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by

Jan R. Høegh-Krohn

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1 Introduction.

In Quantum Theories the scattering operator is of great significance in relating theory with observation. The scattering operator is given relative to a decomposition of the total energy operator H into two parts. The free energy Ho and the interaction energy V .

$$H = Ho + V$$

H and Ho are self adjoint operators on a Hilbert space $\mathcal{H}$. The pure states of the system are represented by elements in $\mathcal{H}$. If at the time zero the state is represented by ψ_o then at the time t it is represented by $e^{itH}\psi_o$. Asymptotical for t very large we expect the system to behave as if there were no interaction, that is as if the energy operator was Ho .

$$e^{itH}\psi_o \sim e^{itH_o}\psi_\pm \quad \text{for} \quad t \to \pm\infty$$

The scattering operator S is now defined by

$$S\psi_- = \psi_+$$

and describe the asymptotic time transition of the system. We see that existence of S corresponds to the exsistence of the asymptotic limits

$$\lim_{t \to \pm\infty} e^{-itH} e^{itH_o} = W_{\pm}$$

since

$$\psi_o = W_{\pm} \psi_{\pm}$$

As is well known there is also another way of decribing scattering, or asymptotic transition from very early to very late times, then by the scattering operator using asymptotic states as described above. That is by using asymptotic operators. The advantages of using asymptotic operators in conenection with Quantum Field Theories were pointed out by Friedrichs in [1] as well as by Y. Kato and N. Mugibayaski in [2] .

In a quantum system the observable quantities are represented by operators on the Hilbert space $\mathcal{H}$, and if the quantity at the time zero is represented by A_o, then at the time t it is represented by $e^{-itH}A_o\, e^{itH}$. Asymptoticaly we expect the system to behave as if H_o were the energy operator, or

$$e^{-itH}A_o\, e^{itH} \sim e^{-itH_o}A_{\pm}\, e^{itH_o} \text{ for } t \to \pm\infty$$

The scattering is now given by the transition

$$A_{-} \to A_{+}$$

describing the asymptotic transition of the system. We see that even if there exist no asymptotic states there still may exists asymptotic operators, and we can therefore still study scattering of the system. We expect the mappings $A_o \longrightarrow A_{\pm}$ to preserve the algebraic relations, so if we substitute $e^{itH_o}A_o\, e^{-itH_o}$ for A_o in the asymptotic relation above we have $e^{-itH}\, e^{itH_o}A_o\, e^{-itH_o}\, e^{itH} \sim A_{\pm}$ for $t \to \pm\infty$

or the following relations

$$\lim_{t \to \pm\infty} e^{-itH} e^{itH_o} A_o e^{-itH_o} e^{itH} = A\pm$$

To gather with the spesification of the sence in which the limit is to be taken, we will use this as a definition of the asymptotic operators $A\pm$.

In two earlier papers [3] the author studied perturbation by annihilation-creation operators, using a technique based on "gentleness". This technique however was not able to deal with the case of V containing pure annihilation and pure creation terms. In that case it is well known that there will be a vacuum renormalization, i.e. a general shift of the whole spectrum of the energy operator, see for instance [1] , and this causes considerable difficulties for the above mentioned technique.

On the other hand Y. Kato and N. Mugibayashi [2] studied perturbations where V contained pure annihilation and pure creation terms, by an adaption of Cook's Method [4] to the study of asymptotic limits of annihilation-creation operators, but with a very strong restriction on V , namely that the kernels of V being finite dimentional.

Some of the ideas of this paper however,were inspired by the work of Y. Kato and N. Mugibayashi.

2 Quantum Field with nonlocal interaction or Perturbation by annihilation-creation operators.

The free energy operator H_o is given with respect to a specific representation of the Hilbert-space $\mathcal{H}$, the so called Fock representation. An element f in $\mathcal{H}$ is given by a sequence f_n of complex valued functions, where f_o is just a complex constant and f_n is a function $f_n(x_1 \ldots x_n)$ of n-variables $x_1 \ldots x_n$, and each x_i is a variable of the Euclidian 3-space E_3 . We will consider only the case of one fermion field interacting with it self. The reason for this is partly one of notational conveniance and partly the fact that if the interaction is more than quadratic in the boson field, some of the proofs will become more complicated. Thus the details of the boson interaction will be given on a later occation in order not to make this paper toolong.

That we consider only one fermion field means that the function $f_n(x_1 \ldots x_n)$ are all antisymmetric i.e.

$$f_n(x_1 \ldots x_n) = \frac{1}{n!} \sum (-1)^{\sigma} f_n(x_{\sigma(1)} \ldots x_{\sigma(n)})$$

where the summation runs over all permutations σ of the indices $1 \ldots n$. The inner product in $\mathcal{H}$ is given by

$$(f, g) = \sum_{n=0}^{\infty} n! \int \cdots \int f_n(x_1 \ldots x_n) \, \bar{g}_n(x_1 \ldots x_n) \, dx_1 \ldots x_n$$

Let Ω be the self adjoint operator in $L_2(E_3)$

$$\Omega = \sqrt{-\Delta + m^2}$$

on its natural domain of definition, where Δ is the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and m a non negative constant called the "mass of the free fermion". H_o is then given by

$$(H_o f)_n (x_1 \ldots x_n) = \sum_{i=1}^{n} \Omega_i f_n(x_1 \ldots x_n)$$

where Ω_i is the operator Ω operating on the variable X_i, and $(H_o f)_o = 0$. H_o is obviously self adjoint on its natural domain of definition. The interaction operator V is now given in terms of the annihilation-creation operators. The annihilation operator $a(x)$ is defined for $x \in E_3$ by

$$(a(x) f)_n(x_1 \ldots x_n) = (n+1) f_{n+1}(x, x_1 \ldots x_n)$$

and the creation operator $a^*(x)$ as the adjoint of $a(x)$. $a(x)$ and $a^*(x)$ are both improper operators but their definition is easely made presice in the following way. Let $h \in L_2(E_3)$ and define

$$a(h) = \int a(x)h(x)dx$$

and $a^*(h)$ as the adjoint of $a(\bar{h})$. The definition of the integral above is of course

$$(a(h)f)_n = (n+1)\int dx\ h(x) f_{n+1}(x,x_1 \ldots x_n)$$

It is easily verified that due to the fact that $f_n(x_1 \ldots x_n)$ is antisymmetric, we have the following anticommutation relations characteristic of a fermion field.

$$
\begin{aligned}
&a(h)a(g) + a(g)a(h) = 0 \\
(1) \qquad &a^*(h)a^*(g) + a^*(g)a^*(h) = 0 \\
&a^*(h)a(g) + a(g)a^*(h) = \int g(x)h(x)dx
\end{aligned}
$$

The last equality implies that

$$\| a(h) \| \leq \| h \|_2$$

so that for $h \in L_2$, $a(h)$ is a bounded operator. The interaction V is now given in the following way

$$V = \sum_{0 \leq k\ l \leq N} V_{kl}$$

where V_{kl} is given in terms of the kernels $V_{kl}(x_1 \ldots x_k | y_1 \ldots y_l)$ antisymmetric in $x_1 \ldots x_k$ and in the following manner

$$V_{kl} = \int \cdots \int V_{kl}(x_1 \ldots x_k \mid y_1 \ldots y_l) a^*(x_1) \ldots a^*(x_k) a(y_1) \ldots a(y_l) \, dx_1 \ldots dx_k \, dy_1 \ldots dy_l$$

V_{kl} may also be defined explicitly by

$$(V_{kl}\ f)_n(x_1 \dots x_n) = \text{asym}\binom{n}{l}l! \int \dots \int dy_1 \dots dy_l$$

$$V_{kl}(x_1 \dots x_k | y_1 \dots y_l)\ f_m(y_1 \dots y_l, x_{k+1} \dots x_n)$$

where $n = k-l+m$ and asym is short for the antisymmetrization with respect to the variables $x_1 \dots x_n$. Since $a(h)$ is bounded for $h \in L_2$, we see that for $V_{kl}\ (x_1 \dots x_k | y_1 \dots y_l)$ smooth enough the operator V will be bounded. We will assume that V is symmetric i.e.

$$V_{kl}\ (x_1 \dots x_k | y_1 \dots y_l) = \bar{V}_{lk}\ (y_1 \dots y_l | x_1 \dots x_k)$$

3 <u>The existence of asymptotic limits of annihilation-creation operators.</u>

By definition of the operator H_o we get that

$$e^{itH_o}\ a(h)\ e^{-itH_o} = a(e^{-it\Omega} h)$$

and

$$e^{itH_o}\ a^*(h)\ e^{-itH_o} = a^*(e^{it\Omega} h)$$

and it is well known that $e^{it\Omega} h$ tends to zero as $|t|^{-3/2}$ for $t \to \pm\infty$ if $h \in C_o^\infty$. This together with the equetions above is the main ingredients in proving that the limits as $t \to \pm\infty$ of

$$a_t(h) = e^{-itH}\ e^{itH_o}\ a(h)\ e^{-itH_o}\ e^{itH}$$

and

$$a_t^*(h) = e^{-itH}\ e^{itH_o}\ a^*(h)\ e^{-itH_o}\ e^{itH}$$

exist .

We get namely up to the question of domains and the question of differentiability, by first derivating and then intergrating that

$$a_t(h) - a(h) = -i \int_0^t ds\ e^{-isH} \left[V,\ e^{isH_o}\ a(h)\ e^{-isH_o}\right] e^{isH}$$

$$a_t^*(h) - a(h) = -i \int_0^t ds\ e^{-isH} \left[V,\ e^{isH_o}\ a^*(h)\ e^{-isH_o}\right] e^{isH}$$

So that the question of existence of the asymptotic limits for $a_t(h)$ and $a_t^*(h)$ is equivalent to the convergence of the integrals on the right hand side. By utilizing that $e^{-it\Omega}h \sim c\,|t|^{-3/2}$ we get that $e^{itH_0}\,a(h)\,e^{-itH_0}$ in some sence behave like $c\,|t|^{-3/2}$, and thus that the integrand behaves like $c\,|t|^{-3/2}$. But this is integrable at infinety and thus we are able to prove that

Theorem 1

$$a_{\pm}(h) = \text{strong}\lim_{t\to\pm\infty} a_t(h)$$

and

$$a_{\pm}^*(h) = \text{strong}\lim_{t\to\pm\infty} a_t^*(h)$$

exist .

The proof of this and later theorems will be published an other place.

From the definition of $a_t(h)$ and $a_t^*(h)$ we get

$$e^{isH}\,a_t(h)\,e^{-isH} = a_{t-s}\,(e^{-is\Omega}h)$$

$$e^{isH}\,a_t^*(h)\,e^{-isH} = a_{t-s}^*\,(e^{is\Omega}h)$$

Taking the limit $t\to\pm\infty$ on both sides we get

$$e^{isH}\,a_{\pm}(h)\,e^{-isH} = a_{\pm}(e^{-is\Omega}h)$$

$$e^{isH}\,a_{\pm}^*(h)\,e^{-isH} = a_{\pm}^*(e^{is\Omega}h)$$

Derivating with respect to s this gives us the following consequence of theorem 1. We state

Theorem 2

$$[H,\ a_{\pm}(h)] = a_{\pm}(-\Omega h)$$

$$[H,\ a_{\pm}^*(h)] = a_{\pm}^*(\Omega h)$$

Now theorem 2 may be used to give some information on the spectral structure of the operator H , if the mass m is strictly positive.
We state

Theorem 3. Under the assumption that the mass m is strictly positive, there exist in $\mathcal{H}$ maksimal closed subspaces $V^{\pm}$ with the property that

$$a_{\pm}(h)V^{\pm} = 0 \quad \text{for} \quad h \in L_2$$

These subspaces $V^{\pm}$ are invariant under H, and the smallest closed subspace containing $V^{\pm}$ and invariant under $a_{\pm}^{*}(h)$ is $\mathcal{H}$.

Let the spectral decomposition of $V^{\pm}$ with respect to H be

$$V^{\pm} = \int V_{\omega}^{\pm} dv^{\pm}(\omega)$$

then $\mathcal{H}$ has the following direct integral decomposition

$$\mathcal{H} = \int \mathcal{H}_{\omega}^{\pm} \, dv^{\pm}(\omega)$$

where $\mathcal{H}_{\omega}^{\pm}$ is the Fock space constructed with the vacuum space $V_{\omega}^{\pm}$.

We have not yet given the conditions on the kernels $V_{kl}\ (x_1 \ldots x_k | y_1 \ldots y_l)$ under which we are able to prove these theorems. There are tree distinct cases in which we can do so

I. Gentle perturbation by annihilation-creation operators with vacuum interaction causing a finite vacuum renormalization. It is, the interaction operator V is of the form

$$V = \sum_{0 \leq k,l \leq N} V_{kl}$$

where the kernels $V_{kl}\ (x_1 \ldots x_k | y_1 \ldots y_l)$ are all smooth functions.

II. Gentle perturbation by annihilation-creation operators, with translation invariant interaction but no vacuum or one particle interaction. It is the interaction operator V is of the form

$$V = \sum_{2 \leq k, l \leq N} V_{kl}$$

and the kernels $V_{kl}(x_1 \ldots x_k | y_1 \ldots y_l)$
are translation invariant

$$V_{kl}(x_1+a \ldots x_k+a | y_1+a \ldots y_l+a) = V_{kl}(x_1 \ldots x_k | y_1 \ldots y_l)$$

III. Local interaction of a relativistic scalar field (meson field) with nonrelativistic particles (nucleons). This model was first investigatied by E. Nelson and he proved, using a canonical transformation due to Gross, that after subtracting an infinit nucleon self energy term the total energy operator existed and defined a self adjoint operator. We are able to prove that the asymptotic meson fields exists as strong limits, so that theorem 1, 2 and 3 holds also in this case. The proves will be given in an forthcomming paper.

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