Differential Information Economies and Incomplete Markets

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Foreword

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ABSTRACT

In a pure exchange economy with differential information, there may be ex-post Pareto-dominant core allocations which are not attainable as Rational Expectations Equilibria because of information verifiability issues. On the other hand, many of the core allocations in the differential information economy do not seem realistic, given incentive constraints. This fundamental tension between missed trading opportunities and moral hazard will be explored using concepts from cooperative game theory and financial economics.

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1 Introduction

When two economic agents negotiate the terms of a state-contingent contract, they prefer the contract to be in terms they can verify. In the presence of asymmetric information, there are deals that are left on the table because of this verifiability constraint. (See, for instance, Flåm (2007)). Often, there are allocations in the core of a differential information economy that are not attainable as a Rational Expectations Equilibrium. This result seems to contradict the First Welfare Theorem, which states that competitive equilibrium, in the absence of externalities, is Pareto-optimal. Here, there are markets with competitive equilibria that are Pareto-dominated by allocations in the core. In an example presented by Allen and Yannelis (2001), autarky is the only competitive equilibrium, but it is strictly Pareto-dominated by allocations in the core. However, this should not be interpreted as a refutation of the First Welfare Theorem, because, when information is distributed asymmetrically, the market is not entirely competitive.

The question then arises what would be an ideal solution concept for differential information economies. An ideal solution concept should be incentive compatible, i.e. it should not invite deviations by individuals or coalitions. Thus, it would have to belong to the core. This suggests that we should look to cooperative games for such a solution. However, while the core describes allocations that are feasible, and individually and coalitionally stable, the description of the underlying cooperative game does not explain *how* the individuals negotiate their way to a given core solution. In other words, it does not take into account the strategic aspects of the game arising from the presence of asymmetric information, such as the possibility of misrepresenting information. The standard core refinements for asymmetric information economies, such as Wilson's coarse and fine cores and Yannelis' private core, do not adequately address this issue either, as I will show.

I suggest two approaches. One is to modify the concepts of coarse core and the

private core to take into account the incentive compatibility of each agents' actions. I call my modifications of the coarse core and the private core, the locally coarse core and action-measurable private core, respectively.

The other approach is to to model trading on information as a financial market, since the invisible hand of the market can often can accomplish what cannot be accomplished through an analytic solution.

A considerable amount of mathematical machinery is required to study these issues formally. I present the mathematical preliminaries in Section 2. In Section 3, I present several core concepts designed for differential information economy and discuss their strengths and weaknesses. In Section 4, I discuss the approach of modeling differential information economy as a financial market. In Section 5, I summarize the results of this paper and present ideas for further research.

2 Preliminaries

2.1 Exchange economies, coalitional games, and the core

A *pure exchange economy* is an economy with no production. I restrict attention to economies with a finite number of goods and a finite set of agents. Each agent is characterized by an initial endowment of goods and an increasing, complete, transitive, continuous, and convex preference relation over the possible bundles of goods. This preference relation is representable by an increasing, continuous, and concave, possibly quasi-linear utility function.

Formally, an exchange economy (without uncertainty) is described by:

$$\langle N, C, (e_i), (u_i) \rangle \qquad (i \in N),$$

where

- N is the finite set of agents
- $C \in \mathbb{Z}_{++}$ is the number of physical goods in the economy, and $\mathbb{R}^C$ is the finite-dimensional commodity space
- $e_i \in \mathbb{R}^C_+$ is the initial endowment of agent $i \in N$
- u_i is the utility function representing agent i's preferences over $\mathbb{R}^C$.

The assumption that the preference relation is increasing assures that every good in the economy is desirable, and that there are no economic "bads." Completeness of preferences ensures that any two bundles of goods can be compared. Transitivity is required for preference maximization: without transitivity there might be bundles of goods that have no best elements, such as in the preference cycle $A \succ B \succ C \succ A$. Continuity of preferences guarantees the existence of a utility representation, and rules out lexicographic and other preferences that are hard to deal with mathematically: it entails that if an agent strictly prefers market basket x to y, there exists a

number $\varepsilon > 0$ such that, for any market basket z, whenever $\|x - z\| < \varepsilon$, the agent also prefers z to y. Finally, concavity of the utility function ensures that agents prefer to smooth consumption.

Initial endowments of goods are assumed to be non-negative: an agent cannot, at least initially, hold a negative amount of a good. In addition, agents are assumed to behave competitively, i.e., to take prices as given, and to try to maximize their utility by choosing the best possible bundle of goods they can afford under their budget constraint.

Utility functions may be, but are not necessarily assumed to be, of the von Neumann-Morgenstern type. Initially, utility functions are assumed to be strictly concave, but later I will make use of quasi-linear utility functions with utility linear (and with the same slope) in the same good for every agent. The good in which utility is linear can be thought as money, and used to shift utility among the agents, without affecting the total utility achieved in the economy. This makes the game associated with the exchange economy a *transferable utility (TU)* game. Transferable utility can be used to satisfy incentive constraints and facilitate trade where it otherwise would not be possible because of information asymmetries or incentive compatibility issues (Forges et al. 2002), and thus lead to more efficient allocations in the economy.

Following Osborne and Rubinstein (1994), I associate the exchange economy with transferable payoff $\langle N, C, (e_i), (u_i) \rangle$ with the coalitional game $\langle N, v \rangle$, where N is the set of agents.

Let $S \in 2^N \backslash \varnothing$ be a nonempty subset of N, termed a *coalition.*[1] Each agent joining a coalition brings with himself or herself a vector of endowments $e_i \in \mathbb{R}^C_+$. The total resources of the coalition are then represented by a vector $\sum_{i \in S} e_i$ of these endowments.

[1]The coalition N of all agents is called the *grand coalition.*

A profile of vectors $(z_i)_{i\in N}$ is called an *allocation* when

$$\sum_{i\in N} z_i = \sum_{i\in N} e_i.$$

That is, a profile of vectors is an allocation when it is a (re)distribution of the total endowment in the economy among all the agents. As with endowments, I will generally require final allocations to be non-negative.

Let v be a function that associates real number, $v(S)$ with each coalition S. Then $v(S)$, called the *value of coalition*, represents the total (maximum) payoff available for division among the members of the coalition S when it forms.

Definition 2.1. Value of coalition (for a TU game):

$$v : S \in 2^N \backslash \varnothing \to \mathbb{R}$$

$$v(S) = \max \left\{ \sum_{i\in S} u_i(z_i) : z_i \in \mathbb{R}^C_+ \text{ and } \sum_{i\in S} z_i = \sum_{i\in S} e_i \right\}$$

Note that because utility is transferable, $v(S)$ is a scalar. The idea is that we are only interested in the total payoff the coalition can attain, and assume that the coalition can divide the transferable utility appropriately among its members to keep them from defecting. When utility is not transferable (NTU), the value of a coalition, $V(S)$ is defined as:

Definition 2.2. Value of coalition (for a NTU game):

$$V : S \in 2^N \backslash \varnothing \to \mathbb{R} \cup \mathbb{R}^2 \cup ... \cup \mathbb{R}^N$$

$$V(S) = \left\{ (u(x_i))_{i\in S} \mid (\forall i \in S) x_i \in \mathbb{R}^C_+, \sum_{i\in S} x_i = \sum_{i\in S} e_i \right\}$$

where $(x_i)_{i\in S}$ is required to be Pareto-optimal within S.

That is, the value of coalition in a NTU game is the set of all utility vectors that are Pareto-optimal and achievable by the coalition on its own, with the additional condition that the allocation uses up the combined endowments of the agents in the coalition.[2]

A *core solution* is an allocation i.e. a (re)distribution of the total endowment in the economy among all the agents which satisfies two conditions: feasibility and stability. Let $(z_i)_{i\in S}$ be a proposed allocation of commodities that the members of the coalition S would get if they join the grand coalition, and let $\sum_{i\in S} u_i(z_i)$ be the total payoff to the coalition S if they join the grand coalition. Feasibility is defined as follows:

Definition 2.3. Feasibility: An allocation $(z_i)_{i\in N}$ is *feasible* in a TU game if

$$\sum_{i\in N} u_i(z_i) = v(N).$$

The allocation $(z_i)_{i\in N}$ is feasible in an NTU game if it is in $V(N)$, as defined in Definition 2.2.

The *stability* condition requires that no individual agent or a coalition of agents S can obtain a higher payoff for themselves if they break away from the grand coalition. Formally, the definition of stability is:

Definition 2.4. Stability: An allocation $(z_i)_{i\in N}$ is *stable* in a TU game if, for every $S \in 2^N \backslash \varnothing$,

$$v(S) \leq \sum_{i\in S} u_i(z_i).$$

The allocation $(z_i)_{i\in N}$ is stable in an NTU game if, for every $S \in 2^N \backslash \varnothing$, and every proposed S-allocation $(\widehat{z}_i)_{i\in S}$ (i.e., every proposed reallocation of the endowments of the members of S), either all the members of S are indifferent between (z_i) and

[2]There are alternative definitions. For example, Osborne and Rubinstein (1994) do not include the utility or Pareto-optimality in their definition of value for a NTU economy.

$(\widehat{z}_i)$, or at least one member of S strictly prefers (z_i) to $(\widehat{z}_i)$. That is,

$$\forall(\widehat{z}_i)_{i\in S} \text{ such that } \sum_{i\in S}\widehat{z}_i = \sum_{i\in S} e_i \text{ and } (\forall i \in S)\widehat{z}_i \geq 0, \text{ if}$$

$$\exists i \in S \text{ such that } u_i(\widehat{z}_i) > u_i(z_i), \text{ then also } \exists j \in S \text{ such that } u_j(z_j) > u_j(\widehat{z}_j).$$

If the stability condition is violated, the coalition S is said to *block* the allocation z, and the allocation z is not in the core.

It is commonly assumed that TU games satisfy a property called *superadditivity*:[3] the addition of agents to coalitions cannot lower the feasible payoffs for the original coalition members. Formally, for all coalitions S and T, if $S \cap T = \varnothing$, then:

$$v(S \cup T) \geq v(S) + v(T).$$

2.2 Uncertainty and asymmetric information

To formally describe economies where agents have asymmetric information, two notions must be introduced: *uncertainty* and *information partitions*. The first economists to introduce uncertainty into the standard economic model were Arrow (1953) and Debreu (1959), who considered a world where goods differ in time or state of the world, and Harsanyi (1967, 1968a,b) who introduced outcomes that depend on choices that nature makes into non-cooperative games. Later, Radner (1968) added asymmetric information into the Arrow-Debreu model. In this paper, I follow Radner's general approach and define:

1. A set of possible states of nature—descriptions of possible future events that might affect an agent's utility.

[3] In the presence of differential information, superadditivity does not always hold (Allen 2006), but I will assume that games are at least *cohesive*. Cohesiveness means that the payoff to the coalition N of all players must be at least as large as the sum of the payoffs of the members of any partition of N. It ensures that it is optimal for the grand coalition to form.

2. A collection of subsets of the state space, representing information that the agent can discern.[4]

3. A probability measure over (2).

Let Ω symbolize the state space, and let $\omega_i \in \Omega$ symbolize an elementary event—a state of nature—that can take place:

$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$$

The following is an example of a state space:

Example 2.1. Weather conditions in Bergen, Norway, on March 20, 2050:

$$\Omega = \{\text{snowy, rainy, sunny, cloudy}\}$$

To describe informational asymmetries among agents, a *partition* $\mathbb{P}_i$ of the state space must be defined for each agent.

Definition 2.5. Partition: A *partition* $\mathbb{P}$ of a set Ω is a collection of subsets of Ω that is:

Exhaustive: For all $\omega \in \Omega$, $\exists P \in \mathbb{P}$ such that $\omega \in P$. That is, every element of the state space must appear in one of the elements (called *atoms*) of $\mathbb{P}$.

Mutually exclusive: For all $P_1, P_2 \in \mathbb{P}$, if $P_1 \neq P_2$, then $P_1 \cap P_2 = \varnothing$. That is, each element of the state space can appear in only one of the atoms of $\mathbb{P}$.

For example, the collection

$$\{\ \{\ \text{sunny, cloudy, snowy}\},\ \{\text{rainy}\}\ \}$$

[4]Note that I assume that the state space is finite. Results that apply to finite state spaces can usually be extended to countable state spaces, as long as each state has a positive probability (Allen 2006). Introducing an uncountable state space raises mathematical subtleties which are beyond the scope of this paper.

is a partition of the state space in Example 2.1, but the collections

$$\{\{\text{sunny}\}, \{\text{snowy, rainy}\}\} \text{ and } \{\{\text{sunny, cloudy, snowy}\}, \{\text{snowy, rainy}\}\}$$

are not. They violate the exhaustiveness and the exclusivity conditions, respectively.

Each *event* (element) of the agent's information partition $\mathbb{P}_i$ represents the states the agent is able to distinguish from other states. For example, given the partition:

$$\mathbb{P}_i = \{\{\text{snowy}\}, \{\text{rainy}\}, \{\text{sunny}\}, \{\text{cloudy}\}\}$$

the agent can tell what the weather is like in Bergen, Norway, on March 20, 2050, with certainty. But, if the information partition is instead:

$$\mathbb{P}_i = \{\{\text{snowy, rainy, cloudy}\}, \{\text{sunny}\}\}$$

the agent can only distinguish fair (sunny) and foul (not sunny) weather.[5]

To completely describe what each agent knows, we define an information field for each agent:

Definition 2.6. Field: Given a finite set Ω, a *field* $\mathcal{F}$ is composed of all possible unions of its atoms, such that:

1. $\varnothing \in \mathcal{F}$.
2. (Closure under unions) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.[6]
3. (Closure under complement) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, where $A^c := \Omega \backslash A$ is the *complement* of A.

From (2) and (3) and the de Morgan laws it follows that fields are closed under intersection, while (1) and (3) imply that $\Omega \in \mathcal{F}$.

[5] In an alternative framework, used by Vohra (1999), each agent's private information is represented by his or her type, and an information state refers to a profile of agents' types.

[6] This gives closure under *finite* unions. Replacing condition (2) by (2′): $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{F}$, one obtains closure under *countable* unions. Such a field is called a σ-field.

Example 2.2. Generating a field from a partition: Given the state space:

$$\Omega = \{\omega_1, \omega_2, \omega_3\}$$

and the agent i's partition:

$$\mathbb{P}_i = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}.$$

the field $\mathcal{F}_i$ generated by the agent's partition is then:

$$\mathcal{F}_i = \{\varnothing, \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_3\}\}.$$

For example, if the agent's information partition is:

$$\mathbb{P}_i = \{\{\text{snowy, rainy, cloudy}\}, \{\text{sunny}\}\}$$

the field $\mathcal{F}_i$ generated by the partition is:

$$\mathcal{F}_i = \{\varnothing,\ \{\text{snowy, rainy, cloudy}\}, \{\text{sunny}\}, \{\text{snowy, rainy, cloudy, sunny}\}\ \}\ .$$

To complete the description of the uncertainty and asymmetric information, one needs to define a *probability measure* $\mu : \mathcal{F} \to \mathbb{R}_+$, representing the agents' subjective probabilities over the various events in their information fields. μ assigns non-negative values to each atom in the agents' information fields. For example, given the field

$$\mathcal{F}_i = \{\varnothing,\ \{\text{snowy, rainy, cloudy}\}, \{\text{sunny}\}, \{\text{snowy, rainy, cloudy, sunny}\}\ \}\ ,$$

The function μ could assign the following probabilities to the events in the field:

Event	Probability
$\mu(\varnothing)$	0
$\mu(\{\text{snowy, rainy, cloudy}\})$	0.9
$\mu(\{\text{sunny}\})$	0.1
$\mu(\{\text{snowy, rainy, cloudy, sunny}\})$	1

The uncertainty in the differential information economy is then fully described by the abstract probability triple:

$$(\Omega, \mathcal{F}, \mu)$$

which is called a *probability space*, provided μ satisfies the following conditions:

Definition 2.7. Probability Space: $(\Omega, \mathcal{F}, \mu)$ is a probability space if:

Subadditivity: $\forall A, B \in \mathcal{F}, \mu(A \cup B) \leq \mu(A) + \mu(B),$ and $A \cap B = \varnothing \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$.

Probability Measure: $\mu(\Omega) = 1$.

The differential information economy is then defined by:

Definition 2.8. Differential Information Economy:

$$\langle \Omega, \mu, e_i, \mathcal{F}_i, u_i \rangle \qquad (i \in N)$$

where:

- N is the finite set of agents.
- $C \in \mathbb{Z}_+$ is the number of physical goods in the economy, and $\mathbb{R}^C$ is the finite-dimensional commodity space.
- $\mathcal{F}_i$ is the private information field of agent i.
- e_i: $\Omega \to \mathbb{R}^C_+$ is the random initial endowment of agent i.
- $u_i : \Omega \times \mathbb{R}^C_+ \to \mathbb{R}$ is the (possibly state-dependent) utility function, representing agent i's preferences over $\mathbb{R}^C$.
- μ is the probability measure representing agents' subjective probabilities attached to their information partitions.[7]

[7] I assume, following Harsanyi (1967, 1968a,b) that agents have a common prior.

Defining the core solution for the differential information economy, however, is more complicated. The standard core concept does not take into account informational differences among agents, and there is, unfortunately, no universally accepted core solution concept for such cases. I will present some alternative definitions of the core adapted to a differential information economy in Section 3.3 and Section 3.4.

2.3 State-contingent contracts and measurability

Once there is uncertainty in the economy, agents have more trading opportunities: they can now write state-contingent contracts. A *state-contingent contract* is a function $f : \Omega \to X$ which specifies a contingent claim to a commodity bundle that may be different in different states of nature. Recalling Example 2.1, an example of a state-contingent contract would be an agent agreeing to give an ice-cream cone to another agent on March 20, 2050, in exchange for some consideration, if the weather in Bergen is sunny, and nothing otherwise.

However, the presence of asymmetric information restricts trade because of the possibility of misrepresentation and coordination failures. Continuing with the ice-cream example, if the two agents had entered into the contract in the previous paragraph, but only the provider of the ice-cream cone could check the weather in Bergen, Norway on March 20, 2050, the provider could decide to keep the ice-cream cone to himself or herself and lie to the other agent about the weather. Knowing this, the recipient would be unlikely to enter into such a contract. Furthermore, if neither party were able to verify that the weather is sunny in Bergen, Norway on March 20, 2050, they would fail to coordinate and trade would not occur.

In technical terms, for the agents to be able to write a contract contingent on weather being sunny in Bergen, Norway on March 20, 2050, the information partition of both agents must contain the singleton {sunny}, at least by the time anything is to be exchanged. It does not matter whether the remaining atoms are {rainy, cloudy, snowy} or {rainy}, {cloudy} and {snowy}, or something else, as long as

each agent can distinguish sunny from other states of nature. This requirement is called *information adaptedness*. Mathematically, information adaptability is related to the concept of *measurability*. To clarify: different agents know different things at different times. Measurability refers to whether the contract depends only on information in a given field; adaptedness requires that, at every point in time, what a contract specifies should happen is adapted to what the parties to the contract know at that point in time.

For example: in the case of the ice-cream cone, suppose the provider is to give the ice-cream cone to the recipient if the weather in Bergen is sunny on March 20, 2050, and the recipient is to pay the provider if and only if the ice-cream cone is to be provided. If the payment is supposed to occur before the parties to the contract know whether the ice-cream cone will be provided (perhaps payment is due one month in advance), then the contract is non-adapted (also called anticipating) — it requires the agents to peek into the future before making present decisions. Nevertheless, once the weather on March 20, 2050 is known to both parties, the contract would still be measurable to them. So adaptedness takes into consideration the fact that the information people have available changes over time, and an adapted contract makes sure that the measurability requirements do not change in ways that conflict with how the available information evolves.

If a contract is based on information that is verifiable by the agent, it is called a *measurable* function. If it is based on information that is not verifiable by the agent, then it is called a *nonmeasurable* function. What is measurable from an agent's viewpoint is described by the field $\mathcal{F}_i$ generated by his or her information partition. Each element of the field $\mathcal{F}_i$ represents a *measurable event*.

It is generally required that contracts must be based on measurable events, so the agent is allowed to contract only on the events contained in his information field $\mathcal{F}_i$. Continuing with Example 2.1, suppose the field $\mathcal{F}_i$ generated by the agent's

information partition is:

$$\mathcal{F}_i = \{\varnothing,\ \{\text{snowy, rainy, cloudy}\},\ \{\text{sunny}\},\ \{\text{snowy, rainy, cloudy, sunny}\}\ \}\ .$$

The weather events that are measurable to the agent are then: fair (sunny), foul (snowy, rainy, or cloudy), and any weather. The agent could agree to buy a warm jacket if the weather is foul ({snowy}, {rainy}, or {cloudy}), an ice-cream cone if the weather is {sunny}, etc. However, it would be more problematic for the agent to agree to buy a pair of skis if the weather is {snowy}, because the event {snowy} is not measurable to him or her.

When a contract is based on information that is nonmeasurable, several issues related to incentive compatibility and enforceability of the contract arise, as in the following example:[8]

Example 2.3. Extended Subaru warranty:
Imagine that you own a Subaru with an extended warranty contract from the dealership. You are driving south on Highway 101 in Shakopee, MN, when suddenly you hear a loud clunk and see a cloud of smoke coming from under the hood. You pull over, exit the car, and notice that the the oil has drained from the engine. You have the car towed to the nearest garage, and the mechanic there tells you that the engine has been damaged by the oil spilling out, and that it should be replaced. You know that a new engine should be covered under the extended warranty, so you have the car towed to your dealership. The dealership assures you that replacing the engine is not necessary, and that the garage mechanic is simply unfamiliar with the particular model of the vehicle. You don't have enough knowledge about car engines to determine what the actual state of nature is. It is unclear how the situation should be resolved.

The Subaru example describes a simple differential information economy, where two agents—the owner and the dealership—have entered into a state-contingent

[8]This really happened to me.

contract—the extended warranty. An event described in the contract—damage to the car—may have been realized, and the question is now whether the contract should be executed—whether the dealership should perform repairs. The problem is that one of the parties—the dealership—has verifiable information about the true state of the world, i.e., whether the car engine is damaged, whereas the second party—the owner—does not. The contract is, therefore, based on information that is measurable to the dealership, but nonmeasurable to the owner of the Subaru.

The dealership's superior information creates a moral hazard, because the dealership has a financial incentive to misrepresent the condition of the engine to the owner. The issue is similar to the classic lemons problem (Akerlof 1970), where an informed seller can discern whether the car he or she is offering for sale is a lemon, but the uninformed buyer can only discern the coarse event consisting of all possible states. [9]

On the other hand, the asymmetric information between the owner and the dealership creates a trading opportunity for the knowledgeable garage mechanic. He or she can act as an intermediary between the owner and the dealership—for instance, as a negotiator or an expert witness in a trial. At the same time, his or her presence creates additional incentive compatibility issues — we need to also consider the garage mechanic's incentives in the situation.

If the contract between the Subaru owner and the dealership would have been written in terms that would have been verifiable by both parties, perhaps the owner would have never gotten into such trouble. As Flåm (2007) points out, few fancy being cheated, taken to court, or involved in litigation, and that most people strongly prefer that future claims be defined in verifiable terms. However, it is also the case

[9] In fact, the buyer in Akerlof's story would presumably eventually find out if the car is a lemon; otherwise, unless the buyer is interested in resale, it is unclear why the buyer should care. So the lemons problem is really one of adaptedness: whether the car is a lemon is measurable with respect to what the buyer eventually knows, and with respect to what the seller knows all along. The problem is that the quality of the car is nonmeasurable with respect to what the buyer can determine at the time of the sale.

that if parties can only contract on information they can personally verify, there will be trading opportunities left on on the table. Information asymmetry is, then, an endogeneous cause of incompleteness of the market.

In the presence of asymmetric information, there is a fundamental tension between measurability and incentive compatibility, and trading opportunities. The core solution concept applied to the differential information economy should then be sensitive to both of these aspects. It should be information measurable, and individually and coalitionally incentive compatible, yet not so restrictive as to prevent trade.

The classical notion of the core does not does not address differences in informational structures, so we need to consider extensions of the core modified to a differential information economy, such as the *coarse core*, the *fine core*, and the *private core*, described in Section 3.3 and Section 3.4. Unfortunately, I find that even these notions of the core do not satisfactorily address incentive compatibility (or, in the case of the coarse core, allow enough welfare-improving trade to occur). Instead, in Section 3.5 I propose my own versions of the private core called *locally coarse core* and *action-measurable private core* based on the requirement of measurability of the agents' actions rather their allocations, as in Yannelis' private core.

2.4 Time dimension and enforceability of contracts

Writing state-contingent contracts adds a time dimension to the model. I restrict attention to models with three time periods. Initially (ex ante), there is uncertainty over the state of nature. Agents write contracts, and subsequently the state of nature is realized. After the state of nature is realized (interim), the agents receive a signal as to what the event containing the realized state of nature is. The ex post period then occurs, when the agents carry out the previously made agreements and consumption takes place (Allen and Yannelis 2001).

Recalling Example 2.1, if the agent's information field ex ante is:

$$\mathcal{F}_i = \{\varnothing,\ \{\text{snowy, rainy, cloudy}\},\ \{\text{sunny}\},\ \{\text{snowy, rainy, cloudy, sunny}\}\ \}\ .$$

and the realized state of nature is {snowy}, at the interim stage the agent would know that:

$$\omega \in\ \{\text{snowy, rainy, cloudy}\}.$$

i.e., that the realized state of nature is one of the states {rainy}, {cloudy} and {snowy}.

In some models, agents learn the true state of nature at the ex post stage, whereas in others they do not.[10] If they do, incentive and measurability constraints are irrelevant, and the only remaining issue is that of enforceability of contracts (de Clippel 2006). In contract theory, the problem of enforceability of contracts at the ex post stage is usually dealt with in one of two ways. Either the contract includes an infinite penalty for misrepresentation, or it is assumed that there is an independent third party who can enforce contracts.[11]

However, the assumption of perfect enforceability in the context of asymmetric information is problematic: if the agents can trust that their contracts will be enforced regardless of what they know, it is not clear why the agents should restrict themselves only to contracts based on terms measurable to them. The addition of the asymmetric information to the model then becomes irrelevant.

Therefore, in this paper, I consider only the latter case: when the true state of nature is not revealed to all the agents, even ex post. Contracts must then designed so that the better-informed parties have incentive to reveal their information; otherwise contracting opportunities are limited to what every party can discern. In

[10] If the agents do not learn anything in the ex post stage that they do not already know in the ex ante stage, then the model can be collapsed into one with two dates, as the interim stage can be considered equivalent to the ex post stage.

[11] There are models in which verifiability of contracts is imperfect, for example in the costly state verification literature due to Townsend (1979) and in the tax compliance literature (Reinganum and Wilde 1986). For papers where verifiability is endogenous, see for example Ishiguro (2002) or Kvaløy and Olsen (2008)

other words, contracts may depend on the private information of the agents only if it gives the right incentives to the agents to reveal their information truthfully (Vohra 1999). Any core solution concept I propose must be in accordance with the revelation principle. I show that this is the case with both the locally coarse core (trivially) and the private action measurability core.

3 Core concepts

3.1 General

As discussed in the previous Section, the final allocation of goods in a differential information economy should satisfy several criteria. It should be feasible, efficient, sensitive to the informational differences among agents, fully incentive compatible, and individually and coalitionally rational, i.e. stable against any deviations by individuals or a group of agents. In addition, the solution concept must not be too restrictive to prevent welfare-improving trades from being made.

That is, we should find the most efficient solution under the following constraints:

Definition 3.1. Pure exchange non-disposal constraints:

$$(\forall i \in N)(\forall \omega \in \Omega) \sum_{i \in N} z_i(\omega) = \sum_{i \in N} e_i(\omega)$$

where z is an allocation.

Definition 3.2. Non-negativity constraints:

$$(\forall i \in N)(\forall \omega \in \Omega) z_i(\omega) \geq 0$$

Definition 3.3. Individual rationality (IR) constraints:

$$(\forall i \in N)(\forall \omega \in \Omega) E[u_i(z_i)] \geq E[u_i(e_i)]$$

Definition 3.4. Coalitional rationality (CR) constraint:

$$\forall (\widehat{z}_i)_{i \in S} \text{ such that } (\forall \omega \in \Omega) \sum_{i \in S} \widehat{z}_i(\omega) = \sum_{i \in S} e_i(\omega) \text{ and}$$

$$(\forall i \in S)(\forall \omega \in \Omega) \widehat{z}_i(\omega) \geq 0,$$

$$\text{whenever } \exists i \in S \text{ with } E[u_i(\widehat{z}_i)] > E[u_i(z_i)],$$

$$\text{then also } \exists j \in S \text{ such that } E[u_j(z_j)] > E[u_j(\widehat{z}_j)].$$

In the TU case, this simplifies to

$$v(S) \leq \sum_{i \in S} E[u_i(z_i)],$$

where $v(S)$ in Definition 2.1 is modified to be the maximized sum of expected utilities.

I illustrate these constraints and different solution concepts using the following example from Allen and Yannelis (2001):

Example 3.1. Three-agent game with a destitute agent:

Agents : $i \in N = \{1, 2, 3\}$

Goods : There is only one good, x.

State space : $\Omega = \{a, b, c\}$

Uncertainty : $\mu(a) = \mu(b) = \mu(c) = 1/3$.

Endowments :

$$e_1(\omega) = \begin{cases} 20, & \text{if } \omega \in \{a, b\} \\ 0, & \text{if } \omega = c \end{cases}$$

$$e_2(\omega) = \begin{cases} 20, & \text{if } \omega \in \{a, c\} \\ 0, & \text{if } \omega = b \end{cases}$$

$$e_3(\omega) = 0, \forall \omega.$$

Preferences : $u(x) = x^{1/2}$.

Information fields:

$$\mathcal{F}_1 = \{\varnothing, \Omega, \{a, b\}, \{c\}\}$$

$$\mathcal{F}_2 = \{\varnothing, \Omega, \{a, c\}, \{b\}\}$$

$$\mathcal{F}_3 = \{\varnothing, \Omega, \{b, c\}, \{a\}\}$$

My objective is to find an allocation:

$$(\forall \omega \in \Omega) z = (z_1(\omega), z_2(\omega), z_3(\omega))$$

that is in the core and satisfies the following constraints:

Pure exchange non-disposal constraints:

$$z_1(a) + z_2(a) + z_3(a) = e_1(a) + e_2(a) + e_3(a) = 40 \tag{1}$$

$$z_1(b) + z_2(b) + z_3(b) = e_1(b) + e_2(b) + e_3(b) = 20 \tag{2}$$

$$z_1(c) + z_2(c) + z_3(c) = e_1(c) + e_2(c) + e_3(c) = 20 \tag{3}$$

Non-negativity constraints:

$$(\forall i \in N)(\forall \omega \in \Omega) z_i(\omega) \geq 0 \tag{4}$$

Individual rationality (IR) constraints:

$$E[u_1(z_1)] \geq E[u_1(e_1)] \tag{5}$$

$$E[u_2(z_2)] \geq E[u_2(e_2)] \tag{6}$$

$$E[u_3(z_3)] \geq E[u_3(e_3)] = 0 \tag{7}$$

Coalitional rationality (CR) constraint:

$$(\forall i, j \in N) \text{ if, } \forall \omega \in \Omega, \widehat{z}_i(\omega) + \widehat{z}_j(\omega) = e_i(\omega) + e_j(\omega) \text{ for some } \widehat{z}_i, \widehat{z}_j \geq 0,$$

$$\text{then whenever } E[u_i(z_i)] < E[u_i(\widehat{z}_i)],$$

$$\text{we must have } E[u_j(z_j)] > E[u_j(\widehat{z}_j)]. \tag{8}$$

3.2 The rational expectations equilibrium

Since agents are maximizing their utility under uncertainty, the natural place to start looking for a solution concept is in the Rational Expectations Equilibrium (REE). However, in this case REE is not the ideal solution concept. When agents have complete information, trade is efficient, and agents arrive at an equilibrium, which is also a core solution (Osborne and Rubinstein 1994). But when information is asymmetric, agents in Example 3.1 cannot make the welfare-improving trades, and the only solution under REE is autarky, even though there are several (in fact, infinitely many) allocations in the core that Pareto-dominate autarky.

The endowments and their expected utilities are as follow:

$$e_1 = (20, 20, 0) \qquad e_2 = (20, 0, 20) \qquad e_3 = (0, 0, 0)$$

$$E[u_1(e_1)] = \frac{1}{3}\sqrt{20} + \frac{1}{3}\sqrt{20} + \frac{1}{3}\sqrt{0} = \frac{2\sqrt{20}}{3} = \frac{4\sqrt{5}}{3}$$

$$E[u_2(e_2)] = \frac{4\sqrt{5}}{3} \text{ (by symmetry).}$$

$$E[u_3(e_3)] = 0$$

A more efficient allocation would be:

$$z_1 = (16, 16, 4) \qquad z_2 = (16, 4, 16) \qquad z_3 = (8, 0, 0)$$

This allocation satisfies the pure exchange non-disposal constraints (1), (2), and (3):

$$\text{State } \{a\} : 16 + 16 + 8 = 40$$

$$\text{State } \{b\} : 16 + 4 + 0 = 20$$

$$\text{State } \{c\} : 4 + 16 + 0 = 20$$

It also satisfies the Individual Rationality (IR) constraints (5), (6) and (7):

$$E[u_1(16, 16, 4)] = \frac{1}{3}\sqrt{16} + \frac{1}{3}\sqrt{16} + \frac{1}{3}\sqrt{4} = \frac{10}{3} > E[u_1(e_1)] = \frac{4\sqrt{5}}{3}$$

$$E[u_2(16,4,16)] = \frac{10}{3} > E[u_2(e_2)] = \frac{4\sqrt{5}}{3} \text{ (by symmetry).}$$

$$E[u_3(8,0,0)] = \frac{1}{3}\sqrt{8} + \frac{1}{3}\sqrt{0} + \frac{1}{3}\sqrt{0} = \frac{2\sqrt{2}}{3} > E[u_3(e_3)] = 0$$

as well as the non-negativity constraints (4), since none of the agents receives a negative allocation in any state. Lastly, since no pair of agents can improve upon this allocation, the allocation satisfies the coalitional rationality constraint (8). The allocation

$$z_1 = (16,16,4) \qquad z_2 = (16,4,16) \qquad z_3 = (8,0,0)$$

is, therefore, in the core.[12]

The situation is analogous to incomplete markets in financial economics: in incomplete financial markets, the equilibrium allocations are almost never efficient. Consider the extreme case, similar to the three-agent game here, suggested by van Zandt (2004), of a market in which there are no financial assets and state-contingent trade is not possible. This would only be efficient if the original endowments were Pareto-optimal and there would be no gains from trade.

However, the allocation $z_1 = (16,16,4)$, $z_2 = (16,4,16)$, $z_3 = (8,0,0)$ is not a rational expectations equilibrium. To see this, consider agent 1's objective function:[13].

$$\begin{aligned}\max E[u(z)] &= \left[\mu(a)(z(a))^{\frac{1}{2}} + \mu(b)(z(b))^{\frac{1}{2}} + \mu(c)(z(c))^{\frac{1}{2}}\right] \\ &= \max\left[\frac{1}{3}(z(a))^{\frac{1}{2}} + \frac{1}{3}(z(b))^{\frac{1}{2}} + \frac{1}{3}(z(c))^{\frac{1}{2}}\right] \\ &\text{s.t. } p_a z(a) + p_b z(b) + p_c z(c) = 20p_a + 20p_b\end{aligned}$$

Dividing all the terms by p_a, the Lagrangian becomes:

$$\max_{z(a),z(b),z(c)} \mathscr{L} = \frac{1}{3}\left[z(a)^{\frac{1}{2}} + z(b)^{\frac{1}{2}} + z(c)^{\frac{1}{2}}\right] - \lambda\left[-20 - 20\frac{p_b}{p_a} + z(a) + \frac{p_b}{p_a}z(b) + \frac{p_c}{p_a}z(c)\right]$$

[12]Note that the core is infinitely large and contains autarky, allocations where agent 3 gets all but ε of the surplus over autarky from agents 1 and 2, etc.

[13]I suppress the subscripts indicating the agent to enhance readability

The first-order conditions are:

$$\frac{\partial \mathscr{L}}{\partial z(a)} = 0 \Rightarrow z(a) = \frac{1}{36\lambda^2}$$

$$\frac{\partial \mathscr{L}}{\partial z(b)} = 0 \Rightarrow z(b) = \left(\frac{p_a}{p_b}\right)^2 \frac{1}{36\lambda^2}$$

In a competitive equilibrium, provided that $p_a, p_b \neq 0$,

$$z(b) = \left(\frac{p_a}{p_b}\right)^2 (z(a))$$

$$\frac{\partial \mathscr{L}}{\partial z(c)} = 0 \Rightarrow z(c) = \left(\frac{p_a}{p_c}\right)^2 \frac{1}{36\lambda^2} = \left(\frac{p_a}{p_c}\right)^2 z(a)$$

If Agent 1's problem has (16,16,4) as an optimal choice, then:

$$z(a) = z(b) = 16 \Rightarrow 16 = \left(\frac{p_a}{p_b}\right)^2 z(a) \Rightarrow p_a = p_b$$

$$z(c) = \left(\frac{p_a}{p_c}\right)^2 z(a) \Rightarrow 4 = 16\left(\frac{p_a}{p_c}\right)^2 \Rightarrow \frac{p_a}{p_c} = \frac{1}{2} \Rightarrow p_c = 2p_a$$

To solve for the equilibrium allocations, consider agent 2's objective function:

$$\max \left[\frac{1}{3}(z(a))^{\frac{1}{2}} + \frac{1}{3}(z(b))^{\frac{1}{2}} + \frac{1}{3}(z(c))^{\frac{1}{2}}\right]$$

$$\text{s.t. } p_a z(a) + p_b z(b) + p_c z(c) = 20p_a + 20p_c$$

Substituting $p_a = p_b$ and $p_c = 2p_a$ in the budget constraint and dividing all the terms by p_a, we obtain the following version of the budget constraint:

$$z(a) + \frac{p_b}{p_a} z(b) + \frac{p_c}{p_a} z(c) = 20 + 20\frac{p_c}{p_a}$$

Since $p_a = p_b$, $\frac{p_b}{p_a} = 1$; and since $p_c = 2p_a$, $\frac{p_c}{p_a} = 2$. So agent 2's problem simplifies to:

$$\max_{z(a),z(b),z(c)} \mathscr{L} = \frac{1}{3}\left[z(a)^{\frac{1}{2}} + z(b)^{\frac{1}{2}} + z(c)^{\frac{1}{2}} - \lambda(z(a) + z(b) + 2z(c) - 60)\right]$$

The first-order conditions are:

$$\frac{\partial \mathscr{L}}{\partial z(a)} = 0 \Rightarrow z(a) = \frac{1}{36\lambda^2}$$

$$\frac{\partial \mathscr{L}}{\partial z(b)} = 0 \Rightarrow z(b) = \frac{1}{36\lambda^2} = z(a)$$

$$\frac{\partial \mathscr{L}}{\partial z(c)} = 0 \Rightarrow z(c) = \frac{1}{144\lambda^2} = \frac{1}{4}z(a)$$

$$\frac{\partial \mathscr{L}}{\partial \lambda} = 0 \Rightarrow z(a) = \frac{80}{3}$$

From this follows that: $z(b) = \frac{80}{3}$ and $z(c) = \frac{20}{3}$. This means that at any prices where agent 1 consumes $(16, 16, 4)$, agent 2 consumes $(\frac{80}{3}, \frac{80}{3}, \frac{20}{3}) \neq (16, 4, 16)$. There are, therefore, no prices where agents 1 and 2 could choose the allocation $z_1 = (16, 16, 4)$, $z_2 = (16, 4, 16)$ and be optimizing.

The above result depends on dividing agent 1's objective function by p_a. It is conceivable that agents could set $p_b = p_c$ and $p_a = 0$. However, if this were to happen in equilibrium, the demand for $z(a)$ would be infinite, unless the agents believed the probability of the state a also to be zero. But since $\mu(a) = \frac{1}{3}$, the expectation that a occurs with probability zero is not rational. Hence, the only possible equilibrium prices where agent 1 consumes (16,16,4) and agent 2 consumes (16,4,16) cannot involve rational expectations.

It is clear, then, that for this example, REE is not the ideal solution concept, at least if we believe that the allocation (16,16,4), (16,4,16), (8,0,0) is plausible. It therefore makes sense to look at different core solution concepts designed for economies with asymmetric information. Two of these are the *coarse core* and *fine core* by Wilson (1978). I turn to these next.

3.3 Fine core and coarse core

Wilson (1978) considered two basic cases of how agents in an economy can share information: one where the coalition pools its information (the *fine* core), and one where the coalition only uses information common to all of its members (the *coarse* core). Applied to the grand coalition, the corresponding fields are the *fine* field and the *coarse* field:

Definition 3.5. Fine field: The *fine field* in an economy with differential information is defined by the *join* $\vee_{i\in N}\mathcal{F}_i$, which is the field generated by the union $\cup_{i\in N}\mathcal{F}_i$ of agents' information fields.

A *join* operator $\vee$ is similar to a union operator but preserves the field property (see Definition 2.6). For example, given the information fields from Example 3.1,

$$\mathcal{F}_1 = \{\varnothing, \Omega, \{a, b\}, \{c\}\}$$

$$\mathcal{F}_2 = \{\varnothing, \Omega, \{a, c\}, \{b\}\}$$

the union of F_1 and F_2:

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\varnothing, \Omega, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$$

is not a field, but the join of F_1 and F_2 is. The join is obtained from the union by adding the missing elements {a} and {b,c}:

$$\mathcal{F}_1 \vee \mathcal{F}_2 = \{\varnothing, \Omega, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\, \{b, c\}\}.$$

The join represents agents' pooled information.

Definition 3.6. Coarse field: The *coarse field* in an economy with differential information is defined by the the *meet* $\wedge_{i\in N}\mathcal{F}_i$, which is a field generated by the intersection of agents' information fields $\cap_{i\in N}\mathcal{F}_i$.

The *meet* operator $\wedge$ is equal to the intersection, and preserves the field property. For example, given the information fields from Example 3.1,

$$\mathcal{F}_1 = \{\varnothing, \Omega, \{a, b\}, \{c\}\}$$

$$\mathcal{F}_3 = \{\varnothing, \Omega, \{b, c\}, \{a\}\}$$

the intersection of F_1 and F_3 equals the meet of F_1 and F_3:

$$\mathcal{F}_1 \cap \mathcal{F}_3 = \mathcal{F}_1 \wedge \mathcal{F}_3 = \{\varnothing, \Omega\}$$

The meet represents the information common to all agents in the economy.

The two opposite extremes of information sharing are then the null communication system, $\mathcal{F}_i (i \in N)$, where each agent only knows his or her private information and communicates nothing, and the full communication system, $\vee_{i \in N} \mathcal{F}_i (i \in N)$, where all the information fields are common information.

Wilson's fine and coarse cores are then core solutions that satisfy the normal feasibility and stability conditions but allow for information sharing either using the fine or coarse field, respectively. Under the fine core, agents can trade on events in the pooled information, whereas in the coarse core, agents can only trade on the information that all of the agents have in common.

It is easy to see that under the fine core, incentive compatibility and moral hazard issues arise. If the agents do not have incentive to share their information truthfully, they will lie as long as it is advantageous for them to do so. For instance, if we recall the ice-cream example from Section 2.3, an agent could promise to give another agent an ice-cream cone in exchange for some consideration, if the weather is sunny, but then lie and tell him the weather is, in fact, rainy.

It is also easy to see that under the coarse core, trade will be limited. In the three-agent game with a destitute agent, the only information partitions the agents have in common are the empty set, $\varnothing$, and the whole state space, Ω. Therefore, the agents can only write incentive-compatible and measurable contracts on the whole

state space. There are no welfare-improving trades that the agents can make under such conditions so the only allocation in the core is autarky. The limitations of the coarse core are illustrated even more dramatically by the following modification of the three-agent game:

Example 3.2. Three-agent game with a destitute agent (Agents 1 and 2 with complete information; agent 3 with no information):

This game is in all other aspects similar to Example 3.1 except that agents 1 and 2 have complete information, and agent 3 has no information:

$$\mathcal{F}_1 = \mathcal{F}_2 = \{\varnothing, \Omega, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

$$\mathcal{F}_3 = \{\varnothing, \Omega\}$$

The coarse field $\cap_{i \in N} \mathcal{F}_i$ equals just $\{\varnothing, \Omega\}$, and the only solution in the coarse core is then autarky, despite two of the agents having complete information, and therefore not needing agent 3 to intermediate.

The fine and coarse cores illustrate the fundamental tension that arises in the economy in the presence of asymmetric information. While the coarse core is generally nonempty, trade based on information that everybody can discern is necessarily limited. The fine core, on the other hand, tends to be empty but offers more trading possibilities, if the agents could somehow coordinate reliably and avoid issues of moral hazard (Wilson 1978).[14]

Thus, a more nuanced core solution concept is desirable. Such a solution concept should allow trade in larger numbers of cases than the coarse core, but avoid the incentive and rationality constraint issues with the fine core. One such possibility is the *private core* suggested by Yannelis (1991).

[14]Numerous authors since Wilson have addressed this question. Forges et al. (2002) proved that although the ex ante incentive compatible core can be empty, with quasi-linearity (i.e., transferable utility), it is generally nonempty. Infinite-dimensional economies have been shown to yield nonempty ex ante incentive compatibile cores, see Allen (2006) for proof that the interim incentive compatible coarse core is nonempty. However, in the case of a finite-dimensional economy and a finite state space the issues still remain. For results with infinetely many agents, see Sun and Yannelis (2008).

3.4 The private core

The private core is defined as the set of all feasible and private information measurable allocations which cannot be dominated, in terms of expected utility, by any coalition's feasible and private information measurable net trades (Yannelis 1991). The requirement for measurability adds another constraint to the model:

Definition 3.7. Measurability constraint: Agents can only trade on events they are able to discern.

In the context of the Allen-Yannelis Example 3.1, this means that:

- Agent 1 can only trade on the events $\Omega, \{a, b\}, \{c\}$.
- Agent 2 can only trade on the events $\Omega, \{a, c\}, \{b\}$.
- Agent 3 can only trade on the events $\Omega, \{b, c\}, \{a\}$.

Consider a model containing three time periods. Initially (ex ante), there is uncertainty over the state of nature. Agents write contracts, and subsequently the state of nature is realized. After the state of nature is realized (interim), the agents receive a signal as to what the event containing the realized state of nature is. The ex post period then occurs, when the agents carry out the previously made agreements and consumption takes place (Allen and Yannelis 2001).

Let $A(\widehat{\omega}) \in \mathcal{F}_i$ be the event that agent i observes when $\widehat{\omega}$ is the realized state. The ex ante and interim expected utilities of agent i of state-dependent consumption z_i are then respectively given by

$$E[u_i(z_i)] = \sum_{\omega \in \Omega} u_i(\omega, z_i(\omega))\mu(\omega), \text{ and}$$

$$E[u_i(z_i)|\omega \in A(\widehat{\omega})] = \frac{1}{\mu(A(\widehat{\omega}))} \sum_{\omega \in A(\widehat{\omega})} u_i(\omega, z_i(\omega))\mu(\omega).$$

For example, suppose that the realized state is a. For agent 1, $A(a) = \{a, b\}$. Now $\mu(a) = \mu(b) = \mu(c) = \frac{1}{3}$, and $\mu(A) = \frac{2}{3}$. So the ex ante expected utility for agent 1 is

$$E[u_i(z_i)] = \frac{1}{3}\left[u_1(a, z_1(a)) + u_1(b, z_1(b)) + u_1(c, z_1(c))\right],$$

and the interim expected utility for agent 1 is

$$E[u_i(z_i)|\omega \in \{a, b\}] = \frac{1}{\frac{2}{3}}\left[u_1(a, z_1(a))\frac{1}{3} + u_1(b, z_1(b))\frac{1}{3}\right]$$

$$= \frac{1}{2}\left[u_1(a, z_1(a)) + u_1(b, z_1(b))\right].$$

Allocation $(z_i)_{i\in N}$ is in the private core if the following three conditions hold:

- Measurability: Each z_i is $\mathcal{F}_i$-measurable
- Feasibility: $\sum_{i\in N} z_i = \sum_{i\in N} e_i$
- Stability: For every coalition S, whenever $(\widehat{z}_i)_{i\in S}$ satisfies

$$(\forall\omega \in \Omega)\sum_{i\in S}\widehat{z}_i(\omega) = \sum_{i\in S} e_i(\omega) \text{ and } (\forall i \in S)(\forall\omega \in \Omega)\widehat{z}_i(\omega) \geq 0,$$

it is the case that, for each $\widehat{\omega} \in \Omega$,

$$(\exists i \in S)\, E[u_i(\widehat{z}_i)|\omega \in A_i(\widehat{\omega})] > E[u_i(z_i)|\omega \in A_i(\widehat{\omega})]$$
$$\Rightarrow\ (\exists j \in S)\, E[u_j(z_j)|\omega \in A_j(\widehat{\omega})] > E[u_j(\widehat{z}_j)|\omega \in A_j(\widehat{\omega})].$$

In other words, the stability condition requires that, if any potential breakaway coalition can achieve something that one of its members prefers to the core allocation, then there is another member of the same coalition who prefers the core allocation.

This gives us the interim private core. The ex ante private core does not depend on the realized state of nature, so the ex ante stability condition is:

$$(\exists i \in S)\, E[u_i(\widehat{z}_i)] > E[u_i(z_i)] \Rightarrow (\exists j \in S)\, E[u_j(z_j)] > E[u_j(\widehat{z}_j)].$$

Allen and Yannelis (2001) state that the private core is not suspectible to the criticism of the traditional rational expectations equilibrium. In particular, they argue that while the rational expectations equilibrium does not provide an explanation as to how prices reflect the information asymmetries in the economy, the private core is sensitive to agent's information state and allows a better-informed agent to have an advantage over worse-informed agents in trading.

However, while allocations in the private core have a number of desirable properties — they satisfy individual and coalitional rationality constraints, for instance — two issues remain. The first is whether the allocations in the private core are *incentive compatible* in the full sense of the term. The second is that, while the agents might prefer to stay with the core allocation once they get there, it is not clear exactly how they would negotiate or trade with each other to get there. Therefore, I propose adding the following constraint to the model:

Definition 3.8. Incentive Compatibility (IC) constraint: Misrepresenting the state cannot be more profitable than telling the truth.

Returning to Example 3.1, the private core allocation:

$$x_1 = (16, 16, 4) \qquad x_2 = (16, 4, 16) \qquad x_3 = (8, 0, 0)$$

does not satisfy the IC constraint, because the transfers between agents 2 and 3, and between agents 1 and 3, are based on information that is not verifiable by agent 3. These incentive compatibility issues arise when the state is b or c. At the interim stage, if the state is b,

- Agent 1 knows that: $\omega \in \{a, b\}$
- Agent 2 knows that: $\omega = b$
- Agent 3 knows that: $\omega \in \{b, c\}$

Agent 1 has an incentive to misrepresent the state as c to agent 3, because in state c, agent 1 receives four units from agent 3, whereas in state b he or she owes four units to agent 3. Agent 2, on the other hand, knows the state is b and will also claim his or her four units from agent 3. Agent 3 has cannot tell which of the agents 1 and 2 is lying, because his or her private information does not reveal whether the state is b or c: the state b is not *privately measurable* to him or her. Agent 3's contract with agent 1 stipulates different behaviors in states b and c, even though agent 3 cannot distinguish between states b and c, and this is problematic. Therefore, agent 2 needs to be present when agents 1 and 3 meet to tell 3 which state they are in. But this would mean pooling information, as in the fine core, with its associated incentive compatibility problems. A similar problem arises with agents 2 and 3, and state c.

Allen and Yannelis (2001) propose a notion they call Transfer Coalitionally Bayesian Incentive Compatibility (TCBIC) to address this issue:

Definition 3.9. Transfer Coalitionally Bayesian Incentive Compatibility (TCBIC):

A feasible allocation z is *TCBIC* if it is not the case that for a coalition S, states a, b with $a \neq b$ and $(\forall i \notin S)\, a \in A(b)$ and a net trade vector m_i such that $\sum_{i \in S} m_i = 0$, and, for all $i \in S$,

$$\frac{1}{\mu(A(a))} \sum_{c \in A(a)} u_i(c, e_i(c) + z_i(b) - e_i(b) + m_i)(\mu(c)) > \frac{1}{\mu(A(a))} \sum_{c \in A(a)} u_i(c, z_i(c))(\mu(c)).$$

The TCBIC condition requires that there is no coalition whose members can agree, possibly through making some transfer payments, to misrepresent its information. If the true state is a and everyone outside a given coalition cannot distinguish states a and b, then the TCBIC condition requires that there is no collective incentive to misrepresent the state as b.

The problem is that the TCBIC only remedies situations where the lie cannot be detected by any of the agents. It does not address a situation where one agent is able to detect the lie but another is not. In the context of the Allen-Yannelis example, suppose the realized state of nature is b. Agent 1 will then misrepresent the state to agent 3 as c. Agent 2 can catch agent 1 lying but agent 3 cannot. The TCBIC does not address the question, because it only rules out lies that no-one can detect, and the incentive compatibility problem remains.

The incentive compatibility issues discussed here have lead many economists to model differential information economies as a noncooperative or hybrid games, or by representing the agent's information partitions as types. In these models, incentive compatibility refers to an agent's willingness to report his type truthfully. Obviously, with such a mechanism, only contracts that are self-enforcing can be implemented. The set of such contracts is known as the set of *incentive compatible contracts*. See Vohra (1999).

Rather than explore all or even most of these attempts, which would be impossible within the scope of this paper, I show that when utility in Example 3.3 is made quasi-linear by the addition of a linear good y, which can be used as money, the incentive constraints can be satisfied by appropriate transfers among agents. Quasi-linear utility has been suggested by many authors, such as Forges et al. (2002) to increase trading opportunities in the economy and allow for more equilibria to emerge. This approach, while satisfactory in some respects, is not unproblematic in others.

3.5 Locally coarse core and action-measurable private core

There are two ways one might try to deal with the incentive compatibility problem. One way would be to reconsider the coarse core. The second would be to redefine the private core to take account into the incentive compatibility constraints.

I call core allocations that the agents can reach by bilateral, measurable exchanges between each other, using only information that is in the coarse field between them, the *locally coarse core*. Allowing this type of exchange, and modifying the Example 3.1 by giving agent 3 complete information about the state of nature at the interim stage, the three agents can trade themselves into the allocation:

$$x_1 = (16, 16, 4) \qquad x_2 = (16, 4, 16) \qquad x_3 = (8, 0, 0)$$

without violating incentive compatibility constraints.

Example 3.3. Three-agent game with a destitute agent (Third agent with complete information):

This game is in all other aspects similar to Example 3.1 except that agent 3 has complete information about the state of nature:

$$\mathcal{F}_3 = \{\varnothing, \Omega, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

The coarse field between agents 1 and 3 is:

$$\mathcal{F}_1 \cap \mathcal{F}_3 = \mathcal{F}_1 \wedge \mathcal{F}_3 = \{\varnothing, \Omega, \{a, b\}, \{c\}\} = \mathcal{F}_1.$$

and the coarse field between agents 2 and 3 is:

$$\mathcal{F}_2 \cap \mathcal{F}_3 = \mathcal{F}_1 \wedge \mathcal{F}_3 = \{\varnothing, \Omega, \{a, c\}, \{b\}\} = \mathcal{F}_2.$$

At the ex ante stage, agent 3 proposes the following contract to agents 1 and 2:

- In event $\{a, b\}$, agent 1 gives 4 to agent 3.
- In state c, agent 3 gives 4 (received from agent 2) to agent 1.
- In event a, c, agent 2 gives 4 to agent 3.
- In state b, agent 3 gives 4 (received from agent 1) to agent 2.

These exchanges lead to the allocation:

$$x_1 = (16, 16, 4) \qquad x_2 = (16, 4, 16) \qquad x_3 = (8, 0, 0)$$

This allocation is feasible, and Pareto-efficient. Agent 3 playes the role of an intermediary, using his or her superior information to facilitate trade between agents 1 and 2 (Allen and Yannelis 2001). The exchanges are based on events in the coarse field between agents 1 and 3, and agents 2 and 3, so they satisfy incentive compatibility and measurability. No individual agent or a pair of agents has incentive to deviate, so this allocation is in the core. By contrast, in the original example, where agent 3 had incomplete information about the state of nature at the interim stage, the locally coarse core is a single point (autarky).

The other way to deal with the incentive compatibility problem is to refine the notion of the private core. I propose changing the focus from the measurability of the agents' *allocations* to the measurability of their actions. This makes the private core an *action-measurable private core*. The idea behind the action-measurable private core is that instead of the agent's allocation being required to be privately measurable, it is his or her *actions* that need to be privately measurable. In other words, the focus in the action-measurable private core is on requiring agents to know what they are supposed to give to someone else in each state, rather than on insisting that they know what they are supposed to receive as their final allocations. For this approach to work, each agent must have incentive to report honestly and disclose what he or she knows. That is, the contracts must be incentive compatible.

This is achieved by adding a second, linear good y to the utility function. The good y is then used as monetary transfers among the agents to satisfy incentive constraints. I show that as long as there is a sufficient amount of good y in the economy, the agents can trade to a private core allocation. Unfortunately, I also find that in the three-agent example, the proposed allocation is not stable — agents

1 and 2 have incentive to break away and form their own coalition. So the action-measurable core may not be the ideal solution for the three agents' problem either, at least if the goal is to explain how agent 3 can act as an intermediary.

Example 3.4. Three-agent game with a destitute agent (TU and two goods):

Agents : $i \in N = \{1, 2, 3\}$

Goods : There are two goods, x and y

State space : $\Omega = \{a, b, c\}$

Uncertainty : $\mu(a) = \mu(b) = \mu(c) = 1/3$.

Endowments :

Agent	State a	State b	State c
1	$(20, 10)$	$(20, 10)$	$(0, 10)$
2	$(20, 10)$	$(0, 10)$	$(20, 10)$
3	$(0, 0)$	$(0, 0)$	$(0, 0)$

Information fields:

$$\mathcal{F}_1 = \{\varnothing, \Omega, \{a, b\}, \{c\}\}$$

$$\mathcal{F}_2 = \{\varnothing, \Omega, \{a, c\}, \{b\}\}$$

$$\mathcal{F}_3 = \{\varnothing, \Omega, \{b, c\}, \{a\}\}$$

Preferences : $u(x) = x^{1/2} + y$.

The utility is quasi-linear: linear in the good y and strictly concave in good x.[15] Each agent can use the linear good y as transfers m to other agents in order to satisfy incentive constraints. Let

$$m_{ij}^k(\omega)$$

[15]Interestingly, the results obtained in this section do not go through if, instead of the second good y, state a utility is made linear.

be the transfer that agent i promises to make to agent j of good k in state ω. For example,

$$m_{13}^2(a)$$

is the number of units of the y good that agent 1 transfers to agent 3 in state a. Each $m_{ij}^k(\omega) \geq 0$, which is to say that the $m_{ij}^k(\cdot)$ are not net transfers, but rather gross payments.

I consider a case where agents make bilateral contracts with each other ex ante (rather than try to arrive at a multilateral contract at once). I relax the assumption of nonnegativity temporarily, and allow agent 3 to sell short, i.e. hold a negative position of a good, as long as he or she is not left holding a negative allocation in the end. The negative position is interpreted as an I.O.U. on the good.

First, assume that agents 1 and 3 sign a contract which stipulates the following transfers:

- Agent 3 gives 4 units of the x good to agent 1 in all states:

$$(\forall \omega \in \Omega)\, m_{31}^1(\omega) = 4.$$

- Agent 1 gives 8 units of the x good to agent 3, provided the event $\{a, b\}$ occurs. In event $\{c\}$, agent 1 owes 0 units of the x good to agent 3:

$$m_{13}^1(a) = m_{13}^1(b) = 8 \qquad m_{13}^1(c) = 0.$$

The net effect of this scheme is to leave agent 1 with $(16, 16, 4)$ of the x good, and to give agent 3 $(4, 4, -4)$ of the x good. So agent 3 takes a short position on the x good in state c, which a trade with someone else—presumably agent 2, since no one else is left—must cover.

Here, agent 3's action, "give 4 units of x to agent 1," is privately measurable for 3, and agent 1's action, "give 8 units of x to agent 3 if the event $\{a, b\}$ occurs, and nothing otherwise," is privately measurable to 1. So both parties know what they

must do in order to honor their bargain. Transfers of the y good can now be used to make the contracts incentive compatible.[16]

There is no concern that agent 3 might lie, as the contract specifies the same action $m_{31}^1(\omega)$ in every state. And it is infeasible for agent 1 to lie in the event $\{c\}$, at least if I assume that 3 is the only intermediary. If agents 1 and 2 only trade with 3, and not directly with each other, then only 3 can feasibly (temporarily) hold a short position. I therefore restrict attention to incentive compatibility for agent 1 in event $\{a, b\}$.

If agent 1 observes that the event is $\{a, b\}$, then his or her expected utility from honest reporting is

$$u(e_1(a) + m_{31}(a) - m_{13}(a)) \cdot \frac{\mu(a)}{\mu(\{a,b\})} + u(e_1(a) + m_{31}(b) - m_{13}(a)) \cdot \frac{\mu(b)}{\mu(\{a,b\})}$$

$$= \left(10 + m_{31}^2(a) - m_{13}^2(a) + \sqrt{20 + m_{31}^1(a) - m_{13}^1(a)}\right)\left(\frac{1/3}{2/3}\right) + \left(10 + m_{31}^2(b) - m_{13}^2(a) + \sqrt{20 + m_{31}^1(b) - m_{13}^1(a)}\right)\left(\frac{1/3}{2/3}\right).$$

Agent 1's transfers must be the same in states a and b, so I label these $(m_{13}^1(a), m_{13}^2(a))$ in either case. Under the terms of the contract, $m_{13}^1(a) = 8$ and $m_{31}^1(a) = m_{31}^1(b) = 4$. So agent 1's expected utility becomes

$$\frac{1}{2}\left(20 + m_{31}^2(a) - 2m_{13}^2(a) + m_{31}^2(b) + 2\sqrt{16}\right) = 14 + \frac{m_{31}^2(a) + m_{31}^2(b)}{2} - m_{13}^2(a).$$

The only state in which there is potential for agent 1 to lie is c, so I guess that there is an incentive compatible contract with $m_{31}^2(a) = m_{31}^2(b) = m_{13}^2(a) = 0$. In that case, the expected utility for agent 1 from honest reporting in event $\{a, b\}$ is 14.

[16] This is similar in spirit to Flåm (2007), where agents have random bilateral meetings and transfer goods in ways that increase their combined utilities. Flåm notes that, if one agent's marginal utility is higher than the other's in every good, then the prescribed transfers would require some sort of monetary compensation. His paper abstracts from how this compensation occurs, as his interest is in other issues (specifically, the long-run behavior of a dynamic model).

If agent 1 lies and reports the event as $\{c\}$, one of two equally likely things occurs. Either the true state is a, in which case agent 3 detects the lie, or the true state is b, in which case the lie is successful. If agent 3 catches agent 1 lying, the contract is void, and both parties end up with what they had under autarky. Otherwise, the agents execute the terms of the contract in the reported state c.

Agent 1's expected utility of lying is therefore

$$u(e_1(a)) \cdot \frac{\mu(a)}{\mu(\{a,b\})} + u(e_1(a) + m_{31}(b) - m_{13}(c)) \cdot \frac{\mu(b)}{\mu(\{a,b\})}$$

$$= \frac{1}{2}\left(10 + 2\sqrt{5} + 10 + 2\sqrt{6} + m_{31}^2(b) - m_{13}^2(c)\right).$$

Assuming $m_{31}^2(b) = 0$, the expected utility for 1 of lying becomes

$$10 + \sqrt{5} + \sqrt{6} - \frac{m_{13}^2(c)}{2}.$$

For the contract to be incentive compatible, it must be at least as good for 1 to tell the truth as it is for 1 to lie:

$$10 + \sqrt{5} + \sqrt{6} - \frac{m_{13}^2(c)}{2} \leq 14$$

$$m_{13}^2(c) \geq 2\left(\sqrt{5} + \sqrt{6} - 4\right) \quad (\approx 1.37).$$

So let $m_{13}^2(c) = 2(\sqrt{5} + \sqrt{6} - 4)$. Agent 1 is then willing to report honestly in event $\{a,b\}$.

By an analogous argument, assume that agents 2 and 3 agree that, in every state, agent 3 will give 4 units of the x good and none of the y good to agent 2. In exchange, agent 2 will give 8 units of the x good and none of the y good to agent 3 in the event $\{a,c\}$, and $2(\sqrt{5} + \sqrt{6} - 4)$ units of the y good and none of the x good in event $\{b\}$. The final scheme of transfers is shown in the following table:

Transfer	State a	State b	State c
m_{13}	$(8,0)$	$(8,0)$	$(0,2(\sqrt{5}+\sqrt{6}-4))$
m_{23}	$(8,0)$	$(0,2(\sqrt{5}+\sqrt{6}-4))$	$(8,0)$
m_{31}	$(4,0)$	$(4,0)$	$(4,0)$
m_{32}	$(4,0)$	$(4,0)$	$(4,0)$

Table 1: Transfers

Everyone's action is now privately measurable in every state, and the final allocations become

Agent	State a	State b	State c
1	$(16,10)$	$(16,10)$	$(4,10-2(\sqrt{5}+\sqrt{6}-4))$
2	$(16,10)$	$(4,10-2(\sqrt{5}+\sqrt{6}-4))$	$(16,10)$
3	$(8,0)$	$(0,10-2(\sqrt{5}+\sqrt{6}-4))$	$(0,10-2(\sqrt{5}+\sqrt{6}-4))$

Table 2: Allocations

The x good is now allocated as in Allen and Yannelis, and the problem of agent 3 knowing which direction to transfer wealth in the event $\{b,c\}$ vanishes.

Yet it is unclear why agents 1 and 2 need agent 3. Suppose instead they were to contract directly with each other. In particular, suppose that agent 1 promises to give agent 3

$$m^1_{12}(a) = m^1_{12}(b) = 10 \qquad m^1_{12}(c) = 0$$

and agent 2 promises to give agent 1

$$m^1_{21}(a) = m^1_{21}(c) = 10 \qquad m^1_{21}(b) = 0.$$

The allocation of the x good for both agents would then be

$$(20, 10, 10)$$

For this scheme to be incentive compatible, agent 1 cannot benefit from reporting the event $\{a,b\}$ as $\{c\}$. In event $\{a,b\}$, agent 1's expected utility from telling the

truth is

$$\frac{1}{2}\left(20 + m_{21}^2(a) + m_{21}^2(b) - 2m_{12}^2(a) + 2\sqrt{5} + \sqrt{10}\right).$$

It seems natural to look for a solution where the transfers of the y good in state a are zero, as the incentive problems are related to state c for agent 1 and state b for agent 2. In this case, the expected utility for agent 1 of telling the truth becomes

$$10 + \frac{2\sqrt{5} + \sqrt{10} + m_{21}^2(b)}{2}.$$

If agent 1 lies, then with probability 1/2, the true state is b, and agent 2 detects the lie. In that case, the contract is void. Also with probability 1/2, the true state is a, and the deception succeeds. Agent 1's expected utility of lying to agent 2 is therefore

$$\begin{aligned} u(e_1(a)) \cdot \frac{\mu(b)}{\mu(\{a,b\})} + u(e_1(a) + m_{21}(a) - m_{12}(c)) \cdot \frac{\mu(a)}{\mu(\{a,b\})} \\ = \frac{1}{2}\left(20 + m_{21}^2(a) - m_{12}^2(c) + \sqrt{20} + \sqrt{30}\right). \\ = 10 + \frac{\sqrt{20} + \sqrt{30} + m_{21}^2(a) - m_{12}^2(c)}{2}. \end{aligned}$$

Continuing with the assumption of no state a transfers of the y good, this becomes

$$10 + \frac{2\sqrt{5} + \sqrt{30} - m_{12}^2(c)}{2},$$

so that the incentive compatibility condition requires

$$10 + \frac{2\sqrt{5} + \sqrt{10} + m_{21}^2(b)}{2} \geq 10 + \frac{2\sqrt{5} + \sqrt{30} - m_{12}^2(c)}{2}. \tag{9}$$

The symmetry of the problem makes it natural to look for a solution where $m_{21}^2(b) = m_{12}^2(c)$, and where the inequality (9) holds with equality. Then,

$$\sqrt{10} + m_{12}^2(c) = \sqrt{30} - m_{12}^2(c) \quad \Rightarrow \quad m_{21}^2(b) = m_{12}^2(c) = \frac{\sqrt{30} - \sqrt{10}}{2}.$$

The allocation is then

Agent	State a	State b	State c
1	$(20, 10)$	$(10, 10 + \frac{\sqrt{30}-\sqrt{10}}{2})$	$(10, 10 - \frac{\sqrt{30}-\sqrt{10}}{2})$
2	$(20, 10)$	$(10, 10 - \frac{\sqrt{30}-\sqrt{10}}{2})$	$(10, 10 + \frac{\sqrt{30}-\sqrt{10}}{2})$
3	$(0, 0)$	$(0, 0)$	$(0, 0)$

With this approach, although it turns out that the allocation

$$x_1 = (16, 16, 4) \qquad x_2 = (16, 4, 16) \qquad x_3 = (8, 0, 0)$$

of the x good is attainable, this allocation is not in the core, because it is not coalitionally stable. Agents 1 and 2 do not need the third agent to trade, so we need to look elsewhere for a solution that would allow agent 3 to use his or her superior information to his or her advantage.

The above TU game is similar in spirit to Flåm (2007), where agents have random bilateral meetings and transfer goods in ways that increase their combined utilities. A difference in his approach is that all individual trades or contracts honored by an agent must be based on information that he can verify, i.e. all trade operates through a sequence of (using my terminology) locally coarse cores.

Bilateral trade with short positions might aid in getting to a stable allocation. This process could be construed as the spontaneous arising of financial instruments. I pursue this idea in the next section, where I introduce financial securities in the market in the Allen-Yannelis example 3.1 and show that the agents can indeed reach the allocation:

$$x_1 = (16, 16, 4) \qquad x_2 = (16, 4, 16) \qquad x_3 = (8, 0, 0)$$

and the solution will be stable as well as incentive compatible.

4 Incomplete Financial Markets

4.1 Differential information economy as a financial market

Introducing uncertainty into the economy and allowing agents to write state-contingent contracts creates a market for financial securities. Indeed, there are many similarities between the differential information economy and financial asset markets. Both specify:

- Consumption goods
- Time
- Uncertainty concerning what the state of the world is
- Information partitions
- Set of agents
- Endowments (which can depend on the time and on the state of nature)
- Utility functions

In addition, financial economies specify:

- Securities (financial assets)

So, whether we want to look at differential information economies as a cooperative game or a financial asset market depends largely on our point of view. (See, for example, Daher et al. (2007) or Koutsougeras (1998)). In cooperative games, we first restrict attention to time, uncertainty, agents, endowments, utility functions, and consumption goods. Then we pick the allocation that we think is reasonable. That is, we do not specify budgets, equilibrium prices, or financial goods. Then we worry about whether financial goods or something else can make this allocation arise. In financial economics, by contrast, financial assets are specified from the outset, and

if the financial market is thick enough, we can reach any core allocation, and all markets are in equilibrium at the same time (general equilibrium). An exception this is in some incomplete asset markets such as Hart (1975).

The basic idea behind financial markets is that people trade securities in order to maximize their utility and reduce risk. The state-contingent contracts allow agents to reduce risk by shifting their wealth between states (i.e., lose money in some states in return for gaining money in other states). Since the agents's utility functions are concave in the consumption good x, agents are risk-averse and prefer to smooth consumption over the possible states of nature.

Returning to the original Allen-Yannelis example 3.1, I introduce three types of financial assets into the model: bonds, stocks, and mutual funds. I allow agents to temporarily hold short positions and require actions to be privately measurable.

Example 4.1. Financial market with short sales:

Agents : $i \in N = \{1, 2, 3\}$

Goods : There is only one good, x.

State space : $\Omega = \{a, b, c\}$

Uncertainty : $\mu(a) = \mu(b) = \mu(c) = 1/3$.

Endowments (in each state) :

$$e_1 = (20, 20, 0) \qquad e_2 = (20, 0, 20) \qquad e_3 = (0, 0, 0)$$

Preferences : $u(x) = x^{1/2}$.

Information fields:

$$\mathcal{F}_1 = \{\varnothing, \Omega, \{a, b\}, \{c\}\}$$

$$\mathcal{F}_2 = \{\varnothing, \Omega, \{a, c\}, \{b\}\}$$

$$\mathcal{F}_3 = \{\varnothing, \Omega, \{b, c\}, \{a\}\}$$

As before, the following allocation is in the private core:

$$x_1 = (16, 16, 4) \qquad x_2 = (16, 4, 16) \qquad x_3 = (8, 0, 0)$$

This allocation is feasible and Pareto-dominates other allocations including autarky, but cannot be reached as a rational expectations equiblirium. However, if a suitable mix of financial assets is introduced to the market, the agents will nevertheless be able to trade to it.

The measurability constraint requires that each agent would only be willing to trade an asset if it has same payoff in states he or she cannot distinguish. Therefore, all three agents would be willing to trade a risk-free asset B (thought as a bond) which pays 1 unit of the consumption good in each state.

$$B(a) = 1 \qquad B(b) = 1 \qquad B(c) = 1$$

Since agents are risk-averse, agents 1 and 2 prefer to insure themselves against the state in which they receive nothing. That is, they would like to buy a financial instrument that pays them one or more units of consumption good x in the state where they receive nothing.

Suppose there are two stocks in the market: S_b, which pays one unit of the consumption good x in state b, and S_c, which pays one unit of the consumption good x in state c. The market then has three types of financial instruments whose payoffs are as follows:

Asset	State a	State b	State c
B	1	1	1
S_b	0	1	0
S_c	0	0	1

Table 3: Payoffs of the financial assets

Note that it is not necessary to introduce a state a stock that pays one unit in state a and nothing in states b and c. The net effect of a state a stock can be

achieved by a mix of the other assets: buying one unit of the bond B and selling one unit each of stocks S_b and S_c:

Asset	State a	State b	State c
S_a	1	0	0
$B - S_b - S_c$	1-0-0	1-1-0	1-0-1
S_c	0	0	1

Table 4: Portfolio generating a state a stock

The three assets B, S_b and S_c create a full set of Arrow securities for each state of nature in Ω.

Since the agents gain information about the realized state of nature after they receive their endowment at the interim stage, the trading of these assets must occur ex ante — before the agents receive their endowment. The question which immediately arises is then how the agents are able buy and sell assets before they have any endowments. One solution would be to introduce a second good, y, into the model to be used as money, just as in the three-agent transferable utility example 3.2, and give each agent an initial endowment of y before they receive their endowment of the consumption good x.

However, a simpler approach is to allow each agent to build a self-financing portfolio of assets through short sales. The only restriction is then that the short sales must generate enough revenue to cover the cost of purchasing the portfolio. This is similar to placing orders with an online brokerage conditional on some other assets selling at a certain price first.[17]

Looking at the problem from agent 1's viewpoint, agent 1 wants to increase his or her state c allocation over his or her endowment:

$$e_1 = (20, 20, 0)$$

[17]Similar models are discussed in multiunit auction theory. A full treatment is beyond the scope of this paper.

and is willing to give up some consumption in the other states in exchange. Agent 1, however, can only trade in the bond B and the state c stock S_c because they are the only securities that are privately measurable to him or her. He or she will therefore issue (sell short) a unit of the bond B and use the revenue to buy two shares of the stock S_c. After the sale, his or her expected payoffs are:

Asset	State a	State b	State c
$2S_c - B$	19	19	1
Balance	-1	-1	1

Repeating this process four times, agent 1 gets to the allocation in Example 3.1:

Asset	State a	State b	State c
$8S_c - 4B$	16	16	4

Similarly, agent 2 issues (sells short) four units of the bond B and uses the revenue to buy eight shares of the stock S_b. His or her final allocation is then:

Asset	State a	State b	State c
$8S_b - 4B$	16	4	16

If agent 3 had complete information, i.e. if his or her information partition were:

$$\mathcal{F}_3 = \{\varnothing, \Omega, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\},$$

as in Example 3.3, he could issue eight shares of S_b and eight shares of S_c, and use the revenue to buy the eight bonds on the market. His or her final allocation would then be (8, 0, 0):

Asset	State a	State b	State c
$8B - 8S_b - 8S_c$	8	$8B - 8S_b$	$8B - 8S_c$

However, in this example, agent 3's information partition is:

$$\mathcal{F}_3 = \{\varnothing, \Omega, \{b, c\}, \{a\}\}.$$

Since he or she cannot distinguish between states b and c, he or she cannot issue the securities S_b and S_c directly. But what he or she can do is to create a mutual fund S_{bc} that pays one unit of the consumption good x in the event $\{b, c\}$. Agents 1 and 2 then buy shares of this mutual fund with the revenue from selling their bond certificates. They split the cost and the profits of the mutual fund evenly.

Agent 3's final allocation will then still be $(8, 0, 0)$:

Asset	State a	State b	State c
$8B - 8S_{bc}$	8	$8B - 8S_{bc}$	$8B - 8S_{bc}$

This allocation is privately measurable to agent 3. Moreover, the payoff of the mutual fund tells agents 1 and 2 whether the state is a, in which case the fund pays nothing, or in $\{b, c\}$. Since both agents 2 and 3 can distinguish b and c once they know that the state is not a, the mutual fund essentially reveals agent 3's private information.

The financial market enables the agents to get to the allocation in Example 3.1. Yet it may seem strange to call this an equilibrium: agent 3 has a self-financing portfolio that gives a positive payoff with nonzero probability. That is, agent 3 is taking advantage of an arbitrage opportunity, which should not coexist with competitive equilibrium.

One possible explanation is that because of asymmetric information, the market is not competitive. Agents 1 and 2 increase their utilities by issuing the bonds and buying the stocks or the mutual fund, enabling them to smooth their consumption. Yet they do not take advantage of the arbitrage opportunity, which yields the state

a good for free, even though their utilities are increasing in the state a good. This is because they cannot. Agent 1 is the only party interested in S_c, and agent 2 is the only party interested in S_b. If agent 1 were to try to issue the fund S_{bc}, he or she could not find a customer in agent 3, and would not know when to pay off on it if agent 2 were to purchase the fund. So the only way agents 1 and 2 can purchase any assets is by issuing the bond, and agent 3 is the only one left to purchase the bonds. That is, agent 3 is a monopsonist in the bond market and a monopolist in the stock market. It is therefore unsurprising that the market does not reach a competitive equilibrium.

This still leaves the problem that agent 3 has a finite demand for an arbitrage opportunity: he or she can get the state a good for free, and has expected utility that is strictly increasing in the state a good. So why be content with eight units of the state a good? The reason is that demand for the stocks, and supply of the bonds, is satiated: agents 1 and 2 only issue eight bond certificates and demand only eight units of the mutual fund S_{bc}. That is, the limitation on arbitrage also reflects that the market is not entirely competitive.

5 Summary of results

There are allocations in the core of a differential information economy that are not attainable as a Rational Expectations Equilibrium. This result seems to contradict the First Welfare Theorem, which states that competitive equilibrium, in the absence of externalities, is Pareto-optimal. This paper discusses various ways of how agents might trade into such allocations.

I examine the coarse core, fine core, and the private core. In addition, I introduce two refinements of the core of a differential information economy: the locally coarse core and the action-measurable private core. The locally coarse core requires that agents have information in common with their trading partners, but not necessarily with everyone. The action-measurable private core requires measurability of the agents' actions rather than their allocations. Each of the core concepts leaves something to be desired, and none achieves the optimal allocation in a three-agent example when information is asymmetric and none of the agents has complete information. Some core concepts fail incentive compatibility, while others violate the stability constraint.

Finally, I introduce financial securities into the differential information economy. Through short sales and self-financing portfolios, the agents in a three-agent economy with asymmetric information are able trade into a private core allocation that is Pareto-optimal, measurable, incentive compatible, and stable.

Thus, the financial market approach suggests that a core allocation might be a reasonable prediction even if it cannot arise as rational expectations equilibrium. There is limited arbitrage in the financial markets example I consider, but because of the information asymmetries, the agents cannot fully exploit it. In other words, the apparent conflict with the First Welfare Theorem reflects the fact that the market is not fully competitive.

Bibliography

G. A. Akerlof. The market for "lemons": Quality uncertainty and the market mechanism. *Quarterly Journal of Economics*, 84(3):488–500, 1970.

B. Allen. Market games with asymmetric information: the core. *Economic Theory*, 29:465–487, 2006.

B. Allen and N. C. Yannelis. Differential information economies: Introduction. *Economic Theory*, 18:263–273, 2001.

K. J. Arrow. The role of securities in the optimal allocation of risk-bearing. *Review of Economic Studies*, 31:91–96, 1953. 1964 translation and reprinting, original article in *Econometrie*.

W. Daher, V. F. Martins-da Rocha, and Y. Vailakis. Asset market equilibrium with short-selling and differential information. *Economic Theory*, 32:425–446, 2007.

G. de Clippel. The type-agent core for exchange economies with asymmetric information. *Journal of Economic Theory*, 135:144–158, 2006.

G. Debreu. *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New Haven: Yale University Press, 1959.

S. D. Flåm. Exchange under asymmetric information. Unpublished manuscript, available from the author, 2007.

F. Forges, J.-F. Mertens, and R. Vohra. The ex ante incentive compatible core in the absence of wealth effects. *Econometrica*, 70(5):1865–1892, 2002.

J. C. Harsanyi. Games with incomplete information played by "Bayesian" players, I–III. Part I. The basic model. *Management Science*, 14(3):159–182, 1967.

J. C. Harsanyi. Games with incomplete information played by "Bayesian" players, I–III. Part II. Bayesian equilibrium points. *Management Science*, 14(5):320–334, 1968a.

J. C. Harsanyi. Games with incomplete information played by "Bayesian" players, I–III. Part III. The basic probability distribution of the game. *Management Science*, 14(7):486–502, 1968b.

O. Hart. On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory*, 11(3):418–443, 1975.

S. Ishiguro. Endogeneous verifiability and optimality in agency. *Journal of Economic Theory*, 105:518–530, 2002.

L. C. Koutsougeras. A two-stage core with applications to asset market and differential information economies. *Economic Theory*, 11:563–584, 1998.

O. Kvaløy and T. E. Olsen. Endogeneous verifiability and bilateral contracting. Working paper, available from the authors, 2008.

M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. The MIT Press, Cambridge, Massachusetts, 1994.

R. Radner. Competitive equilibrium under uncertainty. *Econometrica*, 36(1):31–58, 1968.

J. F. Reinganum and L. L. Wilde. Equilibrium verification and reporting policies in a model of tax compliance. *International Economic Review*, 27(3):739–760, 1986.

Y. Sun and N. C. Yannelis. Ex ante efficiency implies incentive compatibility. *Economic Theory*, 36(1):35–55, 2008.

R. M. Townsend. Optimal contracts and competitive markets with costly state verification. *Journal of Economic Theory*, 21:265–293, 1979.

T. van Zandt. Introduction to the economics of uncertainty and information. Unpublished manuscript, available from the author, 2004.

R. Vohra. Incomplete information, incentive compatibility, and the core. *Journal of Economic Theory*, 83:123–147, 1999.

R. Wilson. Information, efficiency, and the core of an economy. *Econometrica*, 46 (4):807–816, 1978.

N. C. Yannelis. The core of an economy with differential information. *Economic Theory*, 1:183–198, 1991.