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SETS IMPLICITLY DEFINED
BY ARITHMETIC CONDITIONS

by

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This note has been substituted for the original text.

1. A condition $A(X)$ defines a set S implicitly if S is the only set which satisfies A, i.e. $\{X \mid A(X)\} = \{S\}$. We note that if A is an arithmetic condition and S is implicitly defined by A, then $S \in \Delta^1_1$:

$$n \in S \equiv \forall X(A(X) \rightarrow n \in X)$$

$$\equiv \exists X(A(X) \wedge n \in X).$$

We shall discuss the following problem: What is the connection between Δ^1_1 sets and sets defined implicitly by an arithmetic condition ?

The answer is known and is contained in the following two results.

I. $\Delta^1_1 = \mathcal{R}$(Sets implicitly defined by arithmetic conditions).

II. $\Delta^1_1 \neq$ Sets implicitly defined by arithmetic conditions.

(Here $\mathcal{R}(\underline{Q})$ denotes the class of sets recursive in some member of $\underline{Q}$.)

I is due to C. Spector [2] and II was shown by S. Feferman [1]. In this note we give an exposition of these results and present somewhat different proofs. Our proof of I is quite direct, the proof of II which we give is in essence the same proof as the one given by Feferman, but we have adopted a Boolean algebra version which does not explicitly refer to the notion of forcing and generic sets. (The use of Boolean-valued logic to obtain independence results in set theory is due to D. Scott. That the same technique may be used in the first order case is one of the main points of this note. We do not claim that our presentation is simpler than Feferman's, but we do believe that the way which we present the proof may give some insight into the notions of forcing and generic sets.)

2. To prove I we analyze in somewhat more detail the inductive definition of the sets H_b, $b \in O$. It is known that Δ_1^1 equals the class of sets recursive in some H_b. We define:

$$
\begin{aligned}
(*) \qquad A(X,a) \equiv .\ & a = 1 \wedge \forall x(\langle x,1\rangle \in X \equiv .\ x = x) \\
.\mathrm{v}.\ & a = 2^{(a)_0} \wedge (a)_0 \neq 0 \wedge \forall x(\langle x,a\rangle \in X \equiv . \\
& \qquad \exists z T^{\hat{y}(\langle y,(a)_0\rangle \in X)}(x,x,z) \\
.\mathrm{v}.\ & a = 3\cdot 5^{(a)_2} \wedge \forall x(\langle x,a\rangle \in X \equiv . \\
& \qquad \langle (x)_0, \{(a)_2\}(((x)_1)_0)\rangle \in X).
\end{aligned}
$$

We next define:

$$
(**) \qquad A^*(X,a) \equiv .A(X,a) \wedge \forall b(b \in C(a) \rightarrow A(X,b))
$$

Here $b \in C(a)$ is the recursively enumerable relation such that if $a \in O$, then $b \in C(a) \equiv .b <_O a$.

We now prove

1. $a \in O: A^*(X,a) \wedge b <_o a \rightarrow A^*(X,b);$

2. $a \in O: A^*(X,a) \rightarrow \hat{x}(\langle x,a\rangle \in X) = H_a.$

The first statement is trivial by the definition of A^*. The second is proved by induction on $a \in O$:

(i) $a = 1$. Then $A^* \equiv A$, and thus $\hat{x}(\langle x,a\rangle \in X) = N = H_1$.

(ii) $a = 2^{(a)_o} \wedge (a)_o \neq 0$. Then $(a)_o \in O$ and from 1. follows that $A^*(X,(a)_o)$. Hence by induction hypothesis $\hat{x}(\langle x,(a)_o\rangle \in X) = H_{(a)_o}$, and then by definition of A it follows that $\hat{x}(\langle x,a\rangle \in X) = \hat{x}\exists z T^{H_{(a)_o}}(x,x,z) = H_a$.

(iii) $a = 3 \cdot 5^{(a)_2}$. The proof is similar to the one given under (ii) and is therefore omitted.

We may now define

$$(***) \quad A_a(X) \equiv .\ \forall x(x \in X \rightarrow x = \langle (x)_o,(x)_1\rangle \wedge ((x)_1 = a \vee (x)_1 \in C(a)) \wedge A^*(X,a).$$

We see that A_a for each a is an arithmetic condition with one free set variable X. We shall show

3. $a \in O:- A_a(X) \rightarrow x \in X \equiv .(x)_1 \leq_O a \wedge H_{(x)_1}((x)_0).$

To prove this we note the following sharper version of 2: If $a \in O$ and $b \leq_O a$ and $A^*(X,a)$, then $\hat{x}(\langle x,b\rangle \in X) = H_b$. This follows immediately from 1 and 2.

Let now $a \in O$, $A_a(X)$ and $x \in X$. Then $(x)_1 \leq_O a$, and hence, by our remark, $H_{(x)_1}((x)_0)$. Conversely, if $(x)_1 \leq_O a$ and $H_{(x)_1}((x)_0)$, then by the remark we may conclude that $\langle (x)_0,(x)_1\rangle \in X$, which by the assumption $A_a(X)$ yields that $x \in X$.

We also note that the set

$$X_a = \{\langle x,y\rangle ;\ y \leq_O a \wedge H_y(x)\},$$

satisfies $A_a(X)$, hence the condition A_a, $a \in O$, is an implicit definition. And from the known characterization of Δ_1^1 in terms of the hierarchy $\{H_b;\ b \in O\}$ the result I follows.

3. The proof of II is rather more complicated and it will be subdivided in several steps.

A. We first define two languages L and L^*. L is the usual 1 order language for number theory with four predicates $\underline{R}_i$, $i = 1,\ldots,4$, corresponding to the arithmetic relations: "$x = y$", "$x = y'$", "$x+y = z$" and "$x\cdot y = z$". L^* is obtained from L by adding one set symbol $\underline{S}$.

B. We shall give an interpretation of the languages in the complete Boolean algebra of regular open sets in 2^N. If we let $A^* = 2^N - cl(A)$, then the Boolean operations in $\mathcal{B}$ (= regular open sets) are given by: $A \wedge B = A \cap B$, $-A = A^*$ and $\bigwedge_k A_k = (\bigcap_k A_k)^{**}$.

The interpretation of L^* in $\mathcal{B}$ is now defined in the following way:

(i) $\alpha \in \| \underline{R}_i(\bar{k},\ldots) \| \equiv . R_i(k,\ldots)$;

(ii) $\alpha \in \| \underline{S}(\bar{k}) \| \equiv . \alpha(k) = 1$;

(iii) $\| \neg \phi \| = - \| \phi \|$;

(iv) $\| \phi_1 \wedge \phi_2 \| = \| \phi_1 \| \wedge \| \phi_2 \|$;

(v) $\| \exists x \phi(x) \| = \bigvee_k \| \phi(\bar{k}) \|$.

We assume that the basic properties of $\mathcal{B}$ are known, but remark that $\leq$ in $\mathcal{B}$ is the same as set-theoretic inclusion. Further if u is a finite sequens of 0's and 1's, we also let u denote the basic neighborhood in 2^N consisting of all functions α such that $\alpha(i) = u_i$, $i < lh(u)$. - Note that if $\alpha \in \| \phi \|$, then there is some u such that $\alpha \in n \subseteq \| \phi \|$, since each $\| \phi \|$ is open.

C. We next introduce certain transformations on the language L^* and on the algebra $\mathcal{B}$. For each $k \in N$ let $\tau_k \alpha$ be the function which equals α at each point different from k but differs from α at k. τ_k induces a transformation on subsets of 2^N by $\tau_k A = \{\alpha \in 2^N; \tau_k\alpha \in A\}$. We note the following facts: $\tau_k\tau_k = \tau_k$, τ_k commutes with the set-theoretic operations in 2^N.

Lemma. $\tau_k(A^*) = (\tau_k A)^*$.

The point to prove is that $cl(\tau_k A) = \tau_k(cl\ A)$, or equivalently $cl(A) = \tau_k cl \tau_k A$.

(1) $\alpha \in cl\ A \equiv . \forall u(\alpha \in u \to u \cap A \neq \emptyset)$

(2) $\alpha \in \tau_k cl \tau_k A = \forall u(\tau_k\alpha \in u \to u \cap \tau_k A \neq \emptyset)$.

We prove (1) $\to$ (2): Let $\tau_k\alpha \in u$, then $\alpha \in \tau_k u$, hence by (1), $\tau_k u \cap A = \emptyset$. But then $u \cap \tau_k A = \tau_k(\tau_k u \cap A) \neq \emptyset$, which is the conclusion of (2).

From this follows that τ_k commutes with the operations in $\mathcal{B}$.

We now define a transformation, also to be denoted by τ_k, on L^* by replacing each occurrence of $\underline{S}(s)$ in a sentence of L^* by

$$(s \neq \bar{k} \wedge \underline{S}(s)) \vee (s = \bar{k} \wedge \neg \underline{S}(s)).$$

Lemma. $\tau_k \| \phi \| = \| \tau_k \phi \|$.

The proof is by induction on the structure of ϕ. The only interesting case is to show that $\tau_k \| \underline{S}(\bar{n}) \| =$

$\|\tau_k(\underline{S}(\bar{n}))\|$. And this may be verified by a simple direct computation.

D. We note that if x is a numerical variable, then $\tau_x\phi$ is a formula of L^* which we may assume contains x free. Hence $\exists x[\tau_x\phi]$ belongs to L^*.

Let $\phi(\underline{S})$ be an arithmetic conditions on $\underline{S}$, we have the following result:

Lemma. $\| \phi(\underline{S}) \rightarrow \exists x[\tau_x\phi(\underline{S})] \| = 2^N$.

Assume that $\|\phi(\underline{S})\| \wedge \bigwedge_k (-\|\tau_k\phi(\underline{S})\|) > \emptyset$. Then there is some u such that

(i) $u \subseteq \|\phi(\underline{S})\|$;

(ii) $\bigvee_k \|\tau_k\phi(\underline{S})\| \subseteq 2^N - u$; as $cl(u) = u$.

Let $k_o > lh(u)$, then $\tau_{k_o}(u) = u$, and hence $u = \tau_k(u) \subseteq \tau_k\|\phi(\underline{S})\| = \|\tau_k\phi(\underline{S})\| \subseteq 2^N - u$; this contradiction proves the lemma.

E. If U is an ultrafilter in $\mathcal{B}$ which preserves the (countably many) quantifiers in L^*, then the map $\mathcal{B} \rightarrow \mathcal{B}/_U$ combined with the evaluation in $\mathbb{B}$ defines a two-valued model of L^* in which $\phi(\underline{S}) \rightarrow \exists x[\tau_x\phi(\underline{S})]$ is true for all formulas ϕ.

In the model $\underline{S}$ will be interpreted by a set S, defined by the condition

$$n \in S \equiv . \| \underline{S}(\bar{n}) \| \in U.$$

We note that for each k, if S_k is the set defined by $n \in S_k \equiv . \| \tau_k(\underline{S}(\bar{n})) \| \in U$, then $S \neq S_k$, as $k \in S \equiv . k \notin S_k$.

If A is an arithmetic condition on the set S, then there is some formula ϕ such that $\phi(\underline{S})$ is true in the model defined by U. But then $\exists x[\tau_x \phi(\underline{S})]$ is also true, which means that $\tau_k \phi(\underline{S})$ is true for some k. But this means that S_k also satisfies the condition A. Hence, there is no implicit arithmetic definition of the set S.

F. It remains to define an ultrafilter which produces a Δ^1_1-set S.

Let $\forall x \phi_n(x)$ be an (effective) enumeration of all sentences in L^* of the indicated form. One may then define a sequence of numbers $k_1, k_2, \ldots$ such that for all n

$$\bigwedge_{i=1}^{n} (\| \forall x \phi_i(x) \| \vee \| \neg \phi_i(\bar{k}_i) \|) \neq \emptyset.$$

The proof is by a simple induction.

We may then define a function β by the following recursion:

$$\beta(n+1) = \mu k \Big[\bigwedge_{i=1}^{n} (\| \forall x \phi_i(x) \| \vee \| \neg \phi_i(\overline{\beta(i)}) \|) \wedge (\| \forall x \phi_{n+1}(x) \| \vee \| \neg \phi_{n+1}(\bar{k}) \|) \neq \emptyset \Big]$$

Let ϕ_t be an (effective) enumeration of all sentences in L^*. Let U be defined through the following recursion:

$$\phi_{t+1} \in U \equiv . \forall n \Big[\|\phi_{t+1}\| \wedge \bigwedge_{i=1}^{n} (\|\forall x \phi_i(x)\| \vee \|\neg\phi_i(\overline{\beta(i)})\|)$$

$$\wedge \bigwedge_{n \leq t} \{\|\phi_n\| ; \phi_n \in U\} \neq \emptyset .$$

We see that U is an ultrafilter in L^* which preserves the logical quantifiers. (It is a maximal filter which refines the family $\{\|\forall x\phi_n(x)\| \vee \|\neg\phi_n(\overline{k}_n)\|\}$.) But then $U' = \{\|\phi\| ; \phi \in U\}$ is an ultrafilter in the subalgebra of $\mathcal{B}$ which is generated through the evaluation of L^*, and it is precisely this ultrafilter we need to know in order to define the appropriate two-valued model.

G. As a first step toward the evaluation of U we note the following result.

Lemma. $\|\phi\|$ is an arithmetic union of basic open sets, i.e. $\alpha \in \|\phi\| \equiv . \exists u [A_\phi(u) \wedge \alpha \in u]$, where A_ϕ is arithmetic.

The proof is by induction. (One proves that $\alpha \in \|\phi(\overline{k}_1, \ldots, \overline{k}_n)\| \equiv . \exists u [A_\phi(u, k_1, \ldots, k_n) \wedge \alpha \in u]$.) The main point is the following evaluation:

$$\alpha \in 2^N - \mathrm{cl}(\|\phi\|) \equiv . \alpha \notin \mathrm{cl}(\|\phi\|) \equiv .$$

$$\exists u [u \cap \|\phi\| = \emptyset \wedge \alpha \in u].$$

Here: $u \cap \|\phi\| = \emptyset \equiv . \neg(u \cap \|\phi\| \neq \emptyset) \equiv .$

$$\neg(\exists u_1[A_\phi(u_1) \wedge u_1 \cap u_2 \neq \emptyset]).$$

Since $u_1 \cap u_2 \neq \emptyset$ evidently is arithmetic, the conclusion follows.

Next observe that $\|\phi\| \neq \emptyset$ is Δ^1_1 since

$$\|\phi\| \neq \emptyset \equiv . \exists u \forall \alpha [\alpha \in u \rightarrow \alpha \in \|\phi\|]$$
$$\equiv . \exists \alpha [\alpha \in \|\phi\|].$$

(We have used the fact that $\|\phi\|$ is open.)

From this we may conclude that the function β is Δ^1_1, hence the ultrafilter U is also Δ^1_1.

But then the set S is also Δ^1_1 since

$$n \in S \equiv . \underline{S}(\bar{n}) \in U.$$

(The appropriate gödel-numbering of L^* has been assumed carried out.)

REFERENCES.

[1] FEFERMAN, S., Some applications of the notion of forcing and generic sets, Fund.Math , 56(1965), 325-345.

[2] SPECTOR, C., Hyperarithmetical quantifiers, Fund.Math., 48(1960), 313-320.