A Note on a Barrier Exchange Option: The World's Simplest Option Formula?*

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Abstract

The paper analyzes a barrier exchange option that is knocked out the first time the two underlying assets have identical market values. Under rather general conditions regarding the price processes for the underlying assets, probably the world's simplest option pricing formula is derived. It applies both to options of American and European type.

A Barrier Exchange Option

In the standard complete market setting of financial economics, market prices of options are calculated as conditional expected discounted cash-flows, often involving cumbersome calculations. Exotic options are more complex than plain vanilla options, thus, requiring even more cumbersome calculations. One class of exotic options is the so-called barrier options, typically involving the use of the reflection principle or computational demanding numerical methods, see e.g., Reiner and Rubinstein (1991), Boyle

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and Lau (1994), Broadie, Glasserman, and Kou (1997), and Haug and Haug (2002).

Assume the existence of a complete and arbitrage free financial market with continuous and frictionless trading possibilities. There are two risky assets. They are represented by the non-negative continuous stochastic processes $\{S_t^1\}$ and $\{S_t^2\}$, $t \in [0,T]$, where S_t^1 and S_t^2 represent the time t market prices of the assets[1], respectively. They do not pay any dividends and $S_0^1 > S_0^2$. Time T is the maturity date for the options analyzed below.

Exchange options without barriers were first analyzed by Fischer (1978) and Margrabe (1978), and the time T payoff $\hat{\pi}_T$ is given by

$$\hat{\pi}_T = \max(S_T^1 - S_T^2, 0). \tag{1}$$

The knock-out exchange option analyzed here has time T payoff

$$\pi_T = \max(S_T^1 - S_T^2, 0)1\{\min(\frac{S_t^1}{S_t^2}) > 1, \quad t \in [0,T]\}, \tag{2}$$

where $1\{\cdot\}$ is a standard indicator function. The first time S_t^1 hits S_t^2 from above, the option is knocked out. We can think of S_t^2 as a random floor which knocks out the option. Alternatively (and equivalently!), we can think of S_t^1 as a random ceiling that knocks out the option if it is hit by S_t^2 from below. Thus, if the ratio $\frac{S_t^1}{S_t^2}$ hits one, the option is knocked out and gives no payoff.

Proposition. *The time $t \in [0,T]$ market value π_t of an American or European knock-out barrier exchange option as described above, given that the barrier has not been hit prior to time t, is*

$$\pi_t = S_t^1 - S_t^2.$$

Proof. In the absence of arbitrage and intermediate payoffs from the option, the market price of the option is equal to the cost of a self-financing portfolio with the same time T payoff as the option. Consider a portfolio consisting of a long position of one unit of S_t^1 and a short position of one unit of S_t^2. In the case the barrier is not hit before time T, both the option and the portfolio have time T value $S_T^1 - S_T^2 > 0$. In the case where the barrier is hit at some time $t \in (0,T]$, sell S_t^1 and eliminate the short position of S_t^2 by using the proceeds from the sale. Both the option and the portfolio have identical market values (equal to zero) also in this case. Finally, in the "American" case, where early exercise is allowed, the replicating portfolio also duplicates the payoff from early exercise of the option. □

[1]For simplicity we sometimes use the market prices S_t^1 and S_t^2 also to refer to the two risky assets.

The market value of this option is simply the difference between the market values of the two underlying assets. As such, the pricing formula is remarkably simple. In fact, it cannot get much simpler than this and it is also valid for rather general price processes for the underlying assets, requiring only continuity. Even plain vanilla European option pricing formulas become more involved when introducing knock-in/out barriers. The option formulas by Fischer (1978) and Margrabe (1978), where barriers are not included and that depend critically on the assumed log-normality of the underlying price processes, are also more complicated than the formula presented here. Replicating the exchange option in the case with no barriers is also more complicated since it requires continuous rebalancing of the hedge portfolio. It may seem surprising that including a barrier and allowing for more general price processes actually simplify the pricing formula for the option. Also, the replicating strategy is simpler because it only consists of a buy-and-hold strategy, a fact which explains why our formula does not depend on the price dynamics for the risky assets.[2]

Conclusions

We have derived perhaps the simplest option pricing formula possible. The formula prices a barrier exchange option and is applicable for general continuous stochastic price processes for the two underlying assets. The replicating portfolio for the option is a buy-and-hold strategy, and the time t option price is therefore simply the time t price difference between the two underlying assets.

The exchange option analyzed here may have potential use in e.g., capital structure problems in the banking and insurance industry, where also liabilities, in addition to assets are typically random.

References

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[2] Haug and Haug (2002) analyze more general knock-in and knock-out exchange options using the standard log-normal framework, which is more restrictive than our framework. However, they do not explicitly focus on the case considered here, nor do they point out our simple formula.

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