

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

The Whitham Equation as a model for surface water waves

Daulet Moldabayev^a, Henrik Kalisch^{a,*}, Denys Dutykh^b^a Department of Mathematics, University of Bergen, Postbox 7800, 5020 Bergen, Norway^b LAMA, UMR5127, CNRS - Université Savoie Mont Blanc, 73376 Le Bourget-du-Lac Cedex, France

HIGHLIGHTS

- Definition of Whitham scaling regime.
- Derivation of a Hamiltonian Whitham system.
- Asymptotic derivation of the Whitham equation.
- Comparison of Whitham, KdV, BBM and Padé models with inviscid free surface dynamics.

ARTICLE INFO

Article history:

Received 16 October 2014

Received in revised form

9 June 2015

Accepted 24 July 2015

Available online 31 July 2015

Communicated by J. Bronski

Keywords:

Nonlocal equations

Nonlinear dispersive equations

Hamiltonian models

Solitary waves

Surface waves

ABSTRACT

The Whitham equation was proposed as an alternate model equation for the simplified description of uni-directional wave motion at the surface of an inviscid fluid. As the Whitham equation incorporates the full linear dispersion relation of the water wave problem, it is thought to provide a more faithful description of shorter waves of small amplitude than traditional long wave models such as the KdV equation.

In this work, we identify a scaling regime in which the Whitham equation can be derived from the Hamiltonian theory of surface water waves. A Hamiltonian system of Whitham type allowing for two-way wave propagation is also derived. The Whitham equation is integrated numerically, and it is shown that the equation gives a close approximation of inviscid free surface dynamics as described by the Euler equations. The performance of the Whitham equation as a model for free surface dynamics is also compared to different free surface models: the KdV equation, the BBM equation, and the Padé (2,2) model. It is found that in a wide parameter range of amplitudes and wavelengths, the Whitham equation performs on par with or better than the three considered models.

© 2015 The Authors. Published by Elsevier B.V.

This is an open access article under the CC BY-NC-ND license

[\(http://creativecommons.org/licenses/by-nc-nd/4.0/\)](http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

In its simplest form, the water-wave problem concerns the flow of an incompressible inviscid fluid with a free surface over a horizontal impenetrable bed. In this situation, the fluid flow is described by the Euler equations with appropriate boundary conditions, and the dynamics of the free surface are of particular interest in the solution of this problem.

There are a number of model equations which allow the approximate description of the evolution of the free surface without having to provide a complete solution of the fluid flow below the surface. In the present contribution, interest is focused

on the derivation and evaluation of a nonlocal water-wave model known as the Whitham equation. The equation is written as

$$\eta_t + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x + K_{h_0} * \eta_x = 0, \quad (1)$$

where the convolution kernel K_{h_0} is given in terms of the Fourier transform by

$$\mathcal{F}K_{h_0}(\xi) = \sqrt{\frac{g \tanh(h_0 \xi)}{\xi}}, \quad (2)$$

g is the gravitational acceleration, h_0 is the undisturbed depth of the fluid, and $c_0 = \sqrt{gh_0}$ is the corresponding long-wave speed. The convolution can be thought of as a Fourier multiplier operator, and (2) represents the Fourier symbol of the operator.

The Whitham equation was proposed by Whitham [1] as an alternative to the well known Korteweg–de Vries (KdV) equation

$$\eta_t + c_0 \eta_x + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x + \frac{1}{6} c_0 h_0^2 \eta_{xxx} = 0. \quad (3)$$

* Corresponding author.

E-mail addresses: daulet.moldabayev@math.uib.no (D. Moldabayev), henrik.kalisch@math.uib.no (H. Kalisch), Denys.Dutykh@univ-savoie.fr (D. Dutykh).

<http://dx.doi.org/10.1016/j.physd.2015.07.010>0167-2789/© 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

The validity of the KdV equation as a model for surface water waves can be described as follows. Suppose a wave field with a prominent amplitude a and characteristic wavelength l is to be studied. The KdV equation is known to produce a good approximation of the evolution of the waves if the amplitude of the waves is small and the wavelength is large when compared to the undisturbed depth, and if in addition, the two non-dimensional quantities a/h_0 and h_0^2/l^2 are of similar size. The latter requirement can be written in terms of the Stokes number as

$$\delta = \frac{al^2}{h_0^3} \sim 1.$$

While the KdV equation is a good model for surface waves if $\delta \sim 1$, one notorious problem with the KdV equation is that it does not model accurately the dynamics of shorter waves. Recognizing this shortcoming of the KdV equation, Whitham proposed to use the same nonlinearity as the KdV equation, but couple it with a linear term which mimics the linear dispersion relation of the full water-wave problem. Thus, at least in theory, the Whitham equation can be expected to yield a description of the dynamics of shorter waves which is closer to the solutions of the more fundamental Euler equations which govern the flow.

The Whitham equation has been studied from a number of vantage points during recent years. In particular, the existence of traveling and solitary waves has been established in [2,3]. Well posedness of a similar equation was investigated in [4–6], and a model with variable depth has been studied numerically in [7]. Moreover, it has been shown in [8,9] that periodic solutions of the Whitham equation feature modulational instability for short enough waves in a similar way as small-amplitude periodic wave solutions of the water-wave problem. However, even though the equation is routinely mentioned in texts on nonlinear waves [10, 11], it appears that the performance of the Whitham equation in the description of surface water waves has not been investigated so far.

The purpose of the present article is to give an asymptotic derivation of the Whitham equation as a model for surface water waves, and to confirm Whitham's expectation that the equation is a fair model for the description of time-dependent surface water waves. For the purpose of the derivation, we introduce an exponential scaling regime in which the Whitham equation can be derived asymptotically from an approximate Hamiltonian principle for surface water waves. In order to motivate the use of this scaling, note that the KdV equation has the property that wide classes of initial data decompose into a number of solitary waves and small-amplitude dispersive residue [12]. For the KdV equations, solitary-wave solutions are known in closed form, and are given by

$$\eta = \frac{a}{h_0} \operatorname{sech}^2 \left(\sqrt{\frac{3a}{4h_0^3}} (x - ct) \right) \quad (4)$$

for a certain wave celerity c . These waves clearly comply with the amplitude–wavelength relation $a/h_0 \sim h_0^2/l^2$ which was mentioned above. It appears that the Whitham equation – as indeed do many other nonlinear dispersive equations – also has the property that broad classes of initial data rapidly decompose into ordered trains of solitary waves (see Fig. 1). Quantifying the amplitude–wavelength relation for these solitary waves yields an asymptotic regime which is expected to be relevant to the validity of the Whitham equation as a water wave model.

As the curve fit in the right panel of Fig. 1 shows, the relationship between wavelength and amplitude of the Whitham solitary waves can be approximately described by the relation $\frac{a}{h_0} \sim e^{-\kappa(l/h_0)^\nu}$ for certain values of κ and ν . Since the Whitham solitary waves are not known in exact form, the values of κ and ν have to be found numerically. Then one may define a Whitham scaling regime

$$\mathcal{W}(\kappa, \nu) = \frac{a}{h_0} e^{\kappa(l/h_0)^\nu} \sim 1, \quad (5)$$

and it will be shown in Sections 2 and 3 that this scaling can be used advantageously in the derivation of the Whitham equation. The derivation proceeds by examining the Hamiltonian formulation of the water-wave problem due to Zakharov, Craig and Sulem [13,14], and by restricting to wave motion which is predominantly in the direction of increasing values of x . The approach is similar to the method of [15], but relies on the new relation (5).

First, in Section 2, a Whitham system is derived which allows for two-way propagation of waves. The Whitham equation is found in Section 3. Finally, in Section 4, a comparison of modeling properties of the KdV and Whitham equations is given. The comparison also includes the regularized long-wave equation

$$\eta_t + c_0 \eta_x + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x - \frac{1}{6} h_0^2 \eta_{xxt} = 0, \quad (6)$$

which was put forward in [16] and studied in depth in [17], and which is also known as the BBM or PBBM equation. The linearized dispersion relation of this equation is not an exact match to the dispersion relation of the full water-wave problem, but it is much closer than the KdV equation, and it might also be expected that this equation may be able to model shorter waves more successfully than the KdV equation. In order to obtain an even better match of the linear dispersion relation, one may make use of Padé expansions. In the context of simplified evolutions equations, this approach was pioneered in [18]. For uni-directional models, this approach was advocated in [19], and in particular, the equation based on the Padé (2,2) approximation was studied in depth. In dimensional variables, this model takes the form

$$\eta_t + c_0 \eta_x + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x - \frac{3}{20} c_0 h_0^2 \eta_{xxx} - \frac{19}{60} h_0^2 \eta_{xxt} = 0. \quad (7)$$

The dispersion relations for the KdV, BBM and Padé (2,2) models are respectively

$$c(k) = c_0 - \frac{1}{6} c_0 h_0^2 k^2 \quad (\text{KdV}),$$

$$c(k) = c_0 \frac{1}{1 + \frac{1}{6} h_0^2 k^2} \quad (\text{BBM}),$$

$$c(k) = c_0 \frac{1 + \frac{3}{20} h_0^2 k^2}{1 + \frac{19}{60} h_0^2 k^2} \quad (\text{Padé (2,2)}).$$

These approximate dispersion relations are compared to the full dispersion relation in Fig. 2. It appears clearly that the Padé (2,2) approximation remains much closer to the full dispersion relation than the dispersion relations based on either the linear KdV or linear BBM equations. As will be seen in Section 4, solutions of both the Whitham and Padé (2,2) equations give closer approximations to solutions of the full Euler equations than either the KdV or BBM equations in most cases investigated. However, the Whitham equation still keeps a slight edge over the Padé model.

2. Derivation of evolution systems of Whitham type

The surface water-wave problem is generally described by the Euler equations with slip conditions at the bottom, and kinematic and dynamic boundary conditions at the free surface. Assuming weak transverse effects, the unknowns are the surface elevation $\eta(x, t)$, the horizontal and vertical fluid velocities $u_1(x, z, t)$ and $u_2(x, z, t)$, respectively, and the pressure $P(x, z, t)$. If the assumption of irrotational flow is made, then a velocity potential $\phi(x, z, t)$ can be used. In order to nondimensionalize the problem, the undisturbed depth h_0 is taken as a unit of distance, and the

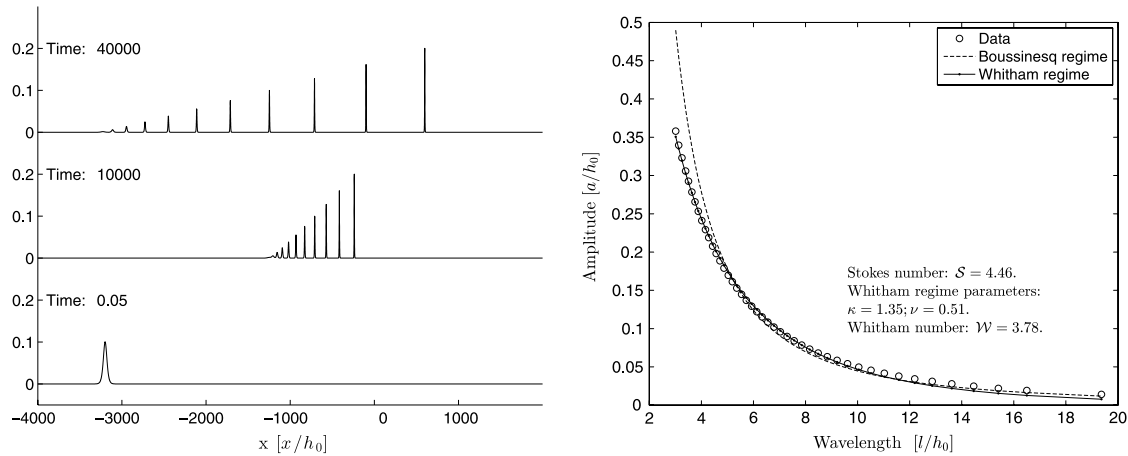


Fig. 1. Left panel: Formation of solitary waves of the Whitham equation from Gaussian initial data. Right panel: Curve fit for the Whitham regime and for the Boussinesq regime to amplitude/wavelength data from Whitham solitary waves. The wavelength is defined as $l = \frac{1}{a} \int_{-\infty}^{\infty} \eta(x) dx$.

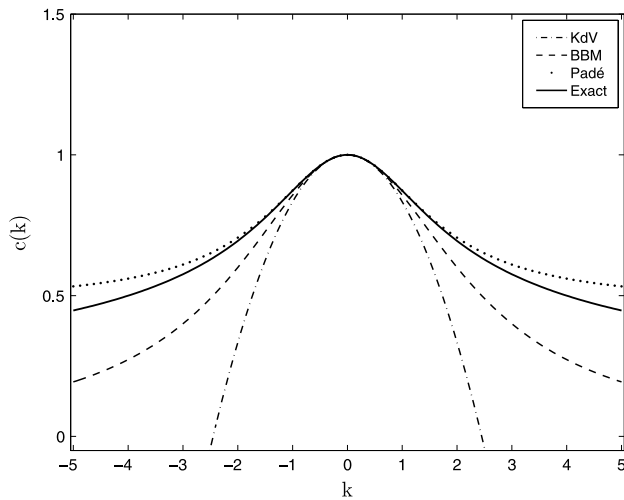


Fig. 2. Approximate models to the exact dispersion relation of the full water-wave problem.

parameter $\sqrt{h_0/g}$ as a unit of time. For the remainder of this article, all variables appearing in the water-wave problem are considered as being non-dimensional. The problem is posed on a domain $\{(x, z)^T \in \mathbb{R}^2 \mid -1 < z < \eta(x, t)\}$ which extends to infinity in the positive and negative x -direction. Due to the incompressibility of the fluid, the potential then satisfies Laplace's equation in this domain. The fact that the fluid cannot penetrate the bottom is expressed by a homogeneous Neumann boundary condition at the flat bottom. Thus we have

$$\begin{aligned} \phi_{xx} + \phi_{zz} &= 0 \quad \text{in } -1 < z < \eta(x, t) \\ \phi_z &= 0 \quad \text{on } z = -1. \end{aligned}$$

The pressure is eliminated with the help of the Bernoulli equation, and the free-surface boundary conditions are formulated in terms of the potential and the surface excursion by

$$\left. \begin{aligned} \eta_t + \phi_x \eta_x - \phi_z &= 0, \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + \eta &= 0, \end{aligned} \right\} \text{on } z = \eta(x, t).$$

The total energy of the system is given by the sum of kinetic energy and potential energy, and normalized such that the potential energy is zero when no wave motion is present at the surface. Accordingly the Hamiltonian function for this problem is

$$H = \int_{\mathbb{R}} \int_0^{\eta} z \, dz dx + \int_{\mathbb{R}} \int_{-1}^{\eta} \frac{1}{2} |\nabla \phi|^2 \, dz dx. \quad (8)$$

Defining the trace of the potential at the free surface as $\Phi(x, t) = \phi(x, \eta(x, t), t)$, one may integrate in z in the first integral and use the divergence theorem on the second integral in order to arrive at the formulation

$$H = \frac{1}{2} \int_{\mathbb{R}} [\eta^2 + \Phi G(\eta) \Phi] \, dx. \quad (9)$$

This is the Hamiltonian formulation of the water wave problem as found in [13,20,14], and written in terms of the Dirichlet–Neumann operator $G(\eta)$. As shown in [21], the Dirichlet–Neumann operator can be expanded in a series of the form

$$G(\eta) \Phi = \sum_{j=0}^{\infty} G_j(\eta) \Phi. \quad (10)$$

In order to proceed, we need to understand the first few terms in this series. As shown in [15,13], the first two terms in this series can be written with the help of the operator $D = -i\partial_x$ as

$$G_0(\eta) = D \tanh(D), \quad G_1(\eta) = D\eta D - D \tanh(D)\eta D \tanh(D).$$

Note that it can be shown that the terms $G_j(\eta)$ for $j \geq 2$ are of quadratic or higher-order in η , and will therefore not be needed in the current analysis.

It will be convenient for the present purpose to formulate the Hamiltonian in terms of the dependent variable $u = \Phi_x$. To this end, we define the operator $\mathcal{K}(\eta)$ by

$$G(\eta) = D\mathcal{K}(\eta)D.$$

As was the case with $G(\eta)$, the operator $\mathcal{K}(\eta)$ can be expanded in a Taylor series around zero as

$$\mathcal{K}(\eta)\xi = \sum_{j=0}^{\infty} \mathcal{K}_j(\eta)\xi, \quad \mathcal{K}_j(\eta) = D^{-1}G_j(\eta)D^{-1}. \quad (11)$$

In particular, note that $\mathcal{K}_0 = \frac{\tanh D}{D}$. In non-dimensional variables, we write the operator with the integral kernel K_{h_0} as $K = \sqrt{\frac{\tanh D}{D}}$, so that we have $\mathcal{K}_0 = K^2$. The Hamiltonian is then expressed as

$$H = \frac{1}{2} \int_{\mathbb{R}} [\eta^2 + u\mathcal{K}(\eta)u] \, dx. \quad (12)$$

The water-wave problem can then be written as a Hamiltonian system using the variational derivatives of H and posing the Hamiltonian equations

$$\eta_t = -\partial_x \frac{\delta H}{\delta u}, \quad u_t = -\partial_x \frac{\delta H}{\delta \eta}. \quad (13)$$

This system is not in canonical form as the associated structure map $J_{\eta,u}$ is symmetric:

$$J_{\eta,u} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}.$$

We now proceed to derive a system of equations which is similar to the Whitham equation (1), but allows bi-directional wave propagation. This system will be a stepping stone on the way to a derivation of (1), but may also be of independent interest. Consider a wavefield having a characteristic wavelength l and a characteristic amplitude a . Taking into account the nondimensionalization, the two scalar parameters $\lambda = l/h_0$ and $\alpha = a/h_0$ appear. In order to introduce the long-wave and small amplitude approximation into the non-dimensional problem, we use the scaling $\tilde{x} = \frac{1}{\lambda}x$, and $\eta = \alpha\tilde{\eta}$. This induces the transformation $\tilde{D} = \lambda D = -\lambda i\partial x$. If the energy is nondimensionalized in accord with the nondimensionalization mentioned earlier, then the natural scaling for the Hamiltonian is $\tilde{H} = \alpha^2 H$. In addition, the unknown u is scaled as $u = \alpha\tilde{u}$. The scaled Hamiltonian (12) is then written as

$$\begin{aligned} \tilde{H} = & \frac{1}{2} \int_{\mathbb{R}} \tilde{\eta}^2 dx + \frac{1}{2} \int_{\mathbb{R}} \tilde{u} \left[1 - \frac{1}{3} \lambda^{-2} \tilde{D}^2 + \dots \right] \tilde{u} dx \\ & + \frac{\alpha}{2} \int_{\mathbb{R}} \tilde{\eta} \tilde{u}^2 dx - \frac{\alpha}{2} \int_{\mathbb{R}} \tilde{u} \left[\lambda^{-1} \tilde{D} - \frac{1}{3} \lambda^{-3} \tilde{D}^3 + \dots \right] \\ & \times \tilde{\eta} \left[\lambda^{-1} \tilde{D} - \frac{1}{3} \lambda^{-3} \tilde{D}^3 + \dots \right] \tilde{u} dx. \end{aligned}$$

Let us now introduce the small parameter $\mu = \frac{1}{\lambda}$, and assume for simplicity that $\alpha = e^{-\kappa/\mu^\nu}$, which corresponds to the case where $\mathcal{W}(\kappa, \nu) = 1$. Then the Hamiltonian can be written in the following form:

$$\begin{aligned} \tilde{H} = & \frac{1}{2} \int_{\mathbb{R}} \tilde{\eta}^2 dx + \frac{1}{2} \int_{\mathbb{R}} \tilde{u} \left[1 - \frac{1}{3} \mu^2 \tilde{D}^2 + \dots \right] \tilde{u} dx \\ & + \frac{e^{-\kappa/\mu^\nu}}{2} \int_{\mathbb{R}} \tilde{\eta} \tilde{u}^2 dx - \frac{e^{-\kappa/\mu^\nu}}{2} \int_{\mathbb{R}} \tilde{u} \left[\mu \tilde{D} - \frac{1}{3} \mu^3 \tilde{D}^3 + \dots \right] \\ & \times \tilde{\eta} \left[\mu \tilde{D} - \frac{1}{3} \mu^3 \tilde{D}^3 + \dots \right] \tilde{u} dx. \end{aligned}$$

Disregarding terms of order $\mathcal{O}(\mu^2 e^{-\kappa/\mu^\nu})$, but not of order $\mathcal{O}(e^{-\kappa/\mu^\nu})$ yields the expansion

$$\begin{aligned} \tilde{H} = & \frac{1}{2} \int_{\mathbb{R}} \tilde{\eta}^2 dx + \frac{1}{2} \int_{\mathbb{R}} \tilde{u} \left[1 - \frac{1}{3} \mu^2 \tilde{D}^2 + \dots \right] \tilde{u} dx \\ & + \frac{e^{-\kappa/\mu^\nu}}{2} \int_{\mathbb{R}} \tilde{\eta} \tilde{u}^2 dx. \end{aligned} \quad (14)$$

Note that by taking μ small enough, an arbitrary number of terms of algebraic order in μ may be kept in the asymptotic series, so that the truncated version of the Hamiltonian in dimensional variables may be written as

$$H = \frac{1}{2} \int_{\mathbb{R}} [\eta^2 + u \mathcal{K}_0^N(\eta)u + u\eta u] dx dz. \quad (15)$$

However, the difference between \mathcal{K}_0 and \mathcal{K}_0^N is below the order of approximation, so that it is possible to formally define the truncated Hamiltonian with \mathcal{K}_0 instead of \mathcal{K}_0^N .

Hence, the Whitham system is obtained from the Hamiltonian (15) as follows:

$$\eta_t = -\partial_x \frac{\delta H}{\delta u} = -\mathcal{K}_0 u_x - (\eta u)_x, \quad (16)$$

$$u_t = -\partial_x \frac{\delta H}{\delta \eta} = -\eta_x - uu_x. \quad (17)$$

One may also derive a higher-order equation by keeping terms of order $\mathcal{O}(\mu^2 e^{-\kappa/\mu^\nu})$, but discarding terms of order $\mathcal{O}(\mu^4 e^{-\kappa/\mu^\nu})$. In this case we find the system

$$\eta_t = -\mathcal{K}_0 u_x - (\eta u)_x - (\eta u_x)_{xx},$$

$$u_t = -\eta_x - uu_x + u_x u_{xx}.$$

3. Derivation of evolution equations of Whitham type

In order to derive the Whitham equation for uni-directional wave propagation, it is important to understand how solutions of the Whitham system (16)–(17) can be restricted to either left or right-going waves. As it will turn out, if η and u are such that $\eta = Ku$, then this pair of functions represents a solution of (16)–(17) which is propagating to the right. Indeed, let us analyze the relation between η and u in the linearized Whitham system

$$\eta_t = -\mathcal{K}_0 u_x, \quad (18)$$

$$u_t = -\eta_x. \quad (19)$$

Considering a solution of the system (18)–(19) in the form

$$\eta(x, t) = Ae^{i(\xi x - i\omega t)}, \quad u(x, t) = Be^{i(\xi x - i\omega t)}, \quad (20)$$

gives rise to the matrix equation

$$\begin{pmatrix} -\omega & \frac{\tanh \xi}{\xi} \\ \xi & -\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (21)$$

If existence of a nontrivial solution of this system is to be guaranteed, the determinant of the matrix has to be zero, so that we have $\omega^2 - \frac{\tanh \xi}{\xi} \xi^2 = 0$. Defining the phase speed as $c = \omega/\xi$, we obtain the dispersion relation

$$c = \pm \sqrt{\frac{\tanh \xi}{\xi}}. \quad (22)$$

The choice of $c > 0$ corresponds to right-going wave solutions of the system (18)–(19), and the relation between η and u can be deduced from (19). Accordingly, it is expedient to separate solutions into a right-going part r and a left-going part s which are defined by

$$r = \frac{1}{2}(\eta + Ku), \quad s = \frac{1}{2}(\eta - Ku).$$

According to the transformation theory detailed in [22], if the unknowns r and s are used instead of η and u , the structure map changes to

$$J_{r,s} = \begin{pmatrix} \frac{\partial F}{\partial(\eta, u)} \end{pmatrix} J_{\eta,u} \begin{pmatrix} \frac{\partial F}{\partial(\eta, u)} \end{pmatrix}^T = \begin{pmatrix} -\frac{1}{2} \partial_x K & 0 \\ 0 & \frac{1}{2} \partial_x K \end{pmatrix}. \quad (23)$$

We now use the same scaling for both dependent and independent variables as before. Thus we have $r = \alpha \tilde{r}$ and $s = \alpha \tilde{s}$. The Hamiltonian is written in terms of \tilde{r} and \tilde{s} as

$$\begin{aligned} \tilde{H} = & \frac{1}{2} \int_{\mathbb{R}} (\tilde{r} + \tilde{s})^2 dx \\ & + \frac{1}{2} \int_{\mathbb{R}} \tilde{K}^{-1}(\tilde{r} - \tilde{s}) \left[1 - \frac{1}{3} \mu^2 \tilde{D}^2 + \dots \right] \tilde{K}^{-1}(\tilde{r} - \tilde{s}) dx \\ & + \frac{\alpha}{2} \int_{\mathbb{R}} (\tilde{r} + \tilde{s}) \left(\tilde{K}^{-1}(\tilde{r} - \tilde{s}) \right)^2 dx \\ & - \frac{\alpha}{2} \int_{\mathbb{R}} \tilde{K}^{-1}(\tilde{r} - \tilde{s}) \left[\mu \tilde{D} - \frac{1}{3} \mu^3 \tilde{D}^3 + \dots \right] (\tilde{r} + \tilde{s}) \\ & \times \left[\mu \tilde{D} - \frac{1}{3} \mu^3 \tilde{D}^3 + \dots \right] \tilde{K}^{-1}(\tilde{r} - \tilde{s}) dx. \end{aligned}$$

Following the transformation rules, the structure map transforms to $J_{\tilde{r},\tilde{s}} = 1/\alpha^2 J_{r,s}$. In addition, the time scaling $t = \lambda\tilde{t}$ is employed. Since the focus is on right-going solutions, the equation to be considered is

$$\lambda\tilde{t}_t = -\frac{1}{2\alpha^2}\lambda\partial_{\tilde{x}}\tilde{K}\left[\frac{\delta(\alpha^2\tilde{H})}{\delta\tilde{r}}\right]. \tag{24}$$

So far, this equation is exact. If we now assume that s is of the order of $\mathcal{O}(\mu^2 e^{-\kappa/\mu^v})$, then the equation for \tilde{r} is

$$\begin{aligned} \tilde{r}_t = & -\frac{1}{2}\partial_{\tilde{x}}\left[1 - \frac{1}{6}\mu^2\tilde{D}^2 + \dots\right]\left\{2\tilde{r} + \frac{\alpha}{2}\left([1 + \frac{1}{6}\mu^2\tilde{D}^2 + \dots]\tilde{r}\right)^2\right. \\ & + \alpha\left[1 + \frac{1}{6}\mu^2\tilde{D}^2 + \dots\right]\left(\tilde{r}\left[1 + \frac{1}{6}\mu^2\tilde{D}^2 + \dots\right]\tilde{r}\right) \\ & - \frac{\alpha}{2}\left(\left[\mu\tilde{D} - \frac{1}{3}\mu^3\tilde{D}^3 + \dots\right]\left[1 + \frac{1}{6}\mu^2\tilde{D}^2 + \dots\right]\tilde{r}\right)^2 \\ & - \alpha\left[\mu\tilde{D} - \frac{1}{3}\mu^3\tilde{D}^3 + \dots\right]\left[1 + \frac{1}{6}\mu^2\tilde{D}^2 + \dots\right] \\ & \times \left(\tilde{r}\left[\mu\tilde{D} - \frac{1}{3}\mu^3\tilde{D}^3 + \dots\right]\left[1 + \frac{1}{6}\mu^2\tilde{D}^2 + \dots\right]\tilde{r}\right)\left.\right\} \\ & + \mathcal{O}(\alpha\mu^2). \end{aligned}$$

As in the case of the Whitham system, we use $\alpha = \mathcal{O}(e^{-\kappa/\mu^v})$, and disregard terms of order $\mathcal{O}(\mu^2 e^{-\kappa/\mu^v})$, but not of order $\mathcal{O}(e^{-\kappa/\mu^v})$. This procedure yields the Whitham equation (1) which is written in nondimensional variables as

$$r_t = -Kr_x - \frac{3}{2}rr_x.$$

As was the case for the system found in the previous section, it is also possible here to include terms of order $\mathcal{O}(\mu^2 e^{-\kappa/\mu^v})$, resulting in the higher-order equation

$$r_t = -Kr_x - \frac{3}{2}rr_x - \frac{13}{12}r_x r_{xx} - \frac{5}{12}rr_{xxx}.$$

4. Numerical results

In this section, the performance of the Whitham equation as a model for surface water waves is compared to the KdV equation (3), the BBM equation (6), and the Padé (2,2) equation (7). For this purpose initial data are imposed, the Whitham, KdV, BBM, and Padé equations are solved numerically, and the solutions are compared to numerical solutions of the full Euler equations with free-surface boundary conditions. We continue to work in normalized variables, such as stated in the beginning of Section 2.

The numerical treatment of the three model equations is by a standard pseudo-spectral scheme, such as explained in [23,24] for example. For the time stepping, an efficient fourth-order implicit method developed in [25] is used. The numerical treatment of the free-surface problem for the Euler equations is based on a conformal mapping of the fluid domain into a rectangle. In the time-dependent case, this method has roots in the work of Ovsyanikov [26], and was later used in [27–30]. The particular method used for the numerical experiments reported here is a pseudo-spectral scheme which is detailed in [31].

Initial conditions for the Euler equations are chosen in such a way that the solutions are expected to be right moving. This is achieved by posing an initial surface disturbance $\eta_0(x)$ together with the trace of the potential $\Phi(x) = \int_0^x \eta_0(x') dx'$. In order to normalize the data, we choose $\eta_0(x)$ in such a way that the average of $\eta_0(x)$ over the computational domain is zero. The experiments are performed with several different amplitudes α and wavelengths

Table 1

Summary of the Stokes number, nondimensional amplitude and nondimensional wavelength of the initial data used in the numerical experiments.

Experiment	Stokes number	α	λ
A	0.2	0.1	$\sqrt{2}$
B	0.2	0.2	1
C	1	0.1	$\sqrt{10}$
D	1	0.2	$\sqrt{5}$
E	5	0.1	$\sqrt{50}$
F	5	0.2	5

λ (for the purpose of this section, we define the wavelength λ as the distance between the two points x_1 and x_2 at which $\eta_0(x_1) = \eta_0(x_2) = \alpha/2$). Both positive and negative initial disturbances are considered. While disturbances with positive main part have been studied widely, an initial wave of depression is somewhat more exotic, but nevertheless important, as shown for instance in [32]. A summary of the experiments' settings is given in Table 1. Experiments run with an initial wave of elevation are labeled as *positive*, and experiments run with an initial wave of depression are labeled as *negative*. The domain for the computations is $-L \leq x \leq L$, with $L = 50$. The function initial data in the *positive* cases is given by

$$\eta_0(x) = \alpha \operatorname{sech}^2(f(\lambda)x) - C, \tag{25}$$

where

$$f(\lambda) = \frac{2}{\lambda} \log\left(\frac{1 + \sqrt{1/2}}{\sqrt{1/2}}\right), \quad \text{and} \quad C = \frac{1}{L} \frac{\alpha}{f(\lambda)} \tanh\left(\frac{L}{f(\lambda)}\right)$$

and C and $f(\lambda)$ are chosen so that $\int_{-L}^L \eta_0(x) dx = 0$, and the wavelength λ is the distance between the two points x_1 and x_2 at which $\eta_0(x_1) = \eta_0(x_2) = \alpha/2$. The velocity potential in this case is given by

$$\Phi(x) = \frac{\alpha}{f(\lambda)} \tanh(f(\lambda)x) - Cx. \tag{26}$$

In the *negative* case, the initial data are given by

$$\eta_0(x) = -\alpha \operatorname{sech}^2(f(\lambda)x) + C.$$

The definitions for $f(\lambda)$ and C are the same, and the velocity potential is

$$\Phi(x) = -\frac{\alpha}{f(\lambda)} \tanh(f(\lambda)x) + Cx.$$

In Figs. 3 and 4, the time evolution of a wave with an initial narrow peak and one with an initial narrow depression at the center is shown. The amplitude is $\alpha = 0.2$, and the wavelength is $\lambda = \sqrt{5}$. In Fig. 3, the time evolution according to the Euler, Whitham, KdV and BBM equations are shown, and in Fig. 4, the time evolution according to the Euler, Whitham, and Padé (2,2) equations are shown.

It appears in Fig. 3 that the KdV equation produces a significant number of spurious oscillations, the BBM equation also produces a fair number of spurious oscillations, and the Whitham equation produces fewer small oscillations than Euler equations. Moreover, while the highest peak in the upper panels in Fig. 3 is underpredicted by the KdV and BBM equation, the Whitham equation produces peaks which are slightly too high. In the case of an initial depression, the Whitham equation also produces some peaks which are too high, but on the other hand, the KdV and the BBM equations introduce phase errors in the main oscillations. As is visible in Fig. 4, the Padé (2,2) equation reproduces the leading wave fairly accurately, but overpredicts the trailing oscillations in the case of a positive disturbance, and underpredicts the trailing oscillations in the case of a negative initial disturbance. Nevertheless, since the Padé (2,2) does not introduce a phase error, the overall performance is better than that of the KdV and BBM equations.

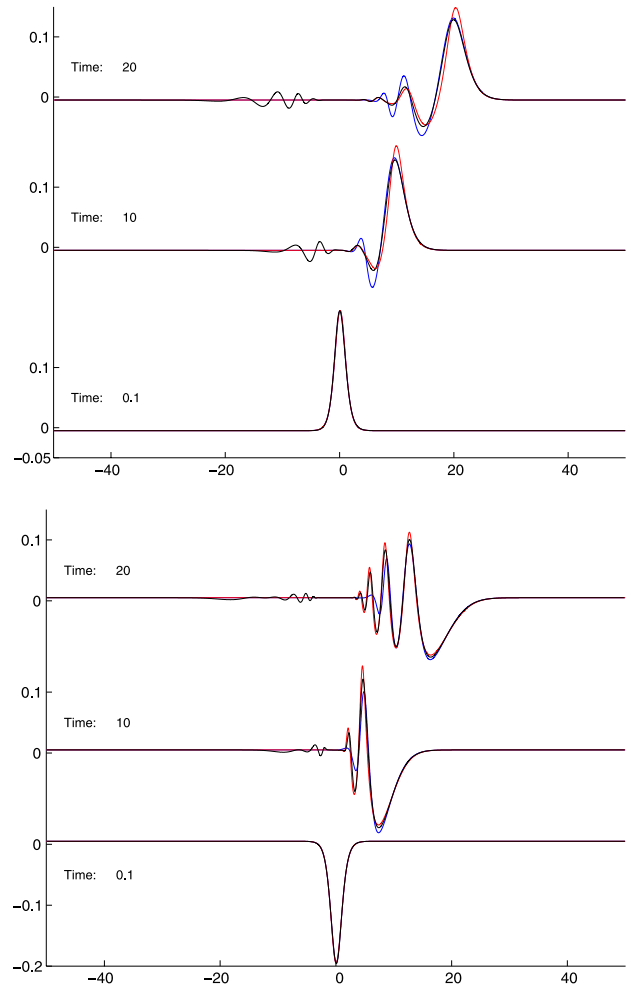
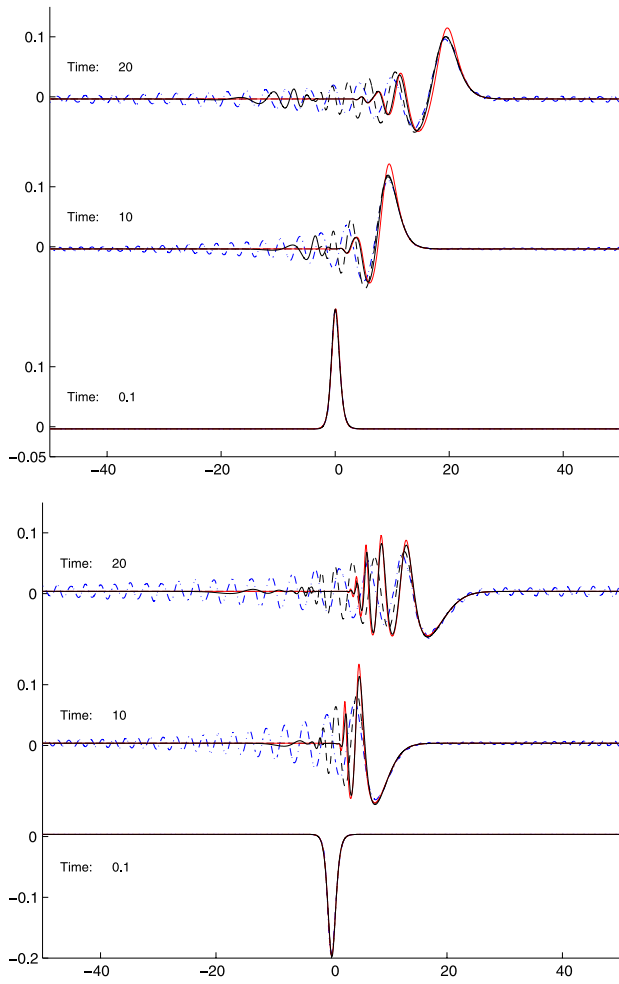


Fig. 3. Wave profiles at three different times: – Euler (black line), --- KdV, - - BBM, – Whitham (red line). Experiment D: $\delta = 1$, $\alpha = 0.2$, $\lambda = \sqrt{5}$. Upper panel: positive case; lower panel: negative case. Horizontal axis: x/h_0 , vertical axis: z/h_0 . Snapshots are given at nondimensional time $t/\sqrt{h_0/g}$.

Fig. 4. Wave profiles at three different times: – Euler (black line), – Padé (blue line), – Whitham (red line). Experiment D: $\delta = 1$, $\alpha = 0.2$, $\lambda = \sqrt{5}$. Upper panel: positive case; lower panel: negative case. Horizontal axis: x/h_0 , vertical axis: z/h_0 . Snapshots are given at nondimensional time $t/\sqrt{h_0/g}$.

In order to compare the performance of the four approximate equations in a more quantitative manner, the discrepancies between solutions of the model equations and the prediction due to solving the Euler equations are measured in an integral norm. In the center right panels of Figs. 5 and 6, the computations from Figs. 3 and 4 are summarized by plotting the normalized L^2 -error between the KdV, BBM, Padé and Whitham, respectively, and the Euler solutions as a function of non-dimensional time. Using this quantitative measure of comparison, it appears that the Whitham equation gives the best overall rendition of the free surface dynamics predicted by the Euler equations.

In the center left panels of Figs. 5 and 6, a similar computation with $\delta = 1$, but smaller amplitude is analyzed. Also in these cases, it appears that the Whitham equation gives a good approximation to the corresponding Euler solutions, and in particular, a much better approximation than either the KdV or the BBM equation. The Padé equation also does better than both KdV and BBM equations, but not better than the Whitham equation.

Figs. 5 and 6 show comparisons in several other cases of both positive and negative initial amplitude, and Stokes numbers $\delta = 0.2$, $\delta = 1$ and $\delta = 5$. In most cases, the normalized L^2 -error between the Whitham and Euler solutions is similar or smaller than the errors between the Euler solutions and the other three model equations. However, the Padé equation generally outperforms both the KdV and the BBM equation by some measure.

The only case in this study in which the KdV, BBM and Padé equations outperform the Whitham equation is in the case of

very long positive initial disturbances. The case when $\delta = 5$ is shown in the lower panels of Fig. 5. However, even in this case, the Whitham equation yields approximations of the Euler solutions which are similar or better than in the case of smaller wavelengths. In addition, in the case of negative initial data, the performance of the Whitham equation is on par with the KdV, BBM and Padé equations in the case when $\delta = 5$ (lower panels of Fig. 5).

5. Conclusion

In this article, the Whitham equation (1) has been studied as an approximate model equation for wave motion at the surface of a perfect fluid. Numerical integration of the equation suggests that broad classes of initial data decompose into individual solitary waves. The wavelength–amplitude ratio of these approximate solitary waves has been studied, and it was found that this ratio can be described by an exponential relation of the form $\frac{a}{h_0} \sim e^{-\kappa(l/h_0)^{\nu}}$. Using this scaling in the Hamiltonian formulation of the water-wave problem, a system of evolution equations has been derived which contains the exact dispersion relation of the water-wave problem in its linear part. Restricting to one-way propagation, the Whitham equation emerged as a model which combines the usual quadratic nonlinearity with one branch of the exact dispersion relation of the water-wave problem. The performance of the Whitham equation in the approximation of

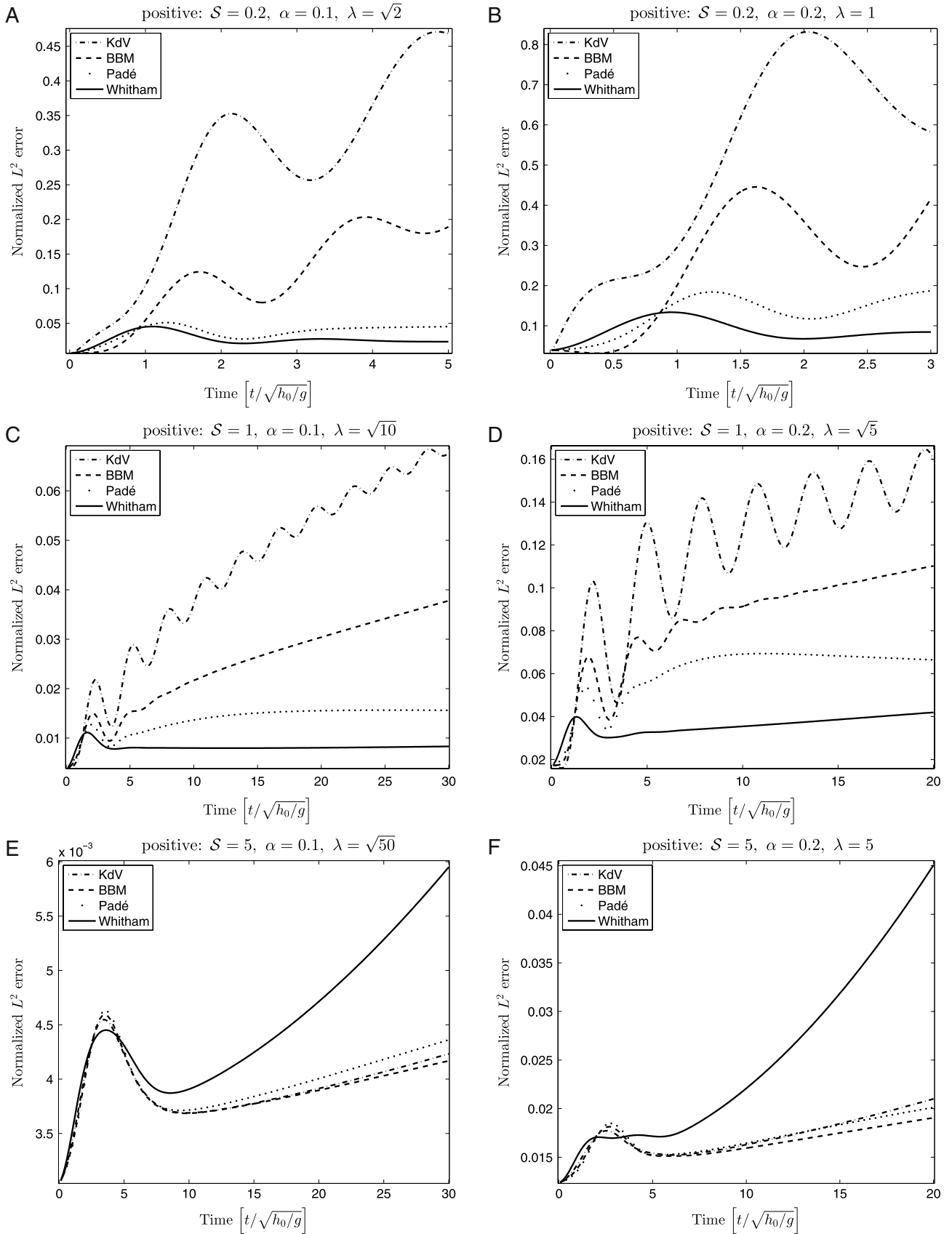


Fig. 5. L^2 errors in approximation of solutions to full Euler equations by different model equations: cases A and B ($\delta = 0.2$), cases C and D ($\delta = 1$), cases E and F ($\delta = 5$), positive.

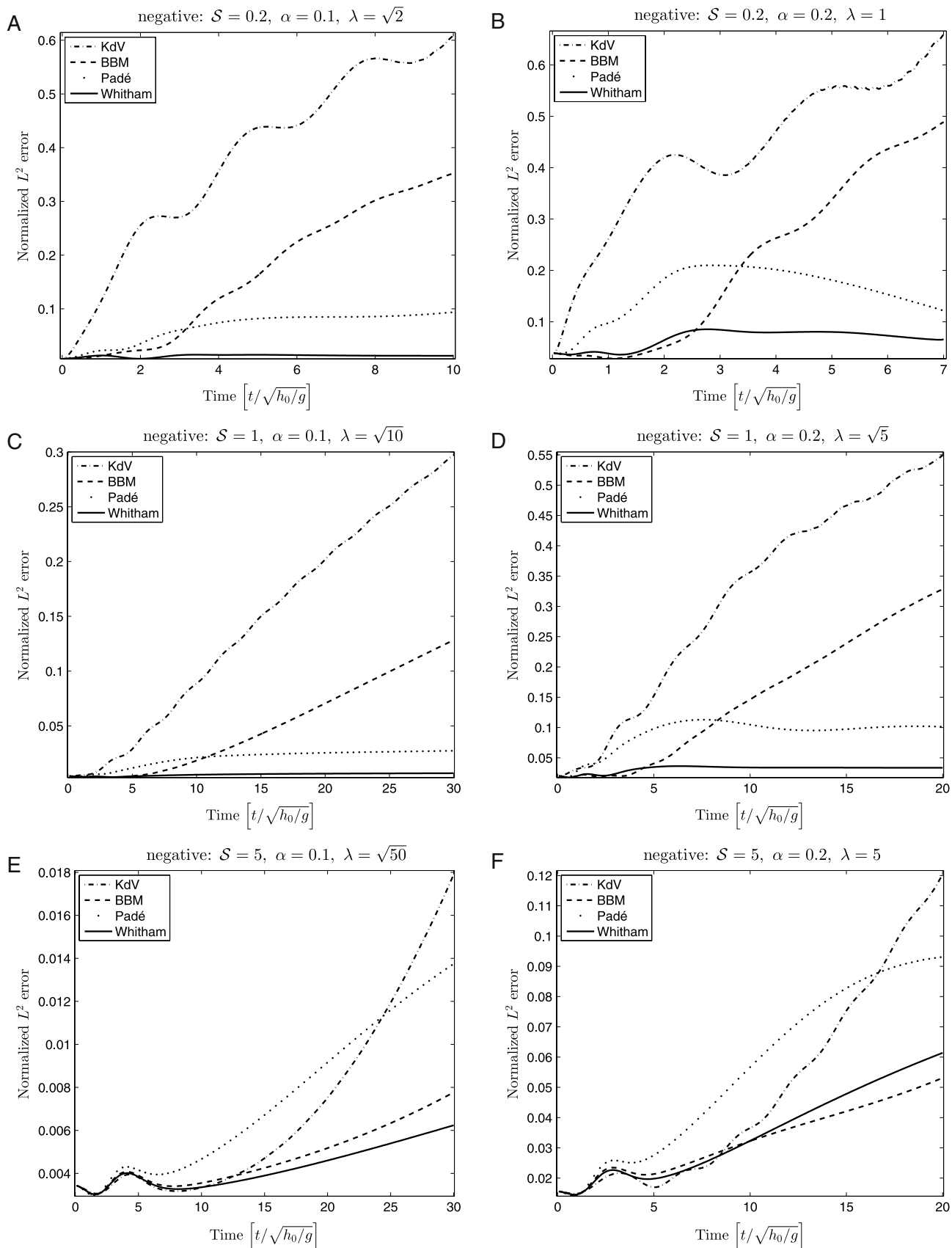


Fig. 6. L^2 errors in approximation of solutions to full Euler equations by different model equations: cases A and B ($S = 0.2$), cases C and D ($S = 1$), cases E and F ($S = 5$), negative.

solutions of the Euler equations free-surface boundary conditions was analyzed, and compared to the performance of the KdV and BBM equations, and to the Padé (2,2) model. It was found that the Whitham equation gives a more faithful representation of the solutions of the Euler equations than either the KdV or the BBM equations, except in the case of very long waves with positive main part. In this last case, the KdV and BBM equations have the upper hand over the Whitham equation. The Padé (2,2) model also outperforms the KdV and BBM equations, but does not quite as well as the Whitham equation for shorter waves and negative disturbances. However, in the case of very long waves with positive main part, the Padé model stays on par with the KdV and BBM equations.

Acknowledgment

This research was supported by the Research Council of Norway (213474/F20).

References

- [1] G.B. Whitham, Variational methods and applications to water waves, *Proc. R. Soc. Lond. Ser. A* 299 (1967) 6–25.
- [2] M. Ehrnström, M.D. Groves, E. Wahlén, Solitary waves of the Whitham equation - a variational approach to a class of nonlocal evolution equations and existence of solitary waves of the Whitham equation, *Nonlinearity* 25 (2012) 2903–2936.
- [3] M. Ehrnström, H. Kalisch, Traveling waves for the Whitham equation, *Differential Integral Equations* 22 (2009) 1193–1210.
- [4] D. Lannes, The Water Waves Problem, in: *Mathematical Surveys and Monographs*, vol. 188, Amer. Math. Soc, Providence, 2013.
- [5] D. Lannes, J.-C. Saut, Remarks on the full dispersion Kadomtsev-Petviashvili equation, *Kinet. Relat. Models* 6 (2013) 989–1009.
- [6] C. Klein, J.-C. Saut, A numerical approach to blow-up issues for dispersive perturbations of Burgers' equation, *Physica D* 295 (2015) 46–65.
- [7] P. Aceves-Sánchez, A.A. Minzoni, P. Panayotaros, Numerical study of a nonlocal model for water-waves with variable depth, *Wave Motion* 50 (2013) 80–93.
- [8] V.M. Hur, M.A. Johnson, Modulational instability in the Whitham equation of water waves, *Stud. Appl. Math.* 134 (1) (2015) 120–143.
- [9] N. Sanford, K. Kodama, J.D. Carter, H. Kalisch, Stability of traveling wave solutions to the Whitham equation, *Phys. Lett. A* 378 (2014) 2100–2107.
- [10] P.G. Drazin, R.S. Johnson, *Solitons: an introduction*, in: *Cambridge Texts in Applied Mathematics*, Cambridge University Press, Cambridge, 1989.
- [11] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.
- [12] M. Ablowitz, H. Segur, Solitons and the inverse scattering transform, in: *SIAM Studies in Applied Mathematics*, vol. 4, SIAM, Philadelphia, 1981.
- [13] W. Craig, C. Sulem, Numerical simulation of gravity waves, *J. Comput. Phys.* 108 (1993) 73–83.
- [14] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.* 9 (1968) 190–194.
- [15] W. Craig, M.D. Groves, Hamiltonian long-wave approximations to the water-wave problem, *Wave Motion* 19 (1994) 367–389.
- [16] D.H. Peregrine, Calculations of the development of an undular bore, *J. Fluid Mech.* 25 (1966) 321–330.
- [17] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. R. Soc. Lond., Ser. A* 272 (1972) 47–78.
- [18] J.M. Witting, A unified model for the evolution of nonlinear water waves, *J. Comput. Phys.* 56 (1984) 203–236.
- [19] R. Fetecau, D. Levy, Approximate model equations for water waves, *Commun. Math. Sci.* 3 (2) (2005) 159–170.
- [20] A.A. Petrov, Variational statement of the problem of liquid motion in a container of finite dimensions, *Prikl. Math. Mekh.* 28 (1964) 917–922.
- [21] D.P. Nicholls, F. Reitich, A new approach to analyticity of Dirichlet-Neumann operators, *Proc. Roy. Soc. Edinburgh Sect. A* 131 (2001) 1411–1433.
- [22] W. Craig, P. Guyenne, H. Kalisch, Hamiltonian long-wave expansions for free surfaces and interfaces, *Comm. Pure Appl. Math.* 58 (2005) 1587–1641.
- [23] M. Ehrnström, H. Kalisch, Global bifurcation for the Whitham equation, *Math. Mod. Nat. Phenomena* 8 (2013) 13–30.
- [24] B. Fornberg, G.B. Whitham, A numerical and theoretical study of certain nonlinear wave phenomena, *Phil. Trans. Roy. Soc. A* 289 (1978) 373–404.
- [25] J. De Frutos, J.M. Sanz-Serna, An easily implementable fourth-order method for the time integration of wave problems, *J. Comput. Phys.* 103 (1992) 160–168.
- [26] L.V. Ovsiannikov, To the shallow water theory foundation, *Arch. Mech.* 26 (1974) 407–422.
- [27] A.I. Dyachenko, E.A. Kuznetsov, M.D. Spector, V.E. Zakharov, Analytical description of the free surface dynamics of an ideal fluid (canonical formalism and conformal mapping), *Phys. Lett. A* 221 (1996) 73–79.
- [28] Y.A. Li, J.M. Hyman, W. Choi, A numerical study of the exact evolution equations for surface waves in water of finite depth, *Stud. Appl. Math.* 113 (2004) 303–324.
- [29] W. Choi, R. Camassa, Exact evolution equations for surface waves, *J. Eng. Mech.* 125 (1999) 756–760.
- [30] P. Milewski, J.-M. Vanden-Broeck, Z. Wang, Dynamics of steep two-dimensional gravity-capillary solitary waves, *J. Fluid Mech.* 664 (2010) 466–477.
- [31] D. Mitsotakis, D. Dutykh, J.D. Carter, On the nonlinear dynamics of the traveling-wave solutions of the Serre equations (2014) submitted for publication, <http://arxiv.org/abs/1404.6725>.
- [32] J.L. Hammack, H. Segur, The Korteweg-de Vries equation and water waves. Part 2. Comparison with experiments, *J. Fluid Mech.* 65 (1974) 289–314.