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J.E. Fenstad and A.M. Nyberg

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The purpose of this note is to discuss the relationship between standard and non-standard concepts in uniform topology. We have, in particular, been interested in the case where the space carries both an algebraic and uniform structure, e.g. as in the case of a topological group.

We assume that the reader is familiar with the standard theory as presented e.g. in [3] and the non-standard theory as presented in [6].

Recently two contributions to the non-standard approach to uniform topology have been published ([4], [5]). There seems to be little overlap with the present discussion. We believe that our emphasize on the notion of a <u>bounded</u> point leads to a very clear understanding of the exact relationship between standard and non-standard concepts.

BOUNDED POINTS

Let $(X, \mathcal{U})$ be a uniform space and let $^*X$ be a non-standard extension of X. By the <u>s-topology</u> on $^*X$ we understand the topology defined by the neighbourhood systems

$N_x = \{ {}^*U(x) \mid U \in \mathcal{U} \}, \quad x \in {}^*X$.

Remark. The uniformity $\mathcal{U}$ can also be defined by an associated family of pseudo-metrics, $\mathcal{D}$. And it is easily shown that the s-topology is generated by the following family of open sets in $*X$,

$$\{S_d(x,r) \mid d \in \mathcal{D},\ x \in *X,\ r \in R^+\},$$

where $S_d(x,r) = \{y \mid \text{st}\, d(x,y) < r\}$.

The monad, $\mu(x)$, of a point $x \in *X$ is defined to be the set

$$\mu(x) = \bigcap_{U \in \mathcal{U}} *U(x) .$$

Remark. Using the associated family of pseudo-metrics we see that $\mu(x) = \{y \in *X \mid d(x,y) \simeq 0 \text{ for all } d \in \mathcal{D}\}$.

One easily notes that the relation $x \in \mu(y)$ is an equivalence relation on $*X$ which we denote by $x \simeq y$. The space X is Hausdorff iff every monad contains at most one standard point.

We call $x \in *X$ near-standard if x belongs to the monad of a standard point.

In the sequel we assume that X is Hausdorff in the associated topology.

DEFINITION. The set B_X of bounded points of $*X$ is defined to be the closure of X in $*X$ with respect to the s-topology,

$$B_X = \text{cl}_s\, X .$$

We shall obtain a characterization of B in terms of Cauchy z-ultrafilters on X .

It is known (see e.g. [2] that to every $x \in {}^*X$ there is associated a unique z-ultrafilter $\mathcal{F}_x$ on X , and to every z-ultrafilter $\mathcal{F}$ on X there corresponds a point $x \in {}^*X$ such that $\mathcal{F} = \mathcal{F}_x$. ($\mathcal{F}_x$ is the unique z-ultrafilter which extends the prime z-filter $\mathcal{F}_x' = \{F \subseteq X \mid x \in {}^*F$ and $F \in Z(X)\}$.)

<u>PROPOSITION</u>. <u>The set of bounded points consists exactly of those</u> $x \in {}^*X$ <u>such that the associated z-ultrafilter is Cauchy, i.e.</u>

$$B_X = \{x \in {}^*X \mid \mathcal{F}_x \text{ is a Cauchy z-ultrafilter}\} .$$

We sketch the proof. Let $x \in cl_s X$, we have to show that $\mathcal{F}_x$ is Cauchy. Pick any $U \in \mathcal{U}$ and choose a closed $V \in \mathcal{U}$ such that $V \circ V \subseteq U$ and such that ${}^*V(x) \cap X \neq \emptyset$. Let $p \in {}^*V(x) \cap X$. We now observe that $V(p) \in \mathcal{F}_x$, since $x \in {}^*V(p)$ and we may assume that $V(p) \in Z(X)$. And obviously $V(p) \times V(p) \subseteq U$, which shows that $\mathcal{F}_x$ is Cauchy.

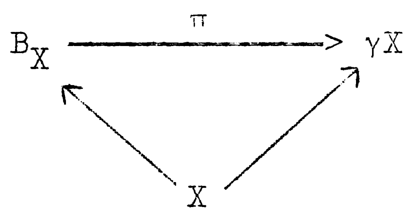
Conversely, let $\mathcal{F}_x$ be Cauchy. Let $U \in \mathcal{U}$ and pick a (symmetric) $U_1 \in \mathcal{U}$ such that $U_1 \circ U_1 \subseteq U$. Since $\mathcal{F}_x$ is Cauchy, there is a $V \in \mathcal{F}_x$ such that $V \times V \subseteq U_1$. It is now possible to pick an $x_0 \in \mu(x)$ such that $x_0 \in {}^*V$. This shows that $(p,x_0) \in {}^*U_1$, hence $p \in {}^*U(x) \cap X$.

<u>Remark</u>. We mention here the following result : X is <u>complete</u> iff every bounded point is near-standard. This generalizes the fact that compactness is equivalent to every (non-standard) point being near-standard.

THE COMMUTATIVE DIAGRAM.

We noted above that if $x \in {}^*X$, then $\mu(x)$ contains at most one standard point. If p is a standard point in $\mu(x)$, then p is uniquely determined. We call p the *standard part* of x , and write $p = st(x)$.

Let γX denote the completion of X . Then X is imbedded in both ${}^*X$ and γX and there is a surjection $\pi\colon B_X \to \gamma X$ such that the following diagram is commutative:



The definition of π is immediate: $\pi(x)$, $x \in B$, is the equivalence class of the Cauchy z-ultrafilter $\mathcal{F}_x$ in γX .

PROPOSITION. *Let* $(X, \mathcal{U})$ *and* $(Y, \mathcal{V})$ *be uniform spaces, and assume that* $(Y, \mathcal{V})$ *is complete*. *Let* $f\colon X \to Y$ *be a uniformly continuous map, hence* f *has an extension to a continuous map* $\hat{f}\colon \gamma X \to Y$. *The following identity is valid for all* $x \in B_X$:

$$st({}^*f(x)) = \hat{f}(\pi(x)) .$$

As a preliminary remark toward a sketch of the proof, we note that if $f\colon X \to Y$ is uniformly continuous, then $f(\mu(x)) \subset \mu(f(x))$ for all $x \in {}^*X$. This follows immediately from the definition of uniform continuity, and, in fact, characterizes this notion.

Let $x \in B_X$ be given. Since $\mathcal{F}_x$ is Cauchy and Y is complete, we see that the z-filter $f^{\#}(\mathcal{F}_x) = \{Z \in Z(Y) \mid f^{-1}(Z) \in \mathcal{F}_x\}$ converges toward the point $\hat{f}(\pi(x))$ in Y. Pick a point x_0 such that $x_0 \in {}^*F$, for all $F \in \mathcal{F}_x$. Then $x_0 \in \mu(X)$ and ${}^*f(x_0) \in {}^*Z$, for all $Z \in f^{\#}\mathcal{F}_x$. Since $f^{\#}(\mathcal{F}_x)$ converges to $\hat{f}(\pi(x))$, we see that

$$ {}^*f(x_0) \in \bigcap_{Z \in f^{\#}(\mathcal{F}_x)} {}^*Z \subseteq \bigcap_{V \in \mathcal{V}} {}^*V(\hat{f}(\pi(x))) = \mu(\hat{f}(\pi(x))) . $$

The uniform continuity of f now implies that ${}^*f(x) \in \mu({}^*f(x_0))$. Hence ${}^*f(x) \in \mu(\hat{f}(\pi(x)))$, which exactly means that $st({}^*f(x)) = \hat{f}(\pi(x))$.

Remark. This generalizes a result in [2] where we considered the relationship between ${}^*X$ and the Store-Čech compactification βX of X.

In the first section of this note we showed that the set B_X, which was defined as the s-closure of X in ${}^*X$, is the set of points x such that $\mathcal{F}_x$ is a Cauchy z-ultrafilter on X. From the observations of this section it follows that

$$ st\ d(x,y) = \hat{d}(\pi(x) , \pi(y)) , $$

for all bounded x and y and all d in the associated family of pseudo-metrics. This identity "explains" why the s-topology as defined by A. Robinson is the appropriate setting for discussing the completion of metric spaces. It also implies that $x \simeq y$ (i.e. $x \in \mu(y)$) iff $\pi(x) = \pi(y)$.

These observations taken together shows that we obtain the completion of X as the set of bounded points modulo monads. Implicit in our observations is also the fact that this non-standard approach is nothing but the "lifting" from βX to $*X$ of a well-known procedure (see e.g. [3]).

But something may be gained. A. Robinson [7] constructed R as Q_f/Q_i , where Q_f is nothing but the bounded points of Q and Q_i are the infinitesimals. The important point here is that the algebraic operations of Q extends to Q_f , i.e. Q_f is a "nice" substructure of $*Q$ which itself is an elementary extension of Q . We will return to this point in the next section.

EXTENDING MAPS FROM X TO γX .

Let as above $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces, and assume that $(Y,\mathcal{V})$ is complete. It is known in the metric case (and fairly straight forward to extend to the uniform case) that a map $f: X \to Y$ is continuous, iff f is s-continuous for all standard x , or, equivalently, iff $f(\mu(x)) \subseteq \mu(f(x))$ for all standard x . Here "standard" can be replaced by "nearstandard".

And a map $f: X \to Y$ is uniformly continuous, iff f is s-continuous for all $x \in *X$, or, equivalently, iff $f(\mu(x)) \subseteq \mu(f(x))$ for all $x \in *X$.

In this section we characterize in a similar way the property of having a continuous extension from X to γX .

PROPOSITION. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces, and assume that $(Y,\mathcal{V})$ is complete. Let f be a map from X to Y . The following three conditions are equivalent.*

(i) f *has a continuous extension to* γX .

(ii) f <u>is s-continuous for all bounded</u> $x \in {}^*X$.

(iii) $f(\mu(x)) \subseteq \mu(f(x))$ <u>for all bounded</u> $x \in {}^*X$.

We first prove that (i) implies (ii) & (iii). Let $\hat{f}$ be the continuous extension of f to γX . We consider X as a dense subspace of γX and we work with non-standard extensions ${}^*X \subseteq {}^*\gamma X$. The map $\hat{f}$ is continuous from γX to Y . Hence ${}^*\hat{f}$ maps near-standard points of ${}^*\gamma X$ to near-standard points in ${}^*Y$. And ${}^*\hat{f}(\hat{\mu}(x)) \subseteq \mu({}^*\hat{f}(x))$, for all near-standard $x \in {}^*\gamma X$. (Here $\hat{\mu}$ denotes the monad in γX.)

First, note that near-standard is the same as bounded in ${}^*\gamma X$ since γX is complete.

Next, note that X is uniformly imbedded as a dense subset of γX . This means, in particular, that $\mu(x)$, the monad of a point $x \in {}^*X$, is the restriction of the monad $\hat{\mu}(x)$ to ${}^*X$. (This follows from the correspondence between <u>entourages</u> in X and γX.) Further, a point $x \in {}^*X$ which is bounded in ${}^*X$ remains bounded in ${}^*\gamma X$.

Putting these things together we see that (i) implies (ii) and (iii). As an example we verify (iii). Thus let x be a bounded point in ${}^*X$. Hence x is bounded, and therefore near-standard in γX . The continuity of $\hat{f}$ then implies that ${}^*\hat{f}(\hat{\mu}(x)) \subseteq \mu(\hat{f}(x))$. Let $y \in \mu(x)$. Then $y \in \hat{\mu}(x)$, hence ${}^*\hat{f}(y) \in \mu({}^*\hat{f}(x))$. But ${}^*f = {}^*\hat{f} \mid {}^*X$, hence ${}^*f(y) \in \mu({}^*f(x))$, which was to be proved.

<u>Remark</u>. Since $\hat{f}$ maps near-standard points of ${}^*\gamma X$ to near--standard points in ${}^*Y$, it follows that f maps bounded points in ${}^*X$ to bounded points in ${}^*Y$.

The s-continuity of f on B_X implies immediately that $f(u(x)) \subset u(f(x))$ for any $x \in B_X$. In fact, let x be bounded and consider an s-open neighbourhood V' of $f(x)$ in $*Y$. It then exists an s-open neighbourhood V of x such that $f(V \cap B_X) \subseteq V'$. As $u(x)$ is the intersection of all s-neighbourhoods of x and $u(x) \subseteq B_X$, we see that $f(u(x)) \subseteq V'$. Since V' is arbitrary, the result follows.

For the final part of the proof assume that $f(u(x)) \subseteq (f(x))$ for all bounded x. Let $\mathcal{F}$ be an arbitrary Cauchy z-ultrafilter on X. We have to show that $f^{\#}\mathcal{F} = \{Z \in Z(Y) \mid f^{-1}(Z) \in \mathcal{F}\}$ is a Cauchy z-filter on Y. Note first that $\mathcal{F} = \mathcal{F}_x$ for some bounded point x.

Consider the family $f(\mathcal{F}_x) = \{f(F) \mid F \in \mathcal{F}_x\}$. It suffices to show that for all $V \in \mathcal{V}$ there is some $F' \in f(\mathcal{F}_x)$ such that $F' \times F' \subseteq V$.

We prove this by contradiction. Assume not: Then there exists a $V \in \mathcal{V}$ such that the following (standard) sentence is true:

$$(\forall F)(F \in \mathcal{F} \rightarrow (f \times f)(F) \not\subseteq V) .$$

A familiar type non-standard argument now gives an <u>internal</u> set $F_o \in *\mathcal{F}$ such that $F_o \subseteq u(x)$ and such that $(f \times f)(F_o) \not\subseteq *V$.

Pick a symmetric $W \in \mathcal{V}$ such that $W \circ W \subseteq V$. It follows that $(f \times f)(F_o) \not\subseteq *W \circ *W$. And since $F_o \subseteq u(x)$, we further obtain that $(f \times f)(u(x)) \not\subseteq *W \circ *W$. We may then choose points $u,v \in u(x)$ such that $(f(u), f(v)) \not\in *W \circ *W$, which easily implies that $f(u(x)) \not\subseteq *W(f(x))$. However,

$u(f(x)) \subset {}^*W(f(x))$, and the result follows.

Remark. We note the following supplementary characterization of the set of bounded points: A point x is bounded, iff for all uniformly continuous $f: X \to R$, $f(x)$ is bounded, or, equivalently, iff for all $f: X \to R$ which has a continuous extension to γX , $f(x)$ is bounded.

It remains to show that if $x \in {}^*X - B_X$, then there is some uniformly continuous $f: X \to R$ such that $f(x)$ is not bounded. Note that X can be imbedded into a product R^I , where $I = \{f: X \to R \mid f$ is uniformly continuous$\}$. If $x \notin B$, then $\mathcal{F}_x$ is a z-ultrafilter which is not Cauchy. Hence there must be some $f \in I$ such that $pr_f(\mathcal{F}_x)$ is a base for a z-ultrafilter which is not Cauchy. But this means that the z-ultrafilter $\mathcal{F}_{f(x)}$ cannot be Cauchy, hence $f(x)$ is not bounded.

Let X now carry both an algebraic and uniform structure, e.g. let X be a topological group. When is γX an algebraic structure of the same kind as X ? The answer must be related to the "degree of continuity" of the algebraic operations. Continuity is known to be too weak and uniform continuity too strong. It turns out that s-continuity of the algebraic operations on the set of bounded points of $^*X$ is the right kind of requirement.

For simplicity assume that X as an algebraic structure has certain operations $f_1, \ldots, f_k$ and that the axioms which X is supposed to satisfy are open, positive sentences (e.g. let X be a group with both group multiplication and inverse operation). We must now consider maps $f_r: X^n \to \gamma X$. (Note that monads and bounded points commute with finite cartesian products, hence our previous results apply.) The s-continuity of the operations f_r implies that each f_r maps bounded points to bounded points. Hence the operations f_r, which can be extended to $^*X$ by general model-theoretic considerations, can be restricted to the bounded points. Since s-continuity means that $f_r(\mu(b_1),\ldots,\mu(b_n)) \subseteq \mu(f(b_1,\ldots,b_n))$, the operations f_r can be further defined on $\gamma X \simeq {}^B/_\mu$.

And since γX is a homomorphic image of a subsystem of $^*X$, which itself is an elementary extension of X, the syntactic form of the axioms implies that they are also valid in γX. Thus in this case the s-continuity of the algebraic operations ensures that γX is an algebraic structure of the same kind as X, and that the extended operations are continuous in the associated topology. And in this case the condition of s-continuity is also necessary.

This includes known results on topological groups and rings. When the axioms are of a more complicated syntactic character, the situation becomes more involved. It is perhaps somewhat doubtful whether a useful general theorem can be stated.

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