59	28.38	2.31	16.70	1841
	8	7.38	3.49	5.54	16.91	9.97	13.10	32.72	134.99	30.29	2.26	17.34	274
	9	7.43	3.50	5.57	17.15	10.05	13.24	33.72	302.86	148.73	2.35	33.41	578
	10	4.21	2.69	3.61	5.87	4.84	5.39	6.88	7.28	7.06	2.05	6.12	56
	11	7.00	3.42	5.34	14.89	9.27	11.87	25.80	105.39	71.07	2.32	219.64	476
	Total	733	43	73	796	145	203	2069	1655	424	25	381	6547