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INVOLUTIVE ALGEBRAS OVER $\mathcal{C}$.

Summary I

by

Erik M. Alfsen

This summary concerns involutive algebras over the complex field $\mathbb{C}$ (*-algebras), which are not supposed to be provided with any norm or topology. In our opinion it presents a natural approach to harmonic analysis based on such properties which are closely related to the underlying group; specifically the multiplication corresponding to the group operation and the involution corresponding to the formation of inverses. Thus the applications are not restricted to the Banach algebras such as $L^{1}(G)$ and $M(G)$. There are connections with operator theory from which rudiments of the theory of states and pure states have been transferred.

The investigation leans heavily on convexity methods, which is natural in view of the fact that (the Hermitian part of) a *-algebra has a natural partial ordering, order and convexity being essentially the same thing in linear spaces.

Paragraph 1 of the present summary treats completely general *-algebras. The paragraphs 2, 3 deal with such *-algebras as we have called unitary and pre-unitary.

In a unitary *-algebra the structure of a ring and of a partially ordered linear space are related by the requirement that there shall be a multiplicative identity which is also an order unit. A *-algebra $\mathfrak{A}$ is pre-unitary if the *-algebra $\mathfrak{A}'$ obtained by adjunction of an identity, is unitary.

The exposition is entirely free of proofs, which will be published elsewhere.

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§1. PURE STATES AND MULTIPLICATIVE FUNCTIONALS

A linear functional p on a $*$-algebra $\mathfrak{A}$ is positive if $p(f^*f) \geqq 0$ for all $f \in \mathfrak{A}$. Every positive linear functional p defines a pseudo inner product $(f,g) = p(g^*f)$ which is "compatible with involution" (i.e. it makes left multiplication by f^* the adjoint of left multiplication by f.). If $\mathfrak{A}$ has an identity e, then every (f,g) compatible with involution can be deduced in this way from a p, namely $p(f) = (f,e)$ for all f

To every positive linear functional p there is in turn associated a semi-norm $N_p(f) = \sqrt{p(f^*f)}$. If $\mathfrak{A}$ has an identity, then p is bounded in its own semi-norm N_p, and it has (functional-) norm equal to $\sqrt{p(e)}$. In the general case p is extendable to a positive functional p' on $\mathfrak{A}'$ if and only if it is bounded in N_p. In that case we shall use the symbol C_p and the term extension coefficient to denote the square of the N_p-norm of p, and we observe that one may define

$$p'(f + \lambda e) = p(f) + \lambda\alpha, \tag{1.1}$$

where α is any number such that $\alpha \geqq C_p$.

The convex cone of positive linear functionals which are extendable in the above sense, will be denoted by $\mathcal{P}^*$. (If $\mathfrak{A}$ has an identity, then every pos. lin. funct. is extendable.) Recall that the (complex) linear span of a face (cf. ((1))) of a cone is called an order ideal. Order ideals are important since they determine (linear) order homomorphisms. The order ideal generated by an element p (rel. to $\mathcal{P}^*$) will be denoted by [p]. Our first result characterizes an order ideal [p] as the set of all linear functionals q for which the conjugate quadratic form $f \rightarrow q(f^*f)$ is N_p-bounded.

P r o p o s i t i o n 1. Let p be a positive linear functional on a *-algebra $\mathfrak{A}$ with identity element. A linear functional q on $\mathfrak{A}$ belongs to [p] if and only if there exists an $\alpha < \infty$ such that for all $f \in \mathfrak{A}$:

$$(1.2) \qquad |q(f^*f)| \leqq \alpha\, p(f^*f) .$$

Recall that a linear operator T on an algebra $\mathfrak{A}$ commutes with left multiplication if

$$(1.3) \qquad T(fg) = f(Tg) ,$$

for all $f,g \in \mathfrak{A}$. Recall also that the associate T^* of an operator T is defined on the (algebraic) dual $\mathfrak{A}^*$ by

$$(1.4) \qquad T^*q\ (f) = q(Tf) ,$$

for $q \in \mathfrak{A}^*$ and for all $f \in \mathfrak{A}$.

P r o p o s i t i o n 2. Let p be a positive linear functional on a *-algebra $\mathfrak{A}$ with identity element. If an N_p-bounded linear operator commutes with left multiplications, then $T^*p \in [p]$.

Specialized to C^*-algebras this gives one half of the proof of the order-isomorphism of [p] and the commutant of the representing algebra over $\mathcal{H}_p$ ([4]). (The other half is an application of the Riesz representation of Hilbert space functionals.)

Out of context we also specialize to L^1-group algebras, for which $\mathcal{P}^*$ may be identified with the cone (in L^∞) of positive definite functions. By Prop. 2, the order ideal of a positive definite function is invariant under translation by central elements of the group. From this one may easily deduce that the extreme, normalized positive definite functions p (for which $[p] = \mathbb{C}\, p$), are characters in the commutative case. However, this fact will follow more directly from our next result.

Adopting the terminology of operator algebras, we shall use the word state to denote a positive and normalized (i.e. $p(e) = 1$) linear functional on a *-algebra with identity, and we shall use the notation pure state to denote an extreme element in the convex set of states. Also we shall use the symbol $\mathfrak{M}_R$ to denote the set of all those non-zero multiplicative linear functionals on a *-algebra, which are real in the sense that they assume real values at Hermitian elements.

To fix the ideas we recall that for a commutative Banach *-algebra, $\mathfrak{M}_R = \mathfrak{M}$ (the latter consisting of all non-zero multiplicative linear functionals) if and only if it is symmetric in the sense that $(e + f^*f)^{-1}$ exists for all elements f (or that $-f^*f$ has a quasi-inverse when the algebra has no identity) ((5, p. 143)). It should be mentioned that many important Banach *-algebras fail to be symmetric in the above sense (e.g. the measure algebra of any non-discrete locally compact group ((8)) , ((9, p. 104)) , ((11)) .

T h e o r e m 1 . The set $\mathfrak{M}_R$ of a commutative *-algebra $\mathfrak{A}$ with identity element consists of all pure states p for which multiplication is separately (and then jointly) continuous in the corresponding semi-norm N_p .

§2. THE RAIKOV-BOCHNER THEOREM FOR PRE-UNITARY *-ALGEBRAS

We shall use the symbol $\mathcal{P}$ to denote the convex cone of all elements f of a *-algebra $\mathfrak{A}$ such that $p(f) = 0$ for every $p \in \mathcal{P}^*$. If $\mathfrak{A}$ has an identity, then $\mathcal{P}$ is the closure of the cone generated by all "conjugate squares" f^*f in the topology defined by the semi-norms N_p . A positive element of a *-algebra $\mathfrak{A}$ is an order unit if it is contained in no proper order ideal.

The concept of a unitary and of a pre-unitary *-algebra were defined in the introduction, and it is easily verified that a *-algebra $\mathfrak{A}$ with identity e is unitary if every $f_0 \in \mathfrak{A}$ admits a finite sequence

$f_1, \dots, f_n \in \mathcal{A}$ and an $\alpha \in \mathbb{R}$ such that

$$\sum_{i=0}^{n} f_i^{*} f_i = \alpha e \quad , \tag{2.1}$$

and similarly that a general *-algebra is pre-unitary if every $f_o \in \mathcal{A}$ admits a finite sequence $f_1, \dots, f_n \in \mathcal{A}$ such that

$$\sum_{i=0}^{n} f_i^{*} \circ f_i = 0 \quad . \tag{2.2}$$

It follows by a standard application of the binormial series of the square root that every Banach algebra with continuous involution is pre-unitary (unitary if identity).

An important necessary and sufficient condition is given in the following

P r o p o s i t i o n 3 . A *-algebra $\mathcal{A}$ is pre-unitary if and only if every $f \in \mathcal{A}$ admits a positive number α such that for all $p \in \mathcal{P}^{*}$

$$p(f^{*}f) \leqq c_p \alpha \tag{2.3}$$

For every element f of a pre-unitary *-algebra we define ν_f to be the infimum value of the square root of the possible bounds α of (2.3) . By definition of ν_f and of c_p

$$| p(f) | \leqq c_p \nu_f \quad , \tag{2.4}$$

for all $f \in \mathcal{A}$, $p \in \mathcal{P}^{*}$; and ν_f is the least number for which (2.4) is valid for all $p \in \mathcal{P}^{*}$.

The normalization condition $p(e) = 1$ in the definition of states may be translated to $C_p = 1$, which is meaningful even if there is no identity. However, the set $\{ p \mid p \in \mathcal{P}^*, C_p = 1 \}$ need not be convex in this case. It turns out that in the general case the appropriate substitute of the set of states is the convex set of sub-states $\mathcal{K} = \{ p \mid p \in \mathcal{P}^*, C_p \leqq 1 \}$. A non-zero extreme point of $\mathcal{K}$ will be called a pure state. This conforms to previous notation since every extreme point p of $\mathcal{K}$ satisfies $C_p = 1$ (hence it is a "state"), and the extension $p \longrightarrow p'$ where

$$(2.5) \qquad p'(f + \lambda e) = p(f) + \lambda ,$$

is an affine isomorphism of $\mathcal{K}$ onto the set of states on $\mathcal{A}'$, carrying the pure states of $\mathcal{K}$ onto the set of all pure states of $\mathcal{A}'$ delected by p_o' where $p_o'(f + \lambda e) = \lambda$.

P r o p o s i t i o n 4 . A *-algebra $\mathcal{A}$ is pre-unitary if and only if the set $\mathcal{K}$ (of "sub-states") is w^*-compact.

P r o p o s i t i o n 5 . The following relation holds for all elements f, g of a commutative pre-unitary *-algebra $\mathcal{A}$ and for any $p \in \mathcal{P}^*$:

$$(2.6) \qquad N_p(fg) \leqq \nu_f \, N_p(g)$$

In particular, multiplication is separately continuous in the semi-norm N_p for every $p \in \mathcal{P}^*$.

From Theorem 1 one may now obtain

T h e o r e m 2 . The set $\mathcal{M}_R$ of a pre-unitary *-algebra consists of all pure states. It is locally compact in the w^*-topology, and it is w^*-compact in the unitary case.

Let $f \to \hat{f}$ be the restriction to $\mathfrak{M}_R$ of the canonical embedding of $\mathfrak{A}$ into $\mathfrak{A}^{**}$, i.e. $\hat{f}(p) = p(f)$ for all $f \in \mathfrak{A}$, $p \in \mathfrak{M}_R$. This gives a functional representation of the customary type. Specifically one has:

P r o p o s i t i o n 6. The mapping $f \to \hat{f}$ is a *-representation of a pre-unitary *-algebra $\mathfrak{A}$ onto a dense subset $\hat{\mathfrak{A}}$ of $C_o(\mathfrak{M}_R)$. If $\mathfrak{A}$ is unitary (and $\mathfrak{M}_R$ compact), then $\hat{\mathfrak{A}}$ is dense in $C(\mathfrak{M}_R)$.

The identification of multiplicative functionals with extreme points makes available the Krein-Milman Theorem, which is most readily applicable in integral form: Every point in a convex compact set for which the set of extreme points is closed, is barycenter of a positive normalized measure on the set of extreme points. (This is of course a mere specialization of the general Choquet Theorem, but it follows from Krein-Milman's Theorem by a simple limit-argument based on "vague" compactness, cf. e.g. ((3, p. 34))). This gives the general form of the Raikov-Bochner Theorem.

T h e o r e m 3. For every extendable positive linear functional p on a commutative pre-unitary *-algebra $\mathfrak{A}$, there is a unique finite positive measure μ_p on $\mathfrak{M}_R$ such that for every $f \in \mathfrak{A}$:

$$p(f) = \int \hat{f} d\mu_p \tag{2.7}$$

Moreover, $p \to \mu_p$ may be extended (by linearity) to a linear order isomorphism of the linear span of $\mathcal{P}^*$ onto the set of all finite measures on $\mathfrak{M}_R$, and $\|\mu_p\| = c_p$.

The content of Theorem 3 may be rephrased by saying that the set $\mathcal{K}$ (of "substates") of a commutative pre-unitary *-algebra $\mathfrak{A}$ is a compact Choquet simplex whose set of extreme points is closed (an r-simplex in the terminology of ((1))). In particular, the linear span of $\mathcal{P}^*$ is a vector-lattice (by a known property of simplexes, or directly by Jordan decomposition of measures).

By the definition of $\mathcal{V}_f$ and by a known maximum principle based on the Krein-Milman Theorem, one has

$$\mathcal{V}_f^2 = \sup_{p \in \mathcal{K}} p(f^*f) = \sup_{p \in \mathcal{M}_R} \widehat{f^*f}(p) = \| \hat{f} \|_\infty^2$$

Hence $\mathcal{V}_f = \| \hat{f} \|_\infty$. By (2.4) , $\hat{f} \to p(f)$ is uniformly continuous on $\hat{\mathcal{A}}$ for every $p \in \mathcal{P}^*$, which gives an alternative proof of Theorem 3 by virtue of the Riesz' Decomposition Theorem. Note however, that both proofs depend on the Krein-Milman Theorem, and they are not essentially different.

It should be noted that in the case of a Banach algebra with continuous involution $\mathcal{V}_f$ is equal to the spectral norm of f . This can easily be verified directly as well. Also $\mathcal{V}_f$ is equal to the norm used by R.V. Kadison in ((9, p. 5)) (i.e. for Hermitian elements).

It is apparent from Theorem 3 that the representation theory for pre-unitary *-algebras is very satisfactory as long as we restrict ourselves to study the (order theoretic) dual, i.e. the linear span of $\mathcal{P}^*$ in $\mathcal{A}^*$. A well behaved representation theory for $\mathcal{A}$ will require stronger axioms (relating ring structure and ordering), and it will be discussed in a subsequent note.

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