RUN-UP OF LONG WAVES ON AN INCLINED PLANE

by

B. Gjevik and G. Pedersen

Abstract

Non-linear wave reflection on inclined planes is studied. Analytical expressions which describe the run-up and the back-wash are obtained both for single and periodic waves. The maximum run-up height for a single harmonic wave compare well with observations of solitary waves on relatively steep slopes. A simple breaking criterion for the collapse of a solitary wave during the back-wash is deviced. The criterion is found to agree well with observations. From the form of the water surface during the run-up we have estimated the possible water spill-over into a reservoir. Energy consideration indicate that less than 30 per cent of the wave energy may be available for power production.

1. Introduction

Large water waves generated either by seismic activity, landslides or avalanches have in many cases caused deaths and widespread destruction. For this reason wave amplification have been subject for numerous studies. Reviews of the litterature are given by LeMéhaute et al. (1968), Meyer and Taylor (1972), Hails and Carr (1975) and Provis and Radok (1977) and in the article by Hibberd and Peregrine (1979).

Power production based on wave run-up has also been considered see for example Mehlum (1978) and Helstad (1980). According to their plans the water in the run-up wedge may be led into a reservoir where energy can be produced by a conventional hydro-electric power plant.

In this article we shall apply a method originally developed by Carrier and Greenspan (1958) in order to study run-up and reflection of non-linear waves on inclined slopes. Analytical expressions for wave run-up heights are obtained both for single waves and periodic waves and the results for a single wave are compared with experimental observations of solitary wave run-up.

Experiments have shown that long waves reflected from relatively steep slopes may break or collapse during the back-wash after the maximum run-up height has been reached. We have deduced a simple breaking criterion for this type of breaking and compared it with obsevations. Finally we have made some estimates of the fraction of wave energy available for power production based on wave run-up.

2. Basic equations

We shall consider the run-up of long two-dimensional water waves on a plane slope which is inclined an angle θ to the horizontal direction. The bottom topography is drawn in figure 1 with the x-axis horizontal and the y-axis in the vertical direction. The undisturbed water has a uniform depth H for $x \leq -l$ and the slope is described by the line $y=-\alpha x$ for $x \leq -l$ where $\alpha = \tan\theta = H/l$. Initially the water is at rest and the water surface is horizontal. For $t=0$ we prescribe, at $x=-L$, the surface elevation $\eta_0(t)$ corresponding to a wave disturbance propagating toward the slope. We will assume that the amplitude of the disturbance is sufficiently small compared to H and that the typical wave length is sufficiently large compared to H so the wave motion may be described by the linearized shallow water equation for $x \leq -l$. Hence the equations of motion and of continuity are

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}(uH) \qquad (2\text{-}1)$$

where η is the surface displacement and u is the horizontal velocity and g denotes the acceleration of gravity. For $x>-l$ where the water depth becomes shallow non-linear effects will be important. In this region the dominant non-linear terms are retained in the equation of motion and in the equation of continuity. Hence

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}[u(h+\eta)] \qquad (2\text{-}2)$$

where $h=-\alpha x$ denotes the water depth at rest. We follow Carrier and Greenspan (1958) and Carrier (1966) and introduce a new set of independent variables σ,λ defined by

$$\frac{\lambda}{2} = u+\alpha g t$$

$$\frac{\sigma}{4} = \sqrt{g(h+\eta)}$$

For $t=0$, $u=0$ and $\lambda=0$. At $x=-1$ η may be neglected compared to H. Hence $\sigma \cong 4c_0$ where $c_0=\sqrt{gH}$. These variables have the remarkable property that u, when regarded as function of σ and λ, is determined by a linear differential equation. If we write

$$u = \frac{1}{\sigma}\frac{\partial\phi}{\partial\sigma} \tag{2-3}$$

it can be shown that the function ϕ of σ and λ satisfies the equation

$$\frac{\partial^2\phi}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial\phi}{\partial\sigma} - \frac{\partial^2\phi}{\partial\lambda^2} = 0 \tag{2-4}$$

The details are given by Carrier and Greenspan (1958).
The surface displacement is determined by

$$\eta = \frac{1}{g}\left(\frac{1}{4}\frac{\partial\phi}{\partial\lambda} - \frac{u^2}{2}\right) \tag{2-5}$$

and x and t as functions of σ and λ are given by the relations

$$x = \frac{1}{2\alpha g}\left(\frac{1}{4}\frac{\partial\phi}{\partial\lambda} - \frac{u^2}{2} - \frac{\sigma^2}{16}\right) \tag{2-6}$$

$$t = \frac{1}{2\alpha g}(\lambda-2u) \tag{2-7}$$

In order to solve eq. (2-4) we introduce a Laplace transformation of ϕ with respect to $t'=\lambda/2\alpha g$ defined by

$$\tilde{\phi} = \int_0^\infty \phi e^{-st'}dt'$$

With the initial conditions $\phi=0$ and $\frac{\partial\phi}{\partial\lambda}=0$ at $t'=0$ we find

$$\frac{d^2\tilde{\phi}}{d\sigma^2} + \frac{1}{\sigma}\frac{d\phi}{d\sigma} - \left(\frac{s}{2\alpha g}\right)^2\tilde{\phi} = 0$$

The solution of this equation bounded for $\sigma=0$ is

$$\tilde{\phi} = CJ_0(2iq\xi) \qquad (2\text{-}8)$$

where J_0 is the zero order Bessel function, C is an integration constant, $i=\sqrt{-1}$, $q=\frac{sl}{c_0}$, and $\xi=\frac{\sigma}{4c_0}$. At $x=-l$ we assume that the amplitude of the disturbance is small and by neglecting the non-linear terms in eq. (2-5) we have

$$\eta = \frac{1}{g}\frac{1}{4}\frac{\partial\phi}{\partial\lambda}$$

By taking the Laplace transformation of this equation we find

$$\tilde{\eta} = \frac{q}{8c_0 g}\tilde{\phi} \qquad (2\text{-}9)$$

at $x=-l$ or $\sigma=4c_0$. The corresponding velocity is

$$\tilde{u} = \frac{1}{\sigma}\frac{\partial\tilde{\phi}}{\partial\sigma} \qquad (2\text{-}10)$$

In consistence with these approximations we also set $t=t'$ at $x=-l$.

The linear set of equations (2-1) is Laplace transformed with respect to t and a solution which represents an incoming and a reflected wave is easily obtained. At $x=-l$ we assume that u and η are continuous. Hence the solutions for $x>-l$ and $x<-l$ are matched by the conditions (2-9) and (2-10). By neglecting terms corresponding to wave reflections at $x=-L$ we obtain

$$\tilde{\phi} = \frac{16c_0 g}{q}\,\frac{\tilde{\eta}_0 J_0(2iq\xi)e^{-q(\frac{L}{l}-1)}}{J_0(2iq) - iJ_1(2iq)}$$

where $\tilde{\eta}_0$ is the Laplace transform of the incoming wave and J_1 is the Bessel function of order one. By using the Laplace inversion theorem we find ϕ and the corresponding expression for $u=\frac{1}{\sigma}\frac{\partial\phi}{\partial\sigma}$ and $\frac{\partial\phi}{\partial\lambda}$ can be written

$$\frac{\partial\phi}{\partial\lambda} = 4gAI_1$$

$$u = -\frac{4gAI_2}{\sigma}$$

where A is the amplitude of the incoming wave, and I_1 and I_2 are functions of σ and λ defined by

$$I_1 = \frac{1}{A}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{2\tilde{\eta}_0 J_0(2iq\xi)e^{sf}}{J_0(iq)-iJ_1(2iq)}ds$$

$$I_2 = \frac{1}{A}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{2\tilde{\eta}_0 iJ_1(2iq\xi)e^{sf}}{J_0(2iq)-iJ_1(2iq)}ds \qquad (2\text{-}11)$$

where $f=\left(\frac{l\lambda}{2c_0}-L+1\right)/c_0$. The integration path in the complex s-plane is chosen so that the real constant γ is larger than the real parts of all singularities of the integrals.

The surface displacement can be expressed by the functions I_1 and I_2 and from (2-5) we find

$$\frac{\eta}{A} = I_1 - \frac{1}{2}\frac{A}{H}\left(\frac{I_2}{\xi}\right)^2 \qquad (2\text{-}12)$$

By using the relations (2-6) and (2-7) η/A is obtained as function of x and t.

At the point where the water surface meets the sloping beach the water depth is zero and $\sigma=0$. For this value $u=\frac{\partial^2\phi}{\partial\sigma^2}$ and from (2-4) $\frac{\partial^2\phi}{\partial\lambda^2}=2u$. Since $\frac{\partial\eta}{\partial t}=\alpha u$ for $\sigma=0$ the maximum and the minimum values of η occur for $u=0$. If we denote the maximum run-up height by $\eta=R$ we have

$$\frac{R}{A} = I_1 \qquad (2\text{-}13)$$

for $\lambda=\lambda_m=2\alpha g t_m$ where λ_m is determined by the condition

$$\frac{\partial^2\phi}{\partial\sigma^2} = 0 \quad \text{for} \quad \sigma=0. \tag{2-14}$$

3. Solution in special cases

In order to discuss some consequencies of the theory we shall evaluate the integrals (2-11) in two cases. First we shall consider an incoming disturbance which consists of a single harmonic wave defined by the surface elevation at $x=-L$

$$\eta_0(t) = \frac{A}{2}(1-\cos\Omega t) \ , \quad \text{for} \quad 0<t<\frac{2\pi}{\Omega} \tag{3-1}$$

where A and Ω are constants. For $t>\frac{2\pi}{\Omega}$ $\eta_0=0$. The Laplace transformed of this function is

$$\eta_0 = \frac{A}{2}\,\frac{\Omega^2(1-e^{-\frac{2\pi s}{\Omega}})}{s(s^2+\Omega^2)}$$

This expression for η_0 leads to poles in the integrals (2-11) for $s=0$ and $s=\pm i\Omega$. The denominators in the integrals are also zero for s_n and $\bar{s}_n$ where $\bar{s}_n$ denotes the complex conjugate of s_n. Values of s_n are given in Appendix 1.

By standard complex integration techniques we find for $0<\Omega f<2\pi$

$$I_1 = 1-\frac{J_0(\kappa\xi)}{J_0^2(\kappa)+J_1^2(\kappa)}\left[J_0(\kappa)\cos\Omega f+J_1(\kappa)\sin\Omega f\right]+r_1(\Omega f) \tag{3-2}$$

$$I_2 = \frac{J_1(\kappa\xi)}{J_0^2(\kappa)+J_1^2(\kappa)}\left[-J_0(\kappa)\sin\Omega f+J_1(\kappa)\cos\Omega f\right]+p_1(\Omega f) \tag{3-3}$$

where $r_1(\Omega f)$ and $p_1(\Omega f)$ represent the contribution from the poles s_n and $\bar{s}_n$. (See Appendix 1)

The parameter κ is defined by

$$\kappa = \frac{2\Omega l}{c_0}$$

For $\Omega f\ 2\pi$ only the poles s_n and $\bar{s}_n$ contribute to the integral and we find

$$I_1 = r_1(\Omega f)-r_1(\Omega f-2\pi) \qquad (3\text{-}4)$$
$$I_2 = p_1(\Omega f)-p_1(\Omega f-2\pi)$$

The front of the disturbance will travel from $x=-l$ to a position x_f where $-l<x_f<0$ in a time span

$$t_f = \int_{-l}^{x_f} \frac{d}{c_0(-\frac{x}{l})^{\frac{1}{2}}} = \frac{2l}{c_0}[1-(-\frac{x_f}{l})^{\frac{1}{2}}]$$

Since u and η are zero ahead of the disturbance

$$\left(\frac{x_f}{c_0}\right)^{\frac{1}{2}} = \frac{\sigma}{4c_0} = \xi$$

and we have

$$\Omega t_f = \kappa(1-\xi)$$

Hence both I_1 and I_2 are zero for $f<t_f$. This means that the series $r_1(\Omega f)$ and $p_1(\Omega f)$ can be expressed by the first terms on the right hand side of (3-2) and (3-3) respectively. It can also be shown that the contributions from the series $r_1(\Omega f)$ and $p_1(\Omega f)$ are negligible except for small values of Ωf and large values of κ. For $\Omega f=\pi$ the terms in the series corresponding to the poles s_1 and $\bar{s}_1$ are deminished by a factor $\exp(-\pi\ \ln(4\pi)/2\kappa)$ compared to the value for $\Omega f=0$. When $\kappa=2$ this factor is 0.14 and when $\kappa<1$ the factor is less than 0.02. Hence we may neglect r_1 and p_1 in (3-2) and (3-3) except in the first stage of the run-up. The maximum run-up height, $\eta=R$, therefore occurs for $f=f_r$ where

$$\tan \Omega f_r = \frac{J_1(\kappa)}{J_0(\kappa)} \tag{3-5}$$

This corresponds to a time

$$t_r = f_r/\Omega + (L-l)/c_0$$

and

$$\frac{R}{A} = 1 + \frac{1}{[J_0^2(\kappa)+J_1^2(\kappa)]^{\frac{1}{2}}} \tag{3-6}$$

The expression (3-6) shows that the ratio R/A depends only on the parameter κ. For $\kappa \to 0$ $R/A \to 2$ which corresponds to reflection at a vertical wall $(l=0)$ or reflection of infinitely long waves $(\Omega=0)$ by a sloping beach. For $\kappa>0$ the ratio R/A is always larger than 2. Similarly we find that the maximum back-wash, $\eta=S$ occurs for $f_s=f_r+\pi$ where f_r is defined by (3-5). Hence we have that

$$\frac{S}{A} = 1 - \frac{1}{[J_0^2(\kappa)+J_1^2(\kappa)]^{\frac{1}{2}}} \tag{3-7}$$

which shows that also the ration S/A depends only on the parameter κ.

The surface displacement at the time of maximum run-up penetration or back-wash is

$$\frac{\eta}{A} = 1 \pm \frac{J_0(\kappa\xi)}{[J_0^2(\kappa)+J_1^2(\kappa)]^{\frac{1}{2}}} \tag{3-8}$$

where $\xi=\frac{\sigma}{4c_0}$ is defined by (2-6). The upper and lower sign in (3-(3-8) refer respectively to the run-up and the back-wash state. Since ξ is a function of $\frac{x}{l}$, κ and $\frac{A}{H}$ the form of the surface displacement at the time of maximum run-up or back-wash height will depend both on κ and the ratio A/H.

The formulas (3-5)-(3-8) lead to approximative values of the parameters R/A, S/A, f_r and f_s which are in good agreement with the values obtained from (3-2)-(3-4) when the series $r_1(\Omega f)$ and $p_1(\Omega f)$ are retained. The two sets of parameters are shown in table 1 for values of κ between 1 and 5.

Table I

κ	Run-up				Back-wash			
	Correct		Approximate		Correct		Approximate	
	$\frac{\Omega f_r}{\pi}$	$\frac{R}{A}$	$\frac{\Omega f_r}{\pi}$	$\frac{R}{A}$	$\frac{\Omega f_s}{\pi}$	$\frac{S}{A}$	$\frac{\Omega f_r}{\pi}$	$\frac{S}{A}$
1.0	1.16	2.13	1.17	2.13	2.16	-0.13	2.17	-0.13
2.0	1.40	2.61	1.38	2.62	2.40	-0.62	2.38	-0.62
3.0	1.72	3.08	1.71	3.34	2.72	-1.26	2.71	-1.34
4.0	2.00	3.50	2.05	3.48	2.96	-1.63	3.05	-1.48
5.0	2.36	3.87	2.34	3.68	3.00	-2.42	3.34	-1.68

The form of the water surface at the time of maximum run-up and maximum back-wash for $\kappa=3$ and A/H=0.1 is shown in figure 2.

In the next case we consider the incoming disturbance consists of a train of periodic waves. Hence the surface elevation at $x=-L$ is

$$\eta_0 = A \sin \Omega t \quad \text{for} \quad t>0 \tag{3-9}$$

the Laplace transform of this function is

$$\eta_0 = A \frac{\Omega}{s^2+\Omega^2}$$

By proceeding in the same way as in the previous case we find

$$I_1 = \frac{2J_0(\kappa\xi)}{J_0^2(\kappa)+J_1^2(\kappa)}[J_0(\kappa)\sin\Omega f - J_1(\kappa)\cos\Omega f] + r_2(\Omega f)$$

$$I_2 = -\frac{2J_1(\kappa\xi)}{J_0^2(\kappa)+J_1^2(\kappa)}[J_0(\kappa)\cos\Omega f + J_1(\kappa)\sin\Omega f] + p_2(\Omega f) \qquad (3\text{-}10)$$

where $r_2(\Omega f)$ and $p_2(\Omega f)$ represent the contribution from the poles s_n and $\bar{s}_n$. r_2 and p_2 are given in Appendix 1. Also in this case the series r_2 and p_2 can be neglected except in the first stage of the run-up.

When the steady state is established the maximum run-up, $\eta=R$, occurs for $f=f_r$ where

$$\tan \Omega f_r = -\frac{J_0(\kappa)}{J_1(\kappa)} \qquad (3\text{-}11)$$

and

$$\frac{R}{A} = \frac{2}{[J_0^2(\kappa)+J_1^2(\kappa)]^{\frac{1}{2}}} \qquad (3\text{-}12)$$

The corresponding surface displacement is

$$\frac{\eta}{A} = \frac{2J_0(\kappa\xi)}{[J_0^2(\kappa)+J_1^2(\kappa)]^{\frac{1}{2}}} \qquad (3\text{-}13)$$

where ξ is a function of x. Again we see that the ratio R/A

depends only on the parameter κ, while the form of the surface at the time of maximum run-up depends both on κ and the ratio A/H. The expression (3-12) also shows that the maximum run-up height for a periodic wave is larger than for the single wave (3-6).

The maximum back-wash, $\eta=S$, is in this case given by $S=-R$ and the surface displacement at the time of the back-wash is given by (3-13) with a change of sign. The water surface at the time of maximum run-up and maximum back-wash for $\kappa=3$ and $A/H=0.1$ is depicted in figure 3.

Experiments show that even relatively long waves often break during the back-wash of the water. In order to establish a criterion for when breaking occurs we shall consider the slope of the water surface at $\xi=0$ which corresponds to the edge of the water wedge. The surface displacement for a single wave is given by (3-8) provided the lower sign is used. We also have that

$$\frac{x}{l} = \frac{A}{H}\left[1-\frac{J_0(\kappa\xi)}{\left(J_0^2(\kappa)+J_1^2(\kappa)\right)^{\frac{1}{2}}}\right] - \xi^2$$

Hence the expression for $\frac{\partial\eta}{\partial x}$ is obtained by differensiating with respect to x. We find

$$\left(\frac{\partial\eta}{\partial x}\right)_{\xi=0} = -\frac{A/l}{\dfrac{4\left[J_0^2(\kappa)+J_1^2(\kappa)\right]^{\frac{1}{2}}}{\kappa^2} - \dfrac{A}{H}}$$

Breaking during back-wash may occur if $\frac{\partial\eta}{\partial x}$ becomes infinite i.e. the breaking criterion for a single wave given by (3-1) may be

$$\frac{A}{H} \geqslant \frac{4\left[J_0^2(\kappa)+J_1^2(\kappa)\right]^{\frac{1}{2}}}{\kappa^2} \tag{3-14}$$

In a similar way we find a breaking criterion for the periodic waves given by (3-9)

$$\frac{A}{H} \geqslant \frac{2\left[J_0^2(\kappa)+J_1^2(\kappa)\right]^{\frac{1}{2}}}{\kappa^2}$$

4. Comparison with experiments

Measurements of the run-up, R, of solitary waves on inclined planes with bottom topography as shown in figure 1 are described by Wiegel (1964), Arntsen (1978) and Langsholt (1981). They found that there is a nearly linear relationship between the amplitude, A, and the run-up height R. Langsholt measured the run-up of solitary waves on inclined planes of different roughness. He found that the run-up was strongly effected by friction especially for small inclination angles. The relation between R and A given by Wiegel and Langsholt can be written

$$\frac{R}{A} = K\left(\frac{A}{H}\right)^{\alpha} \qquad (4\text{-}1)$$

where the parameters K and α are determined by a least-square fit to the data.

Wiegel found that K and α depend on the inclination angle, θ, and he determined K and α for θ values between 5° and 45°. In the recent study by Langsholt K and α is also found to depend on water depth and the roughness parameter of the plane.

On basis of the theoretical predictions in section 3 it is possible to deduce a relation between the run-up and the amplitude for a solitary wave. The surface displacement corresponding to a solitary wave propagating over a uniform depth is

$$\eta = A \cosh^{-2}\phi(t) \qquad (4\text{-}2)$$

where A is the amplitude and

$$\phi = \left(0.75\,\frac{A}{H}\right)^{\frac{1}{2}} \frac{c_0 t}{H}$$

Although the front of a solitary wave is not well defined the time span t_h from the peak to half the peak displacement is easily measured in an experiment. The solitary wave may be approximated by a single harmonic wave (3-1) and we assume that

$$\Omega t_h = \frac{\pi}{2}$$

The matching between the harmonic wave (3-1) and the solitary wave (4-1) is shown in figure 4. Since $\phi(t_h)=0.8814$ we find

$$\kappa = \frac{3.087}{\tan\theta}\left(\frac{A}{H}\right)^{\frac{1}{2}} \qquad (4-2)$$

Hence for given values of water depth, amplitude and inclination angle we may estimate the parameter κ for a solitary wave and use the theory in section 3 to determine the corresponding run-up. The results of these computations are displayed by the graphs in figure 5. It turnes out that for 0.1 A/H 0.4 the relationship between the parameters A/H and R/A can to high degree of accuracy be approximated by the function (4-1). Table II shows values of the parameters K and α computed from this theory and from Wiegel's and Langsholt's experimental data.

Table II

	Theory		Langsholt		Wiegel	
θ	α	K	α	K	α	K
45°	0.15	2.95	-	-	0.15	3.05
30°	0.18	3.81	0.16	3.24	0.13	3.28
20°	0.23	4.94	0.13	3.60	0.15	3.48
15°	0.22	5.58	-	-	0.12	3.62

For inclination angles larger than 30° the theoretical and the experimental results agree well and the difference in R/A is less than 10 per.cent. For inclination angles smaller than 30°.the difference becomes larger and for $\theta=15^\circ$ the theory predicts considerably higher values of R/A than observed. The reason for the discreapancy may mainly be due to frictional effects which becomes more important for small values of θ especially at the edge of the water wedge. This is clearly illustrated by Langsholt who determined the parameters α and K for different water depth, H, between 10 and 25 cm. Langsholt found that for $\theta=30^\circ$ α and K were nearly independent of H provided $H \geqslant 15$ cm. For $\theta=12^\circ$, however, K was found to vary considerably with depth. Another source of error in the theoretical results may be dispersion effects and non-linear effects (for $x<-1$) which are neglected in the present theory. Recently Pedersen (1981) have included these effects and found a better agreement with experimental results for values of θ less than 30°.

Experiments show that a solitary wave reflected from an inclined plane may break during the back-wash state and Arntsen (1978) observed for which values of A/H and θ breaking occured. His results are displayed in figure 6 and these results may be compared with predictions made on basis of the breaking criterion (3-14). For $\kappa>1$ we may use asymptotic expansions for the Bessel functions. Hence the square root in (3-14) may be approximated by $(\frac{2}{\pi\kappa})^{\frac{1}{2}}$. By using this result we finally obtain from (3-14) and (4-2) the following breaking criterion for a solitary wave

$$\frac{A}{H} \geqslant 0.479(\tan\theta)^{\frac{10}{9}}$$

The lower bound on A/H is depicted by dotted line in figure 6 and the close correspondence with Arntsen's observations is streaking.

5. Some consideration regarding power production by wave run-up

The possibilities of power production by water waves has recently been investigated by several research groups. The wave motion may for example be used for pumping water into a reservoir above sea level where electricity is produced by a conventional hydro-electric power plant (Helstad, 1980). It has been argued that as much as 70-80 per cent of the energy in the waves at sea level would be converted to potensial energy at the reservoir. With the present theory we are able to make an estimate of this energy conversion factor. At the time of maximum run-up consider the amount of water which has crossed a vertical plane P which intersect the beach at a level R_0 (see figure 1). The slope of the water surface at the edge of the water ($\xi=0$) can easily be obtained in a similar way as explaned in section 3. For periodic waves given by (3-9)

$$\left(\frac{\partial\eta}{\partial x}\right)_{\xi=0} = p \tan\theta$$

where

$$p = \cfrac{A/H}{A/H + \cfrac{2[J_0^2(\kappa)+J_1^2(\kappa)]^{\frac{1}{2}}}{\kappa^2}}$$

By simple geometrical considerations we find that the amount of water (per. unit length along the wave front) is

$$V = \tfrac{1}{2}(1-p)(R-R_0)^2/\tan\theta$$

If this amount of water fills a large reservoir at a level R_0 it corresponds to a potensial energy (per. unit length) $\Delta E=\rho g V R_0$ where ρ is the density of water. The energy (per.unit width) of the incoming wave front is $E = \frac{1}{2}\rho g A^2 \lambda$ where $\lambda = \frac{2c_0}{\Omega}$ is the wave length. Hence the energy conversion factor may be written

$$\frac{\Delta E}{E} = \frac{\kappa}{4\pi}(1-p)\left(\frac{R-R_0}{A}\right)^2 \frac{R_0}{H}$$

For $R_0=R/3$, $\frac{\Delta E}{E}$ has a maximum value

$$\left(\frac{\Delta E}{E}\right)_m = \frac{\kappa}{27\pi}(1-p)\left(\frac{R}{A}\right)^3 \frac{R}{H}$$

This expression shows that $\left(\frac{\Delta E}{E}\right)_m$ increases with κ and A/H but within the range where this theory is valid we find that $\left(\frac{\Delta E}{E}\right)_m$ is less than 0.3. Let us consider an example which may be relevant for power production by water waves. We take $H=35m$, $\theta=18.7^\circ$, $A/H=0.2$ and the wave period is 10s. In this case $\kappa=7$, $R/A=6.62$ and $\left(\frac{\Delta E}{E}\right)_m=0.28$. For these relatively large values of κ and A/H wave breaking may occur and this effect together with frictional effects most likely reduce the energy conversion factor considerably. Although the estimate relay on very simple arguments it indicates that the energy conversion factor may be well lower 30 per cent in a power plant based on wave run-up.

6. Concluding remarks

In the previous section we found that for inclination angles $\theta<30^\circ$ there are relatively large discreapancies between computed and observed run-up heights. The former values are larger than the latter. For $\theta=30^\circ$ the difference is about 5-10% and for $\theta=15^\circ$ about 20-40%. This discreapancy is mainly due to frictional effects, capillary effects, dispersion and non-linear effects which may be insufficiently modelled by the present theory. Errors may also be induced by approximating the solitary wave by a single harmonic wave component.

Friction is probably the most important source of error for small angles where the discreapancies between theoretical and experimental results are largest. In these cases the run-up wedge ends in a long and very thin layer (compared to H). In experiments, where the water depth H is small, this layer is obviously strongly effected by friction. This is clearly demonstrated by Langsholt's experiments where the ratio R/A is found to increase strongly with H. For this reason one may expect that the computed run-up height will agree better with observation for large scale phenomena where frictional effects (and also cappilary effects) are less important. These cases are probably better described by our theory than by small scaled experiments.

In the theory the solitary wave is approximated by a shorter and steeper harmonic wave component (fig. 4). Since the run-up height is larger for shorter waves than for longer waves (table 1) this may lead to an overestimate of R/A. We have also neglected non-linear terms (when $x<-L$) and dispersion effects. These effects tend to cancel each other for solitary waves and the approximation may be justified when the typical wave length is large compared to 1 (i.e. for small values of κ). The results by Pedersen (1981), based on a numerical study of solitary wave run-up, indicate that the error in R/A due to the latter approximations is less than 10% for $\kappa<4$.

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Appendix 1.

We write the denominator in the integrals (2-11)

$$N(s) = J_0(2i\frac{sl}{c_0}) - iJ_1(2i\frac{sl}{c_0})$$

By using relations for Bessel functions we find

$$N(\bar{s}) = \overline{N(s)} \quad \text{(A-1)}$$

where the bar denotes complex conjugate. If $s = s_n$ is a root of the equation

$$N(s) = 0 \quad \text{(A-2)}$$

it follows from (A-1) that $s = \bar{s}_n$ is a root of the equation.

Approximate values for the roots can be found by using asymptotic expansion of the Bessel functions. We introduce a complex variable defined by

$$z = \frac{2sl}{c_0} e^{-i\frac{\pi}{2}}$$

Hence $N(s)$ can be written

$$N(z) = J_0(z) + iJ_1(z)$$

and the roots of (A-2) are given by z_n and $N(z_n)=0$. By retaining the second order terms in an asymptotic expansion of $J_0(z)$ and $J_1(z)$ for large values of z we find that the equation $N(z_n) = 0$

can be written

$$z_n + \frac{i}{8} = \frac{1}{4} e^{-2iz_n}$$

An approximate solution of this equation is $z_n = R_n e^{i\varepsilon_n}$ where

$$R_n = n\pi$$

and

$$\sin\varepsilon_n = \ln(4R_n)/2R_n$$

and $n=1,2,3\ldots$. The accuracy of the approximation is rather good and for $n=1$ the error in R_n is about 3%. Once the roots of equation (A-2) is known the residual contribution to the integrals (2-11) from the poles s_n and $\bar{s}_n$ can easily be found. We note that

$$\left(\frac{dN}{ds}\right)_{s=s_n} = \frac{J_0(z_n)}{s_n}$$

Hence we find that the residual contributions corresponding to an incoming wave defined by (3-1) are

$$r_1(\Omega f) = 2\mathrm{Re} \sum_{n=1}^{\infty} \frac{\kappa^2}{\kappa^2 - z_n^2} \frac{J_0(z_n\xi)}{J_0(z_n)} e^{\frac{iz_n\Omega f}{\kappa}}$$

and

$$p_1(\Omega f) = 2\mathrm{Im} \sum_{n=1}^{\infty} \frac{\kappa^2}{\kappa^2 - z_n^2} \frac{J_1(z_n\xi)}{J_0(z_n)} e^{\frac{iz_n\Omega f}{\kappa}}$$

The residual contributions corresponding to a periodic wave train (3-7) are

$$r_2(\Omega f) = -4\mathrm{Re} \sum_{n=1}^{\infty} \frac{i\kappa z_n}{\kappa^2 - z_n^2} \frac{J_0(z_n\xi)}{J_0(z_n)} e^{\frac{iz_n\Omega f}{\kappa}}$$

and

$$p_2(\Omega f) = -4\mathrm{Im} \sum_{n=1}^{\infty} \frac{i\kappa z_n}{\kappa^2 - z_n^2} \frac{J_1(z_n\xi)}{J_0(z_n)} e^{\frac{iz_n\Omega f}{\kappa}}$$

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Figure Captions

Figure 1 Geometry for the model.

Figure 2 Surface displacement for single wave $\kappa=3$, $A/H=0.1$, a) maximum run-up, b) back-wash.

Figure 3 Surface displacement for periodic wave $\kappa=3$, $A/H=0.1$, a) maximum run-up, b) back-wash.

Figure 4 Surface displacement for single wave (full drawn line) and solitary wave (dotted line) with $\Omega t_h=\pi/2$.

Figure 5 Wave run-up for solitary wave. Computed run-up height for inclination angles 15°, 20°, 30° and 45° (full drawn lines). Observed run-up height (Wiegel, 1964) for 30° and 45° (dotted lines).

Figure 6 Breaking of solitary waves during back-wash. Arntsen's (1978) observational results: ○ no breaking, △ indication of breaking, □ breaking waves. Breaking criterion with dotted line.

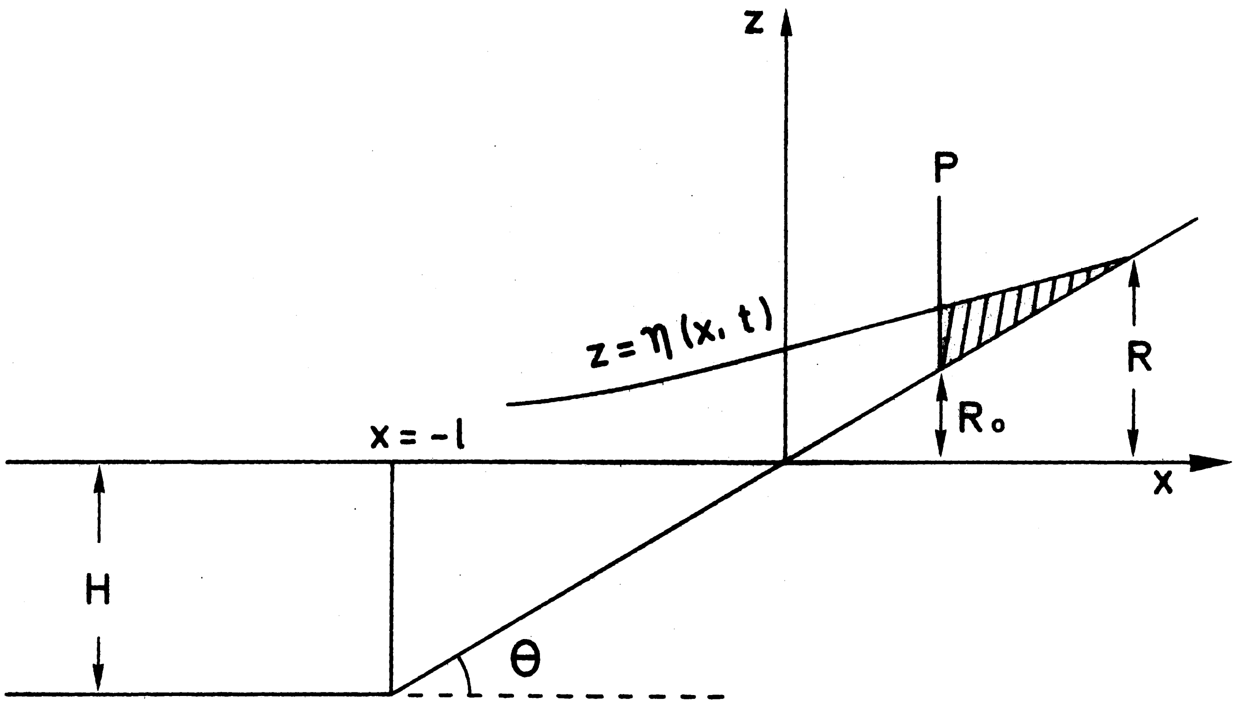

Fig 1

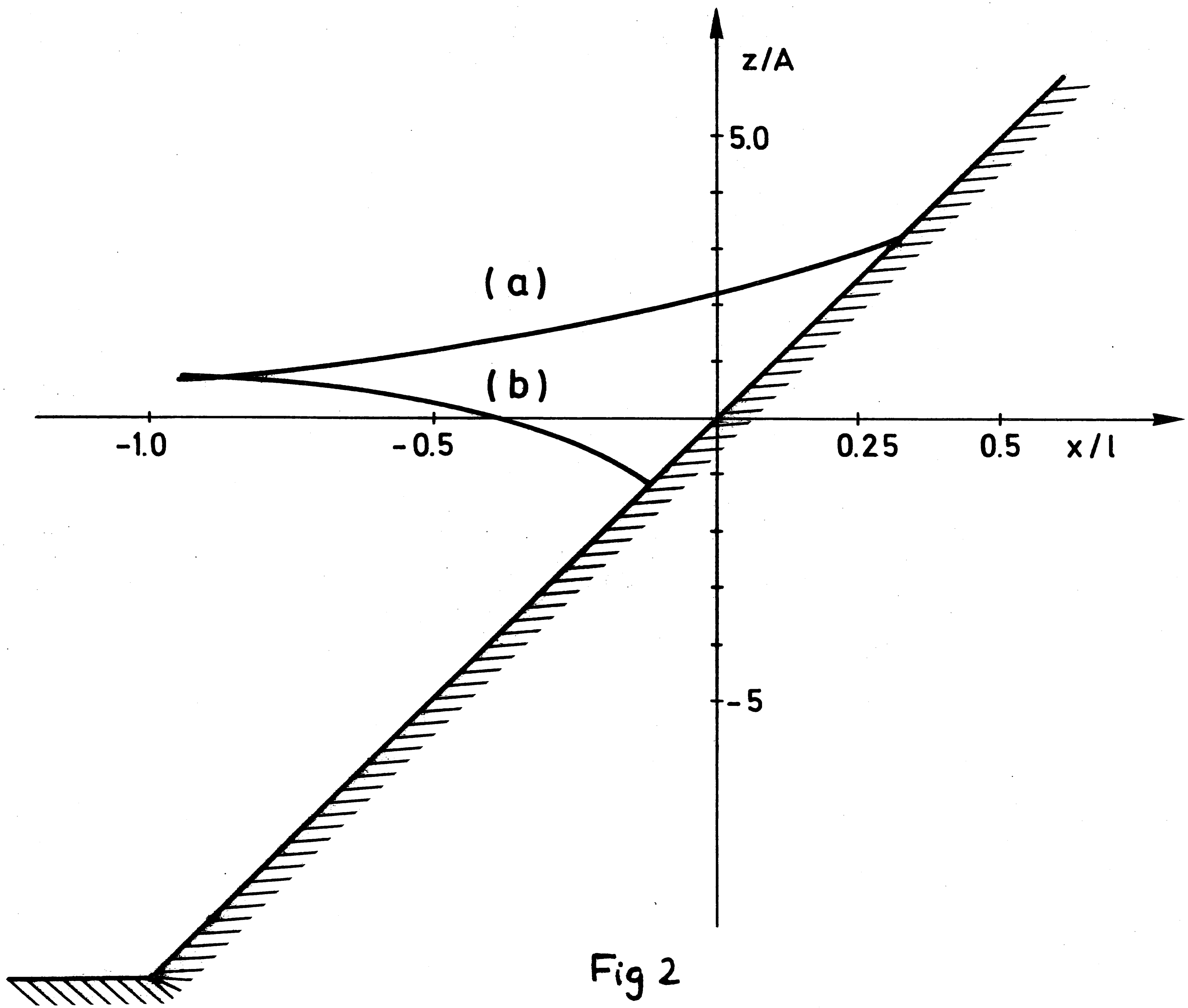

Fig 2

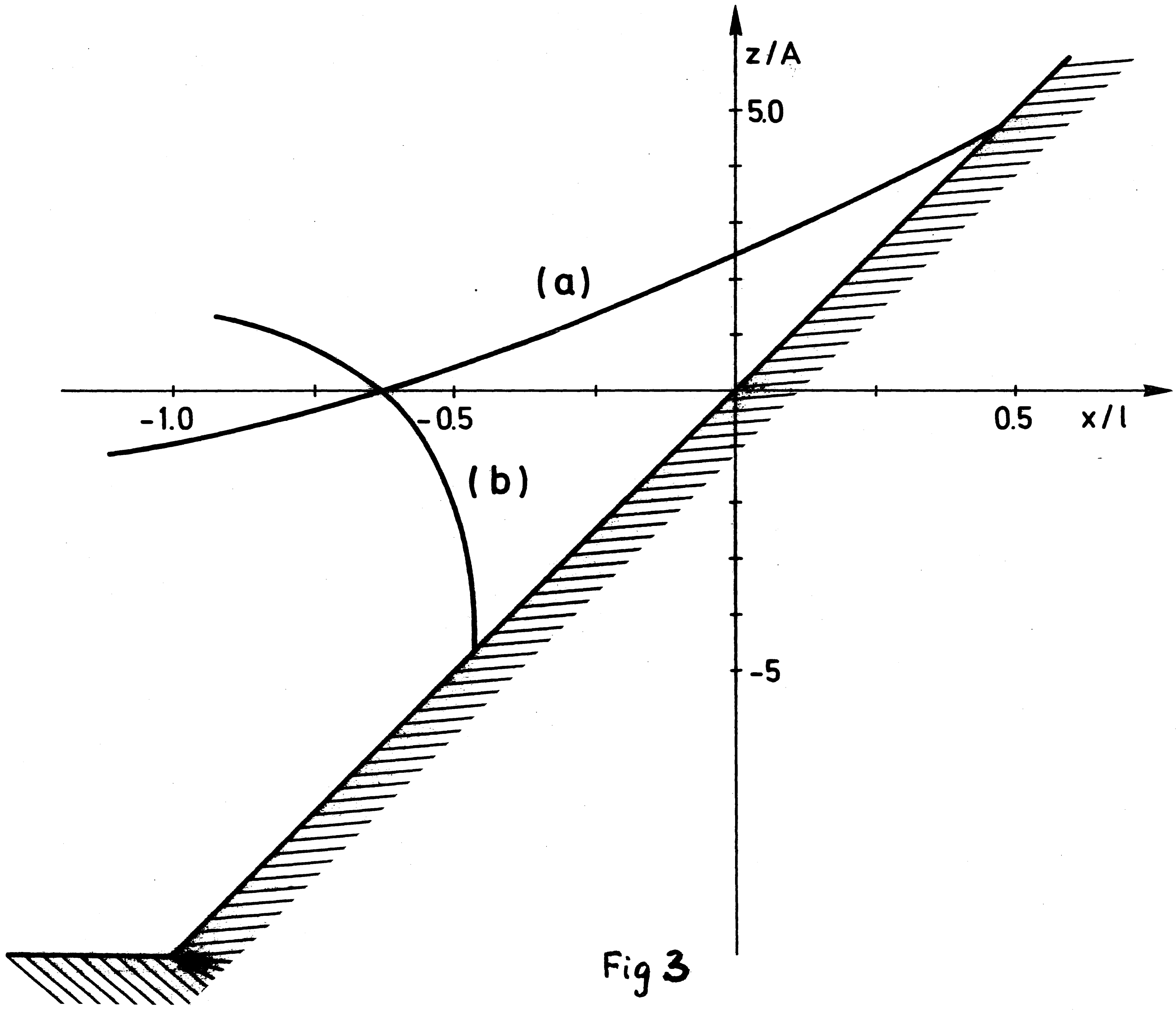

Fig 3

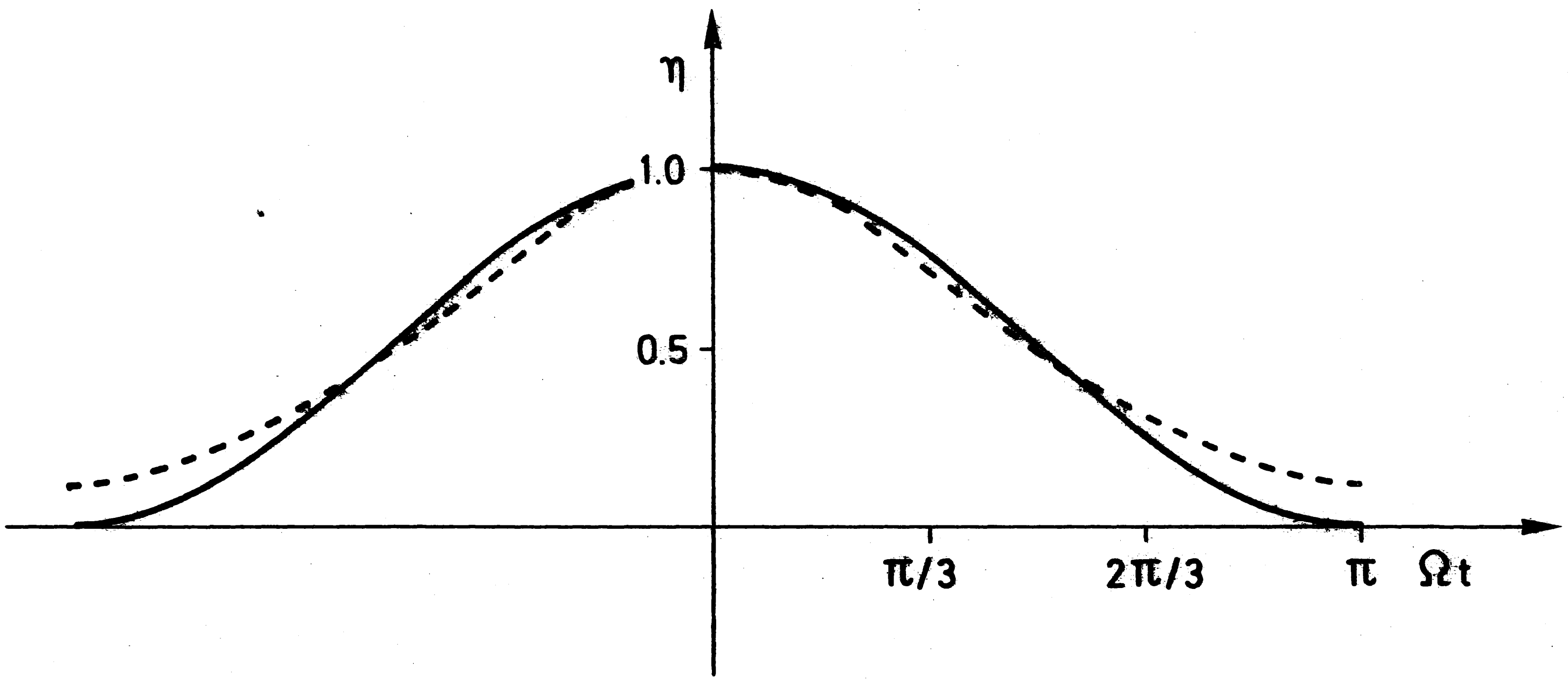

Fig 4

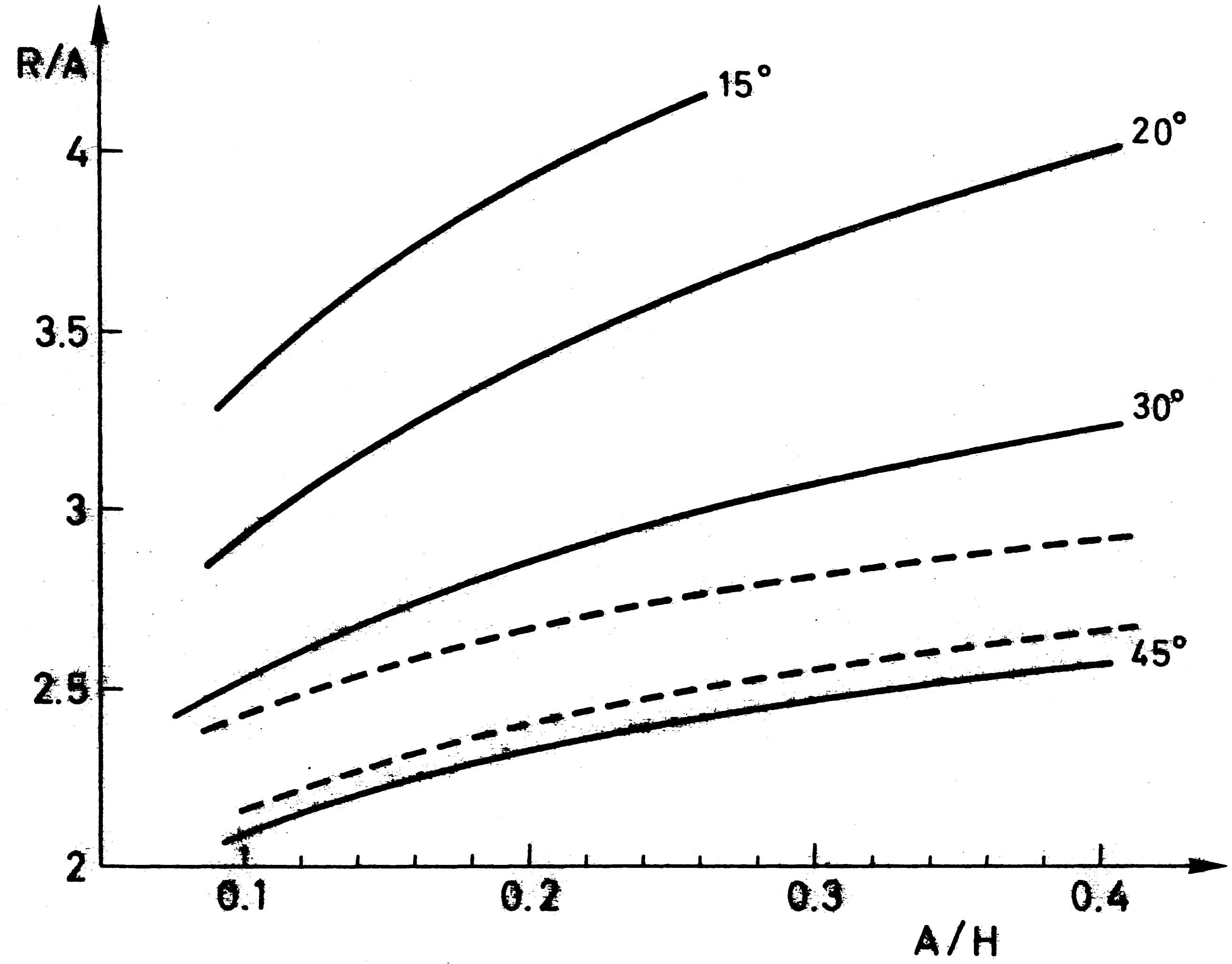

Fig 5

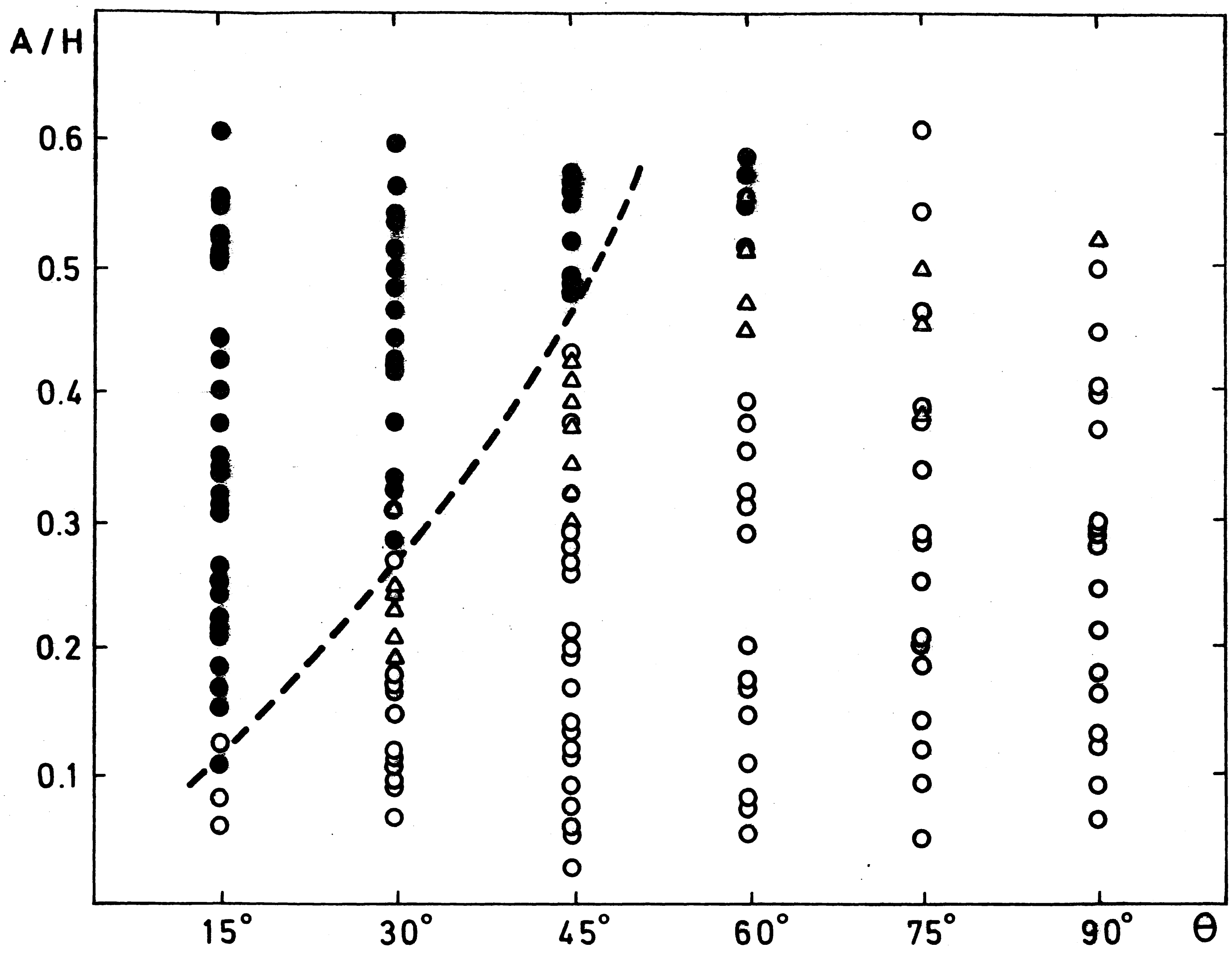

Fig 6