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# Nonlinear Vibration-Rotation Modes Of Inviscid Droplets

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# NONLINEAR VIBRATION-ROTATION MODES OF INVISCID DROPLETS

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## Abstract In Norwegian

Den ikke-lineære dynamikken til en roterende ikke-viskøs dråpe er vist å ha tilknytning til et Korteweg-de Vries(KdV) system med løsninger fra små harmoniske svingninger(linearisert modell), til langsgående cnoidale bølger og solitoner. Ved å bruke ikke-lineære analytiske verktøy ofres lineariteten i løsningene(sfæriske harmoniske) men blir erstattet med en spesiell form-responsiv dynamisk oppførsel, typisk for ikke-lineære systemer. Til dette suppleres en Euler-Lagrange likning hvis en bruker en Euler/Lagrange-formalisme eller i form av en Hamiltonsk likning hvis en bruker en Hamilton-formalisme. I det siste tilfellet er dråpeformen beskrevet, i tredje orden, av en uendelig-dimensional KdV Hamiltonfunksjonal, i enighet med resultatene oppnådd ved en Euler/Lagrange-formalisme. Resultatene er gyldige for dråpelignende systemer, f.eks klaseformasjoner på overflaten av en atomkjerne, dynamikken til en pellet i fusjonsprosesser, stjernemodeller og så videre.

## Abstract In English

The nonlinear dynamics of a rotating inviscid droplet are shown to be described by a Korteweg-de Vries(KdV) system, generating multiscale dynamical behaviour ranging from small harmonic oscillations (linearized model), to travelling cnoidal and solitary waves. Instead of expansions in spherical harmonics, using nonlinear analytic tools sacrifices the linearity of the space but replaces it with a special shape-responsive dynamical behaviour, typical for nonlinear systems. To this we add an Euler-Lagrange equation if using an Eulerian/Lagrangian formalism, or in the form of a Hamiltonian equation if using a Hamiltonian formalism. In the latter case the shapes are described by an infinite-dimensional KdV Hamiltonian system, in agreement with the results finalized using an Eulerian/Lagrangian formalism. The results apply to all droplike systems, from cluster formation in atomic nuclei, the dynamics of the pellet surface in inertial fusion, multiphase disperse flows, stellar models and so forth.



## PREFACE

This thesis is written as a partial requirement to obtain an M.Sc at the Norwegian University of Science And Technology(NTNU). The report is weighted by a full semester and corresponds to over 800 hours of work distributed freely over 20 weeks. The 800 hours allocated are spent on a survey of relevant literature, problem and solution formulation and production of this report.

The overall objective for this thesis is to describe the dynamics of an inviscid rotating droplet under the neglect of gravity and viscous shear on the surface by using a recently developed nonlinear theory. Such results would be interesting for all droplike systems, for instance atomic nuclei, stars, fusion pellets and so on. Since the author has no prior knowledge to the subject, a lot of the time allocated will be spent on the accumulation of literature.

The report is organized into three main parts:

**Part I** presents a thorough overview of previous results. A historical outline is included for motivational reasons and several results regarding our model are presented in a unifying manner.

**Part II** gives an introduction to relevant literature such as Hamiltonian fluid dynamics and the KdV equation. The conventional approach to nonlinear planar water waves which lead to the KdV equation is included since it is a particularly interesting model with respect to our theory.

**Part III** treats the nonlinear liquid drop model where we additionally include the effects of rotation. Some explicit results are presented both analytically and numerically and droplet shapes are shown both qualitatively and quantitatively.

Following the table of contents and list of figures, a list of notation used in this text is included. Finally, references and an appendix regarding the curvature of surfaces, variational calculus and definitions and series expansions of elliptic integrals and Jacobi elliptic functions are included.

I would like to thank my supervisors Prof. Iver H. Brevik and Prof. Johan S. Høye at NTNU for providing useful insight into my equations. Finally, I would like to thank everyone who helped me proofread. Some errors still remain, and I take credit for all of them.

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*July 13, 2009*



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## NOMENCLATURE

### Symbols

$A$	Nucleon number
$\mathbf{B}$	Magnetic field
$c$	Speed of light
$d$	Distance
$\delta$	First variation
$\frac{\delta}{\delta \xi}$	Functional derivative
$\frac{d}{dt}$	Material derivative
$\frac{D}{Dt}$	Material derivative
$\partial_i$	Partial derivative to general coordinate
$\varepsilon$	Smallness parameter
$\varepsilon_{ijk}$	Levi-Civita tensor
$F$	Functional
$\mathcal{F}$	Functional density
$g$	Acceleration of gravity
$G$	Gravitational constant (or functional)
$\hbar$	Reduced Plancks constant
$H$	Hamiltonian
$\mathcal{H}$	Hamiltonian density
$I$	Electric current
$\mathcal{I}$	Angular momentum
$K$	Complete elliptic integral of first kind
$k$	Modulus
$k_B$	Boltzmann constant
$\kappa$	Curvature
$l$	Length
$L$	Lagrangian
$\mathcal{L}$	Lagrangian density
$m, M$	Mass
$\hat{\mathbf{n}}, \hat{\mathbf{N}}$	Unit surface normal
$\mathbf{N}$	Torque
$\eta$	Free-surface elevation
$p$	Pressure
$\mathbf{p}$	Momentum vector
$P_l^m$	Associated Legendre polynomial
$P_l$	Legendre polynomial
$\rho$	Density
$r, \theta, \varphi$	Spherical polar coordinates
$R_0$	Mean radius, nuclear radius
$R$	Mean nuclear radius, radius of curvature
$t$	Time
$T$	Kinetic energy
$U$	Potential energy
$\mathbf{u}$	Fluid velocity field
$\mu_0$	Vacuum permeability
$V$	Angular velocity, travelling wave
$\mathcal{V}$	Volume

$\Omega$	Angular velocity
$\mathbf{\Omega}$	Angular velocity vector
$\omega$	Angular frequency
$\boldsymbol{\omega}$	Vorticity
$\mathbf{x}$	Cartesian coordinates
$\chi$	Magnetic susceptibility
$\gamma$	Coefficient of surface tension
$Y_l^m$	Spherical harmonics
$\Phi$	Scalar potential
$\Psi$	Streamfunction/surface/function
$\xi$	Deformation
$\boldsymbol{\Psi}$	Vector potential
<b>Sub- and Superscripts</b>	
$i, j, k$	Components in general directions
0	Reference value
$\hat{\phantom{x}}$	Normal vector
$\prime$	Perturbed/transformed quantity
*	Dimensionless variable
$\cdot$	Time derivative/material derivative
<b>Abbreviations &amp; non-dimensional groups</b>	
KdV	Korteweg-de Vries
mKdV	Modified Korteweg-de Vries
KP	Kadomtsev-Petviashvili
LDM	Liquid Drop Model
NLDM	Nonlinear Liquid Drop Model
Re	Reynolds number
Ro	Rossby number



## PART I

### INTRODUCTION

A historical outline of the topic of this text is presented together with possible applications. The formulation of this chapter is of a somewhat informal character and is intended to serve as a introductory note. In order to clarify the overall dynamics, the physical concepts of a spinning droplet are presented in section B.

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#### A. HISTORICAL OUTLINE

In 1863, the blind Plateau devised a method for studying the behaviour of liquid drops under the neglect of gravity [Plateau, 1863]. His motivation was his belief that the self-gravity of revolving astronomical objects could be described by the dynamics of a droplet held together by surface tension only. His apparatus consisted of an olive-oil droplet suspended in an alcohol-water mixture, allowing for a match between the buoyancy and weight of the droplet. The droplet was driven by a shaft pierced vertically through the droplet centre and by turning it, Plateau, or his son-in-law and faithful associate Van der Mensbrugghe, could see the effect of rotation. To no surprise, with increasing angular velocity the centrifugal force caused a flattening of the poles with a resulting increase in the equatorial diameter as is required from mass conservation, and the original spherical drop more and more took the shape of an oblate spheroid. Beyond a critical angular velocity the shape of the droplet became *non-axisymmetric* and was well-described in terms of an ellipsoid. A further increase in the angular velocity caused a more profound two-lobed shape (similar to a pear or peanut) to appear, and then, the majority of the liquid broke free from the shaft and into a most remarkable toroidal ring. The simplicity of Plateau's apparatus was also its demise because of the large shape deformations caused by the viscous shear from the surrounding fluid. Because of this, the agreement between theory and Plateau's experiments is hard to quantify.

Plateau's idea inaugurated more than 100 years of study, and in the past two centuries, the simple idea of a freely rotating liquid mass has been in the view of many great physicists and mathematicians. Among them are Maclaurin, Darwin, Riemann, Poincaré, Jacobi and up through Chandrasekhar with his grand work "Ellipsoidal Figures of Equilibrium". That their works consist of beautiful mathematics is simply stating the obvious, but it was not until Chandrasekhar [1965] that the essential physics, although understood long before, were finally proven.

In 1930, after the development of quantum mechanics it was first realized by Gamow [Gamow, 1930] that the spinning droplet and liquid drop model (LDM) also could be used to provide insight into the behaviour of atomic nuclei. To this day, Gamow's idea still stands. A complete description of all the mutual interactions in a nucleus with  $N$  nucleons can contain as many as  $N!$  terms [cf. Krane, 1988]. A direct analytic approach is at best

unrealistic and a different approach must be used; An adequate overall characteristic of the nucleus can be described by use of the static properties of the nucleus such as electric charge, radius, mass, angular momentum, parity, magnetic and electric multipole moments and binding energy. Today it is known [cf. Krane, 1988] from scattering experiments that the distribution of nuclear matter in a nucleus is fairly constant over the nucleus and the density of the nucleus, depending on the nucleon number, drops rapidly for a certain radius  $R$ . It is common to adopt the approximations for the nucleon density and mean nuclear radius by [Krane, 1988]

$$\frac{A}{\frac{4\pi}{3}R^3} \approx \text{const.}; \quad R = R_0 A^{1/3}$$

where  $A$  is the number of nucleons. From electron scattering measurements it is concluded that  $R_0 = 1.2 \sim 1.3 \text{ fm}$ . The main contributor to the attractive force between nucleons is the strong interaction, whereas on the other end of the scale gravity is utterly negligible. Nucleons at the outer rim of a nucleus is surrounded by fewer neighbours than those in the center, making them subject to an attractive force inwards. In a remarkable study of the nuclear binding energy it was found that the loss in binding energy caused by fewer neighbours along the outer rim gave a reasonably good fit towards experimental data. In addition to the strong interactions, a nucleus is electrically charged and its inhabitants suffer from a repulsive Coulomb force. There is a final term which contributes to the nuclear binding energy which is the nucleons ability to make up stable pairs. The liquid drop model, also known as the empirical mass formula, based on the combination of strong interactions, Coulomb energy, pairing of nucleons and the inclusion of *surface effects* is a remarkable benchmark in the study of nuclear structures. Today, it is well-known that the LDM describes particularly well the small vibrations of atomic nuclei. On the other hand, it is also known that there are heavy nuclei in which clusters (of e.g. alpha, carbon, oxygen and so on) are formed on the nucleus' surface and that these lead to the natural decay of such clusters. In the nuclear shell model, only by introducing many-body correlations with as many as  $N!$  terms can such a behaviour be accurately predicted. But is it possible to also describe this phenomena as a collective effect?

Even in the zero-energy vacuum of empty space, random particle-antiparticle processes can cause the creation of a pair

of particles with a small separation distance. In flat spacetime, in order to conserve energy, these particles must annihilate a tiny time later and the phenomena is what is known as virtual processes. The curved spacetime of a black hole spins a different tale, and the particles do not necessarily need to annihilate in order to conserve energy. This is a consequence of the geometry of the black hole: Inside the event horizon, the black hole is space-like and the conserved quantity  $-\xi \cdot \bar{\mathbf{p}}$ , where  $\xi$  is the Killing vector and  $\bar{\mathbf{p}}$  is the four-momentum of the antiparticle, may in fact be negative. Should the quantum fluctuations thus take place on the event horizon of a black hole it may cause one of the particles to form inside the event horizon and the other particle on the outside. The inner particle will be trapped in the gravity of the black hole and cannot escape nor be observed. On the other hand, the outer particle *can* escape, and will give rise to black hole radiation equal to the energy of the escaped particle. This is called Hawking radiation and causes the quantum evaporation of black holes. Remarkably, the radiation is emitted as if the black hole was a black body with the temperature[Hartle, 2003]

$$k_B T = \frac{c^3 \hbar}{8\pi G M}$$

where  $M$  is the mass of the black hole and  $G$  the Newtonian gravitational constant. Since the discovery of Hawking radiation and the inherent endowment of black hole thermodynamic properties such as entropy and temperature, the black hole physics have been described by analogous thermodynamics. In higher-dimensional physics such as e.g. string theory, the thermodynamics analog can be extended to *hydrodynamics*[Cardoso and Gualtieri, 2006, Cardoso, 2006, Kovtun et al., 2005] where the black hole is ascribed additional hydrodynamic properties such as viscosity, diffusion constants and so on. In the same spirit it has been understood that in the high energy limit, field theories have a hydrodynamic description. In such a gravity-hydrodynamic duality, fluids are literally dual to black holes. Therefore, in short, deformed droplets can actually act as direct analogues to the complex behaviour of gravity in higher-dimensional objects (i.e. objects that only exist in a higher number of space-time dimensions) and fluids may actually teach us how gravity works.

Single drop and bubble phenomena are abundant in Nature and are naturally associated with phenomena such as multiphase disperse flows and perhaps more heatedly: single-bubble sonoluminescence. Disperse flows are highly challenging and encountered in modern engineering problems and although a disperse flow consists of thousands or millions of droplets, a fundamental understanding of the dynamical, physiochemical and reactive properties of a single bubble are necessary in order to identify collective phenomena. Single-bubble sonoluminescence may be understood in the following way: If the intensity of a standing acoustic wave in a fluid is high enough, a cavity may grow in a fluid and take the form of a bubble. The same result may be produced by cavitation. When the bubble collapses it will, by the inertia of the inrushing surrounding fluid, be highly compressed and its temperatures may reach many thousand degrees of Kelvin and the bubble will in fact emit a short burst of light with, some researches believe, an intensity high enough to initiate fusion reactions. This is called single-bubble sonoluminescence or single-bubble fusion, and its complete description is at the time of writing one of the unsolved problem of modern physics. Rough estimates of the temperature reveal around

20,000 Kelvin but some even claim to have measured temperatures in the order of one million Kelvin. Thus, the description of the dynamics of a droplet or single bubble will favorably impact the development of methods in multiphase flows and more heatedly, may partly explain the phenomena known as single-bubble sonoluminescence. In the present work we shall treat the fluid inviscid so that our analysis will in the case of single-bubble dynamics be restricted to the natural oscillations prior to the bubble collapse.

The availability of micro-gravity studies in Space, have allowed for more exact experimental investigations[Wang et al., 1986, 1994, Lee et al., 1998] than the original work by Plateau [1863] which by most standards were highly inaccurate. That such studies are very expensive is a given. The ever-advancing research area of superconductors has abled us to diamagnetically levitate organic materials (which mainly consist of water, which is diamagnetic) and hence allow accurate and cheap tabletop studies. Very recently, in December 2008, Hill and Eaves [2008] made such a study with a considerable amount of satisfaction in the excitation of a triangular 3-lobed shape and large-amplitude waves with up to 5 nodes running along the equator of the droplet, dynamics previously unheard of, and thought to define unstable equilibriums. Their method of approach was to diamagnetically levitate (cf. section D) the droplet, lower the surface tension by adding a small quantity of surfactant and rotate the droplet by use of an electromotor.

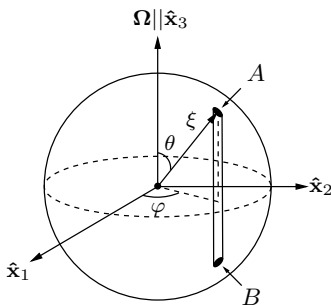
In the late 80's and early 90's, Hamiltonian fluid dynamics, which is the application of Hamiltonian field theory to fluid mechanics, was developed (cf. Salmon [1988], Kuroda [1990], Morrison and Greene [1980]). The reason why Hamiltonian fluid dynamics went so long unexplored (as opposed to Eulerian or Lagrangian fluid mechanics) was the regard of what, in older literature, was thought to be the fundamental statement; the variational principle. However, modern treatments tend to favor the Poisson bracket dynamics, allowing for formulation of non-canonical Hamiltonian fluid dynamics (the Euler equations have a variational principle associated with the Lagrangian variables, but not the reduced Eulerian variables). As Salmon [1988] points out, *any* set of equations may be written in canonical form but while the resulting equations are beautiful in their mathematical structure, the physical meaning attached to the symbols appears almost silly. The most apparent use of the Hamiltonian hydrodynamic description is the analysis of the stability of a fluid system. The single greatest success of Hamiltonian stability theory (not to be confused with Hamiltonian fluid dynamics) is however the proof of the nonlinear stability of the soliton solution to the Korteweg-deVries (KdV) equation.

With motivation from clusters on atomic nuclei, nonlinear effects of droplets were studied recently by Ludu and Draayer [1998a]. Instead of using the conventional spherical harmonics expansion in the scalar potential which is done in the linear model, they developed the potential in terms of nonlinear functions whereupon the Euler equations was shown to reduce to the Korteweg-deVries equation, which has a Hamiltonian structure. By use of the Euler equations *and* Hamilton's equations, their analysis proved the existence of cnoidal waves ranging from small linear oscillations, and up through nonlinear solitary waves. Of course, in the case of linearization, the analysis by Ludu and Draayer [1998a] reduces to the normal modes of the oscillation of a droplet's surface. Including effects of rotation into the analysis by Ludu and Draayer [1998a] is the main topic of this text.

During a light experiment, Plateau kept his eyes fixed too long on the Sun which produced a choroid inflammation in both his eyes. By 1843, Plateau was completely blind. Today, over 150 years later, it is known that freely rotating astronomical bodies sustained by self gravity exhibit the same geometrical form as the centimeter-size drops found by Plateau nearly 150 years ago. The first observation of such an object was by Jacobi [1834] but the accuracy of Jacobi's observation is questionable. More recently and reliably, calculations on the dwarf planet Haumea discovered by Brown et al. [2004] suggest that it is a highly elongated ellipsoid, with its greatest axis at least twice as long as its shortest. Nonetheless, it's self-gravity has together with the centrifugal force achieved hydrostatic equilibrium. Beyond his blindness, Plateau has proven remarkably far-seeing in his early droplet experiments.

## B. PHYSICAL CONCEPTS

A schematic overview of the hydrostatically spinning droplet is shown in Fig. 1.1, although it will in general be deformed by the centrifugal force, it is here shown as a spheroid. The droplet is spinning around some axis which passes through its centre of mass, with a constant angular velocity  $\Omega$ . It is convenient to denote the free surface of the droplet as the distance from the centre of mass by  $\xi = \xi(\theta, \varphi, t)$ , in terms of the usual polar angles  $(\theta, \varphi)$ .



**Fig. 1.1:** Schematic overview of the hydrostatically spinning droplet under the neglect of gravity. The free surface is denoted as the distance from the center of mass by  $r = R_0 + \xi(\theta, \varphi)$ .

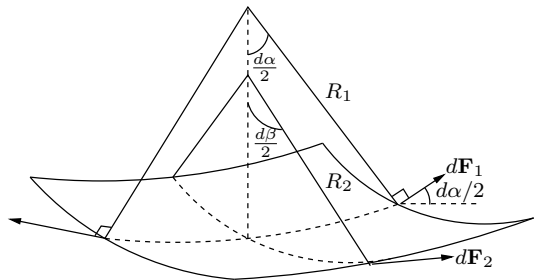
Of course, in the case of axisymmetric droplets the shape of the droplet may be completely described by its meridional section in one of its hemispheres, so that the droplet shape may be regained by rotating the meridional section around the rotational axis and reflecting the droplet across its own equatorial plane. This latter assumption of reflective equatorial symmetry is not necessarily true. Consider the droplet as simply connected and made up of cylindrical columns parallel to the axis of rotation such as is shown in Fig. 1.1 where the droplet is designated  $AB$ . For one such column, the pressure at its endpoints  $A$  and  $B$  must be the same since the two points are at the same distance from the axis of rotation. Thus, the curvature of the surface on these two points must be the same (we will show this very soon) but this in itself does not require reflective equatorial symmetry, only that the curvatures themselves are the same. Nonetheless, it is evident from experimental studies Wang et al. [1986, 1994], Ohsaka and Trinh [1999] that the assumption of equatorial symmetry is well supported. There are however several reports of droplets disobeying this symmetry, but this symmetry break is caused by the presence of wall effects, acoustic flattening, sur-

factants, shear stress from a surrounding fluid etc.

Under the neglect of gravity and viscous shear on the droplet the dynamics are described by the interactions between the capillary force caused by surface tension, and the gyrostatic pressure force exerted on the surface owing to the centrifugal force. Conceptually, the surface tension makes the surface of the droplet behave like an elastic membrane attempting to minimize the surface area. The capillary force caused by surface tension can be described in terms of the coefficient of surface tension  $\gamma$  and the mean curvature by the Young-Laplace equation

$$\Delta p = \gamma \nabla \cdot \hat{\mathbf{n}} \quad [1.1]$$

where  $\hat{\mathbf{n}}$  is the outward unit normal and  $\kappa = \nabla \cdot \hat{\mathbf{n}}$  is the local mean curvature with  $\Delta p$  being the pressure difference across the interface at this point. The derivation of the Laplace-Young equation is straightforward from force balance, or alternatively from variational concepts, on a differential surface element. We will treat them both. Let the differential surface element be described by two principle radii of curvature,  $R_1$  and  $R_2$ . For simplicity, let the surface normal be parallel with the  $\hat{\mathbf{x}}_3$ -axis. The surface tension applies along the edges of the element and gives rise to two orthogonal pairs of forces,  $d\mathbf{F}_1$  and  $d\mathbf{F}_2$ . See Fig. 1.2 for reference.



**Fig. 1.2:** Force balance on an infinitesimal surface element. The angle between  $d\mathbf{F}_2$  and the surface tangent is not indicated but follows the same analysis at  $d\mathbf{F}_1$ . The angles are grossly exaggerated.

The force balance normal to the surface then gives that the gage pressure  $\Delta p$  must satisfy

$$\begin{aligned} \Delta p R_1 R_2 d\alpha d\beta &= (2d\mathbf{F}_1 + 2d\mathbf{F}_2) \cdot \hat{\mathbf{n}} \\ &= 2dF_1 \frac{d\alpha}{2} + 2dF_2 \frac{d\beta}{2} \end{aligned}$$

which is seen from a sine-expansion for small arguments. A definition of the *coefficient of surface tension*  $\gamma \equiv \frac{F_i}{R_i}$  gives after division by the surface area that the gage pressure satisfies

$$\Delta p = \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

This is the same result as equation 1.1 since the principle radii of curvature can be related to the mean curvature by  $\kappa = \nabla \cdot \hat{\mathbf{n}} = \frac{1}{R_1} + \frac{1}{R_2}$  [Arfken and Weber, 2001].

Alternatively one may think of the variation of the energy as the surface gets displaced by some virtual displacement  $\delta\xi$  (formally the surface is displaced *along* its unit normal), the energy required for this operation is then the energy required to change the volume (relates to the pressure) plus the energy required to change the surface (relates to the surface tension). If the surface gets displaced by  $\delta\xi$  and the size of a certain area element is  $df$ , then

$$\delta W = \int -\Delta p \delta\xi df + \gamma \delta f \quad [1.2]$$

where  $\delta f$  is the total change in the area element. It is easy to see that  $\gamma$  is the energy per surface element. Then the arc differentials (the "edges" of the surface Fig. 1.2) are then increased by  $ds_1 = ds_1 \left(1 + \frac{\delta\xi}{R_1}\right)$  and correspondingly for  $ds_2$ . It may be recalled from the previous derivation that  $ds_1 = R_1 d\alpha$ ,  $ds_2 = R_2 d\beta$  and  $df = ds_1 ds_2$ . I.e. the variation of the surface is

$$\delta f = \delta \int df = \int \delta\xi \left(\frac{1}{R_1} + \frac{1}{R_2}\right) df$$

Substituting this last expression into equation [1.2] gives

$$\delta W = \int \left[ \Delta p - \gamma \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \right] \delta\xi df \quad [1.3]$$

The displacement  $\delta\xi$  is arbitrary, hence the bracketed expression is zero (this is a consequence of the fact that  $W$  is stationary). The vivid reader will recognize this derivation as the Euler-Lagrange equation for an effective potential of the drop, with  $\gamma$  being the Lagrangian multiplier corresponding to energy conservation. In the case of rotation of the frame of reference, the gage pressure  $\Delta p$  must account for the pseudo-pressure owing to the presence of an additional centrifugal force, i.e.

$$\gamma \nabla \cdot \hat{\mathbf{n}} = \Delta p_0 + \frac{1}{2} \rho \Omega^2 r_{\perp}^2 \quad [1.4]$$

where  $\rho$  is the mass density of the droplet and  $r_{\perp}$  is the perpendicular distance from the axis of rotation. Should the droplet be submerged in a another fluid  $\rho$  may be replaced by the *difference* in density between the two fluids.

The origin of surface tension is owing to the intermolecular forces, the van der Waal's forces, in the fluid; In the presence of ions, ionic bonding and ion-dipole interactions must also be considered as contributors to the surface tension. We will mention no such cases. At least three types of van der Waal's forces are distinguishable from one another, the permanent dipole - permanent dipole interaction, the permanent dipole - induced dipole interaction and the instantaneous dipole - induced dipole interaction. The first interaction is caused by the electrostatic attraction between permanent dipoles (or in general permanent multipoles), such as the hydrogen bond of water. The second source is the interaction between a permanent multipole on one molecule and the *induced* multipole moment. The third interaction is caused by the attraction between two (originally non-polar) temporarily induced dipoles, the induced dipoles usually disappear in an instant. On the other hand, for astronomical objects the surface tension might be thought of as the self-gravity of the object. In the present work the *origin* of the surface tension is not bothered with, so that the calculations will be quite general.

In all the following we will assume an *ideal* fluid, although this concept requires some clarification. As usual, an ideal fluid implies an inviscid fluid (which also implies an incompressible fluid) but perhaps more importantly we neglect several other effects. In reality there will be a medium surrounding the rotating droplet such that there is a narrow transition layer between the two fluids. We will assume this layer so thin that it, and the phenomena associated with it, may be neglected. The reader may see Hirschfelder et al. [1954] for a lengthy discussion on this topic.

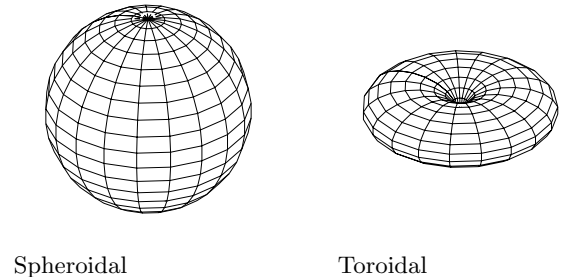
Gone with the wind are also fluid evaporation and its counterpart, entrapping of the surrounding medium (foaming) and the effects of surfactants (surfactants are pollution in the surface

of the drop which lowers its surface tension). We will also assume that the fluid (and the, in principle, surrounding medium) is at a constant temperature so that there is no Marangoni convection at the free surface caused by the temperature gradient. Also, although this is clearly not the case of astronomical objects, we will assume that there is no transition to turbulence. Not all these effects are of equal importance, but we shall see that the complexity of this problem is large even in its purest form.

### C. HYDROSTATIC SHAPES & STABILITY

A point should be made concerning the difference between an isolated and driven drop; An *isolated* drop has a fixed angular momentum. On the other hand, if the drop has a fixed angular velocity (i.e. driven by a shaft[Plateau, 1863] or electro-motor[Hill and Eaves, 2008]) the drop is referred to as a *driven* drop. Although they are highly similiar, the stable shapes of an isolated drop and a driven drop are in general *not* the same. Note that in the following only equilibrium shapes and their stability are dealt with, natural and provoked oscillations (for instance owing to the presence of accoustic fields) are treated in the next section.

The initial shape of the droplet in the case of zero rotation is the spheroid as this represents the minimum surface of a constant volume, and the final shape is an almost toroidal ring. Both these shapes are shown in Fig. 1.3.



**Fig. 1.3:** The initial and final shape of the spinning droplet. To the left is shown the intial spherical shape of the droplet in the case of no rotation, to the right is shown the toroidal-like droplet just before the droplet breaks free and no longer encloses the origin. The figures are not in proportion.

In the following we will derive the axisymmetric hydrostatic figure of equilibrium from both an Eulerian and Lagrangian point of view, as well as summarize the non-axisymmetric results by Brown and Scriven [1980a,b]. The results are systematically represented in Fig. 1.5 in terms of two dimensionless parameters,

$$\Omega^* = \sqrt{\frac{\rho R_0^3 \Omega^3}{8\gamma}}, \quad \mathcal{I}^* = \frac{\mathcal{I}}{4\gamma R^2 (8\gamma R_0^3 \rho)} \quad [1.5]$$

where  $\mathcal{I}$  is the angular momentum of the droplet which is readily found by direct integration over the surface of the droplet. In all the following the reader is referred to Fig. 1.5 for an overview of the equilibrium configurations and stability.

It may be recalled that the equilibrium shape of the droplet is determined from the balance between the surface tension and



the centrifugal force. The droplet energy consists of the energy embedded in the pressure, rotation, surface tension etc. The hydrostatic equilibrium is determined by the point of which there is no mechanical energy available, since no oscillations may then take place. Let an effective potential  $\mathcal{E}$  of the droplet be made up of the pressure-energy, the energy of the surface and the energy of a rigidly rotating body. The droplet is in equilibrium if the first variation of  $\mathcal{E}$  vanishes, i.e.  $\delta\mathcal{E} = 0$ . As will be shown in detail later, the Euler-Lagrange equation that follows from  $\delta\mathcal{E} = 0$  is in fact the Young-Laplace equation [1.4]. We may very well think of the surface area as a potential energy for our problem. As we discussed, the first variation of  $\mathcal{E}$  must be zero, but this in itself does not require that the droplet is in a *stable* equilibrium. A shape is stable to the perturbations that *raises* the potential energy and unstable to perturbations that *lowers* the potential. The second variation of  $\mathcal{E}$  then determines the stability of the equilibrium. To cite Maupertius, the "father" of the least action principle

*Nature is thrifty in all its actions. The laws of movement and of rest deduced from this principle being precisely as those observed in Nature, we can admire the application of it to all phenomena. The movement of animals, the vegetative growing of plants... are only its consequences: and the spectacle of the Universe becomes so much the grander, so much more beautiful, the worthier of its Author, when one knows that a small number of laws, most wisely established, suffice for all movements.*

By using an extension on the method of tensor virials and by assuming small ellipsoidal shape perturbations, Chandrasekhar [1965] showed that there is a point of neutral stability of the axisymmetric shape family at a value of  $\Omega_2^* = 0.566$  (in the following a subscript on  $\Omega^*$  denotes the bifurcation point of the subscripts family). Chandrasekhar actually reports the value  $(\Omega^*)^2 = 0.487$  but he uses the equatorial radius of the *rotating droplet* whereas it has been common to employ the radius of the droplet at rest. The advantage of the Chandrasekhar [1965] analysis is that the small linearized oscillation frequencies of the droplet surface emerge naturally by the method approach and perhaps more importantly his analysis was the first to relate the hydrostatic problem to the problem of self-gravitating masses. Nonetheless, Chandrasekhar [1965] fails to show that the reported neutral stability point  $\Omega_{\max}^* = 0.754$  is actually the bifurcation point of a new family of axisymmetric shapes. This latter fact was pointed out by Ross [1968a] who, by developing the Lagrangian for the revolving drop under the assumption of meridional and equatorial symmetry, showed that a new family of shapes emerges beyond the bifurcation point  $\Omega_{\max}^*$ .

### C.1 THE AXISYMMETRIC EQUILIBRIUM FIGURE

The Lagrangian is the kinetic energy  $T$  minus the potential energy  $U$  and for the hydrostatic case of a revolving drop,  $T$  is therefore is the energy of a rigidly rotating body, and  $U$  is the surface energy. Let  $f(z)$  be the radius from the axis of rotation to the free surface of the droplet as a function of the cylindrical polar component  $z$ , see Fig. 1.4.

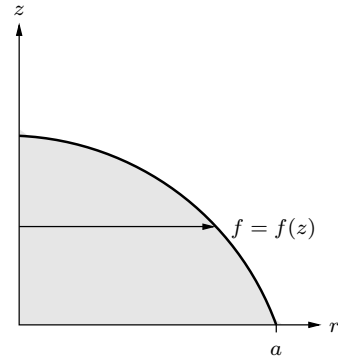


Fig. 1.4: Specification of the free surface of the droplet at  $r = f(z)$ .

The Lagrangian is thus

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}\pi\rho\Omega^2 \int_{\mathcal{V}} r_{\perp}^2 dV - \gamma \int_{\Sigma} d\Sigma \\ &= \pi\rho\Omega^2 \int_{\mathcal{V}^+} r_{\perp}^2 dV - 2\gamma \int_{\Sigma^+} d\Sigma \end{aligned}$$

where  $\mathcal{V}$  denotes the volume of the droplet and the (+) indicates that the integration runs over the positive hemisphere only (that is,  $z$  is positive) and  $f$  is therefore a single-valued function. By  $r_{\perp}$  we understand the perpendicular distance from the axis of rotation to the point at hand, whereas  $d\Sigma$  is an infinitesimal surface element. By using cylindrical polar coordinates  $(r, \theta, z)$  we identify  $r_{\perp} = r$ ,  $dV = r dr d\theta dz$ , and  $d\Sigma = f\sqrt{1+f'^2} d\theta dz$ . By  $f'$  we understand differentiation of  $f$  with respect to  $z$ . An integration over  $r$  and  $\theta$  yields

$$L = \frac{1}{2}\pi\rho\Omega^2 \int_+ f^4 dz - 4\pi\gamma \int_+ f\sqrt{1+f'^2} dz \quad [1.6]$$

The problem now reduces to finding the function  $f = f(z)$  which makes the Lagrangian stationary [Luke, 1967], but subject to the condition that the volume

$$V = \int_{\mathcal{V}} dV = 2\pi \int_+ f^2 dz = \text{const.} \quad [1.7]$$

is known beforehand.

Let  $g = h - 2\pi\lambda f^2$  where  $h$  is the integrand (including the prefactors) of equation [1.6] and  $\lambda$  is a Lagrangian multiplier introduced by the constant-volume constraint that is yet to be determined. The Euler-Lagrange equation corresponding to  $g$  is

$$\frac{\partial g}{\partial f} - \frac{d}{dz} \frac{\partial g}{\partial f'} = 0$$

Indeed, since  $g$  is not explicitly dependent on  $z$ , we may use a first integral of the Euler-Lagrange equation by using  $f'$  as an integration factor. Then

$$g - f' \frac{\partial g}{\partial f'} = \text{const.}$$

Calculating  $\frac{\partial g}{\partial f'}$  gives  $\frac{\partial g}{\partial f'} = -4\pi\gamma \frac{f f'}{\sqrt{1+f'^2}}$ . By inserting this into the relation above one finds (we set, without loss of generality within our framework, the constant equal to zero)

$$\frac{1}{2}\pi\rho\Omega^2 f^4 - 2\pi\lambda f^2 - 4\pi\gamma \frac{f}{\sqrt{1+f'^2}} = 0 \quad [1.8]$$

The mean curvature of an axisymmetric surface is given in the appendix to be  $\nabla \cdot \hat{\mathbf{n}} = \frac{1}{f} \frac{\partial}{\partial f} \left( \frac{f}{\sqrt{1+f'^2}} \right)$ . Thus, the above may be written as

$$\gamma \nabla \cdot \hat{\mathbf{n}} = \frac{1}{2} \rho \Omega^2 f^2 - \lambda \quad [1.9]$$

and hence  $(-\lambda)$  is recognized as the pressure difference  $\Delta p_0$  and, indeed, the Euler-Lagrange equation of the rotating drop is the Young-Laplace equation governing the motion of the interface. We may integrate this as follows: By evaluating the Young-Laplace equation on the point where  $f' = 0$ , typically at the equator, we may write the Young-Laplace equation as

$$\Delta p_0 = \frac{2\gamma}{a} (1 - e), \quad e \equiv \frac{\gamma \rho \Omega^2 a^3}{8} \quad [1.10]$$

Here,  $a$  corresponds to the width  $f = a$  at the point where  $f'$  vanishes, i.e. the equatorial radius. By introducing equation [1.10] into equation [1.8], the Young-Laplace equation may be written

$$\sqrt{1+f'^2} \left[ e \left( \frac{f}{a} \right)^2 - (1-e) \frac{f}{a} \right] = 1 \quad [1.11]$$

Define  $\Psi = \Psi(f)$  as the bracketed expression of the relation above. Then after rearranging, the Young-Laplace equation may be written as

$$\frac{dz}{df} = \frac{\Psi}{\sqrt{1-\Psi^2}} \quad [1.12]$$

The figure of equilibrium is therefore completely described by the function  $z = z(f)$  that is obtained by performing the elliptic integral of equation [1.12]. By making proper substitutions, it also follows that the figure is described in terms of one non-dimensional parameter,  $e$ , in agreement with the Chandrasekhar [1965] analysis. Indeed,  $dz/df = 0$  at  $f \rightarrow 0$  and  $df/dz = 0$  at  $(f/a) \rightarrow 0$  are now two requirements of axial and equatorial symmetry, to which we can make the following important points; Although the surrounding medium has been neglected, we may without loss of generality operate with  $\rho$  as the difference in densities. Therefore, if the drop is rotating in a denser medium  $e$  is negative,  $e < -\frac{1}{2}$  say, it follows that  $\Psi(f)$  has one positive zero less than  $a$ . This means that equation [1.12] takes on complex values, which combined with the fact that equation [1.12] has a first-order pole at  $f/a = 1$  when  $e = -\frac{1}{2}$  leads to the conclusion that the drop takes on the shape of an infinitely long thread as  $e \rightarrow -\frac{1}{2}$ . A consequence of this is that we see that the angular momentum is not monotonically increasing since the inertia of an infinitely thin thread is zero. Therefore, the angular velocity and not the angular momentum is the fundamental quantity to be followed.

Consequently in the other end of the scale, we may examine the figure of equilibrium when  $e > 1$ . In this case,  $\Psi(f) = 0$  still has a real zero when  $f/a = 1$ , and consequently equilibrium forms exist that are partly concave and where the pressure difference at the poles is negative. The pressure reversal at the poles is initiated at  $e = 1$  and for  $e > 1$  it follows that the drop has a tendency to collapse from its poles onto its own equator and become disk- or tori-like. We shall not pursue this topic much further except mention that Ross [1968a] has generalized this further: There is a maximum for  $\Omega^* = \Omega_{\max}^* = 0.7540$  where  $e = 1.6608$ . Thus, in order to make the droplet collapse it is necessary to bring it up to a rotation rate  $\Omega_{\max}^*$  with a following decrease in  $\Omega^*$  in such a way that the angular momentum is increased. This ought to collapse the drop at  $\Omega^* = 0.7071$ . A lengthy paper by Ross [1968a] discusses the stability of these configurations more closely.

It is quite possible to obtain the shape of an axisymmetric equilibrium figure without resorting to tensor virials or methods of classical mechanics. This however is restricted to the linear case, i.e. low rotation rates. The Lagrangian way as we previously discussed is perhaps more general and is valid for all rotation rates. However, we include the following analysis since we later will show that a shift in the natural oscillation frequencies (that are yet to be discussed) may be determined by considering the oscillations as a perturbation from the hydrostatic equilibrium of the rotating droplet. As we have already discussed, the pressure inside the droplet rotating at a rotation rate  $\Omega$  is given by

$$p = p_0 + \frac{1}{2} \rho \Omega^2 r^2 \sin^2 \theta \quad [1.13]$$

where it may be recalled that  $p_0$  is the pressure at the axis of rotation and  $r \sin \theta$  is the perpendicular distance from the axis of rotation. We may operate with the gage pressure so

$$\gamma \nabla \cdot \hat{\mathbf{n}}_0 = \Delta p_0 + \frac{1}{2} \rho \Omega^2 r^2 \sin^2 \theta \quad [1.14]$$

must hold at the free surface.  $\hat{\mathbf{n}}_0$  is here the outward unit normal which may be obtained by prescribing a shape function, and  $\Delta p_0$  is the pressure difference at the poles. For axisymmetric figures and without loss of generality, the shape may be prescribed in terms of e.g. Legendre polynomials. Say that the surface is given implicitly by

$$F_0(r, \theta) = r^3 - R_0^3 \left[ 1 - \varepsilon P_2(\cos \theta) + \varepsilon^2 \sum_{n>2} a_n P_n(\cos \theta) \right] = 0 \quad [1.15]$$

where  $R_0$  corresponds to the droplet radius at rest, i.e. the volume is  $V = \frac{4\pi}{3} R_0^3$ . Because the figure is purely spheroidal in the linear case, the amplitude  $\varepsilon$  of the  $P_2$  component acts as a parameter of the problem and is to be determined. The unit normal vector on the interface  $F_0$  is equal to  $\hat{\mathbf{n}}_0 = \frac{\nabla F_0}{|\nabla F_0|}$ , the reader not convinced by this may want to refer to e.g. [Arfken and Weber, 2001]. Explicitly carrying out this calculation gives

$$\hat{\mathbf{n}}_0 = \frac{\hat{\mathbf{r}} + \frac{R_0^3}{3r^3} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right] \hat{\boldsymbol{\theta}}}{\sqrt{1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}} \quad [1.16]$$

The unit normal has no azimuthal component and the divergence is therefore given by  $\nabla \cdot \hat{\mathbf{n}}_0 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \hat{n}_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \hat{n}_\theta)$ . Carrying out this differentiation is straightforward, we find for the  $\hat{\mathbf{r}}$ -component that

$$(\nabla \cdot \hat{\mathbf{n}}_0)_r = \frac{2}{r \sqrt{1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}} + \frac{R_0^6}{3r^7} \frac{\left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}{\left\{ 1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2 \right\}^{3/2}}$$

Similarly we may carry out the  $\theta$ -part of the operation where we shall use the fact that the Legendre polynomials obey  $\Delta_\Omega P_n + n(n+1)P_n = 0$  where by  $\Delta_\Omega$  we understand the angular part of the Laplacian, equal to  $\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$ . Clearly, the azimuthal part vanishes but it is included for generality since it is in fact the spherical harmonics that obey the given identity, and not the Legendre polynomials. Nevertheless, the zonal spherical harmonics reduce to the Legendre polynomials to which we may use the given identity. After a few rounds of differentiation, this yields

$$(\nabla \cdot \hat{\mathbf{n}}_0)_\theta = \frac{R_0^3}{3r^4} \frac{\varepsilon^2 \sum_{n>2} n(n+1) a_n P_n - 6\varepsilon P_2}{\sqrt{1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}} + \frac{R_0^9}{27r^{10}} \frac{\left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}{\sqrt{1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}} \left[ \varepsilon \frac{\partial^2 P_2}{\partial \theta^2} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial^2 P_n}{\partial \theta^2} \right]$$

The explicit calculation of the divergence is somewhat tardy, but we shall need to complete it in order to properly identify  $\varepsilon$  and thereby determining the shape of the droplet. In collecting all terms we may write down the curvature of the surface to an arbitrary order in  $\varepsilon$  as

$$\nabla \cdot \hat{\mathbf{n}}_0 = \frac{1}{\sqrt{1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}} \left( \frac{2}{r} + \frac{R_0^6}{rr^7} \frac{\left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2}{1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2} + \frac{R_0^3}{3r^4} \left[ \varepsilon^2 \sum_{n>2} n(n+1) a_n P_n - 6\varepsilon P_2 \right] - \frac{R_0^9}{27r^{10}} \frac{\left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2 \left[ \varepsilon \frac{\partial^2 P_2}{\partial \theta^2} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial^2 P_n}{\partial \theta^2} \right]}{1 + \frac{R_0^6}{9r^6} \left[ \varepsilon \frac{\partial P_2}{\partial \theta} - \varepsilon^2 \sum_{n>2} a_n \frac{\partial P_n}{\partial \theta} \right]^2} \right) \quad [1.17]$$

Only terms up to second order in  $\varepsilon$  are interesting. By evaluating the above equation at the interface  $F_0$  we may series expand the denominators and we will keep terms up to second order in  $\varepsilon$ . The final "large" term in the above equation is of third order in  $\varepsilon$  and it can be neglected. In series expanding the rest, we need to make use of the (negative) binomial series  $(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \dots$ . In doing so we find that the curvature is to second order in  $\varepsilon$  given by

$$\nabla \cdot \hat{\mathbf{n}}_0 = \frac{1}{R_0} \left\{ 2 - \frac{4\varepsilon}{3} P_2(\cos \theta) - \frac{20\varepsilon^2}{9} [P_2(\cos \theta)]^2 + \frac{2\varepsilon}{9} \left[ \frac{\partial P_2(\cos \theta)}{\partial \theta} \right]^2 + \varepsilon^2 \sum_{n>2} \frac{n(n+1)-2}{3} a_n P_n(\cos \theta) + \mathcal{O}(\varepsilon^3) \right\} \quad [1.18]$$

By inserting equation [1.18] into the Young-Laplace equation it is verified that  $\varepsilon$ , and thereby the shape, is

$$\varepsilon = \frac{\rho \Omega^2 R_0^3}{4\gamma} + \frac{1}{7} \left( \frac{\rho \Omega^2 R_0^3}{4\gamma} \right)^2 + \dots \quad [1.19]$$

In order to determine the change in oscillation frequencies as an effect of rotation, we will later keep terms up to first order in  $\varepsilon$ .

## C.2 2,3- AND 4-LOBED SHAPES

Brown and Scriven [1980a,b] extended Chandrasekhar's analysis by use of the finite element computing technique where shapes and families of shapes are traced systematically throughout the parameter space. The shapes were calculated as extrema of the droplet energies and their stability determined from the curvature of the energy surface in the neighbourhood of the extremum. As we mentioned, if any perturbation *lowers* the effective potential then the shape is unstable to that perturbation and conversely if the perturbation *raises* the effective potential

then shape is stable to that perturbation. The effective potential energy used by Brown and Scriven [1980a,b] is

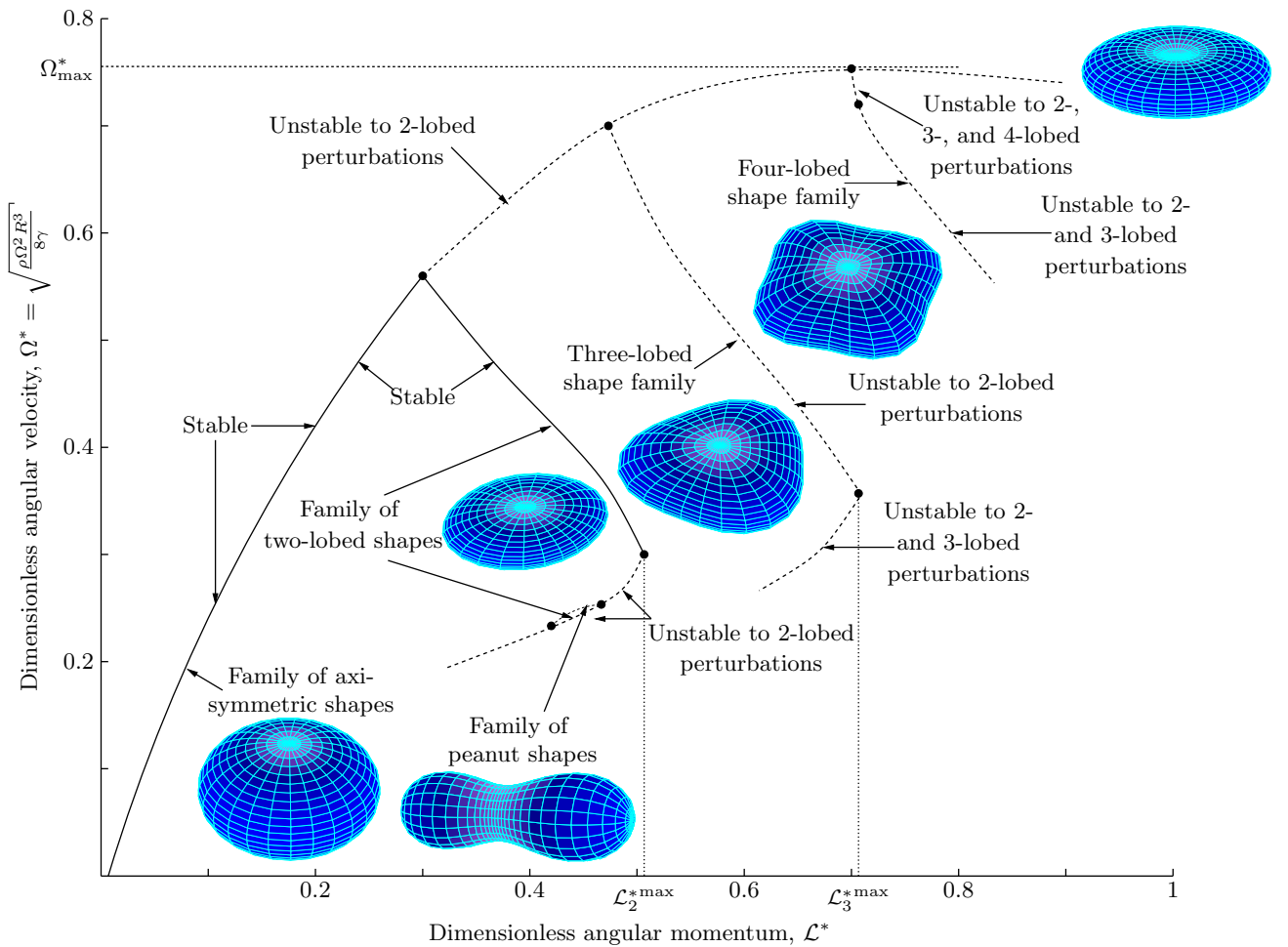
$$\begin{aligned} \mathcal{E} = & \frac{1}{10} \rho \Omega^2 \int \xi^5 \sin^3 \theta \, d\theta \, d\varphi \\ & - \gamma \int \xi \sqrt{(\xi^2 + \xi_\theta^2) + \xi_\varphi^2 \csc^2 \theta} \sin \theta \, d\theta \, d\varphi \\ & + \frac{\lambda}{3} \int \xi^3 \sin \theta \, d\theta \, d\varphi \end{aligned} \quad [1.20]$$

which is the Lagrangian combined with the constant volume constraint. It is readily seen that the first term is the energy of a rigidly rotating body, the second term it's surface energy and the third term the volume times the usual Lagrangian multiplier  $\lambda$ . Should the droplet be isolated, an additional constant-angular momentum constraint may be incorporated by a Legendre transformation to show that  $\mathcal{E}$  reduces to the Routhian of classical mechanics. The Routhian is a "hybrid" of the Lagrangian and the Hamiltonian and also measures the energy of the droplet, albeit in a different way. It may be verified that taking the first variation of  $\mathcal{E}$  in the form above reduces to the Young-Laplace equation. Keep in mind that we are now dealing a non-axisymmetric surface so that the corresponding expression for the curvature is very complicated. To ease the calculations, a section which deals with the general curvature has been included in the appendix. The general Euler-Lagrange equations are also found in the appendix. The reader may want to refer to these before attempting any calculations. In the following section, Fig. 1.5 shows an overview of the stability and bifurcations of different shape families.

For axisymmetric drops, the stability analysis by Brown and Scriven [1980b] conclude that all shapes on the ascending axisymmetric branch (that is,  $0 < \Omega^* < \Omega_{\max}^*$ ) are stable to all axisymmetric deformations. However, a real drop is also subject to asymmetric perturbations. For two-lobed perturbations, Brown and Scriven [1980b] report a neutral stability point of the axisymmetric shape family at  $\Omega_2^* = 0.5599$ , within 1.1% of Chandrasekhar's exact value. This point is found to mark the bifur-

cation point of a family of *two*-lobed shapes. All axisymmetric shapes spinning faster than  $\Omega_2^*$  are found to be unstable to two-lobed perturbations. However, there are more points of neutral stability along the axisymmetric branch. A point  $\Omega_3^* = 0.707$  denotes a bifurcation point where the drop is unstable to a three-lobed perturbation. This point marks the bifurcation of a three-lobed family of droplet shapes. At higher angular momenta the axisymmetric shape is additionally unstable to both two- and three-lobed perturbations. A third bifurcation point is found at  $\Omega_4^*$  where the axisymmetric family is unstable to any four-lobed perturbations. As expected, a four-lobed family branches off from the axisymmetric family at this point. At higher angular momentum the axisymmetric shape family is unstable to these perturbations as well as two- and three-lobed perturbations. Beyond a critical value  $\Omega_{\max}^* = 0.754$  no axisymmetric shape can be calculated by the Brown and Scriven [1980b] analysis. For this,

the centrifugal pressure on the surface of the drop can no longer be counterbalanced by the surface tension and the droplet tends to collapse onto its own equator and the position representation of the drop's surface breaks down. Nonetheless, Brown and Scriven [1980b] reports  $\Omega_{\max}^* = 0.7539$  which is within 0.1% of the exact value by Ross [1968a]. The family of axisymmetric shapes does not end at  $\Omega_{\max}^*$  but bends back on itself towards lower angular velocities and takes on a concave shape at the poles and effectively ends on  $\Omega_{\text{end}}^* = 0.7059$  where the drop no longer encloses the origin. See Fig. 1.5 for reference. However, the Brown & Scriven analysis fails (because of their representation of the surface) well before this happens, at  $\Omega^* = 0.74$ . Nonetheless, the axisymmetric shape family is analytically tractable so that even a correct numerical analysis beyond this point would yield no new results.



**Fig. 1.5:** Stability of shape families and neutral stability points for isolated drops. Solid branches indicate stable shapes, broken branches denote unstable shapes. In the case of driven drops, the two-lobed shape family (see the solid branch after the first neutral stability point) is unstable.

The point of neutral stability  $\Omega_2^*$  marks the subcritical bifurcation of the axisymmetric family into a family of two-lobed shapes. By subcritical we mean that any two-lobed pertur-

bation with  $\Omega^* > \Omega_2^*$  converge back to a axisymmetric shape whereas on the other hand, a perturbation with  $\Omega^* < \Omega_2^*$  converge to a two-lobed shape. When a two-lobed shape has been

attained, decreasing the angular velocity results in growth of the two lobes. At some point a small concavity grows around the axis of rotation and the drop takes on a dumbbell-like form. Eventually with a further decrease in angular velocity, necking occurs and the lobes grow spherical except for a slender amount of fluid at the axis of rotation holding them together. With increasing angular momentum, the Brown and Scriven [1980b] analysis showed that the eigenvalue associated with the curvature of the energy surface is zero at the bifurcation point  $\Omega_2^*$  and rises up through a maximum with increasing angular momentum. Beyond the maximum value the eigenvalue decreases and it is again zero at a value  $\mathcal{L}_2^{*\max} = 0.5075$ . At this point the curvature of the energy surface is negative so, in short, this point marks another turning point for the two-lobed shapes subject to perturbations of the same symmetry. For driven drops, the entire range of two-lobed shapes are found to be unstable. There are found two more secondary bifurcation points on the two-lobed descending branch. These are called secondary since they branch off from a family that has already bifurcated from the main family of axisymmetric shapes. At  $\Omega^* = \Omega_{2(1)}^*$  the two-lobed shape is unstable to a three-lobed perturbation that removes the reflective symmetry across the neck, i.e. the two-lobed shape is unstable to a perturbation that makes one of the lobes grow whereas the other decreases. The resulting shape is very similar to a peanut, and the family ends at  $\Omega^* = \Omega_{2(2)}^* = 0.2335$ . The interested reader is referred to Brown and Scriven [1980c] for the detailed presentation of the peanut-like shape. The entire range of two-lobed shapes is unstable for driven droplets.

A three-lobed family is, in much the same manner as the two-lobed family, found to bifurcate subcritically from the axisymmetric family at the neutral stability point  $\Omega_3^* = 0.707$ . A sample three-lobed shape is shown in Fig. 1.5, but the reader is referred to Brown and Scriven [1980b], Ohsaka and Trinh [1999] for numerically calculated and experimentally observed three-lobed shapes. The three-lobed family is found to be unstable to 2- and 3-lobed perturbations for both isolated and driven drops. The three-lobed shapes have a maximum angular momentum at  $\mathcal{L}_3^{*\max} = 0.71$ . This point marks another point of neutral stability for mechanically isolated drops at which the already unstable shapes that neutrally stable to a 3-lobed perturbation, become additionally unstable to such a perturbation. A driven three-lobed drop is found to be unstable to any two-lobed perturbation. No secondary bifurcations (such as the "peanut"-bifurcation of the two-lobed family) are found for the three-lobed shape family.

The four-lobed family bifurcates subcritically from the axisymmetric family at  $\Omega_4^* = 0.753$ . No supercritical branch has yet been reported for the four-lobed family. With decreasing angular velocity the four lobes grow, although the Brown & Scriven analysis fails before the lobe lengths have peaked. Nevertheless, it is reasonable to believe that the four-lobed family must evolve toward four spheres of equal volume rotating around a central axis. Both the driven and isolated 4-lobed droplet is found to be unstable to 2- and 3-lobed perturbations. Additionally the driven droplet is also unstable to a four-lobed perturbation, but the isolated droplet is stable to these beyond a critical angular velocity  $\Omega^* = 0.7072$ .

Since Plateau's [Plateau, 1863] experiments, the quantification between theoretical predictions (such as the analysis by Brown and Scriven [1980b], Chandrasekhar [1965]) and experiments has been tardy because the experiments simply do not meet the theoretical restrictions. The largest problem has been

the neglect of gravity which has either been resolved by the use of acoustic levitation or submerging the droplet in a density-matched mixture. In either case there will be large shape deformations caused by the experimental setup. The first experiment where the theoretical assumptions are somewhat met was performed in the NASA Space Shuttle by Wang et al. [1986] where the gravity was in the order of  $10^{-3}$  times the gravity at sea level. However, the droplets were subject to acoustic levitation and torque which caused a flattening of the drop. The most obvious discrepancy is the disagreement between the instability point predicted by theory to  $\Omega_2^* = 0.56$ , but where Wang et al. [1986] finds experimentally  $\Omega_2^* = 0.47 \pm 0.04$ .

The discrepancy between the Wang et al. [1986] results and the Brown and Scriven [1980b] analysis were resolved by Wang et al. [1994] during a series of experiments performed aboard the space shuttle Columbia. In this series of experiments the droplet was spun up to the desired rotational speed, and then allowed to spin down to the bifurcation point with all the acoustics turned off. This allowed the bifurcation to take place subcritically *without* the acoustic flattening of the droplet. The bifurcation point estimated by Wang et al. [1994] in the microgravity of the Earth was  $\Omega_2^* = 0.563$ , within 0.5% of theoretically predicted values and therefore in good agreement. The reader is referred to Wang et al. [1994] for a remarkable series of photographs showing the evolution of an axisymmetric droplet bifurcating into a two-lobed droplet, then undergoing necking into the two-sphere limit. Finally, the neck breaks free and droplet fissions into two spheres.

In no experiments found have 3- or 4-lobed shapes been observed in hydrostatic equilibrium, but these *can* be excited by inducing oscillations in the droplet surface, but the droplet is far from hydrostatic equilibrium.

## D. OSCILLATIONS

It is easy to realize that stable oscillations of a non-rotating droplet are allowed since the initial spheroidal shape, as we have already discussed, represents the minimal surface area under the constant-volume constraint. Thus, any real perturbation will tend to increase the surface area of the droplet which results in the capillary action trying to reduce the surface back to its initial optimal shape. The perturbation must add energy, so if there is no viscous dissipation the droplet will tend to oscillate around its optimal shape. We will over the next pages review the oscillations of a non-rotating as well as a rotating droplet. It is reasonable to start with an outline of the linear analysis in a non-rotating frame as this vividly brings forth the concept of internal resonance.

### D.1 THE LINEAR MODEL

If we assume a droplet which exercises irrotational small-amplitude motion, then

$$\rho \frac{\partial \Phi}{\partial t} + p_0 + \gamma \nabla \cdot \hat{\mathbf{n}} = \text{const.}$$

must hold at the free surface. This is the Bernoulli relation for vanishing amplitude waves under the action of surface tension. The curvature of a slightly deformed sphere,  $r = R_0 + \xi$  where  $\xi$  is small compared to  $R_0$ , is found the appendix, therefore

$$\rho \frac{\partial \Phi}{\partial t} - \gamma \left\{ \frac{2\xi}{R_0} + \frac{1}{R_0^2} \left[ \frac{1}{\sin^2 \theta} \frac{\partial^2 \xi}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \xi}{\partial \theta} \right) \right] \right\} = 0$$

must be satisfied at  $r = R_0$ . All constants have been dropped. Taking the time derivative and realizing that  $\frac{\partial \xi}{\partial t}$  is the radial velocity  $u_r = \frac{\partial \Phi}{\partial r}$  gives that

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = \frac{\gamma}{R_0^2} \left\{ \frac{\partial \Phi}{\partial r} + \frac{\partial}{\partial r} \left[ \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) \right] \right\}$$

must be satisfied on the free surface  $r = R_0$ . We seek for solutions in the form of temporally stable oscillations of the surface, i.e. on the form  $\Phi = e^{-i\omega_l^{(0)} t} f(r, \theta, \varphi)$  where  $\omega_l^{(0)}$  is the predicted frequency of oscillation. We keep a subscript (0) because we will later calculate the first-order shift in this frequency as an effect of rotation.  $f$  must satisfy Laplace's equation so that we may expand  $f$  in terms of the spherical harmonics  $Y_{lm}$ , i.e.  $f = Ar^l Y_{lm}(\theta, \varphi)$  where  $l$  is any integer and  $m = 0, \pm 1 \dots \pm l$ . The spherical harmonics are related to the associated Legendre polynomials by  $Y_{lm} = P_l^m(\cos \theta) e^{im\varphi}$  so

$$\Phi = Ar^l P_l^m(\cos \theta) e^{im\varphi} e^{i\omega_l^{(0)} t}$$

This is the general linear solution to  $\Phi$  in spherical coordinates. It is however more convenient to work directly with the spherical harmonics since they satisfy the relation

$$\frac{1}{\sin^2 \theta} \frac{\partial^2 Y_{lm}}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_{lm}}{\partial \theta} \right) + l(l+1) Y_{lm} = 0 \quad [1.21]$$

Inserting the above relation into the free-surface boundary condition gives after a little algebra the frequency relation

$$\left( \omega_l^{(0)} \right)^2 = \frac{\gamma}{\rho R_0^3} l(l-1)(l+2) \quad [1.22]$$

This result was first obtained by Rayleigh [1879]. The frequencies are degenerate since there are  $2l+1$  values of  $m$  for each value of  $l$  so that, in short, each frequency corresponds to  $2l+1$  different modes oscillation. Also,  $l=0$  corresponds to purely radial oscillations of the drop so it vanishes in the incompressible limit and  $l=1$  corresponds to a pure translatory motion of the drop as a whole. Conveniently choosing the origin to follow the center of mass therefore excludes the  $l=1$  mode. The lowest frequency of oscillation is then  $\omega_2^{(0)} = \sqrt{8\gamma/\rho R_0^3}$  which is coupled to the  $l=4$  mode by

$$\omega_4^{(0)} = \pm 3\omega_2^{(0)}$$

In short, there is third-order resonance between the  $l=4$  and  $l=2$  modes of the droplet so that driving the drop in  $l=2$  will tend to excite the  $l=4$  mode. Furthermore, there is quadratic resonance in the terms

$$\omega_8^{(0)} = \pm 2\omega_5^{(0)}, \quad \omega_{16}^{(0)} = \pm 2\omega_{10}^{(0)}$$

These are the only two modes of quadratic resonance of the lower order spherical harmonics. Since the velocity potential and the shape are related by  $\xi_t = \Phi_r$ , an expansion of  $\xi$  in terms of spherical harmonics is also allowed. We will restrict ourselves to the zonal modes of oscillation,  $m=0$ , so that the shape does not depend on latitude and is axisymmetric. For these modes;

$$\xi = \sum_l a_l P_l(\cos \theta) e^{-i\omega_l^{(0)} t}$$

where still  $l \geq 2$  as was required from incompressibility and choosing the origin in the center of mass.  $P_l(\cos \theta)$  is here the

Legendre polynomial of degree  $l$ . Also,  $a_l \ll R_0$  as was required from the linearization.

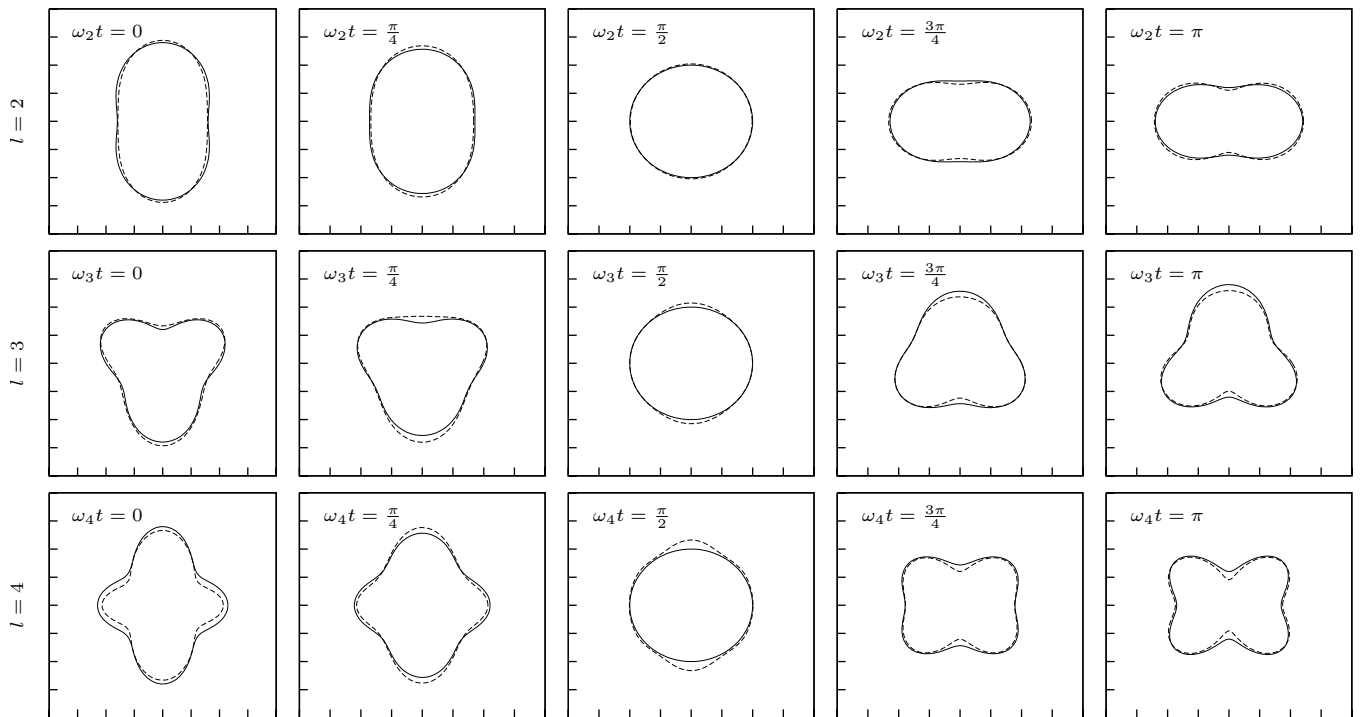
$\xi$  is now a superposition of prolate-oblate oscillations of different degrees, the pure modes  $l=2, 3, 4$  are shown in Fig. 1.6 where also the second-order correction found by Tsamopoulos and Brown [1983] has been included. We do not include the explicit expressions for these modes except mention that they result from a tedious algebraic calculation and are of the form

$$\begin{aligned} \xi_2^{(2)} &= a_2(t) + b_2(t)P_2(\cos \theta) + c_2(t)P_4(\cos \theta) \\ \xi_3^{(2)} &= a_3(t) + b_3(t)P_2(\cos \theta) + c_3(t)P_4(\cos \theta) + d_3P_6(\cos \theta) \\ \xi_4^{(2)} &= a_4(t) + b_4(t)P_2(\cos \theta) + c_4(t)P_4(\cos \theta) + d_4P_6(\cos \theta) \\ &\quad + e_4P_8(\cos \theta) \end{aligned}$$

where the subscript on  $\xi$  indicates the modes 2, 3, 4 and  $a_n, b_n \dots$  are some time-dependent terms. Because of the complicated algebra, Tsamopoulos and Brown [1983] resorts to the use of symbolic manipulators in their original paper. The amplitude of the oscillation/perturbation used in Fig. 1.6 has been set to  $a_l = 0.4R_0$  based on  $\mathcal{O}(a_l^3) \sim 10^{-2}$ . The onset of second-order nonlinear effects deforms the perfect sphere predicted at  $\omega_l t = \frac{\pi}{2}$  into 4, 6 and 8-lobed shapes for  $l=2, 3, 4$  respectively. This is also seen from the form of the second-order corrections.

Large amplitude deformations, up to third/fourth order, has been confirmed by Apfel et al. [1997]. They perform their experiments in the Drop Physics Module at the space ship Columbia and excite the  $l=2, m=0$  axisymmetric mode by means of an intense sound field. Their analysis reveal a prolate-oblate oscillation of the droplet but the onset of nonlinearity makes the oscillations via quadrupole shapes. Although the amplitude of the oscillations found by Apfel et al. [1997] are large (third or fourth-order nonlinear analysis must be used), the oscillations are qualitatively very similar to those predicted by Tsamopoulos and Brown [1983]. Nonlinear analysis beyond a second order expansion has not been found in literature.

Natarajan and Brown [1986, 1987] derived the interaction equations that couple both the quadratic resonance and the third-order resonance (that is, the coupling between the  $l=8, l=5$  modes, the  $l=16, l=10$  modes and the  $l=4, l=2$  modes). They replace the linear theory by letting the spherical harmonics be implicitly dependent on the slow time scale (i.e. the modulated oscillation) and "preaverage" the Lagrangian over a conveniently chosen period. The reader not familiar with such perturbation theory may want to refer to e.g. Nayfeh [1973] or Hinch [1991]. The resulting equations of motion show, for quadratic resonance and non-axisymmetric deformations, a stochastic redistribution of the energy over all the degrees of freedom. That is, the energy is distributed over all the degenerate  $l$ -modes which in this case are about 200-300 possible modes of oscillation. No pattern in the droplet dynamics may be detected so the droplet dynamics are well-described by the word chaotic. Analysis of the third-order resonance between the  $l=2$  and  $l=4$  yields, under the assumption of nonzero angular momentum, stable travelling waves in the azimuthal direction. In the case of zero angular momentum the travelling waves are unstable and the energy is again redistributed over all the degrees of freedom. The multiple time-scale perturbation analysis by Natarajan and Brown [1986, 1987] is very tedious and motivates the use of symbolic manipulators.



**Fig. 1.6:** The first three normal modes of oscillation plotted for a half period  $0 < \omega t < \pi$ . The shape of mode  $l$  is shown as  $r_l = R_0 + aP_l(\theta) \exp(i\omega t)$  where the amplitude  $a$  of the oscillation is  $a = 0.4R_0$  in all plots. The first row shows the  $l = 2$  mode, the second row shows the  $l = 3$  mode and the third row shows the  $l = 4$  mode.

Experimentally, the direct coupling between the axisymmetric  $l = 6$ ,  $l = 3$  as well as non-axisymmetric and axisymmetric  $l = 3, l = 2$  modes were captured by Trinh et al. [1998]. They expand the droplet surface in terms of Legendre polynomials up to sixth order, i.e.  $r = R_0 [1 + \sum_{l=2}^6 c_l(t)P_l(\cos \theta)]$ , levitate and drive the droplet by use of acoustics and find the shape of the droplet by means of an edge-finding camera technique so that each coefficient  $c_l(t)$  may be identified independently. When the drop was driven in the  $l = 3$  axisymmetric mode with frequency  $\omega_3^{(0)}$ , the  $l = 2$  axisymmetric mode was excited and showed a subharmonic frequency of  $\omega_2^{(0)} \approx 0.5\omega_3^{(0)}$  showing that exact harmonic resonance is not required for subharmonic excitation. For comparison, theory reveals  $\omega_2^{(0)}/\omega_3^{(0)} \approx 0.447$ . Also, the  $l = 2$  mode grows at the expense of the  $l = 3$  mode indicating an energy transfer to the lower-order mode but as Trinh et al. [1998] points out, the excited  $l = 2$  mode is not axisymmetric. Exactly which  $l = 2$  mode was excited is not explicitly commented on by Trinh et al. [1998]. The reader should beware that the given frequencies  $\omega_3^{(0)}$  and  $\omega_2^{(0)}$  are oscillation frequencies of free droplets whereas the presence of the sound field will tend to change these somewhat. Intriguingly, the reverse setup where the  $l = 2$  mode is driven in order to excite the  $l = 3$  mode will tend to also excite higher-order harmonics, in this case also  $l = 4, 5, 6$  [Trinh et al., 1998]. In the same spirit, the driven  $l = 6$  mode tends to subharmonically excite the  $l = 3$  mode but the  $l = 3$  mode does not seem to transfer energy to the  $l = 2$  mode as was the case in the latter discussion. Rather, the  $l = 2$

mode seems to be excited very weakly by the driven mode and later more substantially by the  $l = 3$  mode. The conclusion of the Trinh et al. [1998] experiment is that higher-order modes may excite lower-order modes of different symmetry but the reverse coupling is not observed. This is not surprising since the viscous damping associated with higher-order modes is higher than the damping of lower-order modes.

## D.2 THE EFFECT OF ROTATION

Despite being one of the most basic problems in fluid mechanics, the absence of good experimental verification has caused the oscillations of a rotating drop to attract little or no attention theoretically. The effect of rotation on the harmonic frequencies was first examined by Busse [1984]. In doing so, we shall need to make use of the linearized Euler equations in a rotating frame of reference.

### Small-Amplitude Oscillations In A Rotating Flow

The momentum equations in a frame of reference rotating with angular velocity  $\mathbf{\Omega}$  are [Landau and Lifshitz, 1987]

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - 2\mathbf{\Omega} \times \mathbf{u} \quad [1.23]$$

where we have formed the reduced pressure  $p \rightarrow p + \frac{1}{2}(\mathbf{\Omega} \times \mathbf{r})^2$ . The justification of forming the reduced pressure is given in

Part III, section A. To examine the effect of rotation, we nondimensionalize the dependent and independent variables by  $t^* = |\mathbf{\Omega}|t$ ,  $\mathbf{r}^* = \frac{\mathbf{r}}{L}$ ,  $\mathbf{u}^* = \frac{\mathbf{u}}{U}$ ,  $p^* = \frac{p}{\rho|\mathbf{\Omega}|UL^2}$  where  $U$  and  $L$  are characteristic velocities and length scales of the system. The resulting equations of motion are thus

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + Ro(\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* = -\nabla^* p^* - 2\mathbf{\Omega}^* \times \mathbf{u}^* \quad [1.24]$$

where  $Ro = \frac{U}{\Omega L}$  is known as the Rossby number and measures the importance of inertial versus rotational effects. For low Rossby numbers, the above may be linearized so that the convective term is dropped. More formally perhaps, we may expand the dependent variables in a perturbation sequence,

$$\mathbf{u}^* = \sum_{n=0}^{\infty} Ro^n \mathbf{u}_n^*, \quad p^* = \sum_{n=0}^{\infty} Ro^n p_n^* \quad [1.25]$$

and insert this into the basic equations. The continuity equation only yields

$$\sum_{n=0}^{\infty} Ro^n \nabla^* \cdot \mathbf{u}_n^* = 0$$

whereas the momentum equations are written

$$\left[ \frac{\partial \mathbf{u}_n^*}{\partial t^*} + \mathbf{u}_n^* \cdot \nabla^* \sum_{m=0}^{\infty} Ro^{m+1} \mathbf{u}_m^* + \nabla^* p_n^* + 2\mathbf{\Omega}^* \times \mathbf{u}_n^* \right] = 0$$

which is a series of problems in different order of  $Ro$ . The linear theory is recovered from the zeroth order of  $Ro$ , thereby suggesting that we drop the convective term

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho} \nabla p - 2\mathbf{\Omega} \times \mathbf{u} \quad [1.26]$$

where we keep the dimensionality of the equations, and drop the subscript on  $\mathbf{u}$ . We have achieved linearity in the fundamental equations, hence we may directly interpret  $\mathbf{u}$  and  $p$  as the disturbances from the hydrostatic condition. Formally, one would write  $\mathbf{u} = \mathbf{U} + \mathbf{u}'$  and  $p = P + p'$  where  $\mathbf{U}$  and  $P$  are the unperturbed solutions corresponding to hydrostatic conditions and the quantities  $\mathbf{u}'$  and  $p'$  are the perturbed quantities corresponding to an expansion around equilibrium. In the linearity of equation [1.26], both the unperturbed and the primed quantities each fulfill equation [1.26] and there are no "coupling"-terms between these as is normally the case in expansions of the Navier-Stokes/Euler equations such as this. Of course, our expansion is around hydrostatic equilibrium so  $\mathbf{U} = 0$ . Now, the above equation can be shown to possess wave-like solutions in terms of the perturbed quantities. If we write  $\mathbf{u}' = \mathbf{u}'_0 \exp(i\omega t)$  and  $p' = p'_0 \exp(i\omega t)$  where  $\mathbf{u}'_0$  and  $p'_0$  contain the spatial dependence of  $\mathbf{u}'$  and  $p'$  and  $\omega$  is some angular frequency to be specified, the basic equations reduce to solving

$$\nabla \cdot \mathbf{u}'_0 = 0 \quad [1.27a]$$

$$i\omega \mathbf{u}'_0 + \frac{1}{\rho} \nabla p'_0 + 2\mathbf{\Omega} \times \mathbf{u}'_0 = 0 \quad [1.27b]$$

subject to the proper boundary conditions of course. As equation [1.26] stands, the pressure and velocity field may be isolated and expressed in two coupled equations rather than the original three. Formally, it is easier to work with equation [1.26] and later impose the explicit form of the temporal dependency. Taking the divergence of equation [1.26] and using the incompressibility condition on  $\mathbf{u}$  gives

$$\Delta p' = -2\rho \nabla \cdot (\mathbf{\Omega} \times \mathbf{u}')$$

where  $\Delta$  denotes the Laplacian. The right hand side is, by use of the product rule and the Levi-Civita tensor, identified as  $\nabla \cdot (\mathbf{\Omega} \times \mathbf{u}'_0) = \partial_i \varepsilon_{ijk} \Omega_j u'_k = \mathbf{u}' \cdot (\nabla \times \mathbf{\Omega}) - \mathbf{\Omega} \cdot (\nabla \times \mathbf{u}')$ . Hence, taking the partial derivative of the above equation, with respect to time

$$\frac{\partial}{\partial t} \Delta p' = 2\rho \mathbf{\Omega} \cdot \frac{\partial (\nabla \times \mathbf{u}')}{\partial t}$$

where the term on the right hand side is  $\partial_t (\nabla \times \mathbf{u}') = 2(\mathbf{\Omega} \cdot \nabla) \mathbf{u}'$ . This explicit reduction is shown in Part III, we do not recite this except recall that the convective term has been neglected. Thus, taking another partial derivative with respect to time, and using equation [1.26] we find

$$\partial_t^2 \Delta p' = -4\mathbf{\Omega} \cdot [(\mathbf{\Omega} \cdot \nabla) \nabla p' + 2\rho (\mathbf{\Omega} \cdot \nabla) (\mathbf{\Omega} \times \mathbf{u}')] ]$$

The two vector-terms on the right hand side are readily identified by use of the Levi-Civita tensor. The first is shown to be  $(\mathbf{\Omega} \cdot \nabla)^2 p'$  by use of the product rule whereas the second is shown to contain the cross product of  $\mathbf{\Omega}$  with itself, hence it is zero. Therefore, the equation of the disturbance pressure alone is

$$\partial_t^2 \Delta p' = -4(\mathbf{\Omega} \cdot \nabla)^2 p'$$

Then, if the pressure has a temporal dependency in the form  $\exp(i\omega t)$ , it follows directly that

$$\Delta p'_0 = \left( \frac{2\mathbf{\Omega} \cdot \nabla}{\omega} \right)^2 p'_0 \quad [1.28]$$

where it is recalled that  $p' = p_0(x, y, z) \exp(i\omega t)$ . The same decomposition may be performed for the velocity field. We start by taking the cross product of equation [1.26] with  $2\mathbf{\Omega}$  and replace the  $\partial_t 2\mathbf{\Omega} \times \mathbf{u}'$ -term from the original equation. It then follows that

$$\partial_t \left( \partial_t \mathbf{u}' + \frac{1}{\rho} \nabla p' \right) - 4\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{u}') = \frac{2}{\rho} \mathbf{\Omega} \times \nabla p'$$

It follows that the double cross product is

$$\begin{aligned} \varepsilon_{ijk} \Omega_j \varepsilon_{klm} \Omega_l u'_m &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \Omega_j \Omega_l u'_m \\ &= \Omega_i \Omega_j u'_j - \Omega_j^2 u'_i \end{aligned}$$

so we find

$$(4\Omega^2 + \partial_t^2) \mathbf{u}' = \frac{\mathbf{\Omega} \times \nabla p' - \partial_t \nabla p'}{\rho} - 4\mathbf{\Omega} (\mathbf{\Omega} \cdot \mathbf{u}')$$

The final form is obtained by going to equation [1.26] and recognizing  $\mathbf{\Omega} \cdot \mathbf{u}'$  from  $\mathbf{\Omega} \cdot \partial_t \mathbf{u}' = -\mathbf{\Omega} \cdot \nabla p'$  where one imposes a temporal dependency of  $\mathbf{u}'$  and  $p'$  of the form  $\exp(i\omega t)$ . Then

$$\mathbf{u}'_0 = \frac{1}{\rho} \frac{2\mathbf{\Omega} \times \nabla p'_0 - i\omega \nabla p'_0 - \frac{4}{i\omega} (\mathbf{\Omega} \cdot \nabla p'_0) \mathbf{\Omega}}{4\Omega^2 - \omega^2} \quad [1.29]$$

Equations [1.28] and [1.29] will able us to explicitly determine change in oscillation frequencies of the linear model as an effect of rotation. In doing so, it is reasonable to start with the effect of low rotation rates, i.e. to first order in  $\Omega$ .



### Low Rotation Rates

For low rotation rates the drop remains spherical and we may without loss of generality assume that the free-surface is given by the radial function  $r = R_0 + \xi^{(0)}(\theta, \varphi, t)$  where  $R_0$  is the mean radius of the drop, i.e. in terms of its volume  $V = \frac{4\pi}{3}R_0^3$ .  $\xi^{(0)}$  is the deformation of the surface to zeroth order in  $\Omega$ , i.e. since the drop is spheroidal we may make the same assumption of the deformation of the surface as in the non-rotating case. We shall make the expansions such as this of all the dependent variables,

$$p' = p^{(0)} + \dots, \quad \mathbf{u}' = \mathbf{u}^{(0)} + \dots \quad [1.30]$$

We still stick by the notation where a subscript 0 indicates the spatial dependency of the variable, and the temporal terms of the form  $\exp(i\omega t)$  have been taken out. The surface condition gives

$$P + p^{(0)} = \gamma \nabla \cdot (\hat{\mathbf{N}} + \hat{\mathbf{n}}^{(0)}) \quad [1.31]$$

Of course the surface vector is given by,  $\hat{\mathbf{n}} = \hat{\mathbf{N}} + \hat{\mathbf{n}}^{(0)}$  where  $\hat{\mathbf{N}}$  corresponds to the unit normal vector of a perfect sphere and  $\hat{\mathbf{n}}^{(0)}$  relates the the distortion of the surface. The fact that the unit vector and pressure may be described in such a way is a result of linearization, see the appendix. It is clear that  $P$  and  $\nabla \cdot \hat{\mathbf{N}}$  reflect the hydrostatic non-rotating case and therefore complete each other such that  $P = \gamma \nabla \cdot \hat{\mathbf{N}}$ . From the appendix it follows that

$$p^{(0)} = -\frac{2\gamma\xi^{(0)}}{R_0^2} - \frac{\gamma}{R_0^2} \left[ \frac{1}{\sin^2\theta} \frac{\partial^2 \xi^{(0)}}{\partial \varphi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial \xi^{(0)}}{\partial \theta} \right) \right] \quad [1.32]$$

We may expand the distortion in terms of spherical harmonics,

$$\xi^{(0)}(\theta, \varphi, t) = a_l P_l^m(\cos\theta) \exp(im\varphi + i\omega_{l,m}t) \quad [1.33]$$

Here,  $\omega_{l,m}$  is the new frequency of oscillation and we keep it dependent on both  $l$  and  $m$ . Since the spherical harmonics obey relation [1.21] it follows that

$$p^{(0)} = \frac{\gamma}{R_0^2} a_l P_l^m(\cos\theta) e^{(im\varphi + i\omega_{l,m}t)} [l(l+1) - 2], \quad \text{at } r = R_0 \quad [1.34]$$

This of course, also suggests that the pressure may be expanded in spherical harmonics. To first order in  $\Omega$ , the Laplacian of the pressure must vanish and so (see equation [1.28])

$$p^{(0)} = \frac{\gamma}{R_0^2} [l(l+1) - 2] a_l \left( \frac{r}{R_0} \right)^l P_l^m(\cos\theta) e^{(im\varphi + i\omega_{l,m}t)} \quad [1.35]$$

The description is completed by the kinematic relationship

$$i\omega_{l,m}\xi^{(0)} = \mathbf{u}^{(0)} \cdot \hat{\mathbf{r}}, \quad \text{at } r = R_0 \quad [1.36]$$

and allows only radial vibrations. The velocity relation to first order in  $\Omega$  is

$$\mathbf{u}_0^{(0)} = \frac{i\nabla p_0^{(0)}}{\rho\omega_{l,m}} - \frac{2\boldsymbol{\Omega} \times \nabla p_0^{(0)}}{\rho\omega_{l,m}^2} \quad [1.37]$$

By inserting equation [1.37] into equation [1.36] we see that we only need to calculate the radial component of the terms on the right-hand side in equation [1.37]. Calculating these yield

$$\begin{aligned} (\nabla p^{(0)})_r &= \frac{il\beta_l}{R_0} \left( \frac{r}{R_0} \right)^{l-1} P_l^m(\cos\theta) e^{(im\varphi + i\omega_{l,m}t)} \\ (\boldsymbol{\Omega} \times \nabla p^{(0)})_r &= im\Omega\beta_l \left( \frac{r}{R_0} \right)^l P_l^m(\cos\theta) e^{(im\varphi + i\omega_{l,m}t)} \end{aligned}$$

In the axisymmetric case  $m = 0$ , the final term in equation [1.37] vanishes and  $\mathbf{u}^{(0)}$  is simply given by

$$\mathbf{u}_0^{(0)} = \frac{i\nabla p^{(0)}}{\rho\omega_{l,0}} \quad [1.38]$$

For more general  $m$  however, the kinematic relation gives after cancelling all common factors that

$$\begin{aligned} \omega_{l,m}^2 &= \frac{\gamma}{\rho R_0^3} \left[ l(l+1) - 2 + (l(l+1) - 2) \frac{2m\Omega}{\omega_{l,m}} \right] \\ &= (\omega_l^{(0)})^2 \left[ 1 + \frac{2m\Omega}{l\omega_{n,m}} \right] \end{aligned} \quad [1.39]$$

where  $\omega_l^{(0)}$  is the natural frequencies of oscillation, see equation [1.22]. Therefore, as a first approximation, the new frequencies of oscillation are given by

$$\omega_{l,m} = \omega_l^{(0)} \sqrt{1 + \frac{2m\Omega}{l\omega_l^{(0)}}} \quad [1.40]$$

Clearly, the onset of the Coriolis effect causes a splitting of the degeneracy of the frequencies  $\omega_l^{(0)}$ . It is interesting to see that for the axisymmetric oscillations  $m = 0$ , the term proportional to  $\Omega$  does not enter in the analysis.

The fact that terms proportional to  $\Omega$  do not enter in the analysis suggests that we expand our dependent variables in a series of  $\eta = \left( \frac{2\Omega}{\omega_l^{(0)}} \right)^2$ . Thus,

$$\begin{aligned} \mathbf{u}' &= \mathbf{u}^{(0)} + \eta\mathbf{u}^{(1)} + \dots, \quad p' = p^{(0)} + \eta p^{(1)} + \dots, \\ \xi &= \xi^{(0)} + \eta\xi^{(1)} + \dots, \quad \omega_l = \omega_l^{(0)} + \eta\omega_l^{(1)} \end{aligned} \quad [1.41]$$

Here, the terms with superscript (0) are given by equations [1.38], [1.35], [1.33] and  $\omega_l^{(0)}$  is the natural frequency of oscillation as is seen from equation [1.40] with  $m = 0$ . Indeed, the superscript (0) indicates all the results of the first-order effects in  $\Omega$  with  $m = 0$ , which is identical to the linear model. By evaluating equations [1.29] and [1.28] to order  $\eta$  we find

$$\Delta p^{(1)} = (\hat{\boldsymbol{\Omega}} \cdot \nabla) p^{(0)} \quad [1.42a]$$

$$\mathbf{u}^{(1)} = \frac{i}{\rho\omega^{(0)}} \left( \nabla p^{(1)} + (1 - \hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Omega}})\nabla p^{(0)} - \frac{\omega^{(1)}}{\omega^{(0)}} \nabla p^{(0)} \right) \quad [1.42b]$$

If we go to second order in  $\Omega$ , we may use the results of section C.1. This states that the shape is given by  $F_0 = 0 = r^3 - R_0^3 [1 - \varepsilon P_2(\cos\theta)]$  where  $\varepsilon = \frac{\rho\Omega^2 R_0^3}{8\gamma}$ . We shall consider this, and not the spheroidal case, as the unperturbed case. If we consider  $\xi$  as a small perturbation in the normal direction of  $F_0$ , the new interface is described by  $F = F_0 - \xi = 0$ . The kinematic relationship is then  $i\omega\xi = \mathbf{u} \cdot \nabla F_0$  and the calculation is no longer trivial, nor is the equation of the surface pressure. We shall only state the result:

$$\begin{aligned} \frac{\Delta\omega_l^{(1)}}{\omega_l^{(0)}} &= \frac{\rho R_0^3 \Omega^3}{\gamma} \left[ \frac{2}{l(l+2)(2l-1)} + \frac{1}{6(2l-1)(2l+3)} \right. \\ &\quad \left. \times \left( \frac{l^4 + 2l^3 - 4l^2 - 5l + 6}{(l+2)(l+1)} - \frac{(l+1)(l-3)}{4} \right) \right] \end{aligned} \quad [1.43]$$

Although somewhat modified due to the experimental setup, equations [1.40] and [1.43] were experimentally verified by Annamalai et al. [1985] to good precision.

It was concluded in the previous section that the 3-lobed shapes are inherently unstable to two-lobed perturbations and

such a hydrostatic configuration is physically unrealizable. Ohsaka and Trinh [1999] still managed to excite the 3-lobed shape by exciting a two-lobed shape oscillation which prevents the natural growth of the two-lobed shape, thereby suppressing the bifurcation into the two-lobed shape. However, the Ohsaka and Trinh [1999] analysis was by acoustic levitation (resulting in flattening of the drop) and the droplet was suspended in another fluid giving large shape deformations caused by viscous shear on the surface. Nevertheless, Ohsaka and Trinh [1999] showed that the excitation of the 3-lobed shape is possible, although the droplet is far from hydrostatic equilibrium. A remarkable series of photographs of the three-lobed shapes are found in Ohsaka and Trinh [1999].

Hill and Eaves [2008] performed a most intriguing experiment by diamagnetically levitating the droplet. Diamagnetic substances are materials which consists of molecules with no net angular momentum. In an external applied magnetic field this causes the creation of circulating atomic currents which produces a weak magnetic field opposite to the applied field. A permanent magnetic dipole  $\mathbf{m}$  in an external field experiences a force  $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$ , and so if  $\mathbf{m}$  and  $\mathbf{B}$  are opposite there is a repulsive force on  $\mathbf{m}$ . For diamagnets (in vacuum), the form of the induced magnetic moment  $\mathbf{m}$  goes as  $\mathbf{m} = \frac{\chi}{\mu_0} \mathcal{V} \mathbf{B}$  where  $\mathcal{V}$  is the volume of the diamagnetic substance with a susceptibility  $\chi$ .  $\mathbf{B}$  is the external applied magnetic field. With the approximation  $\nabla \mathbf{B}^2 \sim \frac{B^2}{l}$  where  $l$  is the typical length of the magnet the force balance between the repulsive force and gravity is approximately fulfilled by

$$B^2 = \frac{2\mu_0 \rho g l}{\chi}$$

$\mathbf{B}$  is here  $\mathbf{B} \parallel \hat{\mathbf{z}}$ . Water is weakly diamagnetic with a susceptibility  $\chi \sim -10^{-5}$ . Take  $\mathcal{O}(l) = 10^{-1} \text{cm}$ , the external magnetic field is then of an order

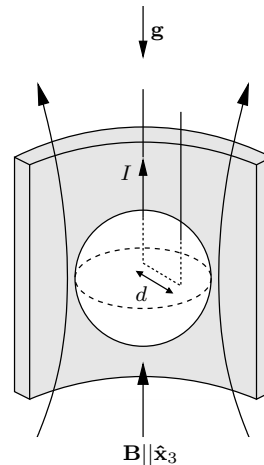
$$B \sim 16T$$

Only superconductors can be used to generate these field strengths. For comparison the field strength of the Earth's magnetic field is of the order  $10^{-6}T$ , and the Sun has a dipole strength of approximately  $10^{-4}T$ . The field strength of the superconducting bore is thereby in the order of certain white dwarfs ( $0.1 - 10^5T$ ). For the interested reader, even though the superconductor is producing field strengths a hundred thousand times stronger than the Sun, it is completely dwarfed by certain types of neutron stars called magnetars, producing field strengths in the order of  $10^9T$ , i.e. a quadrillion times stronger than the magnetic field of the Earth. Levitation of a water droplet can be achieved by using a superconducting vertical-bore to diamagnetically levitate the droplet, see Fig. 1.7

The advantages of diamagnetic levitation of the droplet is the avoidance of the problem with viscous shear on the droplet surface since the droplet can be levitated in vacuum/air instead of being suspended in a high-viscosity fluid mixture. An "electric motor" can be applied to the droplet by passing an electric current  $I$  through the droplet centre by means of a wire through the droplet. One wire vertically pierces the top of the droplet and again protrudes a distance  $d$  from the vertical axis of the droplet. The resulting Lorentz force on a part  $dl$  of the wire is then  $d\mathbf{F} = I dl \times \mathbf{B}$ . This gives rise to a torque  $\mathbf{N}$  on the droplet given by

$$\mathbf{N} = I \int_0^d \mathbf{l} \times [d\mathbf{l} \times \mathbf{B}] = \frac{BI d^2}{2} \hat{\mathbf{x}}_3$$

Depending on the direction of the electric current  $I$ , the torque causes the droplet to rotate clockwise or counterclockwise around its vertical axis.



**Fig. 1.7:** Diamagnetic levitation and rotation of a water droplet, the magnetic field  $\mathbf{B}$ , acceleration of gravity  $\mathbf{g}$  and electric current  $I$  are indicated.

The Hill and Eaves [2008] analysis may be summarized as follows: The suppression of bifurcation to the two-lobed shape is performed by exciting a small-amplitude travelling wave (running opposite to the spin) with 3 nodes. The idea is that the inertia of the travelling wave suppresses the growth of the 2-lobed fluctuation which leads to the 2-lobed shape bifurcation. The initial shape of the spinning droplet is an oblate spheroid, i.e. the droplet is flattened at the poles and bulging at the equator. The three-lobed shape bifurcates from the axisymmetric shape family at  $\Omega_3^* = 0.75$ , for comparison theory reveals a bifurcation point at  $\Omega_3^* = 0.71$ . The agreement between the Hill and Eaves [2008] experiment and theory is good, and because of the absence of viscous shear and gravity the three-lobed shape is very profound. After the onset of the triangular shape, the equatorial shape of the droplet is well described in the inertial frame by

$$r(\varphi, t) = R_0 \{1 + a + b_W \cos(m\varphi) + b_R \cos[m(\varphi - \Omega t)]\}$$

where  $R_0$  is the radius of the droplet at rest and  $m = 3$  the mode of oscillation. The  $b_W$ -term represents a static triangular shape and the  $b_R$ -term represents a triangular rigid-body rotation. Here,  $b_R = b_W \approx 0.3$ . The fact that  $m = 3$  here only reflects the fact that there are three nodes along the equator of the droplet, the expression above is also found to be valid for the spin-up of the two-lobed shape[Hill and Eaves, 2008]. In the *rotating frame* the equatorial shape is found to be travelling waves along the equator with the frequency  $\omega = m\Omega$  and the shape may be described by

$$r(\varphi', t) = R_0 \{1 + a + b_W \cos[m(\varphi' + \Omega t)] + b_R \cos(m\varphi')\}$$

where  $\varphi' = \varphi - \Omega t$ , i.e. relates the azimuthal angle in the rotating frame to the azimuthal angle in the non-rotating frame. This reveals that the static shapes in the inertial frame (by static we mean shapes that do not rotate in the inertial frame) are travelling waves in the rotating frame.

Hill and Eaves [2008] also managed to excite 4- and 5-lobed shapes by the same approach but as they emphasize, these shapes are not caused by hydrostatic equilibrium shapes but rather by the travelling waves in the rotating coordinate system running along the equator of the droplet.

## PART II

### LITERATURE REVIEW

The dynamics of a droplet may be described by a single nonlinear partial differential equation known as the Korteweg-de Vries equation which can be derived from a Eulerian formalism as well as a Hamiltonian formalism. Hence, it is logical to present both a survey of Hamiltonian fluid mechanics and the Korteweg-de Vries equation.

#### A. HAMILTONIAN FLUIDS MECHANICS

Hamiltonian fluid mechanics is the application of Hamiltonian mechanics, i.e. classical mechanics, to the physics of fluids. The reader not familiar with classical mechanics is referred to e.g. Goldstein et al. [2002], Arnold [1978] or any other book on Lagrangian or Hamiltonian mechanics.

##### A.1 FUNDAMENTALS

Let a system consist of  $N$  discrete particles with mass  $m_i$  and Cartesian position  $x_i(t)$  and let the system have a potential energy  $U(\mathbf{x}_1, \dots, \mathbf{x}_N)$ . Then the Lagrangian of the system is

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = \sum_i \frac{1}{2} m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i - U(\mathbf{x}_1, \dots, \mathbf{x}_N) \quad [2.1]$$

where by  $\dot{\mathbf{x}}_i$  we understand the velocity of particle  $i$  and by  $(\mathbf{x}, \dot{\mathbf{x}})$  we understand the complete set of all  $\mathbf{x}_i$ 's and so on. Lagrange's equations of motion emerge from Hamilton's principle,  $\delta \int L dt = 0$  where  $\delta$  indicates an arbitrary variation in the path of the system. We may invoke Hamilton's principle in a modified form, i.e.

$$\delta \int \left( \sum_i \mathbf{p}_i \cdot \dot{\mathbf{x}}_i - H(\mathbf{p}, \mathbf{x}) \right) dt = 0 \quad [2.2]$$

where the conjugate momenta  $\mathbf{p}_i$  is defined

$$\mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{x}}_i} \quad [2.3]$$

and the Hamiltonian

$$H(\mathbf{p}, \mathbf{x}) = \sum_i \mathbf{p}_i \cdot \dot{\mathbf{x}}_i - L(\mathbf{x}, \dot{\mathbf{x}}, t) \quad [2.4]$$

Explicitly taking the variation (see Appendix) gives the canonical equations

$$\dot{\mathbf{x}}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \quad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{x}_i} \quad [2.5]$$

The *Poisson bracket* between two general functions  $F(\mathbf{x}, \mathbf{p})$  and  $G(\mathbf{x}, \mathbf{p})$  is defined

$$\{F, G\} = \sum_i \left( \frac{\partial F}{\partial \mathbf{x}_i} \cdot \frac{\partial G}{\partial \mathbf{p}_i} - \frac{\partial F}{\partial \mathbf{p}_i} \cdot \frac{\partial G}{\partial \mathbf{x}_i} \right) \quad [2.6]$$

and it follows that the Poisson bracket is antisymmetric  $\{F, G\} = -\{G, F\}$  and obeys the Jacobi identity (see e.g. Goldstein et al. [2002]). The equations of motion can be written

$$\frac{dF}{dt} = \{F, H\} \quad [2.7]$$

Instead of Newton's laws of motion, Hamilton's principle [2.2], the Hamiltonian equations of motion [2.5] and the Poisson bracket [2.7] are alternative origins for a theory of classical mechanics. Historically speaking, the variational principle (stationary Lagrangian) has usually been regarded as the fundamental statement. Modern treatments of classical mechanics however tends to favor equation [2.7] for the following reasons:

Let the phase-space be defined by

$$(z^1, \dots, z^M) = (\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{p}_1, \dots, \mathbf{p}_N) \quad [2.8]$$

where it follows that  $M = 6N$  and each  $z_i$  is either a Cartesian position or momentum component of one particular particle. We may then write the Poisson bracket as

$$\{F, G\} = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j} \quad [2.9]$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and  $I$  is the  $3N$ -dimensional unit matrix. A consequence of equation [2.9] is that equation [2.7] and the Jacobi identity are all contravariant under arbitrary transformations of the phase space coordinates. I.e. if  $\bar{z}^i$  are the new coordinates,  $\bar{F}(\bar{z})$  is  $F(z)$  expressed in the new coordinates and

$$\bar{J}^{ij} = \frac{\partial \bar{z}^i}{\partial z^m} J^{mn} \frac{\partial \bar{z}^j}{\partial z^n}$$

transforms as a second-rank contravariant tensor, then equations [2.7], [2.9] and the antisymmetry and Jacobi identity of the Poisson bracket hold with all dependent and independent variables  $v$  replaced by  $\bar{v}$ . The antisymmetry and Jacobi identity may be written in the forms

$$J^{ij} = -J^{ji} \quad [2.10]$$

and

$$J^{im} \frac{\partial J^{jk}}{\partial z^m} + J^{jm} \frac{\partial J^{ki}}{\partial z^m} + J^{km} \frac{\partial J^{ij}}{\partial z^m} = 0 \quad [2.11]$$

where the latter is the Jacobi identity. Thus, a complete specification of a Hamiltonian system is equivalent to a choice of Hamiltonian  $H(z)$  and a contravariant tensor  $J^{ij}$  with the above

two properties. It may very well be that  $J^{ij}$  is singular but still satisfies equations [2.10],[2.11] which typically happens after a transformation from canonical coordinates to a reduced set of fewer coordinates. In this reduced phase space, canonical coordinates do not exist, but all the results of classical mechanics still apply. I.e.  $J^{ij}$  still obeys the necessary antisymmetry and "Jacobi-identity", and the Poisson brackets [2.7], [2.9] still hold.

## A.2 A FORM OF HAMILTON'S PRINCIPLE

The most intuitive generalization of Hamilton's principle is the straight-forward generalization of the Lagrangian from discrete particles to particles distributed continuously in space. Let  $\mathbf{x}(a, b, c, \tau)$  be the location of a fluid particle labelled by  $\mathbf{a} = (a, b, c)$  at a time  $\tau$ . We will keep the labeling coordinates constant following the particles and, indeed, they are analogous to the subscript  $i$  in the previous section. It is convenient to assign the labels as

$$dm = da db dc \quad [2.12]$$

so that, of course

$$\rho = \frac{\partial \mathbf{a}}{\partial \mathbf{x}} \quad [2.13]$$

is the mass density of the fluid. Applying  $\frac{\partial}{\partial \tau}$  (the material derivative) to the above equation then yields

$$\frac{\partial \rho}{\partial \tau} + \rho \nabla \cdot \mathbf{u} = 0 \quad [2.14]$$

where  $\mathbf{u} = \frac{\partial \mathbf{x}}{\partial \tau}$  and  $\nabla = (\partial_x, \partial_y, \partial_z)$ . The natural Lagrangian is  $L = T - U$  where  $T$  is the kinetic energy of the system and  $U$  is the specific internal energy plus the potential energy for external forces. Thus,

$$L = \int \left[ \frac{1}{2} \left( \frac{\partial \mathbf{x}}{\partial \tau} \right)^2 - E \left( \frac{\partial \mathbf{x}}{\partial \mathbf{a}}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) \right] d\mathbf{a} \quad [2.15]$$

Here,  $\Phi(\mathbf{x})$  is the potential for external forces and  $E(\rho^{-1}, S)$  is a *prescribed* thermodynamic function of the specific volume and specific entropy  $S$ . The entropy depends only on the labeling coordinates, so

$$\frac{\partial S}{\partial \tau} = 0$$

Hamilton's principle states that the action is stationary, thus  $\delta \int L d\tau = 0$ . In constructing the Lagrangian equations of motion we thus need  $\frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial \dot{\mathbf{x}}} \right)$  and  $\frac{\partial L}{\partial \mathbf{x}}$ . Explicitly, we find by direct differentiation

$$\frac{\partial^2 \mathbf{x}}{\partial \tau^2} = \frac{1}{\rho} \nabla p - \nabla \Phi \quad [2.16]$$

where

$$p \equiv -\frac{\partial E}{\partial \alpha}$$

which is the usual equation of state relating the pressure and internal energy. Equation [2.16] is recognized as the usual Euler equation. All of Maxwell's thermodynamic relations follow from the differentiation of  $E$ , i.e.

$$dE = \frac{\partial E}{\partial \alpha} d\alpha + \frac{\partial E}{\partial S} dS$$

so that by defining  $T \equiv \frac{\partial E}{\partial S}$  one finds

$$T dS = dE + p d\alpha \quad [2.17]$$

We also note that the velocity is given by  $\mathbf{u} = \frac{\delta L}{\delta \dot{\mathbf{x}}}$  and that Hamilton's principle may be cast in the form

$$\delta \int \left\{ \int \left[ \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - H \right] d\mathbf{a} \right\} d\tau = 0 \quad [2.18]$$

where the Hamiltonian is

$$H(\mathbf{u}, \mathbf{x}) = \int \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + E + \Phi \right] d\mathbf{a} \quad [2.19]$$

Also, the Poisson bracket is

$$\{F, G\} = \int \left[ \frac{\delta F}{\delta \mathbf{x}(\mathbf{a})} \cdot \frac{\delta G}{\delta \mathbf{u}(\mathbf{a})} - \frac{\delta F}{\delta \mathbf{u}(\mathbf{a})} \cdot \frac{\delta G}{\delta \mathbf{x}(\mathbf{a})} \right] d\mathbf{a} \quad [2.20]$$

Clearly, since the fluid motion is determined by  $\mathbf{x} = \mathbf{x}(\mathbf{a}, t)$  so that consequently  $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$ , Hamilton's principle states that action is stationary for arbitrary variations  $\delta \mathbf{a}(\mathbf{x}, t)$ . We may therefore recast the action principle to

$$\delta \int \left\{ \int \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - E \left( \frac{\partial \mathbf{x}}{\partial \mathbf{a}}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) \right] \frac{\partial \mathbf{a}}{\partial \mathbf{x}} d\mathbf{x} \right\} dt = 0 \quad [2.21]$$

for arbitrary variations  $\delta \mathbf{a}$ . In carrying out the variations, the integrand must be expressible in terms of  $\mathbf{a}$  and its derivatives. In doing so, we may take advantage of conservation of mass

$$\frac{\partial \mathbf{a}}{\partial \tau} = \frac{\partial \mathbf{a}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{a} = 0 \quad [2.22]$$

and append this equation as a constraint to the action principle, and then vary both  $\mathbf{a}$  and  $\mathbf{x}$ . Thus

$$\delta \int \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - E \left( \frac{\partial \mathbf{x}}{\partial \mathbf{a}}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) - \mathbf{A} \cdot \frac{D\mathbf{a}}{dt} \right] \frac{\partial \mathbf{a}}{\partial \mathbf{x}} d\mathbf{x} dt = 0 \quad [2.23]$$

where  $\mathbf{A} = (A, B, C)$  is a set of Lagrangian multipliers. The above statement is the form of Hamilton's principle with  $(x, y, z, t)$  as the independent variables. Taking the velocity variation gives

$$\mathbf{u} = A \nabla a + B \nabla b + C \nabla c$$

In using the above to eliminate  $\mathbf{u}$ , we find

$$\delta \int \left[ \mathbf{A} \cdot \frac{\partial \mathbf{a}}{\partial t} + \frac{1}{2} (A_i \nabla a_i)^2 + E + \Phi \right] \frac{\partial \mathbf{a}}{\partial \mathbf{x}} d\mathbf{x} dt = 0 \quad [2.24]$$

for variations  $\delta \mathbf{A}$  and  $\delta \mathbf{a}$ . Furthermore, we may append the continuity equation as an additional constraint and thereby replace  $\frac{\partial \mathbf{x}}{\partial \mathbf{a}}$  by  $\rho$ , then

$$\delta \int \left\{ \rho \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - E(\rho^{-1}, S(\mathbf{a})) - \Phi(\mathbf{x}) - \zeta \cdot \frac{D\mathbf{a}}{Dt} \right] + \phi \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \right\} d\mathbf{x} dt = 0 \quad [2.25]$$

for independent variations  $\delta \rho$ ,  $\delta \mathbf{u}$ ,  $\delta \mathbf{a}$  and  $\delta \phi(\mathbf{x}, t)$ . Here,  $\zeta = (\zeta, \eta, \theta)$  are the Lagrangian multipliers corresponding to equation [2.22] and  $\phi$  the multiplier corresponding to equation [2.14]. Taking the velocity variation one finds

$$\mathbf{u} = \zeta \nabla a + \eta \nabla b + \theta \nabla c + \nabla \phi \quad [2.26]$$

Then, after a few cancellations and integration by parts, the action principle states

$$\delta \int \rho \left[ \zeta \frac{\partial a}{\partial t} + \eta \frac{\partial b}{\partial t} + \theta \frac{\partial c}{\partial t} + \frac{\partial \phi}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + E(\rho^{-1}, S) + \Phi \right] d\mathbf{x} dt = 0 \quad [2.27]$$

That we use  $\mathbf{u}$  is only an abbreviation for equation [2.26]. Taking the variations  $\delta \mathbf{a}$ ,  $\delta \zeta$ ,  $\delta \rho$  and  $\delta \phi$  gives

$$\delta \zeta_i : \frac{Da_i}{Dt} = 0 \quad [2.28a]$$

$$\delta a_i : \frac{D\zeta_i}{Dt} = \frac{\partial E}{\partial S} \frac{\partial S}{\partial a_i} \quad [2.28b]$$

$$\delta \phi : \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad [2.28c]$$

$$\delta \rho : \zeta \cdot \frac{\partial \mathbf{a}}{\partial t} + \frac{\partial \phi}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + E + \frac{P}{\rho} = 0 \quad [2.28d]$$

The eight equations above yield the evolution equations for the eight scalar dependent variables. However, both equation [2.18] and equation [2.24] yield six equations for six dependent variables, which suggests setting gauge conditions on equations [2.28]. Suppose for instance that the entropy is constant everywhere,  $S = 0$ . Salmon [1988] gives a proof that  $c$  may be set formally to zero. We will not recast this proof, but the interested reader may find it in Salmon [1988] or Boozer [1985]. The resulting equations are, for the action principle

$$\delta \int \left[ \zeta \frac{\partial a}{\partial t} + \eta \frac{\partial S}{\partial t} + \frac{\partial \phi}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + E + \Phi \right] dt d\mathbf{x} \rho = 0 \quad [2.29]$$

where

$$\mathbf{u} = \zeta \nabla a + \eta \nabla b + \nabla \phi$$

And the variations yield the following equations

$$\delta \zeta : \frac{Da}{Dt} = 0 \quad [2.30a]$$

$$\delta a : \frac{D\zeta}{Dt} = 0 \quad [2.30b]$$

$$\delta \eta : \frac{DS}{Dt} = 0 \quad [2.30c]$$

$$\delta S : \frac{D\eta}{Dt} = T \quad [2.30d]$$

$$\delta \phi : \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad [2.30e]$$

$$\delta \rho : \zeta \frac{\partial a}{\partial t} + \eta \frac{\partial S}{\partial t} + \frac{\partial \phi}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \Phi + E + \frac{P}{\rho} = 0 \quad [2.30f]$$

Equation [2.29] together with equations [2.30] are the Eulerian versions of Hamilton's principle and they are equivalent to the ideal fluid equations. However, it should be noted that the variational principle used here is only one of an infinite number of possibilities. For instance, Virasoro [1981] and Swaters [2000] uses the vorticity as the momentum coordinate.

### A.3 THE HAMILTONIAN STRUCTURE FOR FREE BOUNDARY PROBLEMS

Hamilton's principle for free surfaces can in fact be put in a fairly simple form as opposed to the perhaps more general previous calculation. We may for instance consider irrotational gravity waves in an ideal liquid. Let  $\mathbf{x}$  and  $y$  be the horizontal and vertical coordinates with the bottom designated by  $y = -d(\mathbf{x})$  and  $y = 0$  for the quiescent free surface. If the velocity potential is given by  $\Phi(\mathbf{x}, y, t)$ , we denote

$$\Phi = \Phi(\mathbf{x}, t), \quad y = \eta(\mathbf{x}, t) \quad [2.31]$$

on the displaced surface. That is,  $\Phi$  is simply the velocity potential evaluated at the free surface  $\eta$ . By introducing a Lagrangian density in  $\mathbf{x}$ -space [Luke, 1967]

$$\mathcal{L} = \Phi \eta_t - \mathcal{H} \quad [2.32]$$

where  $\mathcal{H}$  is the energy density in  $\mathbf{x}$ -space, the action principle states that

$$\delta I = \delta \int_{\mathbf{x}} \mathcal{L} d\mathbf{x} dt = 0 \quad [2.33]$$

The energy density is given by

$$\mathcal{H} = \frac{1}{2} \int_{-d}^{\eta} (\nabla \Phi)^2 dy + \frac{1}{2} g \eta^2 \quad [2.34]$$

The boundary conditions for the problem are stated as

$$\Delta \Phi(\mathbf{x}, y, t) = 0 \quad [2.35a]$$

$$\nabla \Phi(\mathbf{x}, y, t) \cdot \hat{\mathbf{n}} = 0, \quad \text{on } y = \eta \quad [2.35b]$$

They are accompanied by the kinematic and Bernoulli conditions by

$$\Phi_y = \eta_t + \nabla \Phi \cdot \nabla \eta \quad [2.36a]$$

$$\Phi_t + \frac{1}{2} (\nabla \Phi)^2 + g \eta = 0, \quad \text{on } y = \eta \quad [2.36b]$$

Surface tension can of course be incorporated by including it in equation [2.36b] and the equation for the energy density.

The variational principle now gives that

$$\delta I = \int_{\mathbf{x}} \left[ \left( \eta_t - \frac{\delta \mathcal{H}}{\delta \Phi} \right) \delta \Phi - \left( \xi_t + \frac{\delta \mathcal{H}}{\delta \eta} \right) \delta \eta \right] d\mathbf{x} dt \quad [2.37]$$

suggesting the evolution equations

$$\Phi_t = - \frac{\delta \mathcal{H}}{\delta \eta} \quad [2.38a]$$

$$\eta_t = \frac{\delta \mathcal{H}}{\delta \Phi} \quad [2.38b]$$

If only the evolution equations are required, Green's theorem may be applied to the above results to show that the explicit calculation of  $\mathcal{H}$  is not necessary, only the specification of the surface and the velocity potential. Detailed calculations regarding this are found in Miles [1977] and Luke [1967].

The above results are in fact only a special case of a very general Poisson bracket, first derived by Lewis et al. [1986]. It is quite possible to also take into account vorticity in their formalism. We shall mention but the briefest of outlines and cite some deeper results. Define the Poisson bracket as

$$\{F, G\} = \int_{\mathcal{V}} \left\langle \boldsymbol{\omega}, \frac{\delta F}{\delta \mathbf{u}} \times \frac{\delta G}{\delta \mathbf{u}} \right\rangle d\mathcal{V} + \int_{\Sigma} \left( \frac{\delta F}{\delta \Sigma} \frac{\delta G}{\delta \Phi} - \frac{\delta G}{\delta \Sigma} \frac{\delta F}{\delta \Phi} \right) d\Sigma \quad [2.39]$$

where  $F, G$  are two arbitrary functionals and  $\boldsymbol{\omega}$  the vorticity,  $\mathcal{V}$  is the volume bounded by the free surface  $\Sigma$ ,  $\mathbf{u}$  is the incompressible velocity field and  $\langle \cdot, \cdot \rangle$  denotes the inner product.  $\frac{\delta}{\delta \Sigma}$  is the functional derivative with respect to  $\Sigma$  that leaves  $\mathcal{V}$  unchanged. It is now possible to show that the Euler equations with surface tension are equivalent to this bracket with the Hamiltonian being the physical energy. For the free-surface problem, the Euler equations together with the boundary conditions read (conveniently, we set  $\rho = 1$ )

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p \\ \partial_t \Sigma &= \mathbf{u} \cdot \hat{\mathbf{n}} \\ \nabla \cdot \mathbf{u}, \quad \Delta p \Big|_{\Sigma} &= \gamma \nabla \cdot \hat{\mathbf{n}} \end{aligned} \quad [2.40]$$

with  $\hat{\mathbf{n}}$  being the outward unit normal on  $\Sigma$  and  $\gamma$  the constant surface tension. By taking the Hamiltonian to be the physical energy

$$\mathcal{H} = \frac{1}{2} \int_{\mathcal{V}} \mathbf{u} \cdot \mathbf{u} d\mathcal{V} + \gamma \int_{\Sigma} d\Sigma \quad [2.41]$$

then for any functional  $F$  it follows that

$$F_t = \{F, H\} \quad [2.42]$$

By calculating the functional derivatives of  $H$  we find that

$$\frac{\delta H}{\delta \mathbf{u}} = \mathbf{u} \quad [2.43a]$$

$$\frac{\delta H}{\delta \Phi} = \langle \mathbf{u}, \hat{\mathbf{n}} \rangle \quad [2.43b]$$

$$\frac{\delta H}{\delta \Sigma} = \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + \gamma \nabla \cdot \hat{\mathbf{n}} \quad [2.43c]$$

Now,

$$F_t = \int_{\mathcal{V}} \left\langle \frac{\delta F}{\delta \mathbf{u}}, \frac{\partial \mathbf{u}}{\partial t} \right\rangle d\mathcal{V} + \int_{\Sigma} \frac{\delta F}{\delta \Sigma} \frac{\partial \Sigma}{\partial t} d\Sigma \quad [2.44]$$

and by calculating the bracket of equation [2.39] we find that

$$\begin{aligned} \{F, H\} &= \int_{\mathcal{V}} \left\langle \frac{\delta F}{\delta \mathbf{u}}, \mathbf{u} \times \boldsymbol{\omega} \right\rangle d\mathcal{V} + \int_{\Sigma} \left\langle \frac{\delta F}{\delta \Sigma}, \mathbf{u}, \mathbf{u} \right\rangle d\Sigma \\ &\quad - \int_{\mathcal{V}} \left\langle \nabla \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right), \frac{\delta F}{\delta \mathbf{u}} \right\rangle d\mathcal{V} \\ &\quad - \int_{\Sigma} \left\langle \gamma \cdot \hat{\mathbf{n}} \frac{\delta F}{\delta \mathbf{u}}, \hat{\mathbf{n}} \right\rangle d\Sigma \end{aligned} \quad [2.45]$$

Hence, if equations [2.44] and [2.45] are equal for all  $\frac{\delta F}{\delta \mathbf{u}}$  and  $p$  is the solution of the problem

$$\Delta p \Big|_{\Sigma} = \gamma \nabla \hat{\mathbf{n}}, \quad \Delta p = -\nabla \cdot [(\mathbf{u} \cdot \nabla) \mathbf{u}] \quad [2.46]$$

then the final term of equation [2.45] drops out when  $\nabla p$  is subtracted from the vector identity  $-(\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{u} \times \boldsymbol{\omega} - \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u})$ . Equation [2.43] follows and the Poisson bracket of equation [2.39] holds.

Note that the variational derivatives do not necessarily exist as they have been defined here (cf. Lewis et al. [1986]). Moreover, this result has been generalized further into a very complicated bracket, but the result is not interesting for our part.

## B. THE KORTEWEG-DE VRIES EQUATION

The Korteweg-de Vries (KdV) equation protrudes as a natural generalization of unidirectional waves where we account for both dispersion and nonlinear effects, but not dissipation. The KdV equation occurs for shallow water waves, in nonlinear optics, plasma physics, flow in porous media, solid state physics and ever so on. The KdV equation is found in many forms, depending on the physical condition, but is most commonly written

$$\partial_t u + \partial_x^3 u + 6u \partial_x u = 0 \quad [2.47]$$

The factor 6 represents only a convenience when integrating the KdV, as we shall shortly see. However, the variables  $x$  and  $t$  do not necessarily represent the physical spatial and temporal variables, but are usually rescaled versions of these. The coefficients of the KdV equation are not universal, and are changed from one physical situation to the next.

Indeed, the KdV equation and its two-dimensional generalization is a family of fully nonlinear equations with exact solutions expressible in analytic terms. We will thoroughly discuss the solutions and consequences of the KdV equation, but it is nothing but reasonable to start by its derivation.

## B.1 A PHYSICAL SYSTEM LEADING TO THE KDV EQUATION - SHALLOW WATER WAVES

Although one of the most common problems and seemingly easy, water waves can in fact be quite horrid. We shall derive the Korteweg-de Vries (actually we will derive its two-dimensional generalization) for shallow water waves by means of a Stokes transformation (also known as strained/stretched coordinates and so on).

Assume a fixed coordinate system so that the bottom (assumed flat) is at  $z = 0$  and the still-water depth at  $z = h$  so that the free surface is given by  $z = h + \eta(x, y, t)$ . We assume an irrotational flow so  $\mathbf{u}$  is given by  $\mathbf{u} = \nabla \Phi$  where  $\Phi$  is some scalar potential which is yet to be specified. Then the incompressibility condition on  $\mathbf{u}$  gives

$$\Delta \Phi = 0$$

The normal component of the velocity at  $z = 0$  is zero so

$$\left. \frac{\partial \Phi}{\partial z} \right|_{z=0} = 0$$

At the surface, the fluid velocity must equal the velocity of the free surface. Thus,

$$\eta_t + \Phi_x \eta_x + \Phi_y \eta_y = \Phi_z, \quad \text{on } z = h + \eta$$

Finally, the Bernoulli relation on the free surface gives

$$\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + g\eta - \frac{\gamma}{\rho} \frac{\eta_{xx} + \eta_{yy}}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} = 0$$

where we have accounted for surface tension but not dissipation. The higher-order terms in the curvature have been neglected. We know that for vanishing amplitude waves (linear waves), the dispersion relation gives [Graebel, 2007]

$$\omega^2 = (gk + \gamma k^3) \tanh(kh)$$

where  $\omega$  is the frequency and  $k$  is the wave number. This dispersion relation will provide many hints as how to proceed to account for nonlinear effects.

We will profit from making the governing equations non-dimensional, let the depth be measured in  $h$  and the velocity in  $\sqrt{gh}$ . After a minimum of algebra, the governing equations state

$$\Delta \Phi = 0 \quad [2.48a]$$

$$\Phi_z \Big|_{z=0} = 0 \quad [2.48b]$$

$$(\eta_t + \Phi_x \eta_x + \Phi_y \eta_y) \Big|_{z=h+\eta} = \Phi_z \Big|_{z=h+\eta} \quad [2.48c]$$

$$\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \eta - S \frac{\eta_{xx} + \eta_{yy}}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} = 0 \quad [2.48d]$$

where  $S = \frac{\gamma}{\rho gh^2}$ . If the nonlinear wave propagates either to the left or the right, it would seem natural to take a Galilean transformation  $x \pm t$  as a new coordinate. We will additionally "stretch" our coordinates so that the nonlinearities may be taken out in a uniform way.

In our non-dimensional system and for small  $k$ , after series expanding the hyperbolic term the dispersion relation reads (up to some factor)

$$\omega = k \sqrt{1 - k^2}$$

Say that the nonlinear wave propagates along the  $x$ -axis with a small coupled component in the  $y$ -direction. Then  $\mathbf{k} = k_{\parallel} + k_{\perp}$  with  $k_{\perp} \ll k_{\parallel}$  so that  $\omega$  may, after a couple of series expansions, be written

$$\omega = k_{\parallel} + k_{\parallel}^3 + \frac{1}{2} \frac{k_{\perp}^2}{k_{\parallel}}$$

The propagator  $\exp[i\omega t - i\mathbf{k} \cdot \mathbf{x}]$  then reads

$$\exp[i\omega t - i\mathbf{k} \cdot \mathbf{x}] = \exp\left[ik_{\parallel}t + \frac{1}{2}\left(\frac{k_{\perp}^2}{k_{\parallel}} + k_{\parallel}^3\right)t - k_{\parallel}x - k_{\perp}y\right]$$

Without loss of generalization, we may take  $k_{\parallel} = \varepsilon^{1/2}k'_{\parallel}$  and  $k_{\perp} = \varepsilon^p k'_{\perp}$  so that the primed wave numbers  $k'_{\perp}, k'_{\parallel}$  are of order one. The fact that we choose  $\varepsilon^{1/2}$  to be the scaling is merely a convention. We may therefore write the propagator as

$$\exp\left[\frac{1}{2}\left(k'_{\parallel}{}^3 + \frac{k'_{\perp}{}^2}{k'_{\parallel}}\right)\varepsilon^{3/2}t - \varepsilon^{1/2}k'_{\parallel}(x-t) - \varepsilon^p k'_{\perp}y\right]$$

This suggests that  $p = 1$  and introduces a new set of variables by

$$\begin{aligned}\zeta &= \varepsilon^{1/2}(x-t) \\ \sigma &= \varepsilon y \\ \tau &= \varepsilon^{3/2}t\end{aligned}$$

We will use a perturbation expansion of  $\Phi$  and  $\eta$ , i.e.

$$\begin{aligned}\eta &= \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \dots \\ \Phi &= \varepsilon^{1/2}\Phi^{(1)} + \varepsilon^{3/2}\Phi^{(2)} + \dots\end{aligned}$$

We will seek for solutions to  $\Phi$  in the form

$$\Phi = \sum_{n=0}^{\infty} z^n f_n(x, y, t)$$

so that Laplace's equation  $\Delta\Phi = 0$  gives

$$\sum_{n=0}^{\infty} [(n+1)(n+2)z^n f_{n+2} + z^n \Delta f_n] = 0$$

A general solution is a recurrence relation in  $f_n$  where

$$f_{n+2} = -\frac{\Delta f_n}{(n+1)(n+2)}$$

Now, the requirement  $\Phi_z|_{z=0} = 0$  gives that  $f_1 = 0$  and it follows that all odd  $f_n$  are zero. The only unknown is therefore  $f_0$  and our potential may be written

$$\Phi = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \Delta^n f_0$$

In the strained coordinates, the Laplacian is given by

$$\Delta = \varepsilon\partial_{\zeta}^2 + \varepsilon^2\partial_{\sigma}^2$$

and with our new coordinates it also follows that

$$\partial_t = -\varepsilon^{1/2}\partial_{\zeta} + \varepsilon^{3/2}\partial_{\tau}$$

so our potential is

$$\Phi = \varepsilon^{1/2} \left( f_0 + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} [\varepsilon\partial_{\zeta}^2 + \varepsilon^2\partial_{\sigma}^2]^n f_0 \right)$$

To the lowest order in  $\varepsilon$  we find that

$$\Phi_z = -\varepsilon^{3/2} z f_{\zeta, \zeta}$$

The kinematic boundary condition then gives to order  $\varepsilon^{3/2}$

$$\partial_t \eta = -\varepsilon^{3/2} \eta_{\zeta}^{(1)}$$

where we have used  $\partial_t = -\varepsilon^{1/2}\partial_{\zeta} + \varepsilon^{3/2}\partial_{\tau}$ . Therefore, the condition on  $z = 1 + \eta$  gives

$$\eta^{(1)} = \Phi_{\zeta}^{(1)}$$

Going to order  $\varepsilon^{5/2}$ , we find that

$$\eta_t = -\varepsilon^{3/2} \eta_{\zeta}^{(1)} + \varepsilon^{5/2} \eta_{\tau}^{(1)} - \varepsilon^{5/2} \eta_{\zeta}^{(2)}$$

We shall additionally need  $\eta_x, \eta_y, \Phi_x, \Phi_y$ . These are given by

$$\begin{aligned}\eta_x &= \varepsilon^{3/2} \eta_{\zeta}^{(1)} + \varepsilon^{5/2} \eta_{\zeta}^{(2)} \\ \eta_y &= \varepsilon^2 \eta_{\sigma}^{(1)} + \varepsilon^3 \eta_{\sigma}^{(2)}\end{aligned}$$

and by differentiating  $\Phi$  and evaluating at the limit  $z = 1 + \eta$  we find

$$\Phi_x = \varepsilon \Phi_{\zeta}^{(1)} - \frac{\varepsilon^2 (1 + \varepsilon \eta^{(1)})^2}{2} \Phi_{\zeta\zeta}^{(1)} - \dots$$

The terms  $\eta_y \Phi_y$  can be neglected since  $\eta_y$  is to second order in  $\varepsilon$  and  $\Phi_y$  is at least of order  $\varepsilon^{3/2}$  so that the following product is of order  $\mathcal{O}(\eta_y \Phi_y) = \varepsilon^{7/2}$ . The differentiation of  $\Phi$  with respect to  $z$  gives

$$\begin{aligned}\Phi_z &= \varepsilon^{1/2} \partial_z \left[ -\frac{z^2}{2} (\varepsilon \partial_{\zeta}^2 + \varepsilon^2 \partial_{\sigma}^2) f + \frac{z^4}{4!} (\varepsilon \partial_{\zeta}^2 + \varepsilon^2 \partial_{\sigma}^2)^2 f + \dots \right] \\ &= -\varepsilon^{3/2} z f_{\zeta\zeta} - \varepsilon^{5/2} z f_{\sigma\sigma} + \varepsilon^{1/2} \frac{z^3}{6} (\varepsilon \partial_{\zeta}^2 + \varepsilon^2 \partial_{\sigma}^2)^2 f + \dots\end{aligned}$$

Inserting the limit  $z = 1 + \eta$  gives to order  $\varepsilon^{5/2}$

$$\Phi_z = -\varepsilon^{3/2} f_{\zeta\zeta} - \varepsilon^{5/2} \eta^{(1)} f_{\zeta\zeta} - \varepsilon^{5/2} f_{\sigma\sigma} + \frac{1}{6} \varepsilon^{5/2} f_{\zeta\zeta\zeta\zeta}$$

Of course, the terms  $\varepsilon^{3/2}$  only gives  $\eta^{(1)} = \Phi_{\zeta}^{(1)}$  as we have already mentioned. To order  $\varepsilon^{5/2}$ , the kinematic boundary condition gives

$$-\eta_{\zeta}^{(2)} + \Phi_{\tau}^{(1)} + 2\Phi_{\zeta}^{(1)} \Phi_{\zeta\zeta}^{(1)} - \frac{1}{6} \Phi_{\zeta\zeta\zeta\zeta}^{(1)} + \Phi_{\sigma\sigma}^{(1)} = 0 \quad [2.49]$$

We shall need to perform the same analysis on the dynamic boundary condition where the terms  $\Phi_t, \Phi_x^2, \Phi_y^2, \Phi_z^2$  as well as the curvature terms are required. Only terms up to order  $\varepsilon^2$  will be required since in order to couple our two boundary conditions we will need to take the partial derivative with respect to  $x$  of the upcoming equation, thereby introducing an extra  $\varepsilon^{5/2}$  so that the equations are of the same order. To lowest order,  $\Phi_z \sim \varepsilon^{3/2}$  and  $\Phi_y \sim \varepsilon^{3/2}$  so these may be neglected. To second order in  $\varepsilon$ ,  $\Phi_x$  is

$$\Phi_x^2 = f_{\zeta}^2$$

The differentiation of  $\Phi_t$  yields

$$\Phi_t = (\varepsilon^2 \partial_{\tau} - \varepsilon \partial_{\zeta}) \left[ f + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} (\varepsilon \partial_{\zeta}^2 + \varepsilon^2 \partial_{\sigma}^2)^n f \right]$$

so that to second order in  $\varepsilon$  only the term  $\varepsilon^2 \partial_{\tau} f$  and the first term of the sum survives. Therefore,

$$\Phi_t = f_{\tau} + \frac{1}{2} f_{\zeta\zeta}$$

Also required are the terms,  $\eta_x$ ,  $\eta_y$ ,  $\eta_{xx}$ ,  $\eta_{yy}$ . These are readily written down

$$\begin{aligned}\eta_x &= \varepsilon^{3/2}\eta_\zeta^{(1)} + \varepsilon^{5/2}\eta_\zeta^{(2)}, & \eta_y &= \varepsilon^2\eta_\sigma^{(1)} + \varepsilon^2\eta_\sigma^{(2)}, \\ \eta_{xx} &= \varepsilon^2\eta_{\zeta\zeta}^{(1)}, & \eta_{yy} &= 0\end{aligned}$$

where  $\eta_{yy}$  is of order  $\varepsilon^3$ . Therefore, it follows that to order  $\varepsilon^2$ , the surface-tension terms are

$$\frac{\eta_{xx} + \eta_{yy}}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} = \varepsilon^2\eta_{\zeta\zeta}^{(1)}$$

In collecting all terms and recalling that  $\eta^{(1)} = \Phi_\zeta^{(1)}$  and  $\Phi^{(1)} = f$ , the dynamic boundary condition gives

$$\Phi_\tau^{(1)} + \frac{1}{2}\left(\Phi_\zeta^{(1)}\right)^2 + \eta^{(2)} - \left(S - \frac{1}{2}\right)\Phi_{\zeta\zeta\zeta}^{(1)} = 0 \quad [2.50]$$

We shall need to couple equations [2.49] and [2.50]. Indeed, by taking the partial derivative with respect to  $\zeta$  of the latter and adding the two equations together, we find

$$\Phi_{\tau\zeta}^{(1)} + \frac{3}{2}\Phi_\zeta^{(1)}\Phi_{\zeta\zeta}^{(1)} + \frac{1}{6}(1 - 3S)\Phi_{\zeta\zeta\zeta\zeta}^{(1)} + \frac{1}{2}\Phi_{\sigma\sigma}^{(1)} = 0 \quad [2.51]$$

The above equation is known as the Kadomtsev-Petviashvili (KP) equation and is the natural two-dimensional generalization of the Korteweg-de Vries equation. A different form is perhaps more convenient, the velocity components are still given by  $u = \Phi_\zeta$ ,  $v = \Phi_\sigma$  so the KP equation may be written

$$u_\tau + \frac{3}{2}uu_\zeta + \frac{1}{6}(1 - 3S)u_{\zeta\zeta\zeta} + \frac{1}{2}v_\sigma = 0 \quad [2.52]$$

The one-dimensional version of the above equation, i.e. without  $v_\sigma$  is known as the Korteweg-de Vries equation.

Although the KP equation has been rigorously derived on a physical ground, it is easily understood on intuitive grounds also. The usual linear wave equation  $(\partial_t^2 - \Delta)\Phi = 0$  has dispersion-free solutions of the form  $u = a \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$ . This type of solution is very common for small  $k$ , but for larger  $k$  dispersion will come into play. It is perhaps most intuitive for the signal to lag behind the phase so we suggest

$$\omega^2 = c^2k^2 - \beta^2k^4$$

as the new dispersion relation.  $c$  is some velocity specific to the medium and  $\beta$  is some indication on the strength of the dispersion. Then, approximately

$$\omega = ck - \frac{\beta^2}{2c}k^3$$

Assume we rescale our system so that  $c = 1$  and  $\beta^2 = 2$ , then with  $k^2 = k_x^2 + k_y^2$  where  $k_x$  is small but  $k_y^2 \ll k_x^2$  the dispersion relation reads

$$\omega = k_x - k_x^3 + \frac{1}{2}\frac{k_y^2}{k_x} + \dots$$

We may rig our coordinate system in such a way that it moves with constant speed  $c = 1$ , the  $k_x$ -term on the right-hand side vanishes. Therefore, a suggested solution to this dispersion relation is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{1}{2} \frac{\partial u}{\partial y} = 0$$

where we have expressed the time-derivative by the more general convective derivative. Apart from a change of factors (which is partly due to the scaling), this is the same result as equation [2.51].

## B.2 ELEMENTARY SOLUTIONS TO THE KDV

The KdV equation (equation [2.51] without the two-dimensionality) on a general form is

$$u_t - 6uu_x + u_{xxx} = 0 \quad [2.53]$$

Of course, as is highly clear in the previous derivation of the KdV equation for water waves,  $x$  and  $t$  do not necessarily represent the physical spatial and temporal coordinates. Nonetheless, we will assume that by a proper transformation of the dependent variables that the above form has been attained.

We shall show that the KdV equation in its general form is integrable. Assume that  $u$  may be written as  $u = f(\zeta)$  and  $\zeta = x - ct$ . Then, the differentiation of equation [2.53] reads

$$-cf' - 6ff' + f''' = 0 \quad [2.54]$$

where a prime denotes differentiation with respect to  $\zeta$ . By writing  $ff'$  as  $\frac{1}{2}(f^2)'$  we may integrate the above to obtain

$$-cf - 3f^2 + f'' + A = 0 \quad [2.55]$$

where  $A$  is a constant of integration. We shall use  $f'$  as an integrating factor, therefore:  $ff' = \frac{1}{2}(f^2)'$ ,  $f^2f' = \frac{1}{3}(f^3)'$ ,  $f''f' = \frac{1}{2}(f'^2)'$  so upon integration and rearrangement one finds

$$\frac{1}{2}(f')^2 = f^3 + \frac{c}{2}f^2 + Af + B \quad [2.56]$$

where  $B$  is another factor of integration. We may impose the boundary conditions  $f, f', f'' \rightarrow 0$  as  $\zeta \rightarrow \pm\infty$  (later this is shown to reflect the solitary wave solution). Thus  $A$  and  $B$  are zero and we may rearrange the above to

$$(f')^2 = f^2(2f + c) \quad [2.57]$$

This is integrable and

$$\int \frac{df}{f\sqrt{2f+c}} = \pm \int d\zeta \quad [2.58]$$

A suitable substitution is  $f = -\frac{1}{2}c \operatorname{sech}^2\theta$ . Then the square root is

$$\sqrt{2f+c} = c^{1/2} \operatorname{sech}\theta \sinh\theta$$

and the differentiation yields

$$df = c \operatorname{sech}^2\theta \tanh\theta d\theta$$

and hence the entire integral reduces to

$$-\frac{2}{c^{1/2}} \int d\theta = \pm\zeta - x_0 \quad [2.59]$$

where  $x_0$  is a constant of integration. Therefore  $\theta = \frac{1}{2}c^{1/2}(x - ct - x_0)$  and  $f$  is given by

$$f = -\frac{1}{2}c \operatorname{sech}^2 \left[ \frac{1}{2}c^{1/2}(x - ct - x_0) \right] \quad [2.60]$$

The fact that  $f \leq 0$  is only caused by our choice of the KdV equation, we may make the substitution  $u \rightarrow -u$  so that  $f \geq 0$ . Clearly, the phase shift  $x_0$  plays a minor role only, it merely denotes the position of the wave at a time  $t = 0$ .

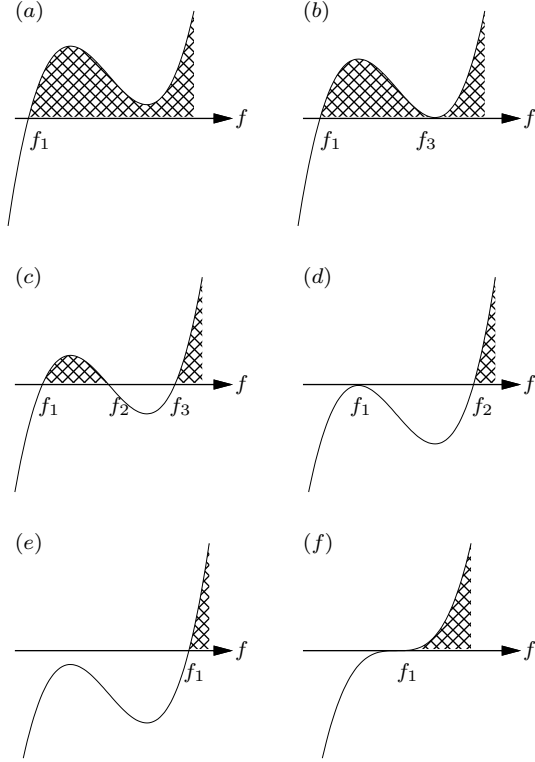
The qualitative nature of the KdV equation is best understood by writing it on the form of equation [2.56]. It is clear



that we are only interested in bounded and real solutions of  $f(\zeta)$ , thus  $(f')^2 \geq 0$ . We may write equation [2.56] as

$$\frac{1}{2} (f')^2 = f^3 + \frac{c}{2} f^2 + Af + B \equiv F(f) \quad [2.61]$$

We shall therefore require that  $F(f) \geq 0$ . Clearly  $F(f)$  has up to three zeros and they may be simple, double or triple zeros. A sketch of  $F(f)$  for different values of  $A, B, c$  is shown in Fig. 2.1



**Fig. 2.1:** Sketches of the graphs of  $F(f)$  for different values of  $A, B, c$ . Real, but not necessarily bounded, solutions where  $(f')^2 \geq 0$  are shown in the shaded regions.

It is evidently clear that if the solution is to remain bounded, the solutions  $a, d, e, f$  must be discarded. We are left with the left-hand parts of the curves in (b) and (c). In order to properly classify the solutions, it is appropriate to Taylor-expand  $F$  around the zeros. For any simple zero  $F(f_1) = 0$  one finds

$$(f')^2 = 2(f - f_1)F'(f_1) + \mathcal{O}[(f - f_1)^2]$$

which is solved iteratively to yield

$$f = f_1 + \frac{1}{2} (\zeta - \zeta_1)^2 F'(f_1) + \mathcal{O}[(\zeta - \zeta_1)^4]$$

so that  $f$  has a local minimum or maximum at  $\zeta = \zeta_1$ . Instead, if  $f_1$  is a double zero then

$$(f')^2 = (f - f_1)^2 F''(f_1) + \mathcal{O}[(f - f_1)^3]$$

and we obtain by the same technique

$$f - f_1 \sim \alpha \exp \left[ \pm \zeta \sqrt{F''(f_1)} \right], \quad \zeta \pm \mp \infty$$

From (b) it is seen that  $F$  has a simple zero at  $f_1$  and a double zero at  $f_2$ , and attains  $f = f_2$  as  $\zeta \rightarrow \pm\infty$ . Therefore,

(b) corresponds to the solitary-wave solution. The case in (c) has a simple zeros at  $f_1$  and  $f_2$ . In fact, since  $F'(f_1) > 0$  and  $F'(f_2) < 0$ ,  $f'$  will change sign at these two points. Both  $f_1$  and  $f_2$  are simple zeros, consequently their behaviour is algebraic so that they are a finite distance apart. A solution can therefore be completely specified by giving  $f$  and  $f'$  at any point between  $\zeta = \zeta_0$  between  $f_1$  and  $f_2$ . The solution will then tend to oscillate between these two points with a frequency [Infeld and Rowlands, 2000]

$$2 \int_{f_1}^{f_2} \frac{df}{f'} = 2 \int_{f_1}^{f_2} \frac{df}{\sqrt{2F(f)}}$$

and the solution is given implicitly by

$$\zeta = \zeta_1 \pm \int_{f_1}^f \frac{df}{\sqrt{2F(f)}} \quad [2.62]$$

In order to venture any further we shall first need to introduce the Jacobian elliptic functions.

We define

$$v = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad [2.63]$$

where  $0 \leq k \leq 1$ . Also we define the proper inverse functions by

$$\text{sn}(v|k) = \sin \varphi \quad \text{cn}(v|k) = \cos \varphi \quad [2.64]$$

These are two of the *Jacobian elliptic functions*. The two special cases  $k = 0, 1$  are expressible in elementary functions. For  $k = 0$  one trivially finds

$$k = 0: \quad v = \varphi, \quad \rightarrow \quad \text{cn}(v|0) = \cos v$$

Consequently for  $k = 1$  one may make the substitution  $u = \cos \theta$  so that  $du = -\sin \theta d\theta = -\sqrt{1 - u^2} d\theta$  and therefore

$$\int_0^\varphi \frac{d\theta}{\cos \theta} = \int_1^{\cos \varphi} -\frac{du}{u\sqrt{1 - u^2}} = \text{asech}(\cos \varphi)$$

Thus,

$$k = 1: \quad v = \text{asech}(\cos \varphi), \quad \rightarrow \quad \text{cn}(v|1) = \text{sech } v$$

$\text{cn}$  and  $\text{sn}$  are therefore periodic functions for  $0 \leq k < 1$  but loses it's periodicity for  $k = 1$ . The period is

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = 4 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

and we define the *complete* elliptic integral of the first kind,

$$K \equiv K(k) \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad [2.65]$$

Clearly  $K(0) = \pi/2$ . It is also verified that

$$\text{cn}^2 + \text{sn}^2 = 1$$

In the same spirit as for trigonometric functions, it also follows that

$$\begin{aligned} \frac{\partial}{\partial v} \text{cn} &= \frac{\partial \varphi}{\partial v} \frac{\partial}{\partial \varphi} \text{cn} \\ &= \sqrt{1 - k^2 \sin^2 \theta} \frac{\partial}{\partial \varphi} \cos \varphi \end{aligned}$$

where we have made use of the basic differentiation rule

$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(t) dt = \frac{\partial b}{\partial x} f(b) - \frac{\partial a}{\partial x} f(a)$$

Thus it is useful to define another elliptic function  $\text{dn}$  by

$$\text{dn}(v|k) = \sqrt{1 - k^2 \sin^2 \varphi} \quad [2.66]$$

so that

$$\frac{\partial}{\partial v} \text{cn} = -\text{sn} \text{dn} \quad [2.67]$$

And it is also obtained that

$$\text{dn}^2 + k^2 \text{sn}^2 = 1 \quad [2.68]$$

We shall see in some greater detail how the KdV equation may be analysed by use of the elliptic functions. We may write our function  $F(f)$  (defined by equation [2.61]) on the form  $F(f) = (f - f_1)(f - f_2)(f - f_3)$  where  $f_1 < f_2 < f_3$  as is seen in Fig. 2.1:(c). Therefore, in integrating from  $f_1$  to  $f$

$$\zeta = \zeta_1 \pm \int_{f_1}^f \frac{dg}{\sqrt{(g - f_1)(g - f_2)(g - f_3)}} \quad [2.69]$$

where of course we have called  $f = g$  in the integrand in order not to confuse the integrand with the limit. A very convenient substitution is  $g = f_1 + (f_2 - f_1) \sin^2 \theta$ , then

$$dg = 2(f_2 - f_1) \sin \theta \cos \theta d\theta$$

After a little algebra one also obtains

$$(g - f_1)(g - f_2)(g - f_3) = (f_2 - f_1)^2 (f_3 - f_1) \cos^2 \theta \sin^2 \theta \times \left[ 1 - \frac{f_2 - f_1}{f_3 - f_1} \sin^2 \theta \right]$$

Therefore letting  $k^2 = \frac{f_2 - f_1}{f_3 - f_1}$  we find

$$\zeta = \zeta_1 \pm \sqrt{\frac{2}{f_3 - f_1}} \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad [2.70]$$

where of course  $\varphi$  follows from

$$f = f_1 + (f_2 - f_1) \sin^2 \varphi = f_2 - (f_2 - f_1) \cos^2 \varphi$$

Thus, the cnoidal function is still defined as the inverse function of the integral and equal to  $\cos \varphi$  so

$$\text{cn}^2 \left[ (\zeta - \zeta_1) \sqrt{\frac{f_3 - f_1}{2}} \middle| k \right] = \cos^2 \varphi = -\frac{f - f_2}{f_2 - f_1}$$

and we are finally able to solve for  $f(\zeta)$  by rearranging,

$$f(\zeta) = f_2 - (f_2 - f_1) \text{cn}^2 \left[ (\zeta - \zeta_1) \sqrt{\frac{f_3 - f_1}{2}} \middle| k \right] \quad [2.71]$$

$f_1, f_2, f_3$  are derived from factorizing and solving  $F(f) = 0$  so that for given initial conditions we may give  $f$  by calculating the cnoidal function. The first order expansions of the elliptic functions are [Abramowitz and Stegun, 1972]

$$\begin{aligned} \text{cn}(v|k) &= 1 - \frac{v^2}{2!} + (1 + 4k^2) \frac{v^4}{4!} \\ &\quad - (1 + 44k^2 + 16k^4) \frac{v^6}{6!} + \dots \end{aligned} \quad [2.72a]$$

$$\begin{aligned} \text{sn}(v|k) &= v - (1 + k^2) \frac{v^3}{3!} + (1 + 14k^4) \frac{v^5}{5!} \\ &\quad - (1 + 135k^2 + 135k^4 + k^6) \frac{v^7}{7!} + \dots \end{aligned} \quad [2.72b]$$

$$\begin{aligned} \text{dn}(v|k) &= 1 - k^2 \frac{v^2}{2!} + k^2(4 + k^2) \frac{v^4}{4!} \\ &\quad - k^2(16 + 44k^2 + k^4) \frac{v^6}{6!} + \dots \end{aligned} \quad [2.72c]$$

Several other properties and series expansions are found in the appendix where we include modular sketches of the Jacobi elliptic functions and define the inverse and reciprocals of these functions. The sine amplitude and incomplete elliptic integrals of first, second and third kind are also included. Sketches of the first and second complete elliptic integrals together with integrals of the Jacobi elliptic functions are also found in the appendix.

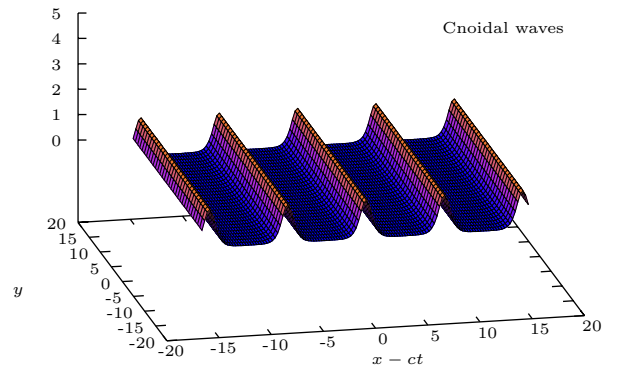
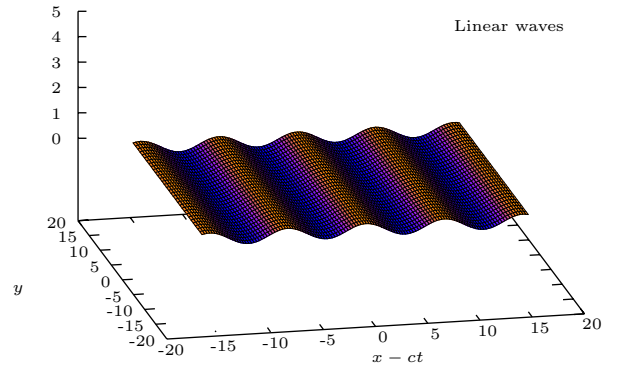
From equation [2.71] it is clear that  $f = f_2$  determines the peak of the wave and that  $f = f_1$  the trough. Therefore  $\frac{1}{2}(f_2 - f_1)$  may be regarded as the relative amplitude of the wave. The wavelength is also found by equation [2.62] and is equal to

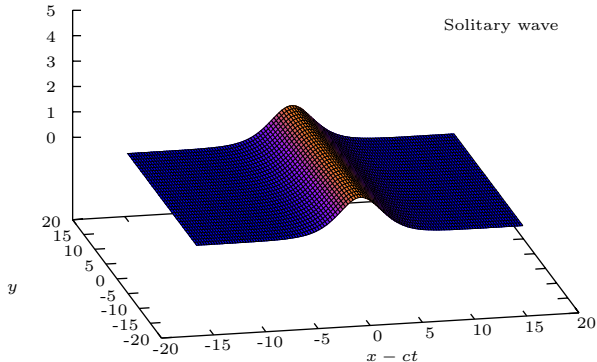
$$2K \sqrt{\frac{2}{f_3 - f_1}} \quad [2.73]$$

By comparing equation [2.56] and the factorization  $F(f) = (f - f_1) \dots$  we see that

$$c = -2(f_1 + f_2 + f_3)$$

Thus the wave, both it's speed, shape and wave length all depend on the amplitude in some complicated manner, but the wave itself is completely described by prescribing these quantities for these will fix  $f_1, f_2$  and  $f_3$ . Fig. 2.2 shows typical wave profiles that are solutions (with different initial conditions) to the KdV equation.





**Fig. 2.2:** Typical cnoidal wave profiles. In the top picture,  $k = 0$  and the nonlinear theory allows small-amplitude linear waves. In the middle,  $k = 0.998$  and the cnoidal wave profile is characterized by very flat troughs and sharp edges. When  $k = 1$  the cnoidal wave profile is a solitary wave, a localized bump travelling without dispersion.

### Limiting Behaviour - Linear and Solitary Waves

There are two special points of interest concerning the solution [2.71]. For vanishing amplitude waves (linear waves), equation [2.71] must be shown to reduce to trigonometric functions and on the other side of the scale where  $m \rightarrow 1^-$ , the same equation must describe solitary waves.

Putting the amplitude equal to  $a = \frac{f_2 - f_1}{2}$  so that  $k^2 = \frac{2a}{f_3 - f_1}$ . The limit  $m \rightarrow 0$  corresponds to the limit  $a \rightarrow 0$ . The cnoidal function obeys

$$\lim_{k \rightarrow 0} \text{cn}(v|k) = \cos v$$

Replacing this limit into equation [2.71] gives directly

$$\lim_{m \rightarrow 0} f = f_2 - 2a \cos^2 \left[ (\zeta - \zeta_3) \sqrt{\frac{f_3 - f_1}{2}} \right] \quad [2.74]$$

If we introduce  $\| = \sqrt{2(f_3 - f_1)}$ ,  $\hat{f}_2 = f_2 - a$  and use the trigonometric relation  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$  and replace all these into the above equation, the solution  $f$  may be written

$$f = \hat{f}_2 - a \cos \left[ \| (x - ct - \zeta_3) \right] \quad [2.75]$$

which describe harmonic oscillations.

On the opposite end of the scale when  $k \rightarrow 1^-$  we have shown that the cnoidal function obeys

$$\lim_{k \rightarrow 1^-} \text{cn}(v|k) = \text{sech } v$$

For the solitar wave, the two zeros  $f_2$  and  $f_3$  must coalesce to form a double zero, see Fig. 2.1:(b,c). We will call this new double zero for  $f_2$ . The solution  $f$  may therefore be written

$$\lim_{k \rightarrow 1^-} f = f_2 - (f_2 - f_1) \text{sech}^2 \left[ (\zeta - \zeta_3) \sqrt{\frac{f_2 - f_1}{2}} \right] \quad [2.76]$$

The amplitude is set to  $\frac{1}{2}a = f_2 - f_1$ . Also,  $c$  is given by  $c = -2(f_1 + 2f_2)$  so that

$$\zeta = x + 2(f_1 + 2f_2)t = x - at + 6f_2t \equiv X - at$$

<sup>1</sup>Representing, of course, one of an infinite number of possibilities.

and  $f$  may be written in the solitary limit as

$$f = f_2 - \frac{1}{2}a \text{sech}^2 \left[ \frac{1}{2}a^{1/2} (X - at - \zeta_3) \right] \quad [2.77]$$

which is consistent with the direct integration, equation [2.60] apart from a translation  $f_2$ . Of course, making a transformation of the form  $u \rightarrow u - f_2$  so that  $f_2$  is fixed at zero renders identical equations.

### B.3 SCATTERING SOLUTIONS TO KDV

The fairly simple type of solutions described in the previous section do not nearly exhaust the possibility of other solutions to the KdV equation. The general initial value problem of the KdV equation has in fact a close connection with the scattering problem of quantum mechanics. The development will revolve around the Sturm-Liouville problem of the form  $\psi_{xx} + (\lambda - u)\psi = 0$  where  $\psi = \psi(x; t)$  is a bounded function,  $\lambda$  the eigenvalue and  $u(x)$  the real potential [Arfken and Weber, 2001]. We accommodate a parametric dependence of  $t$  on  $\psi$  by the semicolon. One simple way of connecting the KdV equation with the Sturm-Liouville problem is to make the substitution  $u = v^2 + v_x$  in the KdV equation, conveniently written in the form  $u_t - 6uv_x + u_{xxx} = 0$ . Then,

$$2vv_t + v_{xt} - 6(v^2 + v_x)(2vv_x + v_{xx}) + 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx} = 0$$

This is rearranged to give

$$\left( 2v + \frac{\partial}{\partial x} \right) (v_t - 6v^2 v_x + v_{xxx}) = 0$$

The term inside the parenthesis:  $v_t - 6v^2 v_x + v_{xxx}$  is commonly called the modified KdV (mKdV) equation because of the extra term  $v^2 v$ . Evidently, if  $v$  is a solution of the mKdV equation, then  $u = v^2 + v_x$  is a solution of the KdV equation. The substitution  $u = v^2 + v_x$  is a Riccati equation for  $v$ , which may be linearized to

$$v = \frac{\psi_x}{\psi}$$

with  $\psi(x, t) \neq 0$ . With this substitution,  $u$  obeys

$$\psi_{xx} - u\psi = 0 \quad [2.78]$$

The connection is completed by noting that the KdV equation is Galilean invariant under the transformation  $u(x, t) = \lambda + u(x + 6\lambda t, t)$ , where  $\lambda = \lambda(t)$  is some arbitrary term. In this spirit, also  $u(x + 6\lambda t, t)$  fulfills equation [2.78], and hence

$$\psi_{xx} - (u - \lambda)\psi = 0 \quad [2.79]$$

which is the Sturm-Liouville equation<sup>1</sup>, with potential  $u$  and eigenvalue  $\lambda$ .

As it turns out, equation [2.79] is only the time-independent Schrödinger equation of quantum mechanics, in one dimension, and the eigenvalue problem is then an analogue of the scattering problem of quantum mechanics, i.e. given  $u(x)$ , determine  $\psi(x; \lambda)$ . This problem may be solved in a wide variety of methods; Fourier transforms, asymptotic expansions, Green's function and ever so on. We shall present no such particular cases, but a Green's function approach is found in e.g. Arfken and Weber [2001].

On the other hand, given the scattering data  $\psi(x; \lambda)$ , an inverse scattering problem can uniquely determine the  $u(x)$  which gave rise to these data. In physical terms, a good analogue

is e.g. the problem of finding the shape of a mechanically vibrating body, from a knowledge of all the sound it makes, both the frequencies, amplitudes and phases. The inverse scattering problem, although well-defined and understood in 1850s, was not completely solved until 1951. However, as we are currently interested in the qualitative behaviour and not the quantitative solutions, a succinct treatment of the problem is not presented here. Only a brief outline follows below.

Let  $u(x, t)$  be the potential function of the Sturm-Liouville problem in the form

$$\psi_{xx} + (\lambda - u)\psi = 0$$

with  $u(x, 0) = f(x)$  a prescribed function. The first step is to solve the above equation for  $u(x, 0)$  (solving the scattering problem). The scattering data are described by the asymptotic behaviour of the eigenfunction  $\psi(x; k; t)$  by

$$\psi_n(x) \sim c_n \exp(-\kappa_n x) \quad [2.80]$$

for  $\lambda < 0$  and  $\kappa_n = i\sqrt{\lambda}$  for each eigenvalue  $n = 1, 2, \dots, N$ , and

$$\hat{\psi}(x; k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as } x \rightarrow \infty \\ a(k)e^{-ikx} & \text{as } x \rightarrow -\infty \end{cases} \quad [2.81]$$

for  $\lambda > 0$ ,  $k = \sqrt{\lambda}$ . The explicit form of  $\hat{\psi}$  is given in Drazin and Johnson [1989]. The time-evolution of the coefficients fulfills [Drazin and Johnson, 1989, G. L. Lamb, 1980, Infeld and Rowlands, 2000]

$$\begin{aligned} \kappa_n &= \text{const.}, & c_n(t) &= c_n(0) \exp(4\kappa_n^3 t) \\ b(k; t) &= b(k; 0) \exp(8ik^3 t) \end{aligned} \quad [2.82]$$

$u(x, t)$  is then found from

$$u(x, t) = -2 \frac{\partial}{\partial x} K(x, x; t) \quad [2.83]$$

where  $K(x, x; t)$  is

$$K(x, z; t) = F(x+z; t) + \int_x^\infty K(x, y; t) F(y+x; t) dy = 0 \quad [2.84]$$

and

$$\begin{aligned} F(X; t) &= \sum_{n=1}^{\infty} c_n^2(0) \exp(8\kappa_n^3 t - \kappa_n X) \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k; 0) \exp(8ik^3 t + ikX) dk \end{aligned} \quad [2.85]$$

This completes the presentation, however brief, of the initial-value problem of the KdV equation. Instead of solving the original nonlinear equation, we have reduced the problem to solving two linear problems, equations [2.84] and [2.85].

#### B.4 OF CONSERVATION LAWS, LAGRANGIANS AND HAMILTONIANS

Conservation laws are abundant in physics, they connect the mathematical equations and the physical phenomena. For two quantities  $T$  and  $X$ , a conservation law is in general

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0 \quad [2.86]$$

but such that neither  $T$  nor  $X$  depend on the derivatives with respect to  $t$ . Incidentally, if this idea is to be applied to a

one-dimensional function  $u(x, t)$ , then  $T$  and  $X$  may depend on  $x, t, u, u_x, u_{xx}, \dots$ , but not  $u_t$ . The KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0 \quad [2.87]$$

is already in conservation form, with  $T = u$  and  $X = u_{xx} - 3u^2$ . In fact, mass, momentum and energy of water waves are readily shown to be conserved quantities. We assume that  $u$  is bounded, i.e.  $X \rightarrow \text{const.}$  as  $x \rightarrow \infty$ . Then, integrating the conservation law once gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} u dx = 0. \quad [2.88]$$

Of course,  $X$  (or  $u(x, t)$ ) can also be periodic, the analogue of the above is then equal to the integral over one period. If we multiply equation [2.87] by  $u$ , the conservation law can be written

$$\frac{\partial}{\partial t} \left( \frac{1}{2} u^2 \right) + \frac{\partial}{\partial x} \left( uu_{xx} - \frac{1}{2} u_x^2 - 2u^3 \right) = 0 \quad [2.89]$$

and thus

$$\frac{d}{dt} \int_{-\infty}^{\infty} u^2 dx = \text{const.} \quad [2.90]$$

and  $u^2$  is therefore a conserved quantity. Consequently, multiplying the KdV equation by  $3u^2$  and adding it to  $u_x \partial_x$  of the KdV equation, we obtain after some algebra

$$\begin{aligned} \frac{\partial}{\partial t} \left( u^3 + \frac{1}{2} u_x^2 \right) + \frac{\partial}{\partial x} \left( -\frac{9}{2} u^4 + 3u^2 u_{xx} - 6u u_x^2 \right. \\ \left. + u_x u_{xxx} - \frac{1}{2} u_{xx}^2 \right) = 0 \end{aligned} \quad [2.91]$$

and consequently

$$\frac{d}{dt} \int_{-\infty}^{\infty} \left( u^3 + \frac{1}{2} u_x^2 \right) dx = 0 \quad [2.92]$$

For water waves, we know that the mass flux goes as  $u$ , the momentum flux as  $u^2$  and the energy flux as  $u^3$  and its derivatives. Thus, we have within the framework of the KdV equation, shown the intrinsic conservation of three fundamental physical quantities; mass, momentum and energy. In any physical system, these three conserved quantities are to be expected. However, by using very laborious methods, Miura et al. [1968] found additionally *eight* conservation laws, each corresponding to a higher power of  $u$ . Eleven conservation laws are more than reasonably expected in general, but as we will shortly see, these eleven are only a small part of an infinity of conservation laws.

The Miura transformation

$$u = v^2 + v_x \quad [2.93]$$

led us to the connection between the Sturm-Liouville problem and the KdV equation. It is now more convenient to work with a different type of transformation, namely

$$v = \frac{1}{2\varepsilon} + \varepsilon w \quad [2.94]$$

where  $\varepsilon$  is a real parameter. Then the Miura transformation becomes

$$u = w + \varepsilon w_x + \varepsilon^2 w^2 \quad [2.95]$$

where we have additionally realized that  $u$  is invariant under Galilean transformations and a constant term proportional to

$\varepsilon^{-2}$  has been neglected. Insertion into the KdV gives, after a minimum of algebra,

$$u_t - 6uu_x + u_{xxx} = \left(1 + \varepsilon \frac{\partial}{\partial x} + 2\varepsilon^2 w\right) \times [w_t - 6(w^2 + \varepsilon^2 w^2)w_x + w_{xxx}] \quad [2.96]$$

Hence,  $u$  fulfills equation [2.95] if  $w$  fulfills

$$w_t - 6(w^2 + \varepsilon^2 w^2)w_x + w_{xxx} = 0 \quad [2.97]$$

but not necessarily vice versa. Whenever  $\varepsilon = 0$ ,  $u = w$ .  $w$  is already in conservation form since

$$w_t = \frac{\partial}{\partial x} (w_{xx} - 3w^2 - 2\varepsilon^2 w^3) = 0 \quad [2.98]$$

and consequently

$$\int_{-\infty}^{\infty} w \, dx = 0 \quad [2.99]$$

The advantage of the transformation  $u = w + \varepsilon w_x + \varepsilon^2 w^2$  lies in the asymptotic behaviour  $w \rightarrow u$  when  $\varepsilon \rightarrow 0$ . Because of this, we may expand  $w$  in an asymptotic sequence

$$w(x, t; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n w_n(x, t) \quad [2.100]$$

with  $\varepsilon \rightarrow 0$ . Then,

$$\int_{-\infty}^{\infty} w_n(x, t; \varepsilon) \, dx = \text{const.} \quad [2.101]$$

for every  $n = 0, 1, 2, \dots$ . If we use the asymptotic expansion in the Gardner transformation, equation [2.95], we find

$$\sum_{n=0}^{\infty} \varepsilon^2 w_n \sim u - \varepsilon \sum_{n=0}^{\infty} \varepsilon^n w_{n,x} - \varepsilon^2 \left( \sum_{n=0}^{\infty} \varepsilon^n w_n \right) \quad [2.102]$$

which reads  $w_0 = u$ ,  $w_1 = -u_x$ ,  $w_2 = u_{xx} - u^2$ ,  $w_3 = -(u_{xx} - u^2)_x + 2uu_x$  etc. We note that  $w_1$  and  $w_3$  are exact differentials with respect to  $x$  and equation [2.101] teaches us nothing. On the other hand,  $w_0$ ,  $w_2$  and  $w_4$  (not explicitly written out), reproduces the conservation laws of mass, momentum and energy. Nonetheless, it appears that if  $n$  is even then  $w_n$  does in fact produce a non-trivial constant of the motion. A neat way of separating the odd and even terms is to associate them with e.g. real and imaginary parts, formally we put  $\varepsilon \rightarrow i\varepsilon$  and the Gardner transformation reads

$$u = W + i\varepsilon W_x - \varepsilon^2 W^2 \quad [2.103]$$

where  $w = W$  is now a complex-valued function. Denoting  $W = \alpha + i\beta$  with  $\alpha$  and  $\beta$  real functions, this is written

$$u = \alpha - \varepsilon\beta_x - \varepsilon^2(\alpha^2 - \beta^2) + i(\beta + \varepsilon\alpha_x - 2\varepsilon^2\alpha\beta)$$

and requiring the  $u$  be real, the second paranthesis on the right hand side must be zero:  $\beta + \varepsilon\alpha_x - 2\varepsilon^2\alpha\beta = 0$ . Then,

$$\alpha = u + \varepsilon\beta_x + \varepsilon^2(\alpha^2 - \beta^2)$$

It is convenient to write  $\alpha \sim A_o + A_e$ ,  $\beta \sim B_o + B_e$  as  $\varepsilon \rightarrow 0$ , denoting an asymptotic expansion in even ( $A_e$ ,  $B_e$ ) and odd ( $A_o$ ,  $B_o$ ) terms. In collecting all the even and odd terms of  $\beta + \varepsilon\alpha_x - 2\varepsilon^2\alpha\beta = 0$ , and equation [B.4], we find

$$B_e(1 - 2\varepsilon^2 A_2) \sim -\varepsilon(A_o)_x + 2\varepsilon^2 A_o B_o \quad [2.104a]$$

$$A_o(1 - 2\varepsilon^2 A_2) \sim -\varepsilon(B_e)_x - 2\varepsilon^2 A_e B_e \quad [2.104b]$$

to all order in  $\varepsilon$ . When  $\varepsilon$  tends to zero, these equations imply  $B_e \sim -\varepsilon(A_o)_x \sim -\varepsilon^2(B_e)_{xx}$  and we see by iteration that  $B_e = A_o = 0$  is an exact solution of the two equations above. Thus,  $\alpha \sim A_e$  (even terms only) and  $\beta \sim B_o$  (odd terms only) which is consistent with the Gardner transformation of equation [2.103]. However,  $\beta + \varepsilon\alpha_x - 2\varepsilon^2\alpha\beta = 0$  may be written

$$\beta = \frac{1}{2\varepsilon} \frac{\partial}{\partial x} (\log |1 - 2\varepsilon^2 \alpha|)$$

and so  $\beta$  is an exact differential to all powers in  $\varepsilon$  and hence do not produce conservation laws. On the other hand, the first iteration of equation [B.4] produces  $(\alpha)_0 = u$ . Every subsequent iteration will involve the terms  $\varepsilon\beta_x + \varepsilon^2\beta^2$  together with a contribution  $\varepsilon^2\alpha^2$ . This latter term, will at each iteration generate terms which do not involve any derivatives and hence is not an exact differential. The specific terms not involving any derivatives are

$$(\alpha)_m = u + \varepsilon^2 [(\alpha)_{m-1}]^2 \quad [2.105]$$

Thus there will be an *infinity of conservation laws* where each density is characterized by inclusion of a term  $u^{n+1}$ .

### Lagrangians And Hamiltonians

It is well known that conservation laws result from variational principles that are invariant under transformations belong to a certain continuous group. For instance, Galilean invariance and rotations are such transformations. The application of Noether's theorem to the Lagrangian density then derives the conservation law corresponding to the transformation at hand. Consider the action integral  $A = \int_{t_1}^{t_2} L(\dot{\mathbf{q}}, \mathbf{q}) \, dt$ , it is invariant to the transformation  $t \rightarrow t + \tau$  such that the action integral transforms to

$$A \rightarrow \int_{t_1+\tau}^{t_2+\tau} L(\dot{\mathbf{q}}, \mathbf{q}) \, dt, \quad [2.106]$$

and the Euler-Lagrange equations are unaltered since  $\tau$  appears in the limits only, and hence the least action principle is unaltered. Although a detailed study of Noether's theorem is out of the scope of this text, some basic properties are readily shown. Noether's approach was to consider  $\tau = \tau(t)$ , but infinitesimal and such that  $\tau(t_1) = \tau(t_2) = 0$ . When  $t \rightarrow t + \tau$ ,  $\dot{\mathbf{q}} = \dot{\mathbf{q}}(1 - \dot{\tau})$  and it follows

$$L(\dot{\mathbf{q}}, \mathbf{q}) \rightarrow L(\dot{\mathbf{q}}, \mathbf{q}) - \left( \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \dot{\tau} \quad [2.107]$$

The action integral then transforms as

$$A \rightarrow \int_{t_1}^{t_2} L(\dot{\mathbf{q}}, \mathbf{q}) \, dt - \int_{t_1}^{t_2} (p_i \dot{q}_i - L) \dot{\tau} \, dt \quad [2.108]$$

where we have used  $dt \rightarrow (1 + \dot{\tau})dt$  and  $p_i = \frac{\partial L}{\partial \dot{q}_i}$ . The corresponding Euler-Lagrange equation (again, see the appendix) is,

$$\frac{d}{dt} (p_i \dot{q}_i - L) = 0$$

or otherwise simply stated as

$$p_i \dot{q}_i - L = \text{constant.} \quad [2.109]$$

which is the Hamiltonian of the system, and corresponds to the equation of conservation of energy.

The KdV equation  $u_t - 6uu_x + u_{xxx} = 0$  can be generated from an appropriate Lagrangian

$$L = \frac{1}{2}\phi_x\phi_t - \phi_x^3 - \frac{1}{2}\phi_{xx}^2 \quad [2.110]$$

where  $u = \phi_x$ . The KdV equation, equation [2.87] follows from the Euler-Lagrange equation of the Lagrangian above, and application of Noether's theorem under translations  $\phi$ ,  $x$  and  $t$  reproduces the conservation laws of mass, momentum and energy. An even deeper result is the fact that conservation of energy for

the KdV equation can be formulated in a Hamiltonian formalism by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ \frac{\delta}{\delta u} \left( u^3 + \frac{1}{2}u_x^2 \right) \right] \quad [2.111]$$

where  $\frac{\delta}{\delta u}$  is the variational derivative. By virtue of the infinity of conserved quantities of the KdV equation, it can be written as an *infinite dimensional Hamiltonian system*. Indeed, in the following shall see explicitly how the energy  $E$  of a liquid drop, taken to be Hamiltonian, produces the KdV equation with parametrically dependent coefficients.

## PART III

### NONLINEAR MODES OF A ROTATING DROPLET

In the present part we develop the dynamical equation for the droplet surface. We start by presenting rudiments of a previous nonlinear model and examine the effects of rotation/flattening later on. Several explicit solutions are finally presented and the part is wrapped up by a discussion and summary of the results.

#### A. GOVERNING EQUATIONS

For a free surface, Luke [1967] showed that the equations of motion in the Eulerian variables may be obtained from an appropriate Lagrangian function which equals the pressure. Also, Natarajan and Brown [1986, 1987] described the oscillations of an inviscid drop in terms of spherical harmonics, they use a Lagrangian of the form

$$\begin{aligned}
 L = & \int_0^{2\pi} \int_0^\pi \int_0^{\xi(\theta, \varphi, t)} \rho \left[ \frac{1}{2} \left( \Phi_r^2 + \frac{1}{r^2} (\Phi_\theta^2 + \Phi_\varphi^2 \csc^2 \theta) \right) \right. \\
 & \left. - \Phi_t \right] r^2 \sin \theta \, dr \, d\theta \, d\varphi \\
 & + \gamma \int_0^{2\pi} \int_0^\pi \int_0^{\xi(\theta, \varphi, t)} \xi \sqrt{\xi^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta} \sin \theta \, d\theta \, d\varphi \\
 & + \Delta p \int_0^{2\pi} \int_0^\pi \left( \frac{1}{3} \xi^3 - \frac{1}{4\pi} V \right) \sin \theta \, d\theta \, d\varphi
 \end{aligned}$$

This is a suitable Lagrangian for a non-rotating droplet exercising irrotational motion, and oscillations and coupled oscillations have been extensively reviewed by Natarajan and Brown [1986, 1987]. Physically, the first term in the equation above is the kinetic energy (the reader may readily draw motivation directly from the Bernoulli equation), the second term is the surface energy which may very well be thought of as the potential energy. The third term is the constant volume constraint with  $\Delta p_0$  as the usual Lagrangian multiplier associated with the requirement that the curvature of the surface must balance the pressure in the interior. However, their analysis focus on the coupling of oscillation modes directly in terms of spherical harmonics. Rotation can be incorporated into the Lagrangian by the same assumption as we will later use. Although the method of classical mechanics offers a beautiful formalism, the starting alone would be a Lagrangian requiring several lines of equations. We shall work with the Euler equations instead.

The Euler equations in a frame rotating with angular velocity  $\boldsymbol{\Omega}$  are [Landau and Lifshitz, 1987]

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) - 2\boldsymbol{\Omega} \times \mathbf{u} \quad [3.1]$$

The second term on the right hand side is the centrifugal force whereas the final term on the right hand side is known as the Coriolis force. Keep in mind that we are restricting ourselves to

a *driven* droplet such that the angular velocity vector  $\boldsymbol{\Omega}$  is constant in magnitude and direction. In the case of mechanically isolated droplet (representing stars, black holes etc. rather than table-top experiments) an additional term known as the Euler force exists on the right hand side of the equation above. This term is caused by the fact that  $\boldsymbol{\Omega}$  is not constant in magnitude. An auxiliary constraint corresponding to constant angular momentum must then also be specified. The centrifugal term is conservative (irrotational) by the following argumentation: The identity  $\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \nabla) \mathbf{a} + (\mathbf{a} \cdot \nabla) \mathbf{b} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$  is readily shown by use of the Levi-Civita tensor. Thus,

$$\begin{aligned}
 \nabla \cdot \left[ \frac{1}{2} (\boldsymbol{\Omega} \times \mathbf{r}) \cdot (\boldsymbol{\Omega} \times \mathbf{r}) \right] &= [(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla] (\boldsymbol{\Omega} \times \mathbf{r}) \\
 &+ (\boldsymbol{\Omega} \times \mathbf{r}) \times [\nabla \times (\boldsymbol{\Omega} \times \mathbf{r})]
 \end{aligned}$$

The second term on the right hand side is zero since

$$[\nabla \times (\boldsymbol{\Omega} \times \mathbf{r})]_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \Omega_l r_m = \varepsilon_{ijk} \varepsilon_{klm} \Omega_l \delta_{jm} = 0$$

by use of basic properties of the Levi-Civita tensor. On the other hand,

$$\begin{aligned}
 [(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla] (\boldsymbol{\Omega} \times \mathbf{r}) &= (\boldsymbol{\Omega} \times \mathbf{r})_j \partial_j \varepsilon_{klm} \Omega_l r_m = \varepsilon_{klm} (\boldsymbol{\Omega} \times \mathbf{r})_j \Omega_l \delta_{jm} \\
 &= \varepsilon_{klj} \Omega_l (\boldsymbol{\Omega} \times \mathbf{r})_j = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})
 \end{aligned}$$

Therefore the centrifugal term may be written as the gradient of  $\frac{1}{2} (\boldsymbol{\Omega} \times \mathbf{r})^2$  and it is conservative. The conservation of the vorticity  $\boldsymbol{\omega}$  of the flow, that is whether a potential flow  $\nabla \Phi$  tends to stay irrotational, is found by taking the curl of the Euler equation. The centrifugal force and the pressure are conservative so they vanish upon the curl operation. The temporal derivative and the curl operator commutes so  $\nabla \times \frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \boldsymbol{\omega}}{\partial t}$ . Therefore we only need to take the curl of the convective terms and the Coriolis term (we do not even bother discussing the Euler-term). The curl of the convective terms follows from the identity  $\nabla \cdot (\frac{1}{2} \mathbf{u}^2) = (\mathbf{u} \cdot \nabla) \mathbf{u} + \mathbf{u} \times \boldsymbol{\omega}$  which is readily proven by use of the Levi-Civita tensor. Thus

$$\begin{aligned}
 \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) &= \varepsilon_{ijk} \partial_j \varepsilon_{klm} u_l \omega_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j (u_l \omega_m) \\
 &= (\omega_j \partial_j) u_i - (u_j \partial_j) \omega_i + u_i (\partial_j \omega_j) - \omega_i \partial (\partial_j u_j) \\
 &= (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u}
 \end{aligned}$$

Where, by the incompressibility condition  $\nabla \cdot \mathbf{u} = 0$  together with the solenoidal condition on  $\boldsymbol{\omega}$ , the two last terms on the second line vanish. For the Coriolis term we find

$$\nabla \times (\boldsymbol{\Omega} \times \mathbf{u}) = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \Omega_l u_m = -(\Omega_j \partial_j) u_i$$

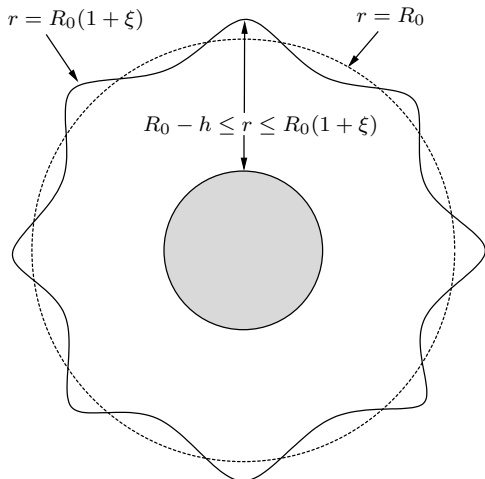
where we have used the usual multiplication property of  $\varepsilon_{ijk}$  and the incompressibility condition on  $\mathbf{u}$ . Therefore, conservation of vorticity of an incompressible flow in a frame of reference rotating with angular velocity  $\boldsymbol{\Omega}$  is

$$\frac{d\boldsymbol{\omega}}{dt} = [(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla] \mathbf{u} \quad [3.2]$$

We shall for the time being keep things irrotational and assume a potential flow. We introduce a convenient geometry so that the radial distance from the center of mass (which lies on the axis of rotation) to the free surface of the droplet is described by the function  $r(\theta, \varphi, t) = R_0 [1 + \xi(\theta, \varphi, t)]$ .  $(\theta, \varphi)$  denotes the usual polar angles, see Fig. 1.1. Guided by experimental evidence suggesting travelling waves with constant transverse profiles we choose

$$r = R_0 [1 + g(\theta)\eta(\varphi - Vt)]$$

i.e.  $\xi(\theta, \varphi, t) = g(\theta)\eta(\varphi - Vt)$  where  $V$  is the angular velocity of the travelling wave.



**Fig. 3.1:** Chosen geometry with the free surface denoted by  $r = R_0 [1 + g(\theta)\eta(\varphi)]$ . We assume a rigid core (shaded region) of radius  $R_0 - h(\theta)$  so that the flow is stratified in the surface layer  $R_0 - h \leq r \leq R_0(1 + \xi)$ .

The flow may be assumed to take place over a rigid core (or in a stratified layer) of some chosen radius, similar to the linear wave analysis often encountered in introductory fluid mechanics where the core/layer would correspond to the bottom of the ocean etc., except that our geometry is spherical and periodic. This is a very generalizing argument since the layer may be chosen arbitrarily small so that it again represents a droplet. Nonetheless, the presence of a rigid core opens the possibility for the study of more complex systems such as multiphase systems, superfluids etc. The two surfaces of the fluid domain will be denoted  $\Sigma_1$  and  $\Sigma_2$  where the former denotes the outer free surface and the latter denotes the inner surface.

The flow is taken potential so the flow velocity field  $\mathbf{u}$  and its incompressibility yields in term of the scalar potential  $\Phi$

$$\mathbf{u} = \nabla \Phi, \quad \Delta \Phi = 0 \quad [3.3]$$

where  $\Delta$  denotes the Laplacian. Under the assumption of a potential flow, the Bernoulli relation holds, thus  $\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{1}{\rho} p = \text{const}$ . This is verified by introducing  $\mathbf{u} = \nabla \Phi$ , rewriting the convective terms in the form  $(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2$  and integration supplemented by the fundamental theorem of differential calculus. The irrotationality of the flow ensures that the above holds between any points in the fluid domain. The constant is arbitrary and time dependent, but choosing it to be equal to the ambient pressure  $p_0$  will prove a good choice. In any case, the time-dependent terms can be absorbed into  $\Phi_t$ . Evaluating this expression on the surface yields the *dynamic boundary condition*

$$\left( \Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{1}{\rho} \Delta p - f \right) \Big|_{\Sigma_1} = 0 \quad [3.4]$$

where  $\Delta p$  is the gage pressure across the interface. Also, we have added a conservative force per unit mass  $\mathbf{f} = \nabla f$  for generalization.

On the free surface  $\Sigma_1$  of the droplet, the radial velocity is given by the kinematic surface condition that the fluid velocity must equal the velocity of the surface, i.e.

$$\dot{r} \Big|_{\Sigma_1} = r_t + r_\theta \dot{\theta} + r_\varphi \dot{\varphi} \Big|_{\Sigma_1} \quad [3.5]$$

which follows from the chain rule  $\frac{dr}{dt} = \frac{\partial r}{\partial t} + \dots$ . This is a very general result that allows for vibrational and travelling wave modes. A potential flow  $\mathbf{u} = \nabla \Phi$  allows the velocity components to be explicitly written as

$$\begin{aligned} (\nabla \Phi)_r &= \Phi_r = \dot{r} \\ (\nabla \Phi)_\theta &= \frac{1}{r} \Phi_\theta = r \dot{\theta} \\ (\nabla \Phi)_\varphi &= \frac{1}{r \sin \theta} \Phi_\varphi = r \sin \theta \dot{\varphi} \end{aligned}$$

By substituting for  $\dot{\theta}$  and  $\dot{\varphi}$ , the free-surface condition may be written

$$\dot{r} \Big|_{\Sigma_1} = R_0 \left( \xi_t + \frac{\xi_\theta}{r^2} \Phi_\theta + \frac{\xi_\varphi}{r^2 \sin \theta} \Phi_\varphi \right) \Big|_{\Sigma_1} \quad [3.6]$$

In the case of linearization, the above reduces to  $\dot{r} = R_0 \xi_t$  (see e.g. Landau and Lifshitz [1987]) and allows only radial vibrations (the LDM).

By assuming that the flow is stratified in a layer  $h(\theta)$  (i.e. at a radius  $r_{\Sigma_2} = R_0 - h(\theta)$ ) introduces a constraint that the flow takes place in the interval  $R_0 - h(\theta) \leq r \leq R_0 [1 + \xi]$ . By requiring no cross-flow through this layer we find that

$$\dot{r} \Big|_{r=R_0-h} = \Phi_r \Big|_{r=R_0-h} = 0 \quad [3.7]$$

The equations [3.4],[3.6],[3.7] together with equation [3.3] serve as the governing equations for the problem. To these we add an Euler-Lagrange equation if using an Eulerian formalism, or in the form of a Hamiltonian equation if we use a Hamiltonian formalism.



## B. NONLINEAR LIQUID DROP MODEL

### B.1 EULERIAN FORMULATION

For a potential flow, the nonlinear dynamics in a droplet is governed by the Euler equations while the potential obeys Laplace's equation. The usual approach to solving Laplace's equation in spherical coordinates is to make use of the spherical harmonics that are direct solutions to Laplace's equation. This is not appropriate for nonlinear systems so we instead suggest the velocity potential

$$\Phi = \sum_{n=0}^{\infty} \left( \frac{r}{R_0} - 1 \right)^n f_n(\theta, \varphi, t) \quad [3.8]$$

where convergence of the series is controlled by the small quantity  $\frac{r}{R_0} - 1 = \max |\xi|$ .

The Laplacian will be denoted  $\Delta = \Delta_r + \frac{1}{r^2} \Delta_\Omega$  where  $\Delta_r$  is the  $r$ -part of the Laplacian and  $\Delta_\Omega$  is, apart from a factor  $\frac{1}{r^2}$ , the  $(\theta, \varphi)$  part of the Laplacian. Laplace's equation gives directly for  $\Phi$  that

$$\begin{aligned} \Delta \Phi &= \frac{1}{r^2} \sum_{n=0}^{\infty} \left( \frac{r}{R_0} - 1 \right)^n \Delta_\Omega f_n + \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} \left( \frac{r}{R_0} - 1 \right)^n f_n \right] \\ &= \frac{1}{r^2} \sum_{n=0}^{\infty} \left\{ \left( \frac{r}{R_0} - 1 \right)^n \Delta_\Omega f_n + n(n-1) \frac{r^2}{R_0^2} \right. \\ &\quad \left. \times \left( \frac{r}{R_0} - 1 \right)^{n-2} f_n + \frac{2nr}{R_0} \left( \frac{r}{R_0} - 1 \right)^{n-1} f_n \right\} \end{aligned}$$

One can now use the trick  $r^2/R_0^2 = (r/R_0 - 1)(r/R_0 + 1) + 1$  to rewrite the second term. It follows that

$$\begin{aligned} \Delta \Phi &= \frac{1}{r^2} \sum_{n=0}^{\infty} \left\{ \left( \frac{r}{R_0} - 1 \right)^n \Delta_\Omega f_n + n(n-1) \left( \frac{r}{R_0} - 1 \right)^{n-2} f_n \right. \\ &\quad \left. + n(n-1) \left( \frac{r}{R_0} + 1 \right) \left( \frac{r}{R_0} - 1 \right)^{n-1} f_n + 2n \frac{r}{R_0} \right. \\ &\quad \left. \times \left( \frac{r}{R_0} - 1 \right)^{n-1} f_n \right\} \end{aligned}$$

Apart from the factors  $n^2$ , the two last terms can be combined by observing that  $-n(r/R_0 + 1) + 2n(r/R_0) = r/R_0 - 1$ . Furthermore the terms  $n^2$  in the third term can be rewritten by using  $r/R_0 + 1 = r/R_0 - 1 + 2$ , and this puts Laplace's equation on the form

$$\begin{aligned} \Delta \Phi &= \frac{1}{r^2} \sum_{n=0}^{\infty} \left\{ \left( \frac{r}{R_0} - 1 \right)^n \Delta_\Omega f_n + n(n-1) \left( \frac{r}{R_0} - 1 \right)^{n-2} f_n \right. \\ &\quad \left. + n \left( \frac{r}{R_0} - 1 \right)^n f_n + n^2 \left( \frac{r}{R_0} - 1 \right)^n f_n \right. \\ &\quad \left. + 2n^2 \left( \frac{r}{R_0} - 1 \right)^{n-1} f_n \right\} \end{aligned}$$

By observing that the variable  $n$  can be incremented by 2 in the second term and by 1 in the final term (this makes no difference since these terms are originally zero), Laplace's equation can be put on a form where the  $(\theta, \varphi)$ -part has been isolated, i.e.

$$\begin{aligned} \Delta \Phi &= \frac{1}{r^2} \sum_{n=0}^{\infty} \left( \frac{r}{R_0} - 1 \right)^n [\Delta_\Omega f_n + (n+2)(n+1)f_{n+2} \\ &\quad + 2(n+1)(n+1)f_{n+1} + n(n+1)f_n] \end{aligned} \quad [3.9]$$

A general solution to the above is a recurrence relation in  $f_n$  so that the bracketed expression is zero for each  $n$ . Thus,

$$f_{n+2} = \frac{-2(n+1)(n+1)f_{n+1} - n(n+1)f_n - \Delta_\Omega f_n}{(n+2)(n+1)} \quad [3.10]$$

Hence the set of  $f_n$  reduce to *two* unknowns,  $f_0$  and  $f_1$ . This is very convenient since in fact there are two boundary conditions that needs to be satisfied. The previous calculation is a well-known way of analyzing nonlinear waves, see e.g. Infeld and Rowlands [2000] for an analysis on nonlinear, shallow water, weak amplitude waves. The first few terms are readily expressed in terms of  $f_0$  and  $f_1$ ,

$$f_2 = \frac{1}{2} (-\Delta_\Omega f_0 + 2f_1) \quad [3.11a]$$

$$f_3 = \frac{1}{6} (4\Delta_\Omega f_0 - \Delta_\Omega f_1 + 6f_1) \quad [3.11b]$$

$$f_4 = \frac{1}{12} \left( \frac{\Delta_\Omega^2}{2} f_0 - 9\Delta_\Omega f_0 + 4\Delta_\Omega f_1 - 12f_1 \right) \quad [3.11c]$$

Ludu et al. [1996] suggests, incorrectly, that by use of a reduced form of the negative binomial series that the explicit calculation of  $f_3$  gives  $f_3 = \frac{1}{6} (4\Delta_\Omega f_0 - 4\Delta_\Omega f_1 + 4f_1 + 2)$ . Also, they suggest that  $f_4 = \frac{1}{12} \left( \frac{\Delta_\Omega^2}{2} f_0 - 14\Delta_\Omega f_0 + 8\Delta_\Omega f_1 - 8f_1 \right)$  which also is incorrect.  $f_1$  and  $f_2$  remain untouched by the error made by Ludu et al. [1996] so to second order in  $\xi$  their analysis is correct. Including the missing algebraic factor in the Ludu et al. [1996] analysis (equation (11) in their paper) yields, after some trivial manipulation, equation [3.10].

We shall need the derivatives of the potential,

$$\Phi_r = \sum_{n=0}^{\infty} \frac{(n+1)}{R_0} \left( \frac{r}{R_0} - 1 \right)^n f_{n+1} \quad [3.12a]$$

$$\Phi_\theta = \sum_{n=0}^{\infty} \left( \frac{r}{R_0} - 1 \right)^n f_{n,\theta} \quad [3.12b]$$

$$\Phi_\varphi = \sum_{n=0}^{\infty} \left( \frac{r}{R_0} - 1 \right)^n f_{n,\varphi} \quad [3.12c]$$

where we have incremented  $n$  by one in the radial term.

The second boundary condition, equation [3.7], is  $\Phi_r|_{\Sigma_2} = 0$ . Inserting the limit  $r = R_0 - h$  into equation [3.12a] yields

$$\sum_{n=0}^{\infty} \frac{n+1}{R_0} \left( -\frac{h}{R_0} \right)^n f_{n+1} = 0 \quad [3.13]$$

which to first order in  $h$  reads  $f_1 = \frac{2h}{R_0} f_2$  so that  $f_0$  may be related to  $f_1$  by substituting for  $f_2$  in equation [3.11a]. Thus,

$$\Delta_\Omega f_0 = - \left( 2 + \frac{R_0}{h} \right) f_1 \quad [3.14]$$

The kinematic boundary condition, equation [3.6], requires the derivatives of the potential evaluated at the surface. To first order in  $\xi$ , equation [3.12] reveals

$$\Phi_r|_{\Sigma_1} = \frac{f_1 + 2\xi f_2}{R_0} + \mathcal{O}(\xi^2) \quad [3.15a]$$

$$\Phi_\theta|_{\Sigma_1} = f_{0,\theta} + \xi f_{1,\theta} + \mathcal{O}(\xi^2) \quad [3.15b]$$

$$\Phi_\varphi|_{\Sigma_1} = f_{0,\varphi} + \xi f_{1,\varphi} + \mathcal{O}(\xi^2) \quad [3.15c]$$

By inserting the three equations above into equation [3.6] together with the requirement of vanishing tangential velocity we find

$$f_1 + 2\xi f_2 = R_0^2 \xi_t + \frac{\xi_\varphi (f_{0,\varphi} + \xi f_{1,\varphi})}{(1 + \xi)^2 \sin^2 \theta} \quad [3.16]$$

We have kept our expansions to first order in  $\xi$  and consequently it is enough to expand  $f_1$  to first order. Then  $f_1 = R_0^2 \xi_t + \mathcal{O}(\xi^2)$  similar to the normal modes of oscillation. We may also make use of  $f_2 = \frac{R_0}{2h} f_1 = \frac{R_0^3 \xi_t}{2h}$ . Solving for  $f_{0,\varphi}$  then gives

$$\xi_\varphi f_{0,\varphi} = -\frac{\sin^2 \theta R_0^3}{h} \xi \xi_t (1 + 2\xi) + \mathcal{O}(\xi^3) \quad [3.17]$$

Together with the requirement  $\xi_\varphi = -V \xi_t$  for the travelling wave,  $f_{0,\varphi}$  reads to second order in  $\xi$

$$f_{0,\varphi} = \frac{V R_0^3 \sin^2 \theta}{h} \xi (1 + 2\xi) + \mathcal{O}(\xi^3) \quad [3.18]$$

The equation above is the description of the connection between the shape of the droplet and the velocity potential, a coupling that is typical for nonlinear systems. The azimuthal velocity is given by  $u_\varphi = \frac{\Phi_\varphi}{r \sin \theta}$  which explicitly reads

$$u_\varphi = \frac{V R_0^2 \sin \theta}{h} \xi \quad [3.19]$$

to first order in  $\xi$ .

In the derivation of the free surface dynamical equation, (equation [3.4]), we will follow the same procedure as for the normal modes but with a correction to higher-order nonlinear terms. We may work from the exact expression of the curvature (see appendix):

$$\nabla \cdot \hat{\mathbf{n}} = \frac{1}{R_0} \frac{\{(1 + \xi)^2 [2(1 + \xi) \sin \theta - \frac{\partial}{\partial \theta} (\xi_\theta \sin \theta)] + \xi_\theta^2 [2(1 + \xi) \sin \theta - \frac{\partial}{\partial \theta} ((1 + \xi) \cos \theta)]\} \sin^2 \theta - \{[\frac{\partial}{\partial \varphi} (\xi_\varphi [(1 + \xi)^2 + \xi_\theta^2]) - 2\xi_{\varphi\varphi} [(1 + \xi)^2 + \xi_\theta^2] \sin \theta - \xi_\varphi^2 \frac{\partial^2}{\partial \theta^2} ((1 + \xi) \sin \theta)]\}}{(1 + \xi) [(1 + \xi)^2 \sin^2 \theta + \xi_\theta^2 \sin^2 \theta + \xi_\varphi^2]^{3/2}} \quad [3.20]$$

The above lets us expand the nonlinearities to the same order as we have previously done for the other boundary conditions. However, it is easier to work directly from the surface energy  $U_{\Sigma_1}$  and equate the first variation of the appropriate expansion to the pressure.  $U_{\Sigma_1}$  is of course

$$U_{\Sigma_1} = \gamma R_0^2 \int_0^{2\pi} \int_0^\pi (1 + \xi) \sqrt{(1 + \xi)^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta} \times \sin \theta \, d\theta \, d\varphi \quad [3.21]$$

Taking the first variation of the exact (that is, no expansion of the square root) surface energy functional yields the local mean curvature given in equation [3.20] of the surface and both methods are equally valid.

It is possible to expand the square root in series in all its three arguments  $\xi, \xi_\theta, \xi_\varphi$ , around 0. We use a multi-variable Taylor expansion but the calculation itself is performed in the next section. The result is

$$\sqrt{(1 + \xi)^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta} \approx (1 + \xi) + \frac{1 - \xi}{2} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta) \quad [3.22]$$

Note that the linear model requires terms up to second order in  $\xi$  since the first variation of the above functional must be calculated. In the same manner, accounting for nonlinearities requires terms up to third order. It is here readily verified that the quadratic terms give  $U_{\Sigma_1} = \gamma R_0^2 \int [(1 + \xi)^2 + \frac{1}{2} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta)] \, d\Omega$ , and corresponds to the linear model in Part I [Landau and Lifshitz, 1987]. Moreover, in equation [3.34] we may neglect the term proportional to  $\eta \eta_\varphi^2$  due to the localization of the solution, i.e. its relative amplitude  $\xi \xi_\varphi^2 \sim \max |\xi|^3 / L^2 \ll 1$  where  $L$  is the angular half-width of the travelling wave, as most experiments show [Trinh and Wang, 1982, Trinh et al., 1996, Holt and Trinh, 1996]. We conclude that the surface energy is, when accounting for nonlinearities,

given by

$$U_{\Sigma_1} = \gamma R_0^2 \int_0^{2\pi} \int_0^\pi \left[ (1 + \xi)^2 + \frac{(1 - \xi^2)}{2} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta) \right] \times \sin \theta \, d\theta \, d\varphi \quad [3.23]$$

To equate the pressure against the surface energy, we need to take the first variation of the above equation. We call the integrand of the above for  $g$ , then

$$\begin{aligned} \frac{\partial g}{\partial \xi} &= [2(1 + \xi) - \xi(\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta)] \sin \theta \\ \frac{\partial}{\partial \theta} \frac{\partial g}{\partial \xi_\theta} &= (1 - \xi^2) \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \xi}{\partial \theta} \right) \right] \sin \theta - 2\xi \xi_\theta^2 \sin \theta \\ \frac{\partial}{\partial \varphi} \frac{\partial g}{\partial \xi_\varphi} &= (1 - \xi^2) \left[ \frac{1}{\sin^2 \theta} \frac{\partial^2 \xi}{\partial \varphi^2} \right] \sin \theta - 2\xi \xi_\varphi^2 \csc^2 \theta \sin \theta \end{aligned}$$

and we identify the first variation to third order in  $\xi$  of the surface energy as

$$\delta U_{\Sigma_1} = \gamma R_0 \int_0^{2\pi} \int_0^\pi [2(1 + \xi) - (1 - \xi^2) \Delta_\Omega \xi + \xi(\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta)] \sin \theta \, d\theta \, d\varphi \quad [3.24]$$

In order to properly identify the first variation we need to have the integral on a form  $\int (\dots) \delta \xi \, d\Sigma_1$  where  $d\Sigma_1$  is the integrand of equation [3.21]. Reconstructing  $d\Sigma_1$  to the lowest order in  $\xi$  corresponds to dividing and multiplying by a factor  $R_0^2 (1 + 2\xi)$  in the above equation (see e.g. Landau and Lifshitz [1987]). Formally,  $d\Sigma_1$  is given by

$$\begin{aligned} d\Sigma_1 &= R_0^2 (1 + \xi) \sqrt{1 + 2\xi + \dots} \approx R_0^2 (1 + \xi) (1 + \xi) \\ &= R_0^2 (1 + 2\xi) \end{aligned}$$

which is found by expanding the square root. Therefore, the above equation may be written

$$\delta U_{\Sigma_1} = \frac{\gamma}{R_0} \int_0^{2\pi} \int_0^\pi \frac{[\dots]}{1 + 2\xi} \delta \xi \, d\Sigma_1$$

where [...] denotes the bracketed expression of equation [3.24]. Thus the gage pressure satisfies

$$\Delta p|_{\Sigma_1} = \frac{\gamma}{R_0} \frac{[\dots]}{1+2\xi} \quad [3.25]$$

where all constants have been dropped (we shall later differentiate this so we drop the constants right away). The pressure has here been identified as the Lagrangian multiplier arising from the requirement of thermodynamic equilibrium [Landau and Lifshitz, 1987]. Upon expanding the denominator this reduces to

$$\Delta p|_{\Sigma_1} = \frac{\gamma}{R_0} [-2\xi - 4\xi^2 - (1 - 2\xi - \xi^2)\Delta_\Omega\xi + \xi(\xi_\theta^2 + \xi_\varphi^2 \csc^2\theta)] + \text{const.} \quad [3.26]$$

In the dynamic boundary condition, i.e. the Bernoulli relation at the free surface  $\Sigma_1$ , we now have all the terms. We will restrict our analysis to second order in  $\xi$  and first order in its derivatives so the  $\nabla\Phi$ -terms reduce to only the azimuthal component which is equal to  $\Phi_\varphi r \sin\theta = \frac{VR_0^2 \sin\theta}{h}\xi + \mathcal{O}(\xi^2)$  at the free surface. For the pressure, we therefore only keep the terms proportional to  $\xi, \xi^2$  and  $\Delta_\Omega\xi$ . The Bernoulli relation reduces to

$$\Phi_t|_{\Sigma_1} = -\frac{1}{2} \frac{V^2 R_0^4 \sin^2\theta}{h^2} \xi^2 + \frac{\gamma}{\rho R_0} (2\xi + 4\xi^2 + \Delta_\Omega\xi) \quad [3.27]$$

In the case of no external forces and linearization the above yields

$$\Phi_t = \frac{\gamma}{\rho R_0} (2\xi + \Delta_\Omega\xi) \quad [3.28]$$

which is exactly the normal mode oscillations treated in Part I (see e.g. Landau and Lifshitz [1987], Rayleigh [1914]).

Differentiating equation [3.27] with respect to  $\varphi$  yields

$$\begin{aligned} \Phi_{\varphi,t}|_{\Sigma_1} &= \frac{V^2 R_0^4 \sin^2\theta}{h^2} g^2 \eta \eta_\varphi + \frac{\gamma}{\rho R_0} (2g\eta_\varphi + 8g^2 \eta \eta_\varphi \\ &\quad - \frac{1}{\sin^2\theta} g \eta_{\varphi\varphi} + \eta_\varphi \Delta_\theta g) \end{aligned}$$

where  $\Delta_\theta$  is only the  $\theta$ -part of the Laplacian. Also, the differentiation of  $\Phi_t$  is to second order in  $\xi$  given by

$$\Phi_{\varphi,t}|_{\Sigma_1} = -\partial_t V f_{0,\varphi} = \frac{VR_0^3 \sin^2\theta}{h} g \eta_t$$

We may rearrange this in terms of  $\eta, \eta_\varphi$  etc. with coefficients depending on  $\theta$  of course. Thus,

$$A(\theta)\eta_t + B(\theta)\eta_\varphi + C(\theta)g(\theta)\eta\eta_\varphi + D(\theta)\eta_{\varphi\varphi} = 0 \quad [3.29]$$

where the coefficients are

$$A = \frac{VR_0^3 \sin^2\theta}{h} \quad [3.30a]$$

$$B = -\frac{\gamma}{\rho R_0} \frac{2g + \Delta_\theta g}{g} \quad [3.30b]$$

$$C = \frac{V^2 R_0^4 \sin^4\theta}{h^2} - \frac{8\gamma}{\rho R_0^2} \quad [3.30c]$$

$$D = -\frac{\gamma}{\rho R_0 \sin^2\theta} \quad [3.30d]$$

Equation [3.29] is the Korteweg-de Vries equation with coefficients depending parametrically on  $\theta$ . For any extra potential force  $\mathbf{f} = \nabla f$ , we may add a term  $f_\varphi$  to the coefficients  $B$  and  $C$ . Higher order terms can be included in the pressure, velocity

and potential. Ludu et al. [1996] suggests that this gives the modified KdV equation with an extra term  $\xi_\theta \xi^2 \cot\theta$  but as is evident from equation [3.24], there are several other terms that must be taken into account. On the other side, several of these terms can likely be neglected because of the high localization of the solution.

## B.2 HAMILTONIAN FORMULATION

The nonlinear liquid drop model has an infinite-dimensional Hamiltonian structure described by a nonlinear Hamiltonian equal to the sum of its potential and kinetic energy. A system such as this is tractable in the same way as a finite-dimensional Hamiltonian system, except the difficulties relating to the definition of the symplectic structure (labelling of particles, different phase spaces etc.). When these difficulties are overcome, the energy  $E$  of the system may be taken Hamiltonian, and all the laws of classical mechanics may be applied. In doing so, we will need to explicitly calculate the potential and kinetic energy and account for nonlinearities. In the following we will keep only one canonical coordinate,  $\eta$ , and leave  $\theta$  decoupled. In this case, the terms dependent on  $\theta$  will be absorbed into the coefficients in the energy, and the energy will be a functional of  $\eta$  only.

In calculating the potential energy, it consists of at least one term: the surface energy (equation [3.21]). We will keep only this term, but evidently this may also be accompanied by e.g. an electrostatic self-energy, self-gravity etc. It is quite easy to Taylor expand the square root of equation [3.21] in all its three arguments  $\xi_\theta, \xi_\varphi$  around 0, so that the surface energy is  $U_{\Sigma_1} = U_{\Sigma_1}^{(0)} + U_{\Sigma_1}^{(1)} + U_{\Sigma_1}^{(2)} + \dots$  where the superscript denotes the power of the Taylor expansion (and not the power of  $\xi$ ). We use a multi-variable Taylor expansion: a function  $f(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{x} \cdot \nabla)^n f(\mathbf{0})$  where  $f$  denotes the square root of the equation above,  $\mathbf{x}$  indicates the set of variables  $\xi, \xi_\theta, \xi_\varphi$  and  $\mathbf{0}$  denotes  $\mathbf{x} = 0$ . The zeroth order term  $f^{(0)}$  is constant, equal to one, and hence

$$U_{\Sigma_1}^{(0)} = \gamma R_0^2 \int_0^{2\pi} \int_0^\pi (1 + \xi) \sin\theta \, d\theta \, d\varphi$$

Consequently, in calculating the first order terms, we will require the derivatives  $\frac{\partial f}{\partial \xi}, \frac{\partial f}{\partial \xi_\theta}, \frac{\partial f}{\partial \xi_\varphi}$ . These are readily found to be

$$\frac{\partial f}{\partial \xi} = \frac{1 + \xi}{\sqrt{(1 + \xi)^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2\theta}} \quad [3.31a]$$

$$\frac{\partial f}{\partial \xi_\theta} = \frac{\xi_\theta}{\sqrt{(1 + \xi)^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2\theta}} \quad [3.31b]$$

$$\frac{\partial f}{\partial \xi_\varphi} = \frac{\xi_\varphi \csc^2\theta}{\sqrt{(1 + \xi)^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2\theta}} \quad [3.31c]$$

Evaluating the derivatives at  $\mathbf{0}$  leaves only the first term which is equal to 1, and hence the first order contribution

$$U_{\Sigma_1}^{(1)} = \gamma R_0^2 \int_0^{2\pi} \int_0^\pi (1 + \xi) \xi \sin\theta \, d\theta \, d\varphi \quad [3.32]$$

The second order expansion is

$$\begin{aligned} f^{(2)} &= \frac{1}{2} \left[ \xi \frac{\partial}{\partial \xi} + \xi_\theta \frac{\partial}{\partial \xi_\theta} + \xi_\varphi \frac{\partial}{\partial \xi_\varphi} \right]^2 f(\mathbf{0}) \\ &= \frac{1}{2} \left[ \xi^2 \frac{\partial^2 f}{\partial \xi^2} + \xi_\theta^2 \frac{\partial^2 f}{\partial \xi_\theta^2} + \xi_\varphi^2 \frac{\partial^2 f}{\partial \xi_\varphi^2} + 2\xi\xi_\theta \frac{\partial^2 f}{\partial \xi \partial \xi_\theta} \right. \\ &\quad \left. + 2\xi\xi_\varphi \frac{\partial^2 f}{\partial \xi \partial \xi_\varphi} + 2\xi_\theta\xi_\varphi \frac{\partial^2 f}{\partial \xi_\theta \partial \xi_\varphi} \right] \end{aligned} \quad [3.33]$$

Indeed, in calculating the derivatives and evaluating them at  $\mathbf{0}$ , only the second and third terms survive, and the derivatives themselves are 1. Thus,

$$f^{(2)} = \frac{1}{2} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta)$$

There are no less than 10 terms that need to be calculated in the third order contribution, but only two survive. We do not explicitly include the calculations, but the result is

$$f^{(3)} = -\frac{1}{2} \xi (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta)$$

However, these terms are not needed in the surface energy. We included them in equation [3.24] since we required one order higher, together with the fact that it was hard to predict which order to expand to.

Evidently we may conclude that  $U_{\Sigma_1}^{(2)} = \gamma R_0^2 \int (1 + \xi)^{\frac{1}{2}} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta) d\Omega$  and it also follows that the surface energy to third order in  $\xi$  is

$$U_{\Sigma_1} = \gamma R_0^2 \int_0^{2\pi} \int_0^\pi (1 + \xi) \left[ (1 + \xi) + \frac{1}{2} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta) \right] \times \sin \theta d\theta d\varphi + \mathcal{O}(\xi^4) \quad [3.34]$$

We shall drop the term proportional to  $\xi \xi_\varphi^2$  by the same argumentation as in the Eulerian derivation. Without loss of generalization we may choose a zero for the potential energy, equal to the surface energy of a non-deformed droplet,  $= \gamma R_0^2 \int \sin \theta d\theta d\varphi$ , which kills the constant term. Thus,

$$U_{\Sigma_1} = \gamma R_0^2 \int_0^{2\pi} \left[ 2S_{1,0}^{1,0} \eta + \left( S_{2,0}^{1,0} + \frac{1}{2} S_{0,2}^{1,0} \right) \eta^2 + \frac{1}{2} S_{1,2}^{1,0} \eta^3 + \frac{1}{2} S_{2,0}^{-1,0} \eta_\varphi^2 \right] d\varphi + \mathcal{O}(\eta^3, \eta_\varphi^3) \quad [3.35]$$

where  $\mathcal{O}(\eta^3, \eta_\varphi^3)$  represents terms of order  $\eta^3$  combined with higher-order derivatives. Also we have introduced the shorthand notation

$$S_{i,j}^{k,l} = \frac{1}{R_0^l} \int_0^\pi h^l g^i g_\theta^j \sin^k \theta d\theta \quad [3.36]$$

The kinetic energy of the system is given by

$$T = \frac{1}{2} \int_{\mathcal{V}} (\nabla \Phi)^2 \rho d\mathcal{V} \quad [3.37]$$

where  $\mathcal{V}$  denotes the volume. The above can be reformulated as  $|\nabla \Phi|^2 = \nabla \cdot (\Phi \nabla \Phi) - \Phi \nabla^2 \Phi$ , so that by use of the divergence theorem and the vanishing Laplacian of  $\Phi$ , the droplet energy is

$$T = \frac{\rho}{2} \oint_{\Sigma_1} (\Phi \nabla \Phi) \cdot d\mathbf{\Sigma}_1 \quad [3.38]$$

where  $d\mathbf{\Sigma}_1$  indicates a differential surface element on the surface  $\Sigma_1$ . Ludu et al. [1996], Ludu and Draayer [1998a,b] suggest that the oriented surface element  $d\mathbf{\Sigma}_1$  is given by  $d\mathbf{\Sigma}_1 = R_0^2 \sin \theta d\theta d\varphi \frac{(1, -\xi_\theta, -\xi_\varphi)}{\sqrt{1 + \xi_\theta^2 + \xi_\varphi^2}}$ , which is incorrect. Likely, Ludu et al. [1996] is fooled by the fact that in Cartesian coordinates, a

position vector can be written  $\mathbf{r} = [x, y, f(x, y)]$  and believes that this also holds in spherical polar coordinates with  $\mathbf{r} = [R_0(1 + \xi), \theta, \varphi]$ , and the surface vector as  $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \varphi}$ . However, this is incorrect since in fact  $\mathbf{r} = R_0(1 + \xi)\hat{\mathbf{r}}$  and has no other components. The oriented surface element is then found by the plane spanned by two differential tangential vectors in  $\Sigma_1$ . Thus,

$$d\mathbf{\Sigma}_1 = \frac{\partial \mathbf{r}}{\partial \theta} d\theta \times \frac{\partial \mathbf{r}}{\partial \varphi} d\varphi$$

with  $\mathbf{r} = R_0[1 + \xi]\hat{\mathbf{r}}$ . The explicit calculation gives  $\frac{\partial \mathbf{r}}{\partial \theta} = R_0 \xi_\theta \hat{\mathbf{r}} + R_0[1 + \xi]\hat{\boldsymbol{\theta}}$  and  $\frac{\partial \mathbf{r}}{\partial \varphi} = R_0 \xi_\varphi \hat{\mathbf{r}} + R_0[1 + \xi] \sin \theta \hat{\boldsymbol{\varphi}}$ . The surface element is thus

$$d\mathbf{\Sigma}_1 = R_0^2 \sin \theta d\theta d\varphi (1 + \xi) \left[ (1 + \xi)\hat{\mathbf{r}} - \xi_\theta \hat{\boldsymbol{\theta}} - \xi_\varphi \csc \theta \hat{\boldsymbol{\varphi}} \right] \quad [3.39]$$

The terms  $\Phi \nabla \Phi$  are

$$\Phi \nabla \Phi|_{\Sigma_1} = \Phi \Phi_r \hat{\mathbf{r}} + \frac{\Phi \Phi_\theta}{R_0(1 + \xi)} \hat{\boldsymbol{\theta}} + \frac{\Phi \Phi_\varphi}{R_0(1 + \xi) \sin \theta} \hat{\boldsymbol{\varphi}} \quad [3.40]$$

Introducing equations [3.39] and [3.40] into equation [3.38], with vanishing tangential velocity, gives

$$T = \frac{\rho R_0^2}{2} \oint_{\Sigma_1} \left[ \Phi \Phi_r (1 + \xi)^2 - \frac{\Phi \Phi_\varphi \xi_\varphi}{R_0 \sin^2 \theta} \right] \sin \theta d\theta d\varphi \quad [3.41]$$

$\Phi$ ,  $\Phi_r$  and  $\Phi_\varphi$  are recalled to be

$$\Phi = f_0 + \xi f_1$$

$$\Phi_r = \frac{f_1 + \xi f_2}{R_0}$$

$$\Phi_\varphi = f_{0,\varphi} + \xi f_{1,\varphi}$$

where

$$f_{0,\varphi} = \frac{V R_0^3 \sin^2 \theta}{h} \xi (1 + 2\xi) \quad [3.42a]$$

$$f_0 = \frac{V R_0^3 \sin^2 \theta}{h} \int_0^\varphi [\xi(\tilde{\varphi}) + 2\xi(\tilde{\varphi})] d\tilde{\varphi} \quad [3.42b]$$

$$f_1 = R_0^2 \xi_t \quad [3.42c]$$

where  $\tilde{\varphi}$  is a dummy variable of integration. The fundamental theorem of calculus ensures that equations [3.42a] and [3.42b] are consistent. By substituting equation [3.42] into equation [3.12] into equation [3.41] and keeping terms up to order three (which was done for the surface energy) gives

$$T = \frac{\rho R_0}{2} \oint_{\Sigma_1} [f_0 f_1 (1 + 2\xi) + 2\xi f_0 f_2 - f_0 f_{0,\varphi} \xi_\varphi \csc^2 \theta] \times \sin \theta d\theta d\varphi \quad [3.43]$$

where the term of order three proportional to  $\eta \eta_\varphi^2$  has been neglected due to the localization of the solution. The second and third term follow from

$$2\xi f_2 = \frac{R_0^3 \xi \xi_t}{h} = -\frac{V R_0^3 \xi_\varphi}{h} \xi = -\xi_\varphi f_{0,\varphi} \sin^{-2} \theta$$

and so only two terms of equation [3.43] need to be calculated. It is readily verified that to third order

$$\oint_{\Sigma_1} f_0 f_1 (1 + 2\xi) \sin \theta d\theta d\varphi = V R_0^5 \oint_{\Sigma_1} \left[ \xi_t \int_0^\varphi [\xi(\tilde{\varphi}) + 2\xi^2(\tilde{\varphi})] d\tilde{\varphi} + 2\xi \xi_t \int_0^\varphi \xi(\tilde{\varphi}) d\tilde{\varphi} \right] \frac{\sin^3 \theta}{h} d\theta d\varphi \quad [3.44]$$

and

$$\oint_{\Sigma_1} f_0 f_{0,\varphi} \xi_\varphi \sin^{-1} \theta d\theta d\varphi = V^2 R_0^6 \int_0^{2\pi} \xi \xi_\varphi \int_0^\varphi \xi(\tilde{\varphi}) d\tilde{\varphi} \frac{\sin^3 \theta}{h^2} d\theta d\varphi \quad [3.45]$$

By definition  $S_{i,j}^{k,l} = R_0^{-l} \int_0^\pi h^l g^i g_\theta^j \sin^k \theta d\theta$  and so together with the condition  $\partial_t = -V\partial_\varphi$  the kinetic energy functional can be written

$$T[\eta] = -\frac{\rho V^2 R_0^5}{2} \int_0^{2\pi} \left[ \left( S_{2,0}^{3,-1} \int_0^\varphi \eta(\tilde{\varphi}) d\tilde{\varphi} + 2S_{3,0}^{3,-1} \int_0^\varphi \eta^2(\tilde{\varphi}) d\tilde{\varphi} \right) \eta_\varphi + (2S_{3,0}^{3,-1} + 2S_{3,0}^{3,-2}) \eta \eta_\varphi \int_0^\varphi \eta(\tilde{\varphi}) d\tilde{\varphi} \right] d\varphi \quad [3.46]$$

Only a heuristic approach of the result derived above is given in Ludu et al. [1996], and their result has several obvious errors. For instance does Ludu et al. [1996] present the result  $T \sim \rho R_0^6 V^2 (S_{2,0}^{3,0} R_0 + S_{3,0}^{5,-2})$  which is seen to be wrong, on several accounts, from basic dimensional analysis. As such, equations [53-62] in Ludu et al. [1996] are, strictly speaking, wrong.

We now make an integration by parts of equation [3.46] and find

$$T[\eta] = \frac{\rho V^2 R_0^5}{2} \int_0^{2\pi} [S_{2,0}^{3,-1} \eta^2 + (4S_{3,0}^{3,-1} + 2S_{3,0}^{3,-2}) \eta^3] d\varphi \quad [3.47]$$

The extra term in equation [3.47] can be discarded by use of the boundary conditions of  $\eta$  (it is also of order  $\eta$  and so does not contribute to the dynamical equation). The entire energy, as a functional of  $\eta$  is

$$E[\eta] = \int_0^{2\pi} [C_1 \eta + C_2 \eta^2 + C_3 \eta^3 + C_4 \eta_\varphi^2] d\varphi \quad [3.48]$$

with coefficients depending parametrically on  $\theta$  in the following way:

$$C_1 = 2\gamma R_0^2 S_{1,0}^{1,0} \quad [3.49a]$$

$$C_2 = \frac{\gamma R_0^2}{2} [2S_{2,0}^{1,0} + S_{0,2}^{1,0}] + \frac{\rho V^2 R_0^6}{2} S_{2,0}^{3,-1} \quad [3.49b]$$

$$C_3 = \frac{\gamma R_0^2}{2} S_{1,2}^{1,0} + \rho V^2 R_0^5 (2S_{3,0}^{3,-1} + S_{3,-0}^{3,-2}) \quad [3.49c]$$

$$C_4 = \frac{\gamma R_0^2}{2} S_{2,0}^{-1,0} \quad [3.49d]$$

If  $E[\eta]$  is interpreted as a Hamiltonian, i.e.  $E[\eta] \rightarrow H[\eta]$ , we may make use of the abstract definition of the Hamiltonian as a subalgebra of the Lie algebra and the Poisson bracket as [Arnold, 1978] (a special form of the bracket in Part II).

$$\{F, G\} = \int_0^\pi \int_0^{2\pi} \left( \frac{\delta F}{\delta \xi} \frac{\partial}{\partial \varphi} \frac{\delta G}{\delta \xi} + \frac{\delta F}{\delta \xi} \frac{\partial}{\partial \theta} \frac{\delta G}{\delta \xi} \right) d\theta d\varphi \quad [3.50]$$

where  $F$  and  $G$  are generic functionals of the form  $\int \int f(\xi) d\theta d\varphi$ . For any functional  $F$  it also follows that

$$F_t = \{F, H\} \quad [3.51]$$

where  $H$  is the Hamiltonian functional. We now take  $F$  to be a functional of  $\eta$  only,  $F = \int_0^{2\pi} \eta(\varphi - Vt) d\varphi$  and the functional derivative is  $\frac{\delta F}{\delta \eta} = 1$ . The functional derivative of  $H[\eta]$  is defined as

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} H[\eta + \varepsilon \delta \eta]$$

but in practice it is easiest to obtain this from the first variation of  $H$ . It is verified that with  $H[\eta] =$

$\int_0^{2\pi} [C_1 \eta + C_2 \eta^2 + C_3 \eta^3 + C_4 \eta_\varphi^2] d\varphi$ , the corresponding first variation is (cf. equation [A.5] in the appendix)

$$\delta H = \int_0^{2\pi} [C_1 + 2C_2 \eta + 3C_3 \eta^2 - 2C_4 \eta_\varphi \varphi] \delta \eta d\varphi$$

and so

$$\frac{\delta H}{\delta \eta} = C_1 + 2C_2 \eta + 3C_3 \eta^2 - 2C_4 \eta_\varphi \varphi \quad [3.52]$$

Since  $H$  is implicitly dependent on  $\varphi$  only, with  $\theta$  acting as parameters of the problem, inserting equation [3.52] into equation [3.51] yields the final result

$$\int_0^{2\pi} \eta_t d\varphi = \int_0^{2\pi} (2C_2 \eta_\varphi + 6C_3 \eta \eta_\varphi - 2C_4 \eta_\varphi \varphi) d\varphi \quad [3.53]$$

Equation [3.53] leads to the KdV equation in the same manner as the Eulerian approach in the previous chapter. Therefore, the energy of the liquid drop is interpreted as an appropriate Hamiltonian of the KdV equation.

## C. THE EFFECTS OF FLATTENING & ROTATION

The effects of rotation on the nonlinear liquid drop model (NLDLM) are hard to quantify because of the onset of the Coriolis effect, equation [3.1]. The Bernoulli relation still holds, but not in the simple form of equation [3.4] but rather along streamlines that change with time. It is possible to a certain extent to examine the effects of rotation, by examining the effects of flattening, be it rotational or acoustical. On this account, we may assume a perturbation  $\xi = g\eta$  around the *flattened droplet* in the form

$$r(\theta, \varphi, t) = R_0 [1 - \varepsilon P_2(\cos \theta) + g(\theta) \eta(\varphi - Vt)] \quad [3.54]$$

where  $\varepsilon$  is a small parameter that describes the flattening, determined from the hydrostatic problem and  $\xi = g(\theta) \eta(\varphi - Vt)$  is a perturbation that leaves the volume of the drop unaltered. We may explicitly check that when  $\xi = 0$ ,

$$\mathcal{V} = \frac{R_0^3}{3} \int_\Omega [1 - \varepsilon P_2]^3 d\Omega \approx \frac{4\pi}{3} R_0^3 - R_0^3 \int_\Omega \varepsilon P_2 d\Omega \quad [3.55]$$

where we have approximated  $(1 - \varepsilon P_2)^3 \approx 1 - 3\varepsilon P_2$  since  $\varepsilon$  is small. By use of the orthogonality condition for the associated Legendre polynomials [Arfken and Weber, 2001]

$$\int_0^\pi P_p^m(\cos \theta) P_q^m(\cos \theta) \sin \theta d\theta = \frac{2}{2q+1} \frac{(q+m)!}{(q-m)!} \delta_{p,q} \quad [3.56]$$

we see that the integral proportional to  $\varepsilon$  is identical to zero, since the integrand may be considered as  $P_2 = P_2 P_0$ . Therefore, the volume is conserved under the expansion  $r = R_0 [1 - \varepsilon P_2]$  and the droplet is therefore flattened at the poles and bulging at the

equator. From Part I we know that  $\varepsilon$  can be related to the rotation rate by  $\varepsilon = \frac{2}{3}\Omega^*$ . We suppose, with some loss of generality, that the travelling wave solutions are situated along the equator  $\theta = \frac{\pi}{2}$ , as most experiments show [Annamalai et al., 1985, Lee et al., 1998, Shi and Apfel, 1995, Hill and Eaves, 2008]. When  $\varepsilon \rightarrow 0$ , this assumption can be made without loss of generality due to the spherical symmetry.

In deriving the dynamical equation for  $\eta$ , we shall have to redo the calculation of section B.1 and B.2 where we additionally account for effects to first order in  $\varepsilon$ . The velocity field is decomposed uniquely into one solenoidal and one conservative part in the form [Wu et al., 2005]

$$\mathbf{u} = \nabla \times \Psi + \nabla \Phi \quad [3.57]$$

By taking the curl and divergence of  $\mathbf{u}$ , the equations to be satisfied are

$$\Delta \Psi = -\omega \quad [3.58a]$$

$$\Delta \Phi = 0 \quad [3.58b]$$

where we have, since  $\mathbf{u}$  has three components and  $\Psi, \Phi$  has four components, imposed the gauge-condition  $\nabla(\nabla \cdot \Psi) = 0$ . If we assume that our system can be described by

$$\mathbf{u} = \Omega \times \mathbf{r} + \nabla \Phi \quad [3.59]$$

then this reproduces the Euler equations in a frame rotating with a constant angular velocity  $\Omega$ . Moreover, this assumes that the droplet has a constant "background vorticity" equal to the vorticity of a rigidly rotating liquid mass. Calculating the vorticity gives

$$\nabla \times \mathbf{u} = 2\Omega$$

and the only problem then relates to the development of the vorticity according to equation [3.2]. By assuming a constant background vorticity as experiments show [Hill and Eaves, 2008], the problem is well-defined: It represents a droplet rotating rigid-body-like with a rotation rate  $\Omega$  and with a superimposed potential wave. In the following we will keep the rotation implicit and treat the problem as if it was a flattened drop only. The

centrifugal and Coriolis terms enter through the Bernoulli relation only and incorporation of these into the final solution is trivial.

### C.1 EULERIAN FORMULATION

All the results of the scalar potential  $\Phi = \sum (r/R_0 - 1) f_n$  still apply, and hence

$$f_{n+2} = -\frac{2(n+1)(n+1)f_{n+1} + n(n+1)f_n + \Delta \Omega f_n}{(n+2)(n+1)} \quad [3.60]$$

which reduces the number of unknowns to only  $f_0$  and  $f_1$  where the rest follows from the recurrence relation of equation [3.60].

The kinematic surface condition that the fluid velocity of the surface equals the velocity of the surface itself must hold, i.e.

$$\dot{r}|_{\Sigma_1} = R_0 \left( \xi_t + \frac{\xi_\theta - \varepsilon P_{2,\theta}}{r^2} \Phi_\theta + \frac{\xi_\varphi}{r^2 \sin \theta} \Phi_\varphi \right) \Big|_{\Sigma_1} \quad [3.61]$$

By evaluating the velocities on the surface, we require to first order in  $\xi$  that

$$\Phi_r|_{\Sigma_1} = \frac{f_1 + 2(\xi - \varepsilon P_2)f_2}{R_0} + \mathcal{O}(\xi^2) \quad [3.62a]$$

$$\Phi_\theta|_{\Sigma_1} = f_{0,\theta} + (\xi - \varepsilon P_2)f_{1,\theta} + \mathcal{O}(\xi^2) \quad [3.62b]$$

$$\Phi_\varphi|_{\Sigma_1} = f_{0,\varphi} + (\xi - \varepsilon P_2)f_{1,\varphi} + \mathcal{O}(\xi^2) \quad [3.62c]$$

where  $\Phi_r = \dot{r}$ . Inserting the relations of equation [3.62] into equation [3.61] we find

$$f_1 + 2(\xi - \varepsilon P_2)f_2 = R_0^2 \left\{ \xi_t + \frac{\xi_\theta - \varepsilon P_{2,\theta}}{r^2} (f_{0,\theta} + [\xi - \varepsilon P_2]f_{1,\theta}) + \frac{\xi_\varphi}{r^2 \sin^2 \theta} (f_{0,\varphi} + [\xi - \varepsilon P_2]f_{1,\varphi}) \right\} \quad [3.63]$$

It is quite possible to arrange this in orders of  $\xi$  and  $\varepsilon$ . In the following we will neglect terms proportional to  $\varepsilon^2$ , but keep terms of order  $\xi, \xi^2, \varepsilon\xi$ . After some tedious, but feasible algebra

$$f_1 + 2[\xi - \varepsilon P_2]f_2 = R_0^2 [\xi_t + \xi_\theta f_{0,\theta} + \xi_\varphi f_{0,\varphi} + \xi\xi_\theta (f_{1,\theta} - 2f_{0,\theta}) + \xi\xi_\varphi (f_{1,\varphi} - 2f_{0,\varphi}) \csc^2 \theta - \varepsilon P_{2,\theta} f_{0,\theta} - \varepsilon P_2 \xi_\theta (f_{1,\theta} - 2f_{0,\theta}) - \varepsilon P_2 \xi_\varphi (f_{1,\varphi} - 2f_{0,\varphi})] \quad [3.64]$$

The second boundary condition gives  $\Phi_r|_{r=R_0 - \varepsilon P_2 - h} = 0$  which explicitly reads to first order in  $h$  that

$$f_1 = \frac{2(h + \varepsilon P_2)}{R_0} f_2 \quad [3.65]$$

We have kept the boundary conditions to first order, hence by taking the linear approximation for  $f_1$ ,

$$f_1 = R_0^2 \xi_t \quad [3.66]$$

introduces the equation

$$2[\xi - \varepsilon P_2]f_2 = \xi_\theta f_{0,\theta} + \xi\xi_\theta (f_{1,\theta} - 2f_{0,\theta}) + \xi_\varphi f_{0,\varphi} \csc^2 \theta + \xi\xi_\varphi (f_{1,\varphi} - 2f_{0,\varphi}) \csc^2 \theta - \varepsilon P_2 \xi_\theta (f_{1,\theta} - 2f_{0,\theta}) - \varepsilon P_2 \xi_\varphi (f_{1,\varphi} - 2f_{0,\varphi}) \csc^2 \theta \quad [3.67]$$

The tangential velocity components vanish. By introducing the

linear approximation for  $f_1$  into equation [3.65] we find

$$f_2 = \frac{R_0}{2(h + \varepsilon P_2)} \xi_t \quad [3.68]$$

and introducing this into equation [3.67] and solving for  $f_{0,\varphi}$  gives

$$\xi_\varphi (1 - 2\xi) f_{0,\varphi} = -\frac{R_0^3 (\xi - \varepsilon P_2) \sin^2 \theta}{h + \varepsilon P_2} \xi_t + \mathcal{O}(\xi^3) \quad [3.69]$$

The travelling wave solution introduces the condition  $\partial_t = -V\partial_\varphi$  which can be used to eliminate  $\xi_t$  and  $\xi_\varphi$ . Indeed, since  $(1 - 2\xi)^{-1} \approx 1 + 2\xi$ , the equation for the unknown  $f_{0,\varphi}$  is

$$f_{0,\varphi} = +\frac{VR_0^3 [\xi - \varepsilon P_2] \sin^2 \theta}{h + \varepsilon P_2} (1 + 2\xi) + \mathcal{O}(\xi^3) \quad [3.70]$$

Equation [3.70] describes in general the connection between the shape of the droplet and the velocity, a behaviour that is typical for nonlinear systems. Evidently, when  $\varepsilon \rightarrow 0$  the above equation reduces to equation [3.18] as it must.

In order to relate the pressure and the shape of the droplet we will need to expand the Young-Laplace equation. If we denote the surface by the radial vector  $\mathbf{r} = R_0 (1 - \varepsilon P_2 + \xi) \hat{\mathbf{r}}$ , the surface energy is given by

$$\begin{aligned} U_{\Sigma 1} &= \gamma \int_\Omega \left| \frac{\partial \mathbf{r}}{\partial \theta} d\theta \times \frac{\partial \mathbf{r}}{\partial \varphi} d\varphi \right| \\ &= \gamma \int \sqrt{(1 - \varepsilon P_2 + \xi)^2 + (\partial_\theta [\xi - \varepsilon P_2])^2 + \xi_\varphi^2 \csc^2 \theta} \\ &\quad \times (1 - \varepsilon P_2 + \xi) \sin \theta d\theta d\varphi \end{aligned} \quad [3.71]$$

It is quite possible to expand the root above in all its three arguments,  $\xi, \xi_\theta, \xi_\varphi$ . The explicit calculation is tedious and given in the next section. By keeping terms up to the third order, the surface energy is

$$\begin{aligned} U_{\Sigma 1} &= \gamma R_0^2 \int_\Omega \left[ 2\xi + \xi^2 + \frac{1+\xi}{2} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta) \right. \\ &\quad \left. - \varepsilon P_2 \left( 2\xi - \frac{7\xi^2}{2} + \xi\xi_\theta \right) \right] \sin \theta d\theta d\varphi \end{aligned} \quad [3.72]$$

For simplicity's sake, terms of order four has been dropped in the above equation. The variational derivative of the above is readily calculated, we find:

$$\frac{\partial[\dots]}{\partial \xi} = \left[ 2(1 + \xi) + \frac{(\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta)}{2} - \varepsilon P_2 (2 - 7\xi + \xi_\theta) \right] \sin \theta \quad [3.73]$$

and

$$\begin{aligned} \frac{\partial}{\partial \theta} \frac{\partial[\dots]}{\partial \xi_\theta} &= \left[ (1 + \xi) \Delta_\theta \xi + \xi_\theta^2 - \varepsilon P_2 \xi_\theta \right. \\ &\quad \left. - \varepsilon \xi \frac{1}{\sin \theta} \partial_\theta (P_2 \sin \theta) \right] \sin \theta \end{aligned} \quad [3.74]$$

The final term required is

$$\frac{\partial}{\partial \varphi} \frac{\partial[\dots]}{\partial \xi_\varphi} = [\Delta_\varphi \xi + \xi_\varphi^2 \csc^2 \theta] \sin \theta \quad [3.75]$$

Here,  $[\dots]$  denotes the integrand of equation [3.72] and  $\Delta_\theta$  and  $\Delta_\varphi$  denotes the  $\theta$  and  $\varphi$  parts of the Laplacian. Collecting all terms gives for the first variation

$$\begin{aligned} \delta U_{\Sigma 1} &= \gamma R_0 \int_\Omega \left[ 2(1 + \xi) - (1 + \xi) \Delta_\theta \xi - \frac{\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta}{2} \right. \\ &\quad \left. - \varepsilon (2 - 7\xi P_2 - \sin^{-1} \theta \partial_\theta [P_2 \sin \theta]) \right] \\ &\quad \times \delta \xi \sin \theta d\theta d\varphi \end{aligned} \quad [3.76]$$

The variational derivative is completed by reconstructing the surface element, which corresponds to multiplying and dividing by a factor  $R_0^2 [1 - 2(\xi - \varepsilon P_2)]$ . This concludes the variational derivative of the surface energy as

$$\delta U_{\Sigma 1} = \frac{\gamma}{R_0} \int_\Omega \frac{[2(1 + \xi) - (1 + \xi) \Delta_\theta \xi - \frac{1}{2} (\xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta) - \varepsilon (2 - 7\xi P_2 - \sin^{-1} \theta \partial_\theta [P_2 \sin \theta])]}{1 + 2(\xi - \varepsilon P_2)} d\xi d\Sigma_1 \quad [3.77]$$

Consequently, the pressure is equal to the integrand of the above, including the prefactor. By expanding the denominator and limiting our analysis to second order in  $\xi$  (or  $\varepsilon \xi$ ) and first order in its derivatives, the pressure is found to be

$$\Delta p \Big|_{\Sigma 1} = \frac{\gamma}{R_0} [-2\xi - 4\xi^2 - (1 + 2\varepsilon P_2) \Delta_\theta \xi + \varepsilon \xi (15P_2 + \sin^{-1} \theta \partial_\theta [\sin \theta P_2])] + const. \quad [3.78]$$

where the constants depend on  $\theta$  also. Note that the above equation deviates from equation [3.26] in that we have dropped all the higher-order terms. Of course, when  $\varepsilon \rightarrow 0$  the model represents exactly the nonlinear liquid drop model [Ludu et al., 1996], and a further linearization gives the liquid drop model, as it must.

It may be recalled that

$$\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{1}{\rho} \Delta p - f = const. \quad [3.79]$$

must hold at the surface.  $f$  denotes the conservative forces acting on the system. It is to a certain extent possible to construct such an expression also for a rotational system. If we assume a constant background vorticity  $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$  then the velocity field can be written  $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r} + \nabla \Phi$  and the Bernoulli relation takes

the form

$$\int_1^2 \left[ \nabla \left( \Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{P}{\rho} - f \right) + (\boldsymbol{\Omega} \times \nabla \Phi) \right] \cdot d\mathbf{s} = const. \quad [3.80]$$

where the points 1 and 2 lie on the *same streamline* and we have formed the reduced pressure  $P = p + \frac{1}{2} (\boldsymbol{\Omega} \times \mathbf{r})^2$ . The Coriolis term does not contribute since  $\nabla \Phi$  is parallel with  $d\mathbf{s}$ . From a mechanical point of view this is obvious since the Bernoulli equation is a force-integrated version of the momentum conservation law. The Coriolis force represents only the observer's lack of awareness to realize that the frame of reference is rotating, in no way does it contribute to the energy. Hence we may replace the above with

$$\Phi_t + \frac{1}{2} |\nabla \Phi|^2 + \frac{P}{\rho} - f = const. \quad [3.81]$$

as long as we restrict our analysis to a certain streamline. The only contribution to the velocity components is the azimuthal component, given by equation [3.70]. Introducing this into equa-

tion [3.79] and keeping terms up to second order in the deformation gives

$$\Phi_t \Big|_{\Sigma_1} + \frac{1}{2} \frac{V^2 R_0^4 \sin^2 \theta}{(h + \varepsilon P_2 \theta)} (\xi - \varepsilon P_2)^2 + \frac{\gamma}{R_0} [-2\xi - 4\xi^2 - (1 + 2\varepsilon P_2) \Delta_\Omega \xi + \varepsilon \xi (15P_2 + \sin^{-1} \theta \partial_\theta [\sin \theta P_2])] + \text{const.} = 0 \quad [3.82]$$

By differentiating with respect to  $\varphi$  we find that by use of

$$\partial_\varphi \left( \Phi_t \Big|_{\Sigma_1} \right) = \frac{V R_0^3 \sin^2 \theta}{(h + \varepsilon P_2)^2} (1 - 2\varepsilon P_2) g \eta_t, \quad [3.83]$$

the free surface elevation equation is

$$\begin{aligned} \frac{V R_0^3 \sin^2 \theta}{(h + \varepsilon P_2)^2} (1 - 2\varepsilon P_2) g \eta_t + \left[ \frac{V^2 R_0^4 \sin^2 \theta}{(h + \varepsilon P_2)^2} \varepsilon P_2 g + \frac{\gamma}{\rho R_0} (2g + (1 + \varepsilon P_2) \Delta_\theta g - \varepsilon (15P_2 + \sin^{-1} \theta \partial_\theta [P_2 \sin \theta]) g) \right] \eta_\varphi \\ + \left[ \frac{8\gamma}{\rho R_0} g^2 - \frac{V^2 R_0^4 \sin^2 \theta}{(h + \varepsilon P_2)^2} g^2 \right] \eta \eta_\varphi + \frac{\gamma(1 + 2\varepsilon P_2)}{\rho R_0 \sin^2 \theta} g \eta_{\varphi\varphi} = 0 \end{aligned} \quad [3.84]$$

This can be simplified to

$$A \eta_t + B \eta_\varphi + C \eta \eta_\varphi + D \eta_{\varphi\varphi} = 0 \quad [3.85]$$

known as the KdV equation. It represents the same physics as the non-flattened case, but with a change of parameters. Traveling wave solutions such as seen in the experiments by Hill and Eaves [2008] are allowed only if the coefficients  $C/(B - AV)$  and  $D/(B - AV)$  do not depend on  $\theta$ . This gives two differential equations for  $g$  and  $h$  that can be solved together with the boundary conditions  $g = h = 0$  for  $\theta = 0$  or  $\pi$ . It is rather easy to incorporate the centrifugal force, but we do not explicitly carry out this term except note that it introduces a term  $\Omega^2 \sin^2 \theta \xi \xi_\varphi$  and so can be incorporated into one of the coefficients of the KdV equation. Explicitly, the coefficients are

$$A(\theta) = \frac{V R_0^3 \sin^2 \theta}{(h + \varepsilon P_2)^2} (1 - 2\varepsilon P_2) g \quad [3.86a]$$

$$\begin{aligned} B(\theta) = - \frac{V^2 R_0^4 \sin^2 \theta}{(h + \varepsilon P_2)^2} \varepsilon P_2 g - \frac{\gamma}{\rho R_0} \frac{2g + (1 + \varepsilon P_2) \Delta_\theta g}{g} \\ + \frac{\gamma \varepsilon}{\rho R_0} (15P_2 + \sin^{-1} \theta \partial_\theta [P_2 \sin \theta]) \end{aligned} \quad [3.86b]$$

$$C(\theta) = \frac{V^2 R_0^4 \sin^2 \theta}{(h + \varepsilon P_2)^2} g - \frac{8\gamma}{\rho R_0} g \quad [3.86c]$$

$$D(\theta) = - \frac{\gamma(1 + 2\varepsilon P_2)}{\rho R_0 \sin^2 \theta} \quad [3.86d]$$

$$[3.86e]$$

It is evidently seen that when  $\varepsilon$  tends to zero, all the results of the nonlinear liquid drop model are recovered as they must.

## C.2 HAMILTONIAN FORMULATION

If we follow the formalism of section B.2, we may again start to derive the energy of the droplet, but now with an additional deformation. Clearly,

$$\begin{aligned} U_{\Sigma_1} &= \gamma \int \sqrt{r^2 + \left( \frac{\partial r}{\partial \theta} \right)^2 + \left( \frac{\partial r}{\partial \varphi} \csc^2 \theta \right)^2} r \sin \theta \, d\theta \, d\varphi \\ &= \gamma R_0^2 \int \sqrt{(1 - \varepsilon P_2 + \xi)^2 + [\partial_\theta (\xi - \varepsilon P_2)]^2 + \xi_\varphi^2 \csc^2 \theta} \\ &\quad \times (1 - \varepsilon P_2 + \xi) \sin \theta \, d\theta \, d\varphi \end{aligned} \quad [3.87]$$

Expanding the surface energy is tardy, but feasible, because of the extra term  $\varepsilon P_2$ . It is possible to expand the square root in the equation above in all its parameter  $\xi, \xi_\theta, \xi_\varphi$  around the hydrostatic condition. Such an expansion was performed for the non-flattened case. By denoting the square root above  $f = \sqrt{q(\xi, \xi_\theta, \xi_\varphi)}$ , we simply Taylor expand

$$\begin{aligned} f(\xi, \xi_\theta, \xi_\varphi) &= \sum_{n=0}^{\infty} \frac{1}{n!} [\xi^* \partial_\xi + \xi_\theta^* \partial_{\xi_\theta} + \xi_\varphi^* \partial_{\xi_\varphi}]^n f(0, 0, 0) \\ &\equiv \sum_{n=0}^{\infty} f^{(n)} \end{aligned} \quad [3.88]$$

where the asterisk only reminds us that these terms are not to be differentiated. The first order term is equal to  $f(0, 0, 0)$  and hence

$$f^{(0)} = \sqrt{(1 - \varepsilon P_2)^2 + \varepsilon^2 [\partial_\theta P_2]^2} \quad [3.89]$$

For the first order terms, we will need to calculate the derivatives  $\partial_\xi f, \partial_{\xi_\theta} f, \partial_{\xi_\varphi} f$ . These are readily found to be

$$\partial_\xi f = \frac{(1 - \varepsilon P_2 + \xi)}{\sqrt{(1 - \varepsilon P_2)^2 + \varepsilon^2 [\partial_\theta P_2]^2}} \quad [3.90a]$$

$$\partial_{\xi_\theta} f = \frac{(\xi_\theta - \varepsilon P_2)}{\sqrt{(1 - \varepsilon P_2)^2 + \varepsilon^2 [\partial_\theta P_2]^2}} \quad [3.90b]$$

$$\partial_{\xi_\varphi} f = \frac{\xi_\varphi \csc^2 \theta}{\sqrt{(1 - \varepsilon P_2)^2 + \varepsilon^2 [\partial_\theta P_2]^2}} \quad [3.90c]$$

By evaluating the derivatives at 0, we find

$$f^{(1)} = \frac{\xi(1 - \varepsilon P_2)}{q_0^{1/2}} - \frac{\xi_\theta \varepsilon P_2}{q_0^{1/2}} \quad [3.91]$$

where we have, only to avoid large and complicated expression, denoted  $q_0 = q(0, 0, 0) = (1 - \varepsilon P_2)^2 + \varepsilon^2 [\partial_\theta P_2]^2$ .



The second order terms requires the explicit calculation of all the derivatives of equation [3.33]. We find

$$\frac{\partial^2 f}{\partial \xi^2} = \frac{1}{q^{1/2}} - \frac{(1 - \varepsilon P_2 + \xi)^2}{q^{3/2}} \quad [3.92a]$$

$$\frac{\partial^2 f}{\partial \xi_\theta^2} = \frac{1}{q^{1/2}} - \frac{(\xi_\theta - \varepsilon P_2)^2}{q^{3/2}} \quad [3.92b]$$

$$\frac{\partial^2 f}{\partial \xi_\varphi^2} = \frac{\csc^2 \theta}{q^{1/2}} - \frac{\xi_\varphi^2 \csc^4 \theta}{q^{3/2}} \quad [3.92c]$$

and the cross terms

$$\frac{\partial^2 f}{\partial \xi \partial \xi_\theta} = -\frac{(\xi_\theta - \varepsilon P_2)(1 - \varepsilon P_2 + \xi)}{q^{3/2}} \quad [3.93a]$$

$$\frac{\partial^2 f}{\partial \xi \partial \xi_\varphi} = -\frac{\xi_\varphi \csc^2 \theta (1 - \varepsilon P_2 + \xi)}{q^{3/2}} \quad [3.93b]$$

$$\frac{\partial^2 f}{\partial \xi_\theta \partial \xi_\varphi} = -\frac{\xi_\varphi \csc^2 \theta (\xi_\theta - \varepsilon P_2)}{q^{3/2}} \quad [3.93c]$$

Evaluation of the derivatives at  $\mathbf{0}$ , and including the appropriate coefficients (equation [3.33]) gives

$$f^{(2)} = \frac{1}{2q^{1/2}} [\xi^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta] - \frac{1}{2q^{3/2}} [\xi(1 - \varepsilon P_2) + \xi_\theta \varepsilon P_2]^2 \quad [3.94]$$

The surface energy is then

$$U_{\Sigma_1} = \gamma R_0^2 \int (1 - \varepsilon P_2 + \xi) [f^{(0)} + f^{(1)} + f^{(2)}] \sin \theta \, d\theta \, d\varphi \quad [3.95]$$

Since we are interested in any motion around the unperturbed hydrostatic state (assumed stable, of course) we may subtract the energy of this state and hence let  $U_{\Sigma_1} \rightarrow U_{\Sigma_1} - \gamma R_0^2 \int (1 - \varepsilon P_2) \sqrt{(1 - \varepsilon P_2)^2 + \varepsilon^2 [\partial_\theta P_2]^2} \sin \theta \, d\theta \, d\varphi$ . Such a subtraction gives

$$\begin{aligned} U_{\Sigma_1} \rightarrow & \gamma R_0^2 \int [(1 - \varepsilon P_2) + \xi] \left\{ \frac{1}{2q_0^{1/2}} [2\xi(1 - \varepsilon P_2) - 2\xi_\theta \varepsilon P_2 \right. \\ & \left. + (\xi^2 + \xi_\theta^2 + \xi_\varphi^2 \csc^2 \theta)] - \frac{1}{2q_0^{3/2}} [\xi(1 - \varepsilon P_2) + \xi_\theta \varepsilon P_2]^2 \right\} \\ & \times \sin \theta \, d\theta \, d\varphi \\ & + \gamma R_0^2 \int \xi q_0^{1/2} \sin \theta \, d\theta \, d\varphi \end{aligned} \quad [3.96]$$

It is possible to absorb the  $\theta$ -dependence into coefficients of the integrals, sorted in powers of  $\eta$ . After some tedious algebra one finds

$$\begin{aligned} U_{\Sigma_1} = & \gamma R_0^2 \int_0^{2\pi} \left\{ \int_0^\pi \left[ \frac{(1 - \varepsilon P_2)^2}{q_0^{1/2}} + q_0^{1/2} g \right] \sin \theta \, d\theta \, \eta + \int_0^\pi \left[ \left( \frac{3(1 - \varepsilon P_2)}{2q_0^{1/2}} - \frac{(1 - \varepsilon P_2)^3}{2q_0^{3/2}} - \frac{\varepsilon^2 P_2^2 (1 - \varepsilon P_2)}{2q_0^{3/2}} \right) g^2 \right. \right. \\ & \left. \left. + \left( \frac{(1 - \varepsilon P_2)}{2q_0^{1/2}} - \frac{\varepsilon^2 P_2^2 (1 - \varepsilon P_2)}{2q_0^{3/2}} \right) g_\theta^2 - \frac{\varepsilon P_2 (1 - \varepsilon P_2)^2}{q^{3/2}} g g_\theta \right] \sin \theta \, d\theta \, \eta^2 + \int_0^\pi \left[ \frac{g^3 + g g_\theta^2}{2q_0^{1/2}} - \frac{(1 - \varepsilon P_2)^2}{2q_0^{3/2}} g^3 \right. \right. \\ & \left. \left. - \frac{\varepsilon^2 P_2^2}{2q_0^{3/2}} g_\theta^2 g - \frac{\varepsilon P_2 (1 - \varepsilon P_2)}{q_0^{3/2}} g^2 g_\theta \right] \sin \theta \, d\theta \, \eta^3 + \int_0^\pi \frac{1 - \varepsilon P_2}{2q_0^{1/2}} g^2 \sin^{-1} \theta \, d\theta \, \eta_\varphi^2 + \int_0^\pi \frac{(1 - \varepsilon P_2)^2}{2q_0^{1/2}} g^3 \sin \theta \, d\theta \, \eta \eta_\varphi^2 \right\} d\varphi \end{aligned} \quad [3.97]$$

which is recast into a more convenient form

$$U_{\Sigma_1} = \int_0^{2\pi} [C_1 \eta + C_2 \eta^2 + C_3 \eta^3 + C_4 \eta_\varphi^2] d\varphi \quad [3.98]$$

where the coefficients  $C$  carry the interpretation of equation [3.97]. The coefficients are quite horrid, but are simplified by recalling that  $\varepsilon$  is small, and making series expansions of the coefficients to first order in  $\varepsilon$ . Thus,

$$C_1 = \int_0^\pi 2g [1 - \varepsilon P_2] \sin \theta \, d\theta \quad [3.99a]$$

$$C_2 = \int_0^\pi \left[ \left( 1 + \frac{7\varepsilon P_2}{2} \right) g^2 + \frac{1}{2} g_\theta^2 - \varepsilon P_2 g g_\theta \right] \sin \theta \, d\theta \quad [3.99b]$$

$$C_3 = \int_0^\pi \left[ \frac{2\varepsilon P_2}{3} g^3 + \frac{1}{2} g g_\theta^2 (1 + \varepsilon P_2) - \varepsilon P_2 g^2 g_\theta \right] \times \sin \theta \, d\theta \quad [3.99c]$$

$$C_4 = \int_0^\pi \frac{1}{2} g^2 \sin^{-1} \theta \, d\theta \quad [3.99d]$$

We observe that when  $\varepsilon \rightarrow 0$ ,  $C_1 \rightarrow 2S_{1,0}^{1,0}$ ,  $C_2 \rightarrow (\frac{1}{2}S_{0,2}^{1,0} + S_{2,0}^{1,0})$ ,  $C_3 \rightarrow \frac{1}{2}S_{1,2}^{1,0}$  and  $C_4 \rightarrow \frac{1}{2}S_{2,0}^{-1,0}$  and thus correspond to the non-flattened case, equation [3.35]. We have dropped the term proportional to  $\eta \eta_\varphi^2$  since it belongs to the fourth order, as discussed

earlier. Evidently, since  $\varepsilon$  is small, this can be further simplified on several accounts by restricting the higher-order terms. E.g. we may write  $C_1 \rightarrow 2S_{1,0}^{1,0} - \varepsilon \int_0^\pi g P_2 \sin \theta \, d\theta$  etc., and drop terms of third or fourth order, depending on the value of  $\varepsilon$ . We do not explicitly carry out this reduction since the coefficients  $C_1, C_2, C_3$  and  $C_4$  only enters the model as parameters of the problem.

The kinetic energy is given by

$$T = \frac{1}{2} \int_V \rho |\nabla \Phi|^2 dV = \frac{\rho}{2} \oint_{\Sigma_1} \Phi \nabla \Phi \cdot d\mathbf{\Sigma}_1 \quad [3.100]$$

where  $d\mathbf{\Sigma}_1$  is the oriented surface element of  $\Sigma_1$ , given by

$$\begin{aligned} d\mathbf{\Sigma}_1 &= \frac{\partial \mathbf{r}}{\partial \theta} d\theta \times \frac{\partial \mathbf{r}}{\partial \varphi} d\varphi \\ &= \left( [1 - \varepsilon P_2 + \xi] \hat{\mathbf{r}} - \partial_\theta [\xi - \varepsilon P_2] \hat{\boldsymbol{\theta}} - \xi_\theta \csc \theta \hat{\boldsymbol{\varphi}} \right) \\ &\quad \times [1 - \varepsilon P_2 + \xi] R_0^2 \sin \theta \, d\theta \, d\varphi \end{aligned} \quad [3.101]$$

The fact that the kinetic energy can be written as the surface integral of  $\Phi \nabla \Phi$  is a result of the chain rule, the divergence theorem and the vanishing Laplacian of  $\Phi$ .  $\Phi \nabla \Phi$  is given by

$$\Phi \nabla \Phi = \Phi \Phi_r \hat{\mathbf{r}} + \frac{\Phi \Phi_\theta}{r} \hat{\boldsymbol{\theta}} + \frac{\Phi \Phi_\varphi}{r \sin \theta} \hat{\boldsymbol{\varphi}} \quad [3.102]$$

Introducing equation [3.101] and equation [3.102] into equation [3.100] we find

$$T = \frac{\rho R_0^2}{2} \oint [\Phi \Phi_r (1 - \varepsilon P_2 + \xi)^2 - \Phi \Phi_\theta \partial_\theta (\xi - \varepsilon P_2) - \Phi \Phi_\varphi (1 - \varepsilon P_2 + \xi) \xi_\varphi \csc \theta] \sin \theta d\theta d\varphi \quad [3.103]$$

Again, with vanishing tangential velocity we find

$$T = \frac{\rho R_0^2}{2} \oint [\Phi \Phi_r (1 - \varepsilon P_2 + \xi)^2 - \Phi \Phi_\varphi (1 - \varepsilon P_2 + \xi) \xi_\varphi \csc \theta] \times \sin \theta d\theta d\varphi \quad [3.104]$$

where  $\Phi, \Phi_r$  and  $\Phi_\varphi$  are to be evaluated at the surface. To third order in  $\xi$  these are

$$\Phi \Phi_r \Big|_{\Sigma_1} (1 - \varepsilon P_2 + \xi)^2 \approx R_0^{-1} [f_0 f_1 (1 + 2\xi + 2\varepsilon P_2) + 2\xi f_0 f_2] \quad [3.105a]$$

$$\Phi \Phi_\varphi \Big|_{\Sigma_1} \xi_\varphi \approx f_0 f_{0,\varphi} \xi_\varphi \quad [3.105b]$$

By inserting these into equation [3.104] we confirm that

$$\begin{aligned} \oint f_0 f_1 (1 + 2\xi + 2\varepsilon P_2) \sin \theta d\theta d\varphi = & -V R_0^5 \int_0^{2\pi} \eta_\varphi \left[ \int_0^\pi \frac{g^3}{h + \varepsilon P_2} \sin^3 \theta d\theta \int_0^\varphi \eta(\tilde{\varphi}) d\tilde{\varphi} + 2 \int_0^\pi \frac{g^3}{h + \varepsilon P_2} \sin^3 \theta d\theta \int_0^\varphi \eta^2(\tilde{\varphi}) d\tilde{\varphi} \right] \\ & - V R_0^5 \int_0^{2\pi} 2\eta\eta_\varphi \int_0^\pi \frac{g^2}{h + \varepsilon P_2} \sin^3 \theta d\theta \int_0^\varphi \eta(\tilde{\varphi}) d\tilde{\varphi} \end{aligned} \quad [3.106]$$

and

$$\oint f_0 f_{0,\varphi} \xi_\varphi \sin^{-1} \theta d\theta d\varphi = V^2 R_0^6 \int_0^{2\pi} \left[ \eta\eta_\varphi \int_0^\pi \frac{g^3 \sin^3 \theta}{(h + \varepsilon P_2)^2} d\theta \int_0^\varphi \eta(\tilde{\varphi}) d\tilde{\varphi} - \eta_\varphi \int_0^\pi \frac{\varepsilon g^2 P_2}{(h + \varepsilon P_2)^2} \sin^3 \theta d\theta \right] d\varphi \quad [3.107]$$

If we define

$$C_{i,j}^{k,l} = R_0^k \int_0^\pi \frac{g^i \sin^k P_2^l}{(h + \varepsilon P_2)^k} d\theta \quad [3.108]$$

then the kinetic energy functional may be written (after a partial integration)

$$T[\eta] = \frac{\rho V^2 R_0^5}{2} \int_0^{2\pi} [(C_{2,3}^{1,0} - 2\varepsilon C_{2,3}^{2,1}) \eta^2 + (C_{3,3}^{1,0} + C_{2,3}^{1,0} + C_{3,3}^{2,0}) \eta^2] d\varphi \quad [3.109]$$

Consequently, the entire energy functional can be constructed by adding together equations [3.98] and [3.109] and so

$$E[\eta] = \int_0^{2\pi} [\mathcal{A}\eta + \mathcal{B}\eta^2 + \mathcal{C}\eta^3 + \mathcal{D}\eta_\varphi^2] d\varphi \quad [3.110]$$

where  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  follows from equations [3.98] and [3.109]. When the energy is taken to be the Hamiltonian of the system, we may construct a functional  $F = \int_0^{2\pi} \eta(\varphi - Vt) d\varphi$  as we have done previously, to show that the corresponding dynamical equation gives

$$\int_0^{2\pi} \eta_t d\varphi = \int_0^{2\pi} [2\mathcal{B}\eta_\varphi + 6\mathcal{C}\eta\eta_\varphi - 2\mathcal{D}\eta_\varphi\eta_\varphi] d\varphi \quad [3.111]$$

which is the KdV equation.

#### D. PHASE SPACE REPRESENTATION

The KdV equation in it's general form (equation [3.29]) is

$$A\eta_t + B\eta_\varphi + Cg\eta\eta_\varphi + D\eta_\varphi\eta_\varphi = 0 \quad [3.112]$$

The reader may observe the difference between the above equation and the KdV equation discussed in Part II in that there is an extra term  $\eta_\varphi$ , this term is removed by changing the variables  $z = \varphi - Vt$ .

A qualitative analysis of KdV equation is obtainable by hierarchical or topological arguments [G. L. Lamb, 1980, Kuksin, 2000], or by using the phase-space portrait of the solutions [Strogatz, 2000, Sagdeev et al., 1988, Infeld and Rowlands, 2000].

The latter gives way to a simple picture of the solutions and, in addition, characterize their periodic and bounded character. In using the relation  $\partial_t = -V\partial_\varphi$ , we may integrate the KdV equation once to obtain  $\eta_{tt} = \frac{(AV-B)V^2}{D}\eta - \frac{CgV^2}{2D}\eta^2 + a$ , where  $a$  is a constant of integration. This is the Newton equation of motion for a particle described by a one-dimensional coordinate  $\eta(t)$ , subject to a force equal to the right hand side of the equation above. By using  $\eta_t$  as an integrating factor, a second quadrature is obtainable and the equation may be written  $\frac{1}{2}\eta_t^2 = \frac{(B-AV)V^2}{2D}\eta^2 - \frac{CgV^2}{6D}\eta^3 + a\eta + b$  where  $b$  is another constant of integration. Therefore, we obtain the potential associated with the force equal to the right-hand side of the relation above. This is "energy conservation" (constant  $b = E$ ) of a particle with kinetic energy  $\frac{1}{2}\eta_t^2$  and a parametrized potential energy  $U(\eta, V)$ , depending on  $V$  as a parameter. The trajectories in the phase-space can be classified through the constant energy  $E$ .  $U(\eta, V)$  is

$$U(\eta, V) = -\frac{CgV^2}{6D}\eta^3 + \frac{(AV-B)V^2}{2D}\eta^2 + a\eta \quad [3.113]$$

It is more convenient to work with fewer coefficient, we put  $t' = \sqrt{\frac{CgV^2}{2D}}t$  so that the force-relation  $\eta_{tt} = \dots$  is written

$$\begin{aligned} \eta_{t't'} = & -\eta^2 + 2\frac{AV-B}{D}\eta - a \\ = & -(\eta - \eta_1)(\eta - \eta_2) \end{aligned} \quad [3.114]$$

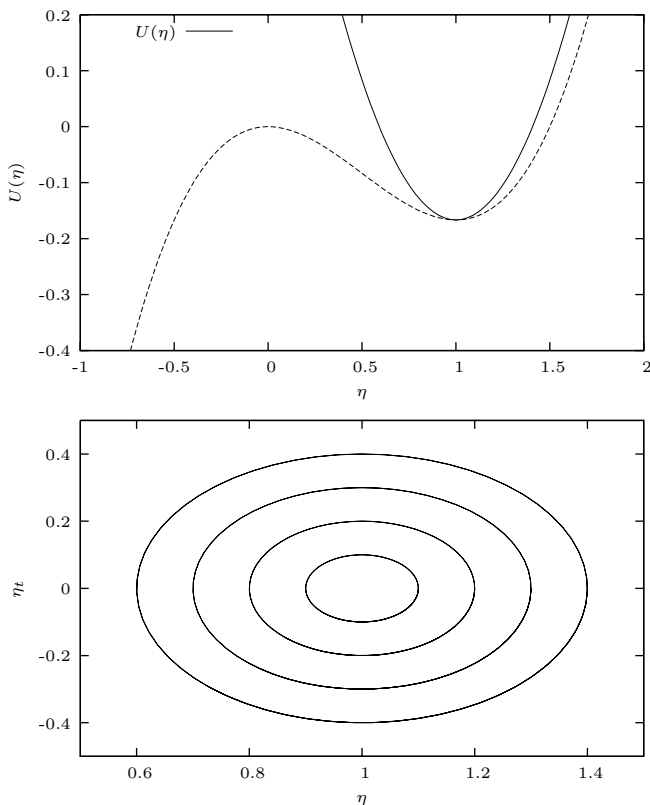
where  $\eta_1$  and  $\eta_2$  mark the zeros of the function  $\eta^2 - \frac{2(AV-B)}{D}\eta + a$ . We have fixed  $\eta_2 > \eta_1$ , and it is also clear that  $\eta_1 + \eta_2 = \frac{AV-B}{D}$

and  $\eta_1\eta_2 = a$  which follows from the factorization. This equation may be analyzed as follows: denote a variable  $u$  as  $\frac{\partial\eta}{\partial t'}$  which implies that  $u_{t'}$  is given by the right-hand side of the relation above. This is a set of two first-order differential equations and may be integrated numerically, we shall use a 4th order Runge-Kutta scheme. The equations of motion are

$$\frac{\partial\eta}{\partial t'} = u \quad [3.115a]$$

$$\frac{\partial u}{\partial t'} = -(\eta - \eta_1)(\eta - \eta_2) \quad [3.115b]$$

There are two fixed points  $\eta_1$  and  $\eta_2$  where the temporal derivatives vanish, but they are not necessarily linearly stable. Assume a small perturbation  $\mu = \eta - \eta_1$  around the fixed point  $\eta_1$ . The equation of motion for  $\mu$  then gives  $\mu_{t'} = u$  so that  $\mu_{t't'} = -\mu^2 + \mu(\eta_2 - \eta_1)$ . As long as  $\mu \ll (\eta_2 - \eta_1)$  it follows that the fixed point  $\eta_1$  has temporally unstable (and stable, depending on the perturbation) solutions of the form  $\mu = \mu_{01}e^{\sqrt{\eta_2 - \eta_1}t'} + \mu_{02}e^{-\sqrt{\eta_2 - \eta_1}t'}$ , and is therefore a saddle-point.  $\mu_{01}$  and  $\mu_{02}$  are arbitrary constants. In Fig. 3.2, we have shown in the top the effective quadratic potential when the model is linearized and the corresponding phase plane. The oscillatory solutions are described in terms of harmonic functions and corresponds to the traditional linear liquid drop model. It is easy to see the harmonic solutions in Fig. 3.2 as concentric circles in the phase map.



**Fig. 3.2:** The linear approximation of the KdV equation gives a quadratic potential and harmonic solutions. The original potential is shown with broken lines

On the other hand, performing a similar linearization around the fixed point  $\eta_2$  gives that  $\mu_{t't'} = -\mu(\eta_2 - \eta_1)$  and therefore  $\mu$  obeys normal mode oscillations of the form  $\mu = \mu_{01}e^{i\sqrt{\eta_2 - \eta_1}t'} +$

$\mu_{02}e^{-i\sqrt{\eta_2 - \eta_1}t'}$ . It follows directly that  $u = \eta_t$  has harmonic solutions. The same result is seen from the potential, by the same substitution  $t'$  it is given by  $U(\eta, V) = -\eta(\eta - \eta_1)(\eta - \eta_2)$ . Whenever  $\eta$  is close to  $\eta_2$ , the effective potential is no longer cubic, but quadratic since the term  $\eta - \eta_2$  is small. Thus there is a point  $\eta \approx \eta_2 = \frac{2(AV-B)}{D}$  which allows periodic harmonic solutions, corresponding to the linear model thoroughly discussed in Part I where the governing equation is the Helmholtz equation.

Whenever  $\eta_1 \rightarrow \eta_2$  or  $\mu$  becomes comparable to  $(\eta_2 - \eta_1)$ , the limit  $\mu \ll (\eta_2 - \eta_1)$  is no longer valid and the term  $\mu^2$  becomes dominating, and nonlinear effects must be taken into account. In this case we may no longer analyze equations [3.115] by use of easy algebraic relations. Instead, we numerically integrate this using a 4th order Runge-Kutta scheme and thereby construct the phase map  $\eta, \eta_t$ , similar to Fig. 3.2. We use the vector version of the RK numerical scheme so notationwise we shall call the two equations above for

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

and we interpret  $x_1$  as  $\eta$  and  $x_2$  as  $u$ . More compactly in vector notation we write this as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

We will use a 4th order RK scheme so [Press et al., 1997]

$$\mathbf{x}_{n+1} = \frac{1}{6}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \quad [3.117]$$

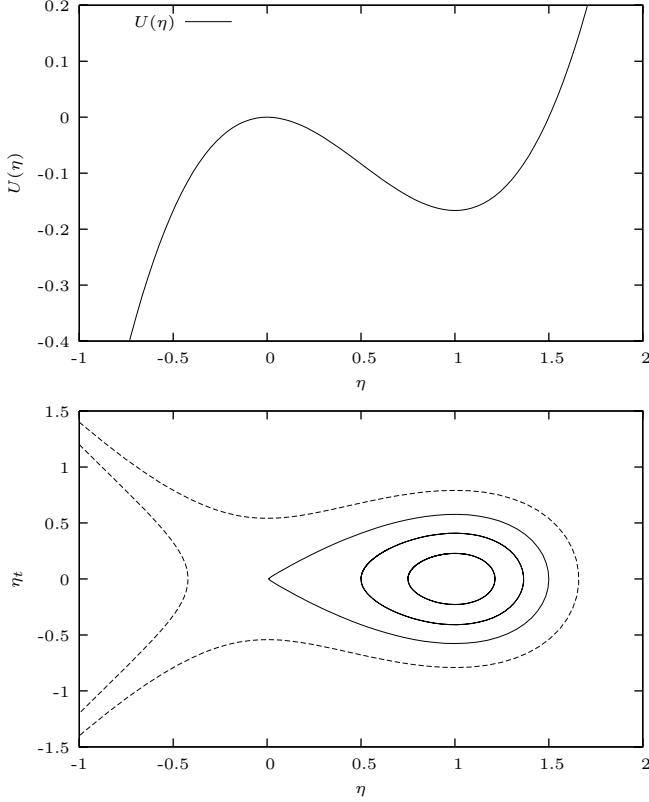
where the subscript on  $\mathbf{x}$  denotes the time-step. Also,

$$\mathbf{k}_1 = \mathbf{f}(\mathbf{x}_n)\Delta t, \quad \mathbf{k}_2 = \mathbf{f}\left(\mathbf{x}_n + \frac{1}{2}\mathbf{k}_1\right)\Delta t$$

$$\mathbf{k}_3 = \mathbf{f}\left(\mathbf{x}_n + \frac{1}{2}\mathbf{k}_2\right)\Delta t, \quad \mathbf{k}_4 = \mathbf{f}(\mathbf{x}_n + \mathbf{k}_3)\Delta t$$

and we also integrate into the past thereby completely determining the trajectories. It is convenient to make a substitution  $\eta' = \eta - \eta_1$  normalize it by a factor  $\sqrt{\eta_2 - \eta_1}$  so that the effective force is written  $-\eta'(\eta' - 1)$  with a similar expression for the potential.

Fig. 3.3 shows the cubic potential of the KdV equation, and the associated trajectories in the phase plane. Here one can directly see the effect of nonlinearities, the harmonic oscillator potential gains a pocket and a saddle and oscillations develop from harmonic to anharmonic cnoidal wave solutions. The harmonic solutions are seen as the concentric circles around the phase plane coordinate 1, 0. At the supreme boundary of the pocket, where the energy equals the potential energy at the saddle, the solution becomes aperiodic (the period approaches infinity) and therefore describe the soliton solution. Solutions with energy higher than this are shown with broken lines, but they are not bounded solutions and therefore not taken into account here.



**Fig. 3.3:** Top: The effective normalized cubic potential of the KdV equation. Bottom: The trajectories in a phase-space  $\eta, \eta_t$  corresponding to a constant energy  $E$ . The broken lines are not admissible solutions as they are not bounded, nor travelling waves.

### E. CNOIDAL & SOLITARY WAVE SOLUTIONS

In its general form, the equation

$$A\eta_t + B\eta_\varphi + C\eta\eta_\varphi + D\eta_{\varphi\varphi} = 0 \quad [3.119]$$

is an exactly integrable equation with 3 classes of solutions: periodic, aperiodic and singular solutions. The solutions are classified by the phase-portrait in Fig. 3.3, but we shall see how the periodic and aperiodic solutions are obtainable directly from the KdV equation. In integrating the KdV equation, an algebraic convenience is to denote the variables  $z = \varphi - Vt$  so that the KdV is written  $(B - AV)\eta_z + C\eta\eta_z + D\eta_{zzz} = 0$ . A first quadrature is obtainable, giving

$$\eta_{zz} = \left( \frac{AV - B}{D} \right) \eta - \frac{C}{2D} \eta^2 + a \quad [3.120]$$

Evidently, if travelling wave solutions are to exist, the coefficients must be constant and hence  $D/(B - AV)$  and  $C/(B - AV)$  do not depend on  $\theta$ . This introduces two differential equations for  $g$  and  $h$  that can be solved together with the boundary conditions  $g = h = 0$  for  $\theta = 0, \pi$ . For instance the solution  $h = R_0 \sin \frac{\theta}{2}$  is a particularly interesting solution as it results in a pressure drop in the form of  $V^2 \rho R_0^2 \sin^2 \frac{\theta}{2}$  and corresponds to the pressure drop predicted by Busse [1984] when  $V \rightarrow \Omega$ .

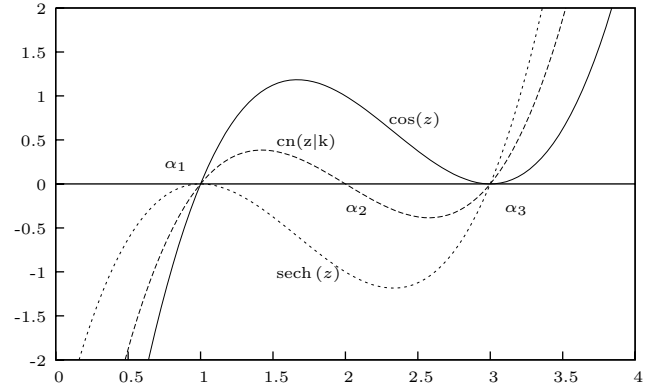
A second quadrature is obtainable by using  $\eta_z$  as an integrating factor. Then,  $\eta_{zz}\eta_z = \partial_z \frac{\eta_z^2}{2}$ ,  $\eta\eta_z = \partial_z \frac{\eta^2}{2}$  etc. Thus it follows that

$$\frac{1}{2} \eta_z^2 = -\frac{C}{6D} \eta^3 + \frac{AV - B}{2D} \eta^2 + a\eta + b \quad [3.121]$$

In order to fully integrate this into cnoidal functions, we make the substitution  $z \rightarrow \sqrt{\frac{C}{6D}} z$ . This substitution renders an equation identical to equation [2.61], except for the sign of the third-order term, i.e.

$$\frac{1}{2} \eta_z^2 = -\eta^3 + \frac{1}{2} \frac{6(AV - B)}{C} \eta^2 + a\eta + b = E - U(\eta, V). \quad [3.122]$$

Since  $\eta_z$  must be real, and for finite amplitude oscillations, the right-hand side must be positive. The negative sign on the  $\eta^3$  term then only reflects the domain of the solution. The solution will oscillate between the two largest zero's of the right-hand side,  $\alpha_2$  and  $\alpha_3$  (in Part II we called the zeros for  $f_1, f_2, f_3$ , here it is  $\alpha_1, \alpha_2, \alpha_3$ ). We may then use all the results of Part II, section B.2 except note that the solution oscillates between the two mentioned points.



**Fig. 3.4:** The effective potential for the KdV equation. The solution oscillates between the two zeros  $\alpha_2$  and  $\alpha_3$ . The two limits  $\alpha_1 \rightarrow \alpha_2$  and  $\alpha_2 \rightarrow \alpha_3$  corresponds to harmonic oscillations and the soliton limit respectively.

We shall not recite the derivation of the solution, except mention the following outline: The right-hand side of equation [3.122] is factorized into  $\frac{1}{2} \eta_z^2 = -(\eta - \alpha_1)(\eta - \alpha_2)(\eta - \alpha_3)$ . After two appropriate substitutions,  $\eta \rightarrow -\eta$ ,  $\beta_i \rightarrow -\alpha_i$ , the factorization is on the form  $\eta_z^2 = (\eta - \beta_1)(\eta - \beta_2)(\eta - \beta_3)$  with  $\beta_3 < \beta_2 < \beta_1$  and the solution oscillates between  $\beta_2$  and  $\beta_3$ . This allows an elliptic integral to be defined (similar to Part II, section B.2) and an inverse cnoidal solution by

$$\eta(z) = \beta_2 - (\beta_2 - \beta_3) \text{cn}^2 \left[ \sqrt{\frac{\beta_1 - \beta_3}{2}} z \middle| k \right] \quad [3.123]$$

where the modulus  $k$  is given by  $k^2 = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}$ . Consequently, snoidal and dnoidal solutions are also possible. By reverting our substitutions,  $\beta_i \rightarrow -\alpha_i$ ,  $\eta \rightarrow -\eta$ ,  $z \rightarrow \sqrt{\frac{C}{6D}} (\varphi - Vt)$ , we confirm that  $\eta(\varphi - Vt)$  obeys

$$\eta(\varphi - Vt) = \alpha_2 + (\alpha_3 - \alpha_2) \text{cn}^2 \left[ \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}} (\varphi - Vt) \middle| k \right] \quad [3.124]$$

and has cnoidal wave solutions.  $k$  is

$$k = \sqrt{\frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_1}} \quad [3.125]$$

In the factorization  $(\eta - \alpha_1)(\eta - \alpha_2)(\eta - \alpha_3)$ , we observe that the velocity  $V$  fulfills

$$V = \frac{C(\alpha_1 + \alpha_2 + \alpha_3)}{3A} + \frac{B}{A} \quad [3.126]$$

The period of the solutions is found by making the substitution  $v = \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}}$  in the cnoidal functions. The period is then  $2K(k)$  (the elliptic functions  $\text{cn}(v|k)$  have period  $4K$  so that e.g.  $\text{cn}^2(v|k)$  has a period  $2K$ ) so that substituting back gives the period

$$T = K \sqrt{\frac{C(\alpha_3 - \alpha_1)}{3D}} \quad [3.127]$$

where we of course understand  $K = K(k)$  is the complete elliptic integral of first kind.

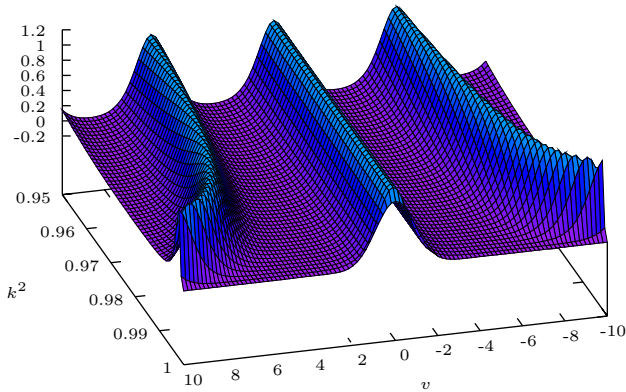
In the supreme limit when  $\alpha_2$  approaches  $\alpha_1$ ,  $k$  approaches 1 and the period tends to infinity. In this case, a localized solution is

$$\eta(\varphi - Vt) = \alpha_2 + (\alpha_3 - \alpha_2) \text{cn}^2 \left( \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}} (\varphi - Vt) \middle| 1 \right)$$

which is a bump of angular half-width  $L = \sqrt{\frac{12D}{C(\alpha_3 - \alpha_2)}}$  and amplitude  $\alpha_3 - \alpha_2$ . In the limit when  $\alpha_2 = 0$  (or translating our system so that  $\alpha_2 = 0$ ), we can write

$$\eta(\varphi - Vt) = \eta_0 \text{sech}^2 \left[ \sqrt{\frac{\eta_0 C}{12D}} (\varphi - Vt) \right] \quad [3.128]$$

which is exactly the soliton solution, i.e. equation [2.77], with a velocity given by  $V = \frac{\eta_0 C}{3A} + \frac{B}{A}$ . Fig. 3.5 shows the smooth transition from periodic cnoidal waves to aperiodic soliton solutions. The soliton is seen as a localized "bump" around  $v = 0$  and the period can be seen to grow to infinity. Among other wave-patterns, the soliton has a special shape-kinematic dependence,  $\eta_0 \sim V \sim 1/\sqrt{L}$ . In contrast to e.g. linear waves, the height, width and speed are connected: a higher soliton is narrower and moves faster. The travelling soliton solution, called "roton" the present analysis, only exists under the assumption of non-zero angular momentum [Tsamopoulos and Brown, 1983, Natarajan and Brown, 1987].



**Fig. 3.5:** The cnoidal function  $\text{cn}(v, k)$  plotted against its argument  $v \in [-10 : 10]$ , and its free parameter  $k^2 \in [-0.95, 1.0]$ . The periodic cnoidal solution becomes aperiodic as  $k \rightarrow 1^-$  which is responsible for the soliton solution. The soliton is seen as the single "bump" around  $v = 0$  when  $k = 1$ .

As mentioned on several occasions, whenever  $\alpha_2 \rightarrow \alpha_3$ ,  $k \rightarrow 0$  and  $K \rightarrow \frac{\pi}{2}$ . In this case, the cnoidal function reduces to a harmonic function,

$$\eta(\varphi - Vt) = \alpha_2 + (\alpha_3 - \alpha_2) \cos^2 \left[ \sqrt{\frac{Cg(\alpha_3 - \alpha_1)}{12D}} (\varphi - Vt) \right] \quad [3.129]$$

In this case,  $\eta$  is very small. In principle, it is a point situated on  $\alpha_3$  in the potential valley of Fig. 3.4 (The KdV equation was derived to second order, so it is in fact a small domain rather than a point and so the behaviour around this point is algebraic.). In conclusion, only by modifying the initial conditions and the parameter  $V$  can periodic solutions and localized bumps be obtained within the same equation. Such different choices introduce different potentials, as seen in Fig. 3.4. The solutions depend of the three free parameters  $\alpha_1, \alpha_2, \alpha_3$ ,  $V$  follows from equation [3.126] subject to two physical constraints. The solution goes into its final form when enforcing the constant-volume constraint  $4\pi = \int_{\Sigma} [1 + g(\theta)\eta(\varphi, t)]^3 d\Omega$  and requiring that  $\eta(\varphi, t)$  is periodic in the sense  $\eta(\varphi, t) = \eta(\varphi + 2\pi, t)$ . This latter introduces the constraint  $2nK = 2\pi\sqrt{C(\alpha_3 - \alpha_1)/12D}$  so consequently

$$nK = \pi \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}} \quad [3.130]$$

where  $n = 1, 2, \dots, N$ .  $K$  bounded from below where  $K(0) = \frac{\pi}{2}$  which requires  $N \leq 2\sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}}$ . In the roton limit this condition becomes a quasiperiodic one because of the rapidly decaying amplitude of the roton solution. The volume conservation condition reads

$$\frac{4\pi R_0^3}{3} = \int_{\mathcal{V}} d\mathcal{V} = R_0^3 \int_{\Omega} (1 + \xi)^3 d\Omega \quad [3.131]$$

which explicitly reads

$$\int_0^{2\pi} [S_{3,0}^{1,0} \xi^3 + 3S_{2,0}^{1,0} \xi^2 + 3S_{1,0} \xi] d\varphi = 0 \quad [3.132]$$

Consequently the whole model contains only one free parameter that can be chosen to be one of the  $\alpha_i, V, \eta_0$  or the droplet energy  $E$ . The conditions of equation [3.132] and equation [3.130] then fixes the remaining parameters. As is such, we can draw the potential shape  $U[\eta]$  as a two-dimensional surface, where each cross-section corresponds to a certain initial condition, i.e. an explicit choice of  $\alpha_i$ . Since  $K$  is bounded from below, i.e.  $K$  increases monotonically from  $\frac{\pi}{2}$  to  $\infty$  as  $k = 0 \rightarrow 1$ , this allows for only a finite number of cnoidal modes. In all practice, giving one of the parameters and enforcing the conditions of equations [3.130] and [3.132] must be done numerically because the equation governing the volume conservation condition is very complicated. Several formulas that are used in the deduction of the next equation are found in the appendix. Taking equation [3.132] and making the substitution  $z = \sqrt{C(\alpha_3 - \alpha_1)/12D}$  and collecting terms of order  $\text{cn}^2, \text{cn}^4, \text{cn}^6$  gives

$$\begin{aligned} & 2\pi \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}} [3S_{1,0}^{1,0} \alpha_2 + 3S_{2,0}^{1,0} \alpha_2^2 + S_{3,0}^{1,0} \alpha_2^3] + [3S_{1,0}^{1,0} (\alpha_3 - \alpha_2) + 6S_{2,0}^{1,0} \alpha_2 (\alpha_3 - \alpha_2) + 2S_{3,0}^{1,0} \alpha_2^2 (\alpha_3 - \alpha_2)] C_2 \Big|_0^{2\pi \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}}} \\ & + [3S_{2,0}^{1,0} (\alpha_3 - \alpha_2)^2 + 3S_{3,0}^{1,0} \alpha_2 (\alpha_3 - \alpha_2)^2] C_4 \Big|_0^{2\pi \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}}} + S_{3,0}^{1,0} (\alpha_3 - \alpha_2)^3 C_6 \Big|_0^{2\pi \sqrt{\frac{C(\alpha_3 - \alpha_1)}{12D}}} = 0 \end{aligned}$$

where  $C_2, C_4$  and  $C_6$  are all functions of  $(\varphi, k)$  and given by equations [A.33c] [A.33e] [A.33f]. Additionally we need to solve equation [3.130]. By taking the value for the coefficients  $A, B, C, D$  for  $\theta = \frac{\pi}{2}$  (the equatorial cross-section) with prescribed  $h$  and  $g$ , one can determine all constants as a function of  $V$  only. By solving the equations [3.126],[3.130],[3.132] one can uniquely determine  $\alpha_1, \alpha_2, \alpha_3$  with e.g.  $V$  being the prescribed factor.

One of many ways of doing this is to use a multidimensional Newton-Raphson algorithm. If the equations to be solved are denoted  $F_i = F_i(x_j) = 0$ , then Taylor-expanding all equations to first order gives

$$F_i(\mathbf{x} + \delta\mathbf{x}) = F_i(\mathbf{x}) + \frac{\partial F_i}{\partial x_j} \delta x_j$$

and by setting  $F_i(\mathbf{x} + \delta\mathbf{x}) = 0$ , i.e. following the tangentline produced in  $\mathbf{x}$  until the function on the left hand side reaches zero, we obtain the equation

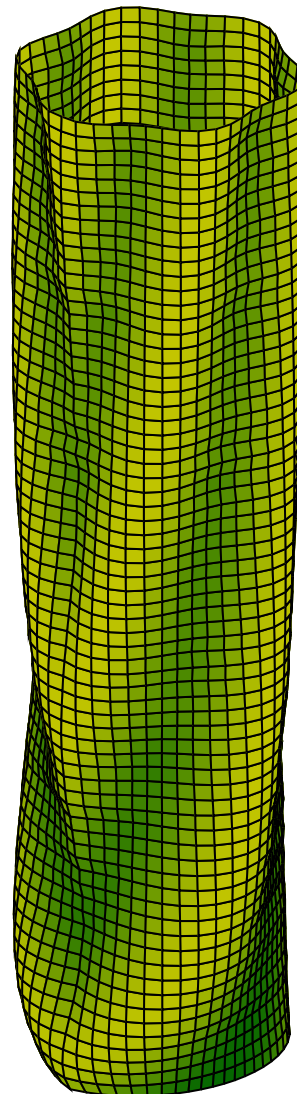
$$\frac{\partial F_i}{\partial x_j} \delta x_j = -F_i(\mathbf{x})$$

This equation set can be solved numerically by e.g. a Gauss-Jordan elimination, Gauss-Seidel iterative technique, LU decomposition and so forth. Indeed, since our equation set represents a Jacobian of dimensions  $3 \times 3$ , an analytic reduction is also possible. When the solution  $\delta x_j$  is obtained, we add it to the final solution

$$\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}} + \delta\mathbf{x}$$

and the procedure is repeated until the solution is reached. Performing the multi-dimensional Newton-Raphson algorithm for our equation set requires use of several series expansions and numerical integrations (the first elliptic integral has a very diverging behaviour when the modulus approaches one and the series expansions are no longer fit). These expansions are given in the appendix.

Because of the complicated expressions for the coefficients in the equations to be solved, it is easier to show the volume conservation condition from the original KdV equation given in Part II with the corresponding cnoidal solution. The original KdV equation presented in Part II represents essentially the same physics as the KdV we are currently discussing. The only difference lies in the parametric dependence of the terms. For the original KdV equation, the solution is even-valued and hence we may use a quasi-periodic condition by matching the two ends of the solution,  $\varphi = -\pi$  and  $\varphi = \pi$ . By prescribing two of the  $\alpha$ 's we use a form of the volume conservation so that it corresponds to conservation of the area of the equatorial cross-section, i.e.  $\int_0^{2\pi} [\eta^2 + 2\eta] d\varphi = 0$  and use this equation to solve for the final  $\alpha$ . We show such an evolution in Fig. 3.6 where the transition from a harmonic  $l = 6$  mode to a final solitonic form is shown.



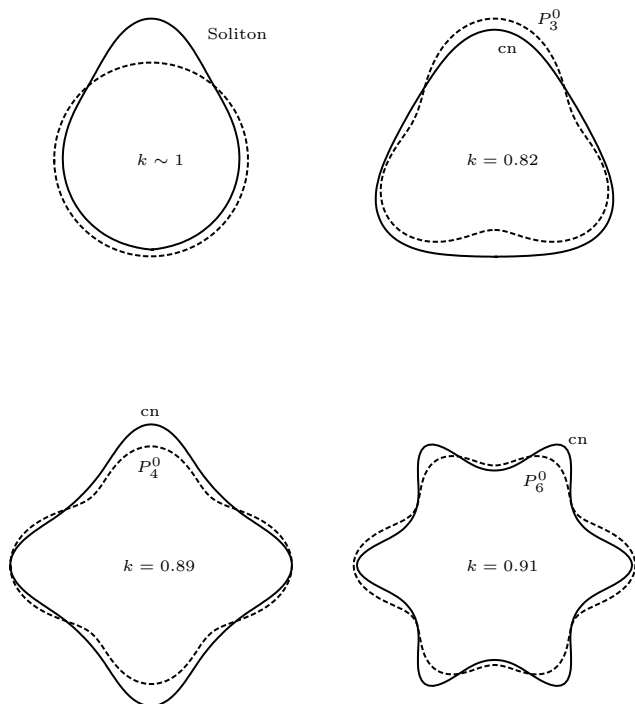
**Fig. 3.6:** Evolution of the equatorial cross-section as a function of  $\alpha_3$  with  $\alpha_1, \alpha_2$  being fixed through the volume conservation condition and a quasi-periodic condition. The smooth transition from an  $l = 6$  mode to a final solitonic form is shown.

The many resonances of the  $l = 2$  mode of liquid drops can also be understood as a consequence of the multivalley profile of the KdV equation:  $\eta_z^2 = a\eta + b\eta^2 + c\eta^3$ . The frequency shift predicted by Busse [1984] (equation [1.40] in Part I) can be reproduced in our framework by letting  $h \rightarrow R_0 \sin \frac{\theta}{2}$ . It gives a pressure drop of the form  $\rho V^2 R_0^2 \sin^2(\theta/2)$  from equation [3.27] and hence leads to a similar result. Moreover, it would be interesting to see whether equation [3.84] reproduces another frequency shift given by Busse [1984] (equation [1.43] in this text).

When the stratification layer  $h$  thins,  $h \rightarrow 0$ , the coefficient  $C$  in equation [3.30] become zero on average. This produces a break in the travelling wave solution ( $L$  is singular). Such "turbulence" from capillary waves has been observed by Holt and Trinh [1996]. The abrupt onset of wave turbulence exper-

imentally noticed can be related to the present theory when  $h \approx 15 - 25 \mu\text{m}$ ,  $V \approx 2.1 - 2.5 \text{ms}^{-1}$ . The results are qualitatively in agreement.

Travelling wave modes that can be described by shapes close to the tesseral spherical harmonics have been found previously by use of multiple time-scale nonlinear analysis [Tsamopoulos and Brown, 1983, 1984]. Such modes are stable under the assumption of non-zero angular momentum. In Fig. 3.7 we show the equatorial cross-section of the soliton and cnoidal solutions together with the closest tesseral harmonics under the volume conservation and periodicity constraints. The shapes are qualitatively the same, and hence the results by Tsamopoulos and Brown [1983, 1984] are reproduced in the present theory.



**Fig. 3.7:** The cnoidal solution for the equatorial cross-section. The soliton limit and 2, 3, 4 mode cnoidal solutions are shown together with the corresponding tesseral spherical harmonics (broken lines). The modulus of the cnoidal solution together with the labels  $l, m$  are given.

## F. SUMMARY & DISCUSSION

The NLD model treated in this text proves that travelling wave modes exist as cnoidal wave on a slightly oblate spheroid. By introducing a geometry of the surface as  $r = R_0 [1 - \varepsilon P_2(\theta) + g(\theta)\eta(\varphi - Vt)]$  and introducing a potential  $\Phi = \sum_{n=0}^{\infty} (r/R_0 - 1) f_n(\theta, \varphi, t)$ , Laplace's equation sets up a recurrence relation between each  $f_n$  that can be identified by use of the boundary conditions that *i*) the flow is stratified in a surface layer  $h(\theta)$  and *ii*) a kinematic condition on the surface of the droplet. Each  $f_n$  is a linear combination of lower order  $f_n$ 's and hence the convergence of the series in  $\Phi$  is controlled by the value  $\max |\xi|$ . The value of  $\varepsilon$  depends on the physical situation, e.g. in the case of rotation it follows that  $\varepsilon = \frac{2}{3} \frac{\rho R_0^3 \Omega^2}{8\gamma} \equiv \frac{2}{3} \Omega^*{}^2$

but it is small. Increasing the value of  $\varepsilon$  increases the error of the solution and the model breaks completely down when  $\varepsilon \approx 0.2$  since this marks the bifurcation point of the axisymmetric family into a family of two-lobed shapes. If the present theory is to be continued for higher values of  $\varepsilon$ , the entire model must be redone by prescribing the elliptic figure of equilibrium given in this text. By assuming a constant background vorticity equal to the vorticity of a rotating liquid  $\omega = 2\Omega$  and equating the pressure from the Young-Laplace equation we are able to show that along streamlines that run azimuthally around the droplet, the differentiated form of the Bernoulli relation is in fact a form of a KdV equation for  $\eta$ . Travelling wave solutions exist as long as the parameters of the KdV equation meet certain restrictions. These restrictions lead to two differential equations for  $g(\theta)$  and  $h(\theta)$  that can be solved with the Cauchy conditions  $h(\pi/2) = g(\pi/2) = 0$  with  $h(0), g(0)$  as free parameters of the solution. By integrating the KdV equation, cnoidal solutions exist for  $\eta$  that describe the smooth transition from small linear oscillations [Landau and Lifshitz, 1987] into full-blooded nonlinear solitary waves [Lamb, 1953, Infeld and Rowlands, 2000].  $\eta$  has three free parameters (constants of integration), but the entire model is a one-parameter model only since by supplying one value, the other two constants of integration are completely determined by requiring that the solution for  $\eta$  is periodic and satisfies volume conservation. By using the definition of the Hamiltonian as a subalgebra of the Lie algebra [Arnold, 1978] we are able to embed the entire model into a Hamiltonian framework, and show that this too leads to the KdV equation when the energy of the droplet is taken Hamiltonian. We introduce several mathematical corrections to previous nonlinear models. The same model can also be applied to bubbles (in the case of no rotation) but with an additional far-field conditions on the bubble surface.

It is well known that the LDM with a shell-energy correction describes excellently well the range of small-amplitude harmonic oscillations of atomic nuclei. The NLD model covers the entire range of oscillations from small linear oscillations to rotons (solitary waves) and such solutions can be used to provide further insight into the behaviour of atomic nuclei, stars, multiphase disperse flows, sonoluminescence and so forth. Clusters that are formed on nuclei surfaces cannot be predicted by use of linear analytic tools but can be excellently described in the present theory by using a tailored KdV potential. Rotons that exist on a droplet surface are then literally dual to the formation of such clusters (e.g. alpha formation on heavy nuclei close to the magic nuclei  $^{208}\text{Pb}$ ,  $^{100}\text{Sn}$ ).

The present theory predicts the existence of travelling wave modes on a rotating droplet, in agreement with recent results. Such behaviour was experimentally observed by Hill and Eaves [2008], but their rotation rate far exceeds our limitations. Nonetheless, it is more than reasonable to expect that an equation qualitatively similar to the KdV equation exists also for higher rotation rates (Hill and Eaves [2008] also observes travelling wave modes bifurcating into two-lobed shapes). The entire model rests only on the assumption of a constant background vorticity but the stability of such a system has not been a subject in this text and remains a topic for future investigations. It would also be very interesting to see a continuation to higher rotation rates where the figure of equilibrium is determined by an elliptic integral.

Apart from the obvious applications to droplets and atomic nuclei, the results of this model can be applied to a number of other

physical situations. For instance can the model be used to describe the dynamics of a pellet surface in inertial fusion, ink-jet delivery systems, stellar models, sonoluminescence and so forth.

It would seem natural to generalize this work in several directions. The potential flow assumption is the bottleneck of the model and may not be appropriate for many physical systems. It is however ideal for examining hydrodynamical effects only. A similar nonlinear model also exists in literature [de la Llave and Panyaotaros, 1996, Panyaotaros, 2001], but they do not derive explicit solutions and moreover, their restoring force is gravitational attraction from a rigid core whereas we focus on surface tension only. Extra restoring forces (such as e.g. gravity) can readily be incorporated into the present model by proper additions to the dynamic boundary condition.

The powerful theory of a Hamiltonian formulation for free boundary problems first presented in Lewis et al. [1986] proposes

a general framework for our problem. This formalism leans on decomposing the velocity field into a solenoidal part and conservative part and write the canonical formulation for the solenoidal part. However, the description of the velocity field by two functions is lost. In all respect, the same results can likely be obtained from the by far easier method tensor virials by Chandrasekhar [1965], that lean only on certain symmetry properties of the Euler equations. The possibilities are nearly exhausted, Chandrasekhar [1965], Rosenkilde [1967] focuses on small amplitude motion around any rotation rate, whereas we have focused on large-amplitude motion for low rotation rates. Still, considering the revived interest of Hamiltonian based mechanics formulated for infinite-dimensional mechanical problems, the most probable course of action is to pursue the work of Lewis et al. [1986].



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## APPENDIX

### A. CALCULUS OF VARIATIONS

A general functional  $J$  consisting of a set of independent variables  $y_i$  and a set of dependent variables  $x_j$  is

$$J = \int f(y_i, y_{ij}, x_j) dx_j, \quad y_{ij} \equiv \frac{\partial y_i}{\partial x_j} \quad [\text{A.1}]$$

Say that  $y_i(x, 0)$  determines the path that ensures the extremum of  $J$ , but such that it is not known beforehand. We then vary the path to

$$y_i(x_j, \alpha) = y_i(x_j, 0) + \alpha \eta_i(x_j)$$

and each  $y_i$  is varied independently. Then,

$$\delta J = \alpha \left. \frac{\partial J}{\partial \alpha} \right|_{\alpha=0} = 0 \quad [\text{A.2}]$$

where now  $J = \int f[y_i(x_j, \alpha), y_{ij}(x_j, \alpha), x_j] dx_j$ .

By introducing a set of constraints  $\varphi_k(y_i, y_{ij}, x_j)$  on  $J$ , the set of  $y_i$  is not independent of each other. The constraints may be in the form

$$\int \varphi_k(y_i, y_{ij}, x_j) dx_j = \text{const.} \quad [\text{A.3}]$$

Clearly the variation of a constant is zero, which allows for adding a constant times each constraint to equation A.2 to obtain

$$\delta \int \left[ f(y_i, y_{ij}, x_j) + \sum_k \lambda_k \varphi_k(y_i, y_{ij}, x_j) \right] dx_j = 0 \quad [\text{A.4}]$$

Treating the entire integrand as a new function  $g(y_i, y_{ij}, x_j) = f + \sum_k \lambda_k \varphi_k$  the variation (equation A.2) is given by the product rule to be

$$\delta \int g(y_i, y_{ij}, x_j) dx_j = \alpha \int \sum_i \left[ \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial \alpha} + \sum_j \frac{\partial g}{\partial y_{ij}} \frac{\partial y_{ij}}{\partial \alpha} \right]$$

By recognizing  $\frac{\partial y_i}{\partial \alpha} = \eta_i$  and by integrating by parts one finds

$$\begin{aligned} \delta \int g dx_j &= \int \sum_i \left[ \frac{\partial g}{\partial y_i} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial g}{\partial y_{ij}} \right] \delta y_i dx_j \\ &+ \int \sum_{i,j} \frac{\partial}{\partial x_j} \left[ \frac{\partial g}{\partial y_{ij}} \delta y_i \right] dx_j \end{aligned} \quad [\text{A.5}]$$

By use of the fundamental theorem of calculus and the requirement that there is no variation in the end-points the second integral vanishes. Additionally the paths  $\delta y_i$  are independent

of each other so if  $y_i(x_j, 0)$  ensures the extremal value of  $J$ , the terms in the first integrand are each zero, i.e.  $g$  satisfies

$$\frac{\partial g}{\partial y_j} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial g}{\partial y_{ij}} = 0, \quad i = 1, 2, \dots, n \quad [\text{A.6}]$$

where  $i$  runs over the  $n$  dependent variables and  $j$  over all the independent variables. This equation is known as the Euler-Lagrange equation and ensures the extremal value of a functional  $J$  with the constraints given by equation A.3. Also, the integrand of equation [A.5] is the variational derivative of  $J$ .

The Lagrangian equations of motion is the resulting Euler-Lagrange equation of the above if one interprets the variables as

$$\begin{aligned} x_j &\rightarrow t \\ f(y_i, y_{ij}, x_j) &\rightarrow L(q_i, \dot{q}_i, t) \end{aligned}$$

where a dot indicates the temporally derivative of the generalized coordinates and  $L(q_i, \dot{q}_i, t)$  as the Lagrangian. Then the least action principle states that

$$\delta \int \left[ L(q_i, \dot{q}_i, t) + \sum_k \lambda_k \varphi_k(q_i, t) \right] dt = 0$$

so that the resulting lagrangian equations of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_k \lambda_k \frac{\partial \varphi_k}{\partial q_i} \quad [\text{A.7}]$$

Note that this latter equation requires that constraint is not dependent on  $\dot{q}_i$ .

### B. MEAN CURVATURE OF A SURFACE

When studying the equilibrium configurations of a liquid drop under the action of a capillary pressure caused by surface tension, the curvature of the drop's surface is required by the Young-Laplace equation  $\Delta p = \nabla \cdot \hat{\mathbf{n}}$ . Most literature takes the sphere as the initial shape and follows up with a perturbation expansion to first order. Here, the exact curvature of a surface expressed in terms of the polar angles is derived with a subsequent linearization shown to reduce to the expressions of e.g. Landau and Lifshitz [1987]

Let  $\xi(\theta, \varphi)$  denote the distance from the origin (conveniently placed at the center of mass of the droplet) and  $\theta$  and  $\varphi$  the meridional and azimuthal angles. See Fig. 1.1 for reference. Then in Euclidean 3-space, the surface is described by

$$\mathbf{r}(\theta, \varphi) = (\xi \sin \theta \cos \varphi, \xi \sin \theta \sin \varphi, \xi \cos \theta) \quad [\text{A.8}]$$

At any point on the surface, a tangential plane spanned by the two tangential vectors  $\frac{\partial \mathbf{r}}{\partial \theta}$  and  $\frac{\partial \mathbf{r}}{\partial \varphi}$ . We will call these vectors  $\mathbf{t}_1$  and  $\mathbf{t}_2$ , of which the explicit calculation gives

$$\begin{aligned}\mathbf{t}_1 &= \frac{\partial \mathbf{r}}{\partial \theta} = ([\xi_\theta \sin \theta + \xi \cos \theta] \cos \varphi, \\ &\quad [\xi_\theta \sin \theta + \xi \cos \theta] \sin \varphi, \xi_\theta \cos \theta - \xi \sin \theta) \\ \mathbf{t}_2 &= \frac{\partial \mathbf{r}}{\partial \varphi} = ([\xi_\varphi \cos \varphi - \xi \sin \varphi] \sin \theta, \\ &\quad [\xi_\varphi \sin \varphi + \xi \cos \varphi] \sin \theta, \xi_\varphi \cos \theta)\end{aligned}$$

or put more simply in polar coordinates

$$\mathbf{t}_1 = \xi_\theta \hat{\mathbf{e}}_r + \xi \hat{\mathbf{e}}_\theta \quad [\text{A.9a}]$$

$$\mathbf{t}_2 = \xi_\varphi \hat{\mathbf{e}}_r + \xi \sin \theta \hat{\mathbf{e}}_\varphi \quad [\text{A.9b}]$$

as is seen by constructing the polar base vectors and reducing the original two expressions for the tangential vectors.

Subsequently the unit normal is found as the normalized orthogonal vector to the plane spanned by  $\mathbf{t}_1$  and  $\mathbf{t}_2$ , i.e.

$$\hat{\mathbf{n}} = \frac{\mathbf{t}_1 \times \mathbf{t}_2}{|\mathbf{t}_1 \times \mathbf{t}_2|} \quad [\text{A.10}]$$

where an outward unit normal has been chosen. The explicit calculation gives

$$\hat{\mathbf{n}} = \frac{\xi \sin \theta \hat{\mathbf{e}}_r - \xi_\theta \sin \theta \hat{\mathbf{e}}_\theta - \xi_\varphi \hat{\mathbf{e}}_\varphi}{[(\xi^2 + \xi_\theta^2) \sin^2 \theta + \xi_\varphi^2]^{1/2}} \quad [\text{A.11}]$$

This is the general result for the outward unit normal. In the case of a sphere where  $\xi(\theta, \varphi) = R$  this trivially reduces to  $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r$  as expected.

The divergence of the above is readily calculated in the polar coordinates  $(\xi, \theta, \varphi)$ , it is verified that

$$\begin{aligned}\frac{1}{\xi^2} \frac{\partial}{\partial \xi^2} (\xi^2 \hat{\mathbf{n}}_r) &= \frac{1}{\xi [(\xi^2 + \xi_\theta^2) \sin^2 \theta + \xi_\varphi^2]^{3/2}} \left\{ 2\xi^3 \sin^3 \theta \right. \\ &\quad \left. + 3\xi \xi_\theta^2 \sin^3 \theta + \frac{3}{2} \frac{\partial}{\partial \theta} (\xi_\varphi \xi^2 \sin \theta) \right. \\ &\quad \left. - 3\xi_{\varphi\varphi} \xi^2 \sin \theta \right\}\end{aligned}$$

Also needed is

$$\begin{aligned}\frac{1}{\xi \sin \theta} \frac{\partial \hat{\mathbf{n}}_\varphi}{\partial \varphi} &= \frac{1}{\xi [(\xi^2 + \xi_\theta^2) \sin^2 \theta + \xi_\varphi^2]^{3/2}} \left\{ -\frac{1}{2} \frac{\partial}{\partial \varphi} [\xi_\varphi (\xi^2 \right. \\ &\quad \left. + \xi_\theta^2) \sin \theta] + \frac{3}{2} \xi_{\varphi\varphi} (\xi^2 + \xi_\theta^2) \sin \theta \right\}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\xi \sin \theta} \frac{\partial \hat{\mathbf{n}}_\theta}{\partial \theta} &= \frac{1}{\xi [(\xi^2 + \xi_\theta^2) \sin^2 \theta + \xi_\varphi^2]^{3/2}} \left\{ [(\xi^2 + \xi_\theta^2) \sin \theta \right. \\ &\quad \left. + \frac{\xi_\varphi^2}{\sin \theta} \frac{\partial}{\partial \theta} (\xi_\theta \sin^2 \theta) - \frac{1}{2} (\xi_\theta \sin \theta) \frac{\partial}{\partial \theta} [(\xi^2 \right. \\ &\quad \left. + \xi_\theta^2) \sin^2 \theta + \xi_\varphi^2] \right\}\end{aligned}$$

The three expressions above are the terms contained in the divergence of  $\hat{\mathbf{n}}$ , collecting them gives the general curvature by

$$\begin{aligned}\nabla \cdot \hat{\mathbf{n}} &= \frac{\{\xi^2 [2\xi \sin \theta - \frac{\partial}{\partial \theta} (\xi_\theta \sin \theta)] + \xi_\theta^2 [2\xi \sin \theta - \frac{\partial}{\partial \theta} (\xi \cos \theta)]\} \sin^2 \theta}{\xi [(\xi^2 + \xi_\theta^2) \sin^2 \theta + \xi_\varphi^2]^{3/2}} \\ &\quad - \left\{ \frac{\partial}{\partial \varphi} (\xi_\varphi [\xi^2 + \xi_\theta^2]) - 2\xi_{\varphi\varphi} (\xi^2 + \xi_\theta^2) \sin \theta - \xi_\varphi^2 \frac{\partial^2}{\partial \theta^2} (\xi \sin \theta) \right\} \quad [\text{A.12}]\end{aligned}$$

The reader may explicitly check that  $\xi(\theta, \varphi) = R$  gives  $\nabla \cdot \hat{\mathbf{n}} = \frac{1}{R}$  as expected. In order to recover the linearized expression by Landau and Lifshitz [1987], say that the surface is a slightly deformed sphere, i.e.  $\xi(\theta, \varphi) = R + \zeta(\theta, \varphi)$  where  $\zeta$  is small compared to  $R$ . Then the above equation is linearized to

$$\nabla \cdot \hat{\mathbf{n}} = \frac{2}{R} - \frac{2\zeta}{R} - \frac{1}{R^2} \left[ \frac{1}{\sin^2 \theta} \frac{\partial^2 \zeta}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \zeta}{\partial \theta} \right) \right]$$

which is the expression found in e.g. Landau and Lifshitz [1987].

The expression for the curvature of an axisymmetric surface is much easier. Say that the surface is described by a single-valued function  $r = f(z)$ , that is we describe the distance from the  $z$ -axis to the free surface by  $f(z)$ . Any such axisymmetric surface may be considered a two-dimensional surface where the three-dimensional surface is obtained by rotating around the symmetry axis. Therefore, the tangential vector associated with the 2D surface is

$$\mathbf{t} = (f', 1) \quad [\text{A.13}]$$

where a prime denotes partial differentiation with respect to  $z$ . Subsequently the normal unit vector is

$$\hat{\mathbf{n}} = \frac{(1, -f')}{\sqrt{1 + f'^2}} \quad [\text{A.14}]$$

Calculating the divergence gives directly

$$\nabla \cdot \hat{\mathbf{n}} = \frac{1}{f} \frac{\partial}{\partial f} \left( \frac{f}{\sqrt{1 + f'^2}} \right) \quad [\text{A.15}]$$

## C. ELLIPTIC FUNCTIONS

### Canonical Forms Of Elliptic Integrals

The normal elliptic integral *of the first kind* is

$$F(\varphi, k) = \int_0^\varphi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta \quad [\text{A.16}]$$

The normal elliptic integral *of the second kind* is

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} d\theta \quad [\text{A.17}]$$

The normal elliptic integral *of the third kind* is

$$\Pi(\varphi, \alpha^2, k) = \int_0^\varphi \frac{1}{(1 - \alpha^2 \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} d\theta \quad [\text{A.18}]$$

When  $\varphi = \frac{\pi}{2}$  the integrals said to be complete, we then define

$$K(k) \equiv F\left(\frac{\pi}{2}, k\right) \quad [\text{A.19a}]$$

$$K(k') \equiv F\left(\frac{\pi}{2}, k'\right) \quad [\text{A.19b}]$$

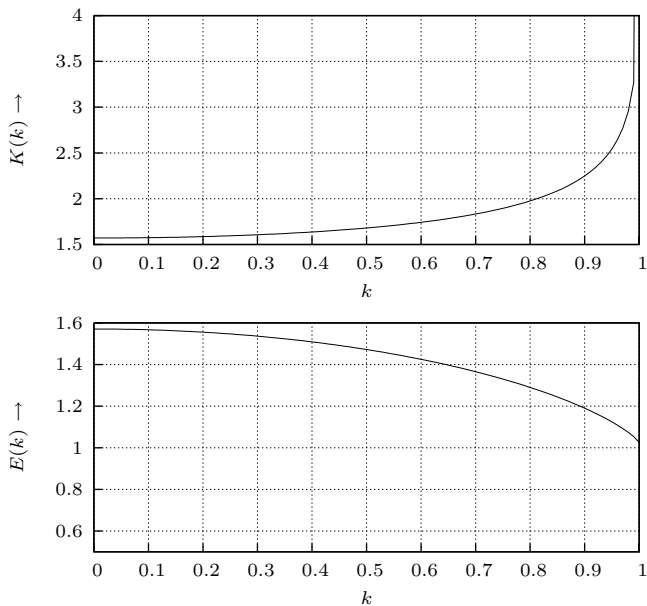
$$E(k) \equiv E\left(\frac{\pi}{2}, k\right) \quad [\text{A.19c}]$$

$$\Pi(\alpha^2, k) \equiv \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) \quad [\text{A.19d}]$$

The reciprocals and quotients of the snu, cnu and dnu are designated

$$\begin{array}{lll} ns\ v \equiv \frac{1}{sn\ v} & tn\ v \equiv \frac{sn\ v}{cn\ v} & sd\ v \equiv \frac{sn\ v}{dn\ v} \\ nc\ v \equiv \frac{1}{cn\ v} & cs\ v \equiv \frac{cn\ v}{sn\ v} & cd\ v \equiv \frac{cn\ v}{dn\ v} \\ nd\ v \equiv \frac{1}{dn\ v} & ds\ v \equiv dn\ v & dc\ v \equiv \frac{dn\ v}{cn\ v} \end{array}$$

and there are in total twelve Jacobi elliptic functions. The functions are doubly periodic, having real periods like the trigonometric functions and imaginary period like the hyperbolic functions. sn, cn and dn have periods  $(4K, 2iK')$ ,  $(4K, 2K + 2iK')$  and  $(2K, 4iK')$  respectively. Sketches of the functions sn, cn and dn are shown in Fig. C.2.



**Fig. C.1:** Sketches of the complete elliptic integrals of first and second kind.

### The Modulus

The number  $k$  is called the *modulus* and may take any complex number. It is not uncommon in physics to restrict  $0 < k < 1$ , or use the parameter  $m = k^2$ . The complementary modulus is given by

$$k' = \sqrt{1 - k^2} \quad [\text{A.20}]$$

### Jacobi Elliptic Functions

We define the amplitude  $u$ , cnoidal, snoidal and dnoidal functions by the inverse functions of the integral

$$v = \int_0^\varphi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \quad [\text{A.21}]$$

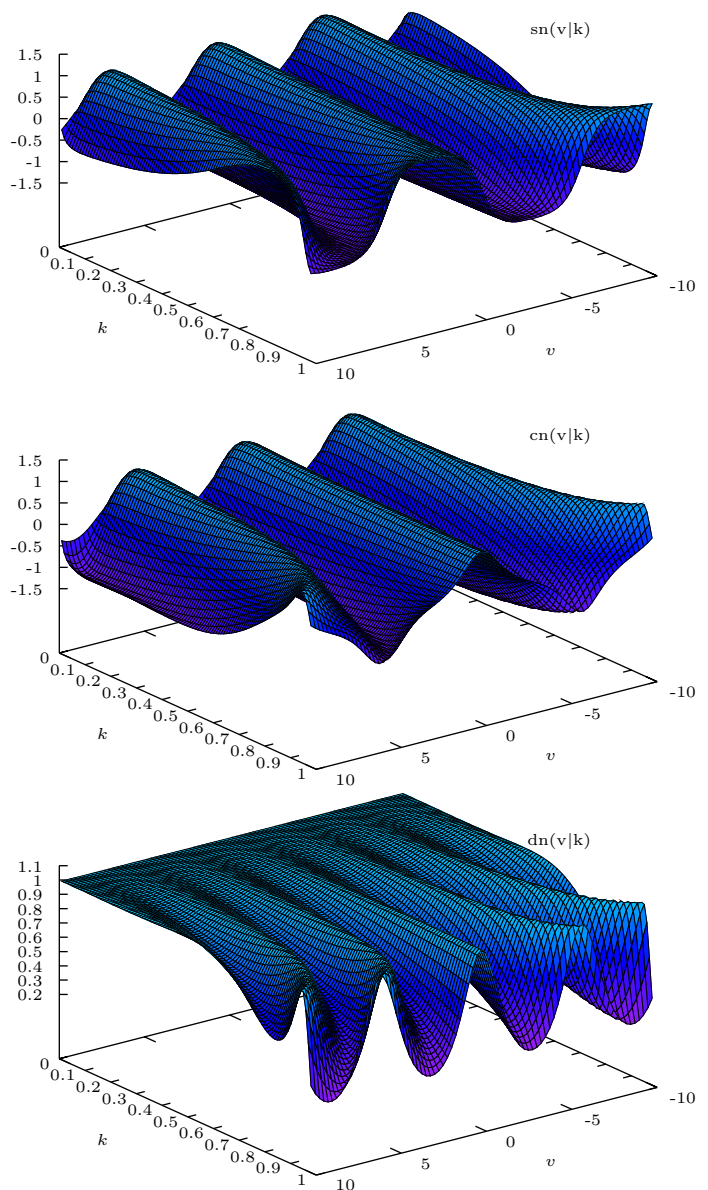
so that

$$\text{am}(v|m) = \varphi \quad [\text{A.22a}]$$

$$\text{cn}(v|m) = \cos \varphi \quad [\text{A.22b}]$$

$$\text{sn}(v|m) = \sin \varphi \quad [\text{A.22c}]$$

$$\text{dn}(v|m) = \sqrt{1 - k^2 \sin^2 \varphi} \quad [\text{A.22d}]$$



**Fig. C.2:** Modular sketches of the Jacobi elliptic functions sn, cn, dn for  $k \in [0, 1]$ ,  $v \in [-10, 10]$ .

### Developments Of Elliptic Integrals Of First And Second Kind

$$F(\varphi, k) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-k^2)^n t_{2n}(\varphi) \quad [\text{A.23}]$$

$$E(\varphi, k) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-k^2)^n t_{2n}(\varphi) \quad [\text{A.24}]$$

where

$$t_0(\varphi) = \varphi, \quad t_2(\varphi) = \frac{1}{2} [\varphi - \sin \varphi \cos \varphi]$$

$$t_4(\varphi) = \frac{1}{8} [3\varphi - \sin \varphi \cos \varphi (3 + 2 \sin^2 \varphi)]$$

$$t_{2n}(\varphi) = \frac{2n-1}{2n} t_{2(n-1)}(\varphi) - \frac{1}{2n} \sin^{2n-1} \varphi \cos \varphi$$

$$\binom{a}{n} = \prod_{i=1}^n \frac{a-n+i}{i}$$

where  $a$  is any complex number and  $n$  is any natural number.

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n! n!} k^{2n} \quad [\text{A.25}]$$

$$E(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{1}{1-2n} \binom{-\frac{1}{2}}{n} \binom{-\frac{1}{2}}{n} k^{2n} \quad [\text{A.26}]$$

### Developments Of Jacobi Elliptic Functions

The cnoidal, snoidal and dnoidal functions have the following Maclaurin series [Abramowitz and Stegun, 1972, Byrd and Friedman, 1954]

$$\begin{aligned} \text{am}(v|k) = & v - k^2 \frac{v^3}{3!} + (4+k^2)k^2 \frac{v^5}{5!} \\ & - (16+44k^2-k^4)k^2 \frac{v^7}{7!} \end{aligned} \quad [\text{A.27a}]$$

$$\begin{aligned} \text{cn}(v|k) = & 1 - \frac{v^2}{2!} + (1+4k^2) \frac{v^4}{4!} \\ & - (1+44k^2+16k^4) \frac{v^6}{6!} + \dots \end{aligned} \quad [\text{A.27b}]$$

$$\begin{aligned} \text{sn}(v|k) = & u - (1+k^2) \frac{u^3}{3!} + (1+14+k^4) \frac{u^5}{5!} \\ & - (1+135k^2+135k^4+k^6) \frac{u^7}{7!} + \dots \end{aligned} \quad [\text{A.27c}]$$

$$\begin{aligned} \text{dn}(v|k) = & 1 - k^2 \frac{u^2}{2!} + k^2(4+k^2) \frac{u^4}{4!} \\ & - k^2(16+44k^2+k^4) \frac{u^6}{6!} + \dots \end{aligned} \quad [\text{A.27d}]$$

Define  $q = e^{-\pi K'/K}$ , the following Fourier series expansions are valid:

$$\text{am}(v|k) = \frac{\pi v}{2K} + 2 \sum_{n=0}^{\infty} \frac{q^{n+1}}{(n+1)[1+q^{2(n+1)}]} \sin \left[ (n+1) \frac{\pi v}{K} \right] \quad [\text{A.28a}]$$

$$\text{cn}(v|k) = \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1+q^{2m+1}} \cos \left[ (2n+1) \frac{\pi v}{2K} \right] \quad [\text{A.28b}]$$

$$\text{sn}(v|k) = \frac{2\pi}{kK} \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{1-q^{2m+1}} \sin \left[ (2n+1) \frac{\pi v}{2K} \right] \quad [\text{A.28c}]$$

$$\text{dn}(v|k) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{q^{n+1}}{1+q^{2n+1}} \cos \left[ (n+1) \frac{\pi v}{K} \right] \quad [\text{A.28d}]$$

The following infinite products hold:

$$\begin{aligned} \text{cn}(v|k) = & \cos \left( \frac{\pi v}{2K} \right) \prod_{n=1}^{\infty} \left[ \frac{(1-q^{2n-1})^2}{(1+q^{2n})^2} \right. \\ & \left. \times \frac{1+2q^{2n} \cos(\pi v/K) + q^{4n}}{1-2q^{2n-1} \cos(\pi v/K) + q^{4n-2}} \right] \end{aligned} \quad [\text{A.29a}]$$

$$\begin{aligned} \text{sn}(v|k) = & \frac{2K}{\pi} \sin \left( \frac{\pi v}{2K} \right) \prod_{n=1}^{\infty} \left[ \frac{(1-q^{2n-1})^2}{(1-q^{2n})^2} \right. \\ & \left. \times \frac{1-2q^{2n} \cos(\pi v/K) + q^{4n}}{1-2q^{2n-1} \cos(\pi v/K) + q^{4n-2}} \right] \end{aligned} \quad [\text{A.29b}]$$

$$\begin{aligned} \text{dn}(v|k) = & \prod_{n=1}^{\infty} \left[ \frac{(1-q^{2n-1})^2}{(1+q^{2n-1})^2} \right. \\ & \left. \times \frac{1+2q^{2n-1} \cos(\pi v/K) + q^{4n-2}}{1-2q^{2n-1} \cos(\pi v/K) + q^{4n-2}} \right] \end{aligned} \quad [\text{A.29c}]$$

When  $k \ll 1$ ,

$$\text{cn}(v|k) \approx \cos v + k^2 \sin v (v - \sin v \cos v)/4 \quad [\text{A.30a}]$$

$$\text{sn}(v|k) \approx \sin v - k^2 \sin v (v - \sin v \cos v)/4 \quad [\text{A.30b}]$$

$$\text{dn}(v|k) \approx 1 - (k^2 \sin^2 v)/2 \quad [\text{A.30c}]$$

When  $k$  is near one,

$$\begin{aligned} \text{cn}(v|k) \approx & \text{sech } v - \frac{1-k^2}{4} \tanh v \text{sech } v \\ & \times (\sinh v \cosh v - v) \end{aligned} \quad [\text{A.31a}]$$

$$\text{sn}(v|k) \approx \tanh v + \frac{1-k^2}{4} \text{sech}^2(\sinh v \cos v - v) \quad [\text{A.31b}]$$

$$\begin{aligned} \text{dn}(v|k) \approx & \text{sech } v + \frac{1-k^2}{4} \tanh v \text{sech } v \\ & \times (\sinh v \cosh v + v) \end{aligned} \quad [\text{A.31c}]$$

### Integrals Of Jacobi Elliptic Functions

Define

$$C_n \equiv \int \text{cn}^n(v|k) \text{d}u \quad [\text{A.32}]$$

In the following we denote  $\text{cn}(v|k)$  by  $\text{cn}v$  only, with  $k$  acting implicitly. The following integrals hold

$$C_0 = F(\varphi, k) \quad [\text{A.33a}]$$

$$C_1 = \frac{\cos(\text{dn}v)}{k} \quad [\text{A.33b}]$$

$$C_2 = \frac{1}{k^2} [E(\varphi, k) - (1-k^2)v] \quad [\text{A.33c}]$$

$$\begin{aligned} C_3 = & \frac{1}{2k^3} [(2k^2-1) \sin(k \text{sn}v) + k \text{sn}v \text{dn}v] \\ & (2-3k^2)k'^2 + 2(2k^2-1)E(\varphi, k) \end{aligned} \quad [\text{A.33d}]$$

$$\begin{aligned} C_4 = & \frac{+k^2 \text{sn}v \text{cn}v \text{dn}v}{3k^4} \\ & 2n(2k^2-1)C_{2n} + (2n-1)(1-k^2)C_{2n-2} \end{aligned} \quad [\text{A.33e}]$$

$$\begin{aligned} C_{2n+2} = & \frac{+ \text{sn}v \text{dn}v \text{cn}^{2n-1}v}{(2n+1)k^2} \\ & (2n+1)(2k^2-1)C_{2n+1} + 2nk'^2 C_{2n-1} \end{aligned} \quad [\text{A.33f}]$$

$$\begin{aligned} C_{2n+3} = & \frac{+ \text{sn}v \text{dn}v \text{cn}^{2n}v}{2(n+1)k^2} \\ & \text{with similar expressions for the other 11 Jacobi functions, see} \end{aligned} \quad [\text{A.33g}]$$

Byrd and Friedman [1954].