Thermodynamics of a two-component Porous Medium. The Poroelastic Potential and Gassmann's Equation.

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Acknowledgment

Contents

1　Introduction

This thesis has three main parts. First part (General Principles and Continuum Mechanics) provides a very short and non-complete introduction to thermodynamics and continuum mechanics. Both theories are developed over several centuries and a brief introduction is actually not possible.

The classical continuum theories fluid mechanics, elasticity, viscosity etc. resides in the one-component one-phase continuous medium. I extend the classical one-component one-phase medium into a two-conponent two-phase medium to set the grounds for the next section (Poromechanics) where I study a chemically reactive porous medium. This grand continuum model also includes poroviscoelasticity rheology.

The last section (Poroelsticity) investigates thermodynamics of the poroelastic non-reactive medium. It will be good ida to have a copy of [6] at hand while reading that section. I manage to first to derive a new formula: the poroelastic potential and it is demonstrated that it is strongly connected to Gassmann's equation, which I derive after.

All the equations are written in general notation that works equally well in 1D, 2D or 3D. However, the entire thesis should be interpreted only in 1D, as then, Cauchy's stress tensor for the matrix degenerates to a scalar, and the deviatoric stress becomes zero. The elstic moduli that are usually two, for isotropic elastic media, degenerate to one modulus: bulk modulus. I some times call what should be called stress, in higher dimensions, pressure. Any boldface letter in the equations represents a vector or tensor, although never interpreted as such.

2 General Principles

2.1 Thermodynamics

Thermodynamics is a universal concept that has extremely wide range of applicability. It has been said

"Thermodynamics can say something about anything, but never everything about something."

This observation about thermodynamics does not give much real insight, hence, there is no way around the laws of thermodynamics, we must go through them. Let's get started!

2.1.1 Conservation of energy

Conservation of energy is the first law of thermodynamics. It states that there is a quantity in the world, called energy, that remains constant whatever happens. In high school, this law is stated as: Energy can neither be created nor destroyed, but it can be converted into different forms.

Let us imagine a bounded region Ω that is separated from the rest of the world by its boundary $\partial\Omega$. We call the region Ω together with its content *system*, and the rest of the world *surroundings*.

We are interested in how the energy E of the system changes when the properties of the system change. In the simplest case, assuming that the system consists of a pure substance, e.g. an ideal gas, the properties will be entropy S, volume V, and amount of matter N.

Recall from mechanics that *work = force · displacement*, and that *work* is a form of energy. We can think of a change in entropy, volume or amount of matter as a generalised displacement.

Given changes dS, dV and dN for entropy, volume and amount of matter respectively, we get that the change in energy dE for the system is

$$dE = TdS - PdV + \mu dN$$

where T, $-P$ and μ are generalized forces for the generalized displacements dS, dV and dN respectively.

T, P and μ are known as absolute temperature, pressure and chemical potential respectively. Absolute temperature T is never negative.

2.1.2 Production of entropy

The second law of thermodynamics can be stated in many equivalent ways and we will use the following:

The total amount of entropy in the world can never decrease.

A process will either produce entropy or not produce entropy. A processes that produces entropy is called *irreversible*, and a processes that produces no entropy is called *reversible*.

Let us look at an example. You can take a bottle of fizzy lemonade out of the fridge on a hot summer day. As we all have experienced, with time it becomes hotter. Everybody knows that heat flows from hot to cold. Of course, you can bring it back to the fridge and wait a bit so that is becomes cold again and say, "now the fizzy lemonade is back to its original cold state, and I have reversed the process thermal diffusion." Ok, but the world has changed forever. Although the amount of entropy in the fizzy lemonade is the same now as before, the total entropy in the world has increased. All refrigerators releases heat on its back, and thereby increasing the entropy of its surroundings. Although it can lower the entropy of a fizzy lemonade emplaced in it, the amount of entropy released to its surroundings while dooing that will always be greater than the loss of entropy in the fizzy lemonade. The process thermal diffusion is an irreversible process.

In reality there exist no reversible processes, they are only imaginary, but serve as guiding lines since they are the limit of irreversible processes that produces very little entropy.

Personally, I find the book [7] as a very good introduction to thermodynamics. It covers its historical development and many modern areas of application, yet it is not very technical and above all fascinating and entertaining. For a more technical book see [2].

2.2 Newtonian mechanics

The purpose of the model presented here is to study geological processes and it suffices to use Newtonian mechanics.

2.2.1 Conservation of mass

According to Einstein $E = Mc^2$. Since this kind of energy conversion is not included in Newtonian mechanics, we must have that: *Mass is conserved.*

2.2.2 Conservation of momentum

Newton's second law tells how a body of mass M changes its velocity $\mathbf{v}$ when subject to a net force $\sum \mathbf{F}$:

$$M\frac{\partial \mathbf{v}}{\partial t} = \sum \mathbf{F} \ ,$$

and this relation can be interpreted as a conservation law.

A consequence of having conservation of mass and conservation of momentum in our theory is that

$$M\frac{1}{2}\mathbf{v}^2$$

is a part of the body's total energy E. In other words

$$E = M\frac{1}{2}\mathbf{v}^2 + U$$

where U is the rest of the body's total energy E. Thus U contains all kinds of energy in the body but the energy due to the body's motion. U is called *internal energy.*

3 Continuum Mechanics

Continuum mechanics is a branch of physics where it is assumed that mass completely and continously fills space.

Its mathematical foundation is calculus of integrals and differentials, developed independently by Gottfried W. Leibniz (1646-1716) and Sir Isaac Newton (1643-1727). Two very important contributors to the development of continuum mechanics are Leonhard P. Euler (1707-1783) and Augustin L. Cauchy (1789-1857). For more about the historical development of continuum mechanics see [4].

For simplicity, we assume that all the functions entering our theory are continous and differentiable everywhere. This enebles us to formulate the balance equations, in so called local form, in terms of differentials instead of integrals.

Moving from Newtonian mechanics to continuum mechanics, the concept of mass M of a body will be substituted by the concept of density ρ for an infinitely small volume. Such an infinitely small volume is called a *particle*.

Formally, we define the density of the particle occupying position x in our Eulerian formulation as

$$\rho_x = \lim_{V(\Omega_x) \to 0} \frac{M(\Omega_x)}{V(\Omega_x)} \ .$$

In this expression Ω_x is a region that contains the point x. $V(\Omega_x)$ is the volume of the region, and $M(\Omega_x)$ is the mass contained in the region. If $M(\Omega_x)$ is a contionus, differentiable function, the density ρ_x is well defined.

The fundamental theorem of calculus gives that the amount of mass M_V in the volume V is

$$M_V = \int_V \rho(x) \ dV \ .$$

For a complete introduction to the beautiful theory of continuum mechanics see [9] or [10].

3.1 General balance equations for a one-phase medium

In Eulerian coordinates the general principles takes the following form:
Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0$$

Conservation of momentum

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla(\rho \mathbf{v}^2) = \nabla \mathbf{q_v}$$

Conservation of energy

$$\frac{\partial(\rho \phi e)}{\partial t} + \nabla(\rho \phi e \mathbf{v}) = \nabla \mathbf{q}_e$$

Production of entropy

$$\frac{\partial(\rho \phi s)}{\partial t} + \nabla(\rho \phi s \mathbf{v}) = \nabla \mathbf{q}_s + Q_s$$

$$Q_s \geq 0 \ .$$

$\mathbf{q_v}, \mathbf{q}_e$ and $\mathbf{q}_s$ are fluxes of momentum, energy and entropy respectively. Q_s is entropy production. Becuse of the sources, the equations will no longer be called conservation equations but balance equations. However, the sum of the balance equations, for a given quantity mass, momentum or energy, of both phases will take the form of a conservation equation.

3.2 Local thermodynamic equilibrium

The expression

$$dE = TdS - PdV + \mu dN$$

has meaning only when it measures the difference in energy between two states that are in *global thermodynamic equilibrium*. This means that the temperature T is equal over the whole system, pressure P is equal over the whole system and that mass N is equally distributed over the whole system. If this is not the case the system is *not* in equilibrium.

We will impose the following assumption on our continuum formulation: each particle is always in thermodynamic equilibrium. This assumption is referred to as *local thermodynamic equilibrium* (LTE).

3.3 Two-phase continuous medium

We have two continua (phases) that occupies simultaneously every point in space. The phases may have different velocities, and there may exist

8

interaction between them. These interacions will appear as sources Q in the conservation equations.

Given a volume V of the two-phase medium, both phases will be present in that volume. Since we can separate the phases, we assume that in the volume V, one phase occupies a volume V_s and the other phase occupies a volume V_f and that these two volumes completely fill the total volume:

$$V = V_s + V_f .$$

We define

$$\phi_s = \frac{V_s}{V}$$

$$\phi_f = \frac{V_f}{V}$$

and we have

$$\phi_s + \phi_f = 1 .$$

This is called the *volume fraction concept*.

3.4 General balance equations for a two-phase medium

The above discussion gives for the first phase:
balance of mass

$$\frac{\partial(\rho_s \phi_s)}{\partial t} + \nabla(\rho_s \phi_s \mathbf{v}_s) = Q_{\rho_s} ,$$

balance of momentum

$$\frac{\partial(\rho_s \phi_s \mathbf{v}_s)}{\partial t} + \nabla(\rho_s \phi_s \mathbf{v}_s^2) = -\nabla \mathbf{q}_{\mathbf{v}_s} + Q_{\mathbf{v}_s} ,$$

balance of energy $(e_s = u_s + \mathbf{v}_s^2/2)$

$$\frac{\partial(\rho_s \phi_s e_s)}{\partial t} + \nabla(\rho_s \phi_s e_s \mathbf{v}_s) = -\nabla \mathbf{q}_{e_s} + Q_{e_s} ,$$

and balance of entropy

$$\frac{\partial(\rho_s \phi_s s_s)}{\partial t} + \nabla(\rho_s \phi_s s_s \mathbf{v}_s) = -\nabla \mathbf{q}_{s_s} + Q_{s_s} .$$

For the second phase:
balance of mass

$$\frac{\partial(\rho_f \phi_f)}{\partial t} + \nabla(\rho_f \phi_f \mathbf{v}_f) = Q_{\rho_f} ,$$

9

balance of momentum

$$\frac{\partial(\rho_f \phi_f \mathbf{v}_f)}{\partial t} + \nabla(\rho_f \phi_f \mathbf{v}_f^2) = -\nabla \mathbf{q}_{\mathbf{v}_f} + Q_{\mathbf{v}_f} \ ,$$

balance of energy $(e_f = u_f + \mathbf{v}_f^2/2)$

$$\frac{\partial(\rho_f \phi_f e_f)}{\partial t} + \nabla(\rho_f \phi_f e_f \mathbf{v}_f) = -\nabla \mathbf{q}_{e_f} + Q_{e_f} \ ,$$

and balance of entropy

$$\frac{\partial(\rho_f \phi_f s_f)}{\partial t} + \nabla(\rho_f \phi_f s_f \mathbf{v}_f) = -\nabla \mathbf{q}_{s_f} + Q_{s_f} \ .$$

Since the sum of balance equations of mass, momentum and energy respectively must have the form of a conservation equation we must have

$$Q_{\rho_s} + Q_{\rho_f} = 0 \ ,$$

$$Q_{\mathbf{v}_s} + Q_{\mathbf{v}_f} = 0 \ ,$$

and

$$Q_{e_s} + Q_{e_f} = 0 \ .$$

The entropy production principle becomes:

$$Q_{s_s} + Q_{s_f} \geq 0 \ .$$

3.5 Mass balance equations for a two-component two-phase medium

Assume now that each phase is composed of the same two components. Lets say we look at a volume $V = V_s + V_f$ of the two phase medium. Let N_s be total mass in V_s, and say that $N_s = N_s^1 + N_s^2$, where N_s^1 is the mass of component 1 and N_s^2 is the mass of component 2. We introduce now the *mass fraction concept*: Define

$$c_s = \frac{N_s^1}{N_s} \ .$$

Then

$$1 - c_s = \frac{N_s^2}{N_s} \ ,$$

and we can formulate one mass balance equation for each component in the first phase:

$$\frac{\partial(\rho_s \phi_s c_s)}{\partial t} + \nabla(\rho_s \phi_s c_s \mathbf{v}_s) = -\nabla \mathbf{q}_{c_s} + Q_{c_s}$$

and

$$\frac{\partial(\rho_s \phi_s (1 - c_s))}{\partial t} + \nabla(\rho_s \phi_s (1 - c_s) \mathbf{v}_s) = -\nabla \mathbf{q}_{1-c_s} + Q_{1-c_s} .$$

The flux terms appear because the two components have the ability to move relatively to each other. Summing these two equations gives

$$\frac{\partial(\rho_s \phi_s)}{\partial t} + \nabla(\rho_s \phi_s \mathbf{v}_s) = -\nabla \mathbf{q}_{c_s} + Q_{c_s} - \nabla \mathbf{q}_{1-c_s} + Q_{1-c_s}$$

which must be equal to the original balance of mass equation for the first phase. Consequently we must have

$$\nabla \mathbf{q}_{c_s} + \nabla \mathbf{q}_{1-c_s} = 0$$

which means that the two components does not move indepently. Furthermore we must have that

$$Q_{c_s} + Q_{1-c_s} = Q_{\rho_s}$$

which means that total mass exchange is the sum of mass exchange for each component. We realise that using the two equations

$$\frac{\partial(\rho_s \phi_s c_s)}{\partial t} + \nabla(\rho_s \phi_s c_s \mathbf{v}_s) = -\nabla \mathbf{q}_{c_s} + Q_{c_s}$$

and

$$\frac{\partial(\rho_s \phi_s (1 - c_s))}{\partial t} + \nabla(\rho_s \phi_s (1 - c_s) \mathbf{v}_s) = -\nabla \mathbf{q}_{1-c_s} + Q_{1-c_s}$$

is equivalent to using the two equations

$$\frac{\partial(\rho_s \phi_s)}{\partial t} + \nabla(\rho_s \phi_s \mathbf{v}_s) = Q_{\rho_s}$$

and

$$\frac{\partial(\rho_s \phi_s c_s)}{\partial t} + \nabla(\rho_s \phi_s c_s \mathbf{v}_s) = -\nabla \mathbf{q}_{c_s} + Q_{c_s} .$$

We will of course use the last set of equations.

The exact same argument can be used for the second phase. We see that extending the model to two components in each phase adds one mass balance equation to each phase, and in each phase we get a new type of process: relative motion of components. This process is known as diffusion.

After running through the same argument for the second phase, we get the two mass balance equations for the second phase:

$$\frac{\partial(\rho_f \phi_f)}{\partial t} + \nabla(\rho_f \phi_f \mathbf{v}_f) = Q_{\rho_f}$$

and

$$\frac{\partial(\rho_f \phi_f c_f)}{\partial t} + \nabla(\rho_f \phi_f c_f \mathbf{v}_f) = -\nabla \mathbf{q}_{c_f} + Q_{c_f} \ .$$

Still we must have conservation of total mass

$$Q_{\rho_s} + Q_{\rho_f} = 0$$

and obviously also

$$Q_{c_s} + Q_{c_f} = 0$$

which is conservation of mass of component 1.

4   Poromechanics

"Poromechanics is a branch of physics and specifically continuum mechanics and acoustics that studies the behaviour of fluid-saturated porous media. A porous medium or a porous material is a solid (often called matrix) permeated by an interconnected network of pores (voids) filled with a fluid (liquid or gas). Usually both solid matrix and the pore network (also known as the pore space) are assumed to be continuous, so as to form two interpenetrating continua such as in a sponge. Many natural substances such as rocks, soils, biological tissues, and man made materials such as foams and ceramics can be considered as porous media. Porous media whose solid matrix is elastic and the fluid is viscous are called poroelastic. A poroelastic medium is characterised by its porosity, permeability as well as the properties of its constituents (solid matrix and fluid).

The concept of a porous medium originally emerged in soil mechanics, and in particular in the works of Karl von Terzaghi, the father of soil mechanics. However a more general concept of a poroelastic medium, independent of its nature or application, is usually attributed to Maurice Anthony Biot (1905-1985), a Belgian-American engineer. In a series of papers published between 1935 and 1957 Biot developed the theory of dynamic poroelasticity (now known as Biot theory) which gives a complete and general description of the mechanical behaviour of a poroelastic medium." (http://en.wikipedia.org/wiki/Poromechanics)

Karl von Terzaghi's contemporary scientist Paul Fillunger was also working in the field of soil mechanics. Reint de Boer has done extensive research on the development of theory of porous media and writes ([3]):

"To conclude, it can be stated that the discovery of fundamental mechanical effects in saturated porous solids, namely uplift, friction, capillarity and effective stresses, is due to two professors from Vienna, Paul Fillunger and Karl yon Terzaghi."

These two highly gifted scientists have given rise to two different "schools" of studying porous media. Terzaghi's approach have been developed further specially by Biot and is known as Biot's Theory (BT). Fillungers approach have been followed by Bowen [1] and de Boer [4] and is now called Theory of Porous Media (TPM).

Martin Schanz and Stefan Diebels [11] compares BT and TPM with respect to elastic wave propagation. They find, by dooing numerical simulations, that the two theories give quite different speed of propagation for the

slow P-wave.

Boris Gurevich [8] investigates the consistency of the two theories with Gassmann's equation. Gurevich concludes that BT is consistent with the Gassmann equation, whereas TPM is not.

The continuity equations and motion equations in the two-phase model presented here are conceptually identical to Fillungers equations in [5]. Only difference is that Fillunger assumes fluid and pure solid is incompressible, and he assumes no mass exchange between the phases.

Hence we can say that the model presented here belongs to TMP. The result we get in the last section contradicts Gurevich's conclusion about TPM's consistency with Gassmann's equation.

4.1 Internal energy

A crucial part of the thermodynamic approach is to investigate the entropy production in the model. Thus we have to solve the total system of equations with respect to

$$Q_{s_s} + Q_{s_f}$$

and make sure this quantity never decreases. This will imply restrictions on the fluxes and sources in the balance equations. Of great importance is to decide which processes that do produce entropy and which that do not produce entropy.

But first we need to decide which variables are governing the change in internal energy for each phase. We start with the fluid phase.

We will treat the fluid as a pure substance made of two components. Hence we assume that the change dU_f in internal energy is goverened by change in the variables S_f, V_f, N_f^1, N_f^2. Introducing the conjugated thermodynamic variable for each variable extensive we get

$$dU_f = T_f dS_f - P_f dV_f + \mu_f^1 dN_f^1 + \mu_f^2 dN_f^2 \ .$$

Since we have specific quantities in our continuum formulation we must divide every variable by the total amount of mass $N_f = N_f^1 + N_f^2$. We have that specific internal energy

$$u_f = \frac{U_f}{N_f}$$

and specific volume

$$v_f = \frac{V_f}{N_f} \ .$$

14

For the components, we define

$$c_f = \frac{N_f^1}{N_f}$$

and as a consequence we get

$$1 - c_f = \frac{N_f^2}{N_f} \ .$$

We realize that the variables governing change in specific internal energy u_f is s_f, v_f and c_f. We get that

$$\frac{d^f u_f}{dt} = T_f \frac{d^f s_f}{dt} - P_f \frac{d^f v_f}{dt} + m_f \frac{d^f c_f}{dt}$$

where $\frac{d^f}{dt}$ is the material time derivative with respect to the fluid phase. There is a connection between μ_f^1 and μ_f^2, and m_f. Consult [2] for details about this connection. We refer to this equation as the LTE-equation for the fluid phase.

The solid phase, i.e. the matrix, in poromechanics is more delicate. The matrix in a volume V of the continuum is made of two parts: a pure solid (made of the same two components as the fluid) that occupies the portion V_s of V, and voids, i.e. the pore space, that occupies the portion V_{pores} of V. Together they fill the volume V:

$$V = V_s + V_{\text{pores}} \ .$$

In the case of poroelasticity we have that change in the matrix' pore space produces no entropy, but for poroviscoelasticity a change in pore space will be split in two parts: one part that is reversible i.e. produces no entropy, and one part that is irreversible i.e. produces entropy: For a function f, $0 \le f \le 1$,

$$dV_{\text{pores}} = f dV_{\text{pores}} + (1 - f) dV_{\text{pores}} \ .$$

We define

$$dV_{\text{pores}}^{\text{rev}} = f dV_{\text{pores}}$$

and

$$dV_{\text{pores}}^{\text{irr}} = (1 - f) dV_{\text{pores}} \ ,$$

15

and get
$$dV_{\text{pores}} = dV_{\text{pores}}^{\text{rev}} + dV_{\text{pores}}^{\text{irr}} .$$

Thus the variables governing change in internal energy U_s for the solid phase, i.e. the matrix, are S_s, V_s, N_s^1, N_s^2 and $V_{\text{pores}}^{\text{rev}}$. Introducing appropriate conjugated thermodynamic variables for each extensive variable we get

$$dU_s = T_s dS_s - P_s dV_s + \mu_s^1 dN_s^1 + \mu_s^2 dN_s^2 + \tau dV_{\text{pores}}^{\text{rev}} .$$

Again, we have to formulate the above equation in specific quantities. Specific pore space does not make sense, but volume fraction ϕ_f does. The LTE-equation for the solid phase is

$$\frac{d^s u_s}{dt} = T_s \frac{d^s s_s}{dt} - P_s \frac{d^s v_s}{dt} + m_s \frac{d^s c_s}{dt} + \frac{\tau v_s}{\phi_s} \frac{d^s \phi_f^{\text{rev}}}{dt}$$

where the specific quantities u_s, s_s, v_s and c_s are defined, analogously as in the fluid phase, by dividing U_s, S_s, V_s, N_s^1 and N_s^2 by $N_s = N_s^1 + N_s^2$, and $\frac{d^s}{dt}$ is the material derivative with respect to the solid phase. We have $V_{\text{pores}} = V_f$ and can write
$$\frac{V_{\text{pores}}}{N_s} = \frac{V_f}{N_s} \frac{V}{V} \frac{V_s}{V_s} = \frac{V_f}{V} \frac{V}{V_s} \frac{V_s}{N_s} = \phi_f \frac{v_s}{\phi_s}$$

which provides a hint for how to get the LTE-equation for the solid phase. See [12] for details. Likewise we have that

$$\frac{d^s \phi_f}{dt} = f \frac{d^s \phi_f}{dt} + (1 - f) \frac{d^s \phi_f}{dt} = \frac{d^s \phi_f^{\text{rev}}}{dt} + \frac{d^s \phi_f^{\text{irr}}}{dt} .$$

We are ready to investigate the entropy production of our model.

4.2 Entropy production

We need to investigate the total entropy production of our model and make sure it never decreases. In other words, we must make sure that

$$Q_{s_s} + Q_{s_f} \geq 0 .$$

Our final expression for total entropy production will not be the most general one, as we will have the following assumptions in our model:

- The two phases are in thermal equilibrium

16

- The two phases are in chemical equilibrium

This means that we can set

$$T_f = T_s = T$$

and that we have

$$\mu_f^1 = \mu_s^1 \quad \text{and} \quad \mu_f^2 = \mu_s^2 \ ,$$

where μ_p^i is the chemical potential of component i in phase p. The last set of equalities are equivalent to

$$m_f = m_s \quad \text{and} \quad \mu_f = \mu_s$$

where $m = \frac{\partial u}{\partial c}$ and μ is the chemical potential of component 1 or 2.

We will first, for each phase p, find the expression for TQ_{s_p} and investigate the entropy production related to processes concerning only the phase p. This will give us expressions for the fluxes $\mathbf{q}_{s_p}, \mathbf{q}_{\mathbf{v}_p}, \mathbf{q}_{c_p}$ and $\mathbf{q}_{e_p}$.

Since the only difference between the equations for two phases is the term

$$\frac{\tau}{\rho_s \phi_s} \frac{d^s \phi_f^{\text{rev}}}{dt}$$

appearing in the LTE-equation for the solid phase, we will drop the super- and subscripts which identify the actual phase. The above term will artificially seem to appear also in the fluid phase, but we multiply the term by δ where $\delta_s = 1$ and $\delta_f = 0$.

Using the equation for conservation of mass, we can simplify the other four balance equations as

$$\rho\phi\frac{da}{dt} = -\nabla\mathbf{q}_a + Q_a - aQ_\rho,$$

for $a = s, \mathbf{v}, e, c$, and the equation for conservation of mass can be rewritten as

$$\phi\frac{d\rho}{dt} = Q_\rho - \rho(\frac{d\phi}{dt} + \phi\nabla\mathbf{v}) \ .$$

Multiplying the LTE-equation by $\rho\phi$ and rearranging gives

$$\rho\phi T\frac{ds}{dt} = \rho\phi\frac{du}{dt} + \rho\phi P\frac{d1/\rho}{dt} - \rho\phi m\frac{dc}{dt} - \delta\tau\frac{d^s \phi_f^{\text{rev}}}{dt} \ .$$

Substituting $u = e - \mathbf{v}^2/2$ and using $\frac{d1/\rho}{dt} = \frac{-1}{\rho^2}\frac{d\rho}{dt}$ and $\frac{d\mathbf{v}^2/2}{dt} = \mathbf{v}\frac{d\mathbf{v}}{dt}$ we get

$$T\left(\rho\phi\frac{ds}{dt}\right) = \left(\rho\phi\frac{de}{dt}\right) - \mathbf{v}\left(\rho\phi\frac{d\mathbf{v}}{dt}\right) - m\left(\rho\phi\frac{dc}{dt}\right) - \frac{P}{\rho}\left(\phi\frac{d\rho}{dt}\right) - \delta\tau\frac{d^s\phi_f^{\text{rev}}}{dt}$$

and we are ready to substitute the five rewritten balance equations into the LET-equation:

$$\begin{aligned}
T(-\nabla\mathbf{q}_s + Q_s - sQ_\rho) = \quad & (-\nabla\mathbf{q}_e + Q_e - eQ_\rho) \\
& -\mathbf{v}(-\nabla\mathbf{q}_\mathbf{v} + Q_\mathbf{v} - \mathbf{v}Q_\rho) \\
& -m(-\nabla\mathbf{q}_c + Q_c - cQ_\rho) \\
& -\frac{P}{\rho}\left(Q_\rho - \rho(\frac{d\phi}{dt} + \phi\nabla\mathbf{v})\right) - \delta\tau\frac{d^s\phi_f^{\text{rev}}}{dt} \quad .
\end{aligned}$$

After collecting related terms we get

$$\begin{aligned}
TQ_s = \quad & T\nabla\mathbf{q}_s - \nabla\mathbf{q}_e + \mathbf{v}\nabla\mathbf{q}_\mathbf{v} + P\phi\nabla\mathbf{v} + m\nabla\mathbf{q}_c \\
& + (Ts - (u + \mathbf{v}^2/2) + \mathbf{v}^2 - P/\rho + mc)Q_\rho \\
& + Q_e - \mathbf{v}Q_\mathbf{v} - mQ_c \\
& + P\frac{d\phi}{dt} - \delta\tau\frac{d^s\phi_f^{\text{rev}}}{dt} \quad ,
\end{aligned}$$

and we see that only the first line in the expression concerns processes that are internal to the phase. Using the product rule of differentiation we rewrite the first line:

$$T\nabla\mathbf{q}_s - \nabla\mathbf{q}_e + \mathbf{v}\nabla\mathbf{q}_\mathbf{v} + P\phi\nabla\mathbf{v} + m\nabla\mathbf{q}_c =$$

$$\nabla(T\mathbf{q}_s - \mathbf{q}_e + \mathbf{v}\mathbf{q}_\mathbf{v} + m\mathbf{q}_c) - \mathbf{q}_s\nabla T - (\mathbf{q}_\mathbf{v} - P\phi)\nabla\mathbf{v} - \mathbf{q}_c\nabla m$$

Setting

$$\begin{aligned}
\mathbf{q}_s &= -\frac{\lambda}{T}\phi\nabla T \\
\mathbf{q}_\mathbf{v} &= P\phi - \eta\phi\nabla\mathbf{v} \\
\mathbf{q}_c &= -D\phi\nabla m \\
\mathbf{q}_e &= T\mathbf{q}_s + \mathbf{v}\mathbf{q}_\mathbf{v} + m\mathbf{q}_c
\end{aligned}$$

and assuming that λ, η and D are non-negative functions ensures that the first line is never negative.

18

In the second line, the factor $(Ts - u + \mathbf{v}^2/2 - P/\rho + mc)$ is the chemical potential μ^2 component 2.

This is the end of treating each phase separately. Now we will look at the sum of entropy production for the two phases. The sum of entropy productions is

$$
\begin{aligned}
T(Q_{s_s} + Q_{s_f}) \;=\; & \frac{\lambda_s \phi_s}{T}(\nabla T)^2 + \frac{\lambda_f \phi_f}{T}(\nabla T)^2 \\
& + \eta_s(\nabla \mathbf{v}_s)^2 + D_s \phi_s(\nabla m_s)^2 \\
& + \eta_f(\nabla \mathbf{v}_f)^2 + D_f \phi_f(\nabla m_f)^2 \\
& + (\mu_s^2 - \mu_f^2)Q_{\rho_s} - (m_s - m_f)Q_{c_s} \\
& - (\mathbf{v}_f - \mathbf{v}_s)Q_{\mathbf{v}_f} + P_f \frac{d^f \phi_f}{dt} + P_s \frac{d^s \phi_s}{dt} - \tau \frac{d^s \phi_f^{\mathrm{rev}}}{dt}
\end{aligned}
$$

where we have inserted the derived expressions for the fluxes in each phase, and used $Q_{\rho_f} = -Q_{\rho_s}$, $Q_{c_f} = -Q_{c_s}$, $Q_{\mathbf{v}_f} = -Q_{\mathbf{v}_s}$ and $Q_{e_f} = -Q_{e_s}$.

Leaving out now all terms that are positive or zero (remember our assumptions $\mu_f = \mu_s$ and $m_f = m_s$), we have left to investigate

$$
-(\mathbf{v}_f - \mathbf{v}_s)Q_{\mathbf{v}_f} + P_f \frac{d^f \phi_f}{dt} + P_s \frac{d^s \phi_s}{dt} - \tau \frac{d^s \phi_f^{\mathrm{rev}}}{dt} \;.
$$

After rewriting the material derivative

$$
\frac{d^f \phi_f}{dt} = \frac{d^s \phi_f}{dt} + (\mathbf{v}_f - \mathbf{v}_s)\nabla \phi_f
$$

and using

$$
\frac{d^s \phi_s}{dt} = -\frac{d^s \phi_f}{dt}
$$

we get:

$$
-(\mathbf{v}_f - \mathbf{v}_s)Q_{\mathbf{v}_f} + P_f(\mathbf{v}_f - \mathbf{v}_s)\nabla \phi_f + (P_f - P_s)\frac{d^s \phi_f}{dt} - \tau \frac{d^s \phi_f^{\mathrm{rev}}}{dt}.
$$

We have

$$
\frac{d^s \phi_f}{dt} = \frac{d^s \phi_f^{\mathrm{rev}}}{dt} + \frac{d^s \phi_f^{\mathrm{irr}}}{dt}
$$

which gives

$$
-(\mathbf{v}_f - \mathbf{v}_s)(Q_{\mathbf{v}_f} - P_f \nabla \phi_f) + (P_f - P_s)\frac{d^s \phi_f^{\mathrm{irr}}}{dt} + (P_f - P_s - \tau)\frac{d^s \phi^{\mathrm{rev}}}{dt}.
$$

19

Since reversible porosity change does not produce entropy we get

$$\tau = P_f - P_s \ .$$

Setting

$$\frac{d^s \phi_f^{\mathrm{irr}}}{dt} = \eta_\phi (P_f - P_s)$$

for some positive function η_ϕ gives entropy production from irreversible porosity change. Finally setting

$$Q_{\mathbf{v}_f} = P_f \nabla \phi_f - \theta(\mathbf{v}_f - \mathbf{v}_s)$$

for some positive function θ gives entropy production from momentum exchange.

Define $\lambda^{\mathrm{eff}} = \lambda_s \phi_s + \lambda_f \phi_f$ and we have established the total entropy production

$$T(Q_{s_s} + Q_{s_f}) = \frac{\lambda^{\mathrm{eff}}}{T}(\nabla T)^2 + D_s \phi_s (\nabla m_s)^2 + D_f \phi_f (\nabla m_f)^2$$

$$+ \eta_s (\nabla \mathbf{v}_s)^2 + \eta_f (\nabla \mathbf{v}_f)^2 + \theta(\mathbf{v}_f - \mathbf{v}_s)^2 + \eta_\phi (P_f - P_s)^2$$

for our model.

4.3 Stress in the porous medium

Gassmann states the following about the total stress P in a porous medium:

Let the volume fraction of fluid be ϕ_f, the pressure in the fluid be P_f and the stress in the matrix be P_m. Then total stress

$$P = \phi_f P_f + P_m \ .$$

He argues further that, since the fluid is everywhere present in the matrix, the stress in the matrix can be split as

$$P_m = \phi_s P_f + P_{\mathrm{eff}} \ ,$$

and we get that total stress

$$P = \phi_f P_f + \phi_s P_f + P_{\mathrm{eff}} = P_f + P_{\mathrm{eff}} \ .$$

Gassmann also assumes: deformation of the porous medium caused by a stress variation

$$\Delta P = \Delta P_f + \Delta P_{\mathrm{eff}}$$

is the sum of deformation caused by ΔP_f and deformation caused by ΔP_{eff}.

5 Poroelasticity

Recall that a poroelastic medium is a porous medium that consists of an elastic matrix and a viscous fluid. In reality things are a bit more complicated.

Studying Gassmanns paper [6] we see that the following assumptions are made:

- Temperature is constant.

- There is no mass exchange between the phases.

- Fluid is elastic.

- Pure solid is elastic.

- Matrix is elastic.

5.1 Thermodynamics of poroelasticity

The thermodynamic equivalents of these assumptions are:

- Temperature is constant.

- There is no mass exchange between the phases.

- Deformation of fluid does not produce entropy.

- Deformation of pure solid does not produce entropy.

- Porosity change does not produce entropy.

We simplify the expression for total entropy production to

$$T(Q_{s_s} + Q_{s_f}) = \eta_f (\nabla \mathbf{v}_f)^2 + \eta_s (\nabla \mathbf{v}_s)^2 + \eta_\phi (P_f - P_s)^2 + \theta (\mathbf{v}_f - \mathbf{v}_s)^2$$

by leaving out the terms related to diffusion and using the assumption that temperature T is constant.

Because deformation of the fluid does not produce entropy we must set $\eta_f = 0$ which gives

$$\mathbf{q}_{\mathbf{v}_f} = P_f \phi_f \ .$$

Likewise, because deformation of solid phase does not produce entropy we must set $\eta_s = 0$ which gives

$$\mathbf{q_{v_s}} = P_s \phi_s .$$

Summing now the two equations for conservation of momentum we get that total pressure

$$P = P_f \phi_f + P_s \phi_s .$$

Using the fact that $\tau = P_f - P_s$ we also have that

$$P = P_f \phi_f + P_s \phi_s = P_f - \phi_s(P_f - P_s) = P_f - \phi_s \tau$$

which gives that effective pressure

$$P_{\text{eff}} = -\phi_s \tau .$$

Because porosity change does not produce entropy we must set $\eta_\phi = 0$, which gives

$$0 = \eta_\phi(P_f - P_s) = \frac{d^s \phi_f^{\text{irr}}}{dt} = (1 - f)\frac{d^s \phi_f}{dt}$$

and we have to set $f = 1$, which gives that

$$\frac{d^s \phi_f^{\text{rev}}}{dt} = f\frac{d^s \phi_f}{dt} = \frac{d^s \phi_f}{dt} .$$

This means that all porosity change is reversible in a poroelastic medium, and as a consequence of this we will assume that the expression

$$\frac{d^s u_s}{dt} = -P_s \frac{d^s v_s}{dt} - \frac{\tau v_s}{\phi_s}\frac{d^s \phi_s}{dt}$$

is an exact differential. There are several equivalent ways to ensure that the above expression is an exact differential, and we will use the following: We assume that $u_s = u_s(v_s, \phi_s)$ is a continous, differentiable function. And then, by definition, we have

$$-P_s(v_s, \phi_s) = \frac{\partial u_s}{\partial v_s} \qquad \text{and} \qquad -\frac{\tau(v_s, \phi_s)v_s}{\phi_s} = \frac{\partial u_s}{\partial \phi_s} .$$

Finally we have to capture the effect of fluid viscosity when the fluid moves relatively to the matrix. Viscous processes do produce entropy and

we must assume that $\theta > 0$. The momentum equation for the fluid phase, after neglecting inertial forces, is

$$0 = -\nabla \mathbf{q}_{\mathbf{v}_f} + Q_{\mathbf{v}_f} \ .$$

Inserting the expressions for momentum exchange

$$Q_{\mathbf{v}_f} = P_f \nabla \phi_f - \theta(\mathbf{v}_f - \mathbf{v}_s)$$

and momentum flux

$$\mathbf{q}_{\mathbf{v}_f} = P_f \phi_f$$

we get

$$0 = -\nabla(P_f \phi_f) + P_f \nabla \phi_f - \theta(\mathbf{v}_f - \mathbf{v}_s)$$

which is the good old Darcy flow:

$$\theta(\mathbf{v}_f - \mathbf{v}_s) = -\phi_f \nabla P_f \ .$$

A common interpretation of θ is

$$\theta = \frac{k}{\mu}$$

where k is the permeability of the matrix and μ is the dynamic viscosity of the fluid.

Now that we are done with the thermodynamics of poroelasticity, we return to Gassmann's paper in more detail.

5.2 Definitions of poroelastic constants

Gassmann defines three elastic constants for the poroelastic medium: Bulk modulus K_f for the fluid

$$\frac{\Delta V_f}{V_f} = -\frac{\Delta P_f}{K_f} \ ,$$

bulk modulus K_s for the pure solid

$$\frac{\Delta V_s}{V_s} = -\frac{\Delta P_s}{K_s} \ ,$$

and for constant fluid pressure ($\Delta P_f = 0$), bulk modulus K_d for the matrix

$$\frac{\Delta V}{V} = -\frac{\Delta P_{\text{eff}}}{K_d} \ .$$

23

Gassmann also assumes that for a pressure variation $\Delta P_f \neq 0$ and $\Delta P_{\text{eff}} = 0$, we have that $\Delta \phi_s = 0$.

This is the end of making assumptions, now it is time to derive some results.

5.3 Equation of state

The porosity evolution equation

$$\Delta P_{\text{eff}} = K_\phi \Delta \phi_s$$

where

$$K_\phi = \left(\frac{\phi_s}{K_d} - \frac{1}{K_s} \right)^{-1}$$

holds for a general pressure variation ΔP.

Proof

Because we can decompose a general pressure variation $\Delta P = \Delta P_f + \Delta P_{\text{eff}}$ it will suffice to prove that the equation of state holds for the two cases

$$\Delta P_f \neq 0 \quad \text{and} \quad \Delta P_{\text{eff}} = 0$$

and

$$\Delta P_f = 0 \quad \text{and} \quad \Delta P_{\text{eff}} \neq 0 \ .$$

For the first case, the equation of state is trivial by the assumption in the previous section.

For the second case, use the definition of K_d:

$$\frac{\Delta V}{V} = -\frac{\Delta P_{\text{eff}}}{K_d} \ ,$$

the following integration theorem for elstic solid which holds for $\Delta P_f = 0$ (see [6]):

$$\frac{\Delta V_s}{V} = -\frac{\Delta P_{\text{eff}}}{K_s} \ ,$$

and

$$\Delta V_s = \Delta(\phi_s V) = \phi_s \Delta V + V \Delta \phi_s \ .$$

(We assume that small differences respect the producut rule of differentiation.)

5.4 Proposition

Now we will merge the thermodynamic approach and Gassmann's approach to poroelasticity and get very important information about the function τ :

$$\tau(v_s, \phi_s) = \tau(\phi_s) .$$

This means that τ is not a function of v_s.

Proof

Inserting $P_{\text{eff}} = -\phi_s \tau$ in the equation of state we get

$$\Delta(\phi_s \tau) = -K_\phi \Delta \phi_s$$

and this equation is equivalent to

$$\Delta \tau = -\frac{K_\phi + \tau}{\phi_s} \Delta \phi_s .$$

By definition, we also have

$$\Delta \tau = \frac{\partial \tau}{\partial \phi_s} \Delta \phi_s + \frac{\partial \tau}{\partial v_s} \Delta v_s .$$

Comparing these two equations, we see that

$$\frac{\partial \tau}{\partial v_s} = 0$$

which means that τ is not a function of v_s.

5.5 The poroelastic potential $u_s(v_s, \phi_s)$

The poroelastic energy potential u_s is

$$u_s(v_s, \phi_s) = K_d \frac{v_s}{\phi_s} \left((\phi_s \kappa - 1) \ln(\phi_s \kappa - 1) - (\phi_s \kappa) \ln(\phi_s \kappa) + \gamma \right)$$

$$+ K_s v_s (\ln(v_s) + \beta) + \alpha$$

where $\kappa = K_s/K_d$, and γ, β and α are arbitrary constants.

It can be found by solving the following set of partial differential equations

$$\frac{\partial^2 u_s}{\partial v_s \partial \phi_s} = \frac{1}{v_s}\frac{\partial u_s}{\partial \phi_s}$$

$$\frac{\partial^2 u_s}{\partial \phi_s^2} = \frac{K_\phi v_s}{\phi_s^2} - \frac{2}{\phi_s}\frac{\partial u_s}{\partial \phi_s}$$

$$\frac{\partial^2 u_s}{\partial v_s^2} = \frac{K_s}{v_s} \ .$$

These three partial differential equations are results from our investigation of the thermodynamics of poroelasticity. The analytic expression for the poroelastic energy potential is a novelty, and does not to my knowledge appear in published literature.

The next section, about Gassmann's equation, shows that the poroelastic potential is closely related to Gassmann's equation, but while Gassmanns equation is valid only under the assumption $\mathbf{v}_f = \mathbf{v}_s$, the poroelastic potential does not have this restriction.

Some words must be said about the bulk modulus K_d for the matrix. The elastic constants K_f, K_s and K_d are all defined for small pressure variations and can be considered as constants only over small range of pressure variation. In addition, for K_d, it can be considered constant only over a small range of porosity variation.

In fact K_d must be dependent on porosity. In the limit when $\phi_f \to 0$ i.e. when the porous medium becomes a pure solid, we must have $K_d \to K_s$. In the other limit, when $\phi_f \to 1$, the matrix ceases to behave as a solid. Consider a porous rock made of solid grains. If the porosity ϕ_f exceeds a certain limit $\tilde{\phi}_f < 1$, the grains cannot be in contact anymore and the matrix becomes fluidized.

5.5.1 Derivation of the partial differential equations

Differential equation 1

$$\frac{\partial^2 u_s}{\partial v_s \partial \phi_s} = \frac{1}{v_s}\frac{\partial u_s}{\partial \phi_s}$$

Proof

By definition

$$-\frac{\tau v_s}{\phi_s} = \frac{\partial u_s}{\partial \phi_s} \; .$$

Differentiating both sides with respect to v_s gives

$$\frac{\partial}{\partial v_s}\left(-\frac{\tau v_s}{\phi_s}\right) = \frac{\partial}{\partial v_s}\left(\frac{\partial u_s}{\partial \phi_s}\right) \; .$$

Using the fact that τ is not a function of v_s and again the definition of $\frac{\partial u_s}{\partial \phi_s}$ we are done:

$$\frac{\partial}{\partial v_s}\left(-\frac{\tau v_s}{\phi_s}\right) = -\frac{\tau}{\phi_s}\frac{\partial v_s}{\partial v_s} = -\frac{\tau}{\phi_s} = \frac{1}{v_s}\frac{\partial u_s}{\partial \phi_s} \; .$$

Differential equation 2

$$\frac{\partial^2 u_s}{\partial \phi_s^2} = \frac{K_\phi v_s}{\phi_s^2} - \frac{2}{\phi_s}\frac{\partial u_s}{\partial \phi_s}$$

Proof

Starting with the equation of state

$$\Delta P_{\text{eff}} = K_\phi \Delta \phi_s$$

and using that $P_{\text{eff}} = -\phi_s \tau$ and that τ is a function of ϕ_s the following equation makes sense:

$$\frac{\partial(-\phi_s \tau)}{\partial \phi_s} = K_\phi \; .$$

Further we multiply both sides with v_s to get

$$\frac{\partial(-\phi_s \tau v_s)}{\partial \phi_s} = K_\phi v_s$$

and one again using the definition of $\frac{\partial u_s}{\partial \phi_s}$ gives:

$$\frac{\partial}{\partial \phi_s}\left(\phi_s^2 \frac{\partial u_s}{\partial \phi_s}\right) = K_\phi v_s \; .$$

Differential equation 2 follows using the product rule of differentiation and dividing by ϕ_s^2.

Differential equation 3

$$\frac{\partial^2 u_s}{\partial v_s^2} = \frac{K_s}{v_s}$$

Proof

By definition

$$-P_s = \frac{\partial u_s}{\partial v_s} \; .$$

Differentiating both sides wrt. to v_s gives

$$\frac{\partial(-P_s)}{\partial v_s} = \frac{\partial}{\partial v_s}\left(\frac{\partial u_s}{\partial v_s}\right) \; .$$

On the other hand, the definition of bulk modulus for the pure solid

$$\frac{\Delta V_s}{V_s} = -\frac{\Delta P_s}{K_s}$$

is equivalent to

$$\frac{\partial(-P_s)}{\partial v_s} = \frac{K_s}{v_s}$$

because

$$\frac{\Delta V_s}{V_s} = \frac{\Delta(V_s/N_s)}{V_s/N_s} = \frac{\Delta v_s}{v_s}$$

where N_s is the amount of matter in the volume V_s.

5.6 Gassmann's equation

We have built a porous medium out of two media where each medium has its own velocity. The process in the porous medium concerning relative motion, i.e. when $\mathbf{v}_f \neq \mathbf{v}_s$, is Darcy flow.

However, it exist conditions that can prevent the fluid to move relatively to the solid. This may happen either if the pores are not connected or if the boundary of the volume V is closed for fluid flow. In this context $\mathbf{v}_f = \mathbf{v}_s$ and we can think of the porous medium as a one-phase medium.

The natural definition of bulk modulus K for the poroelastic medium under the additional assumption that $\mathbf{v}_f = \mathbf{v}_s$ is

$$\frac{\Delta V}{V} = -\frac{\Delta P}{K}$$

where total pressure $P = P_f + P_{\text{eff}}$.

Becauce K is a material property we would expect that it is a function of already introduced material properties. This is indeed the case as K is a function of K_s, K_d, ϕ_f and K_f.

This relation is known as Gassmann's equation and is often encountered in literature in the form

$$K = K_d + \frac{\left(1 - \frac{K_d}{K_s}\right)^2}{\frac{\phi_f}{K_f} + \frac{1-\phi_f}{K_s} - \frac{K_d}{K_s^2}} \; .$$

5.6.1   Application to seismics

"Seismic techniques are commonly used to determine site geology, stratigraphy, and rock quality. These techniques provide detailed information about subsurface layering and rock geomechanical properties using seismic waves. Different lithologies transmit seismic waves at different velocities, bending and reflecting waves at interfaces. By configuring seismic sources and receivers into specific arrays and measuring the time spent for compressional (P) or shear (S) waves to travel from the source to the receivers, interpretation techniques resolve the layer velocities and interface depths. Seismic surveys provide detailed diagrams illustrating geologic structure, layering, and layer velocities beneath the area of exploration."
(http://www.greatgeophysics.com/seism.htm)

It is well known that the speed c_p of propagation of P-waves in elastic media is

$$c_p = \sqrt{\frac{K}{\rho}}$$

where K is the bulk modulus, and ρ is the density of the medium.

Gassmanns equation is often called Gassmanns equation for fluid substitution. The reason for this is that for a given dry porous rock (i.e. the matrix), with material properties K_s, K_d and ϕ_f, the bulk modulus K will change if we change the bulk modulus K_f of the fluid.

As an example, let us imagine we have a column of homogenous reservoir rock in which the pores, at different depths, are saturated with gas and oil. Since the bulk modulus of gas and oil differ greatly, the gas-oil contact will be an interface that separates regions wich differ in bulk modulus K. Although the density ρ of the porous medium may also change when the pore saturant is changed, the gas and oil region of the column will probably have different speeds c_p for propagation of P-waves, and consequently the interface may be detected by seismic methods.

5.6.2 Derivation of K

In the context when $\mathbf{v}_f = \mathbf{v}_s$ the continuity equations describing the volume change ΔV and the volume fraction change $\Delta\phi$ of the porous medium are

$$\frac{1}{v_f}\frac{dv_f}{dt} = \frac{\Delta V}{V} + \frac{1}{\phi_f}\frac{d\phi_f}{dt}$$

and

$$\frac{1}{v_s}\frac{dv_s}{dt} = \frac{\Delta V}{V} + \frac{1}{\phi_s}\frac{d\phi_s}{dt} \quad .$$

These and the following three equations

$$-\frac{dP_f}{dt} = \frac{K_f}{v_f}\frac{dv_f}{dt}$$

$$\frac{dP_s}{dt} = -\frac{K_s}{v_s}\frac{dv_s}{dt} + \frac{P_f - P_s}{\phi_s}\frac{d\phi_s}{dt}$$

$$\frac{d(P_f - P_s)}{dt} = -\frac{K_\phi + P_f - P_s}{\phi_s}\frac{d\phi_s}{dt}$$

are needed to derive the expression for K.

We will first prove two lemmas to be used in the derivation, and after the derivation we will show how to find the three equations.

Lemma 1

$$-\frac{dP_f}{dt} = K_f\left(\frac{\Delta V}{V} + \frac{1}{\phi_f}\frac{d\phi_f}{dt}\right) \quad .$$

Proof

By substituting the first continuity equation into the first equation, lemma 1 follows.

Lemma 2

$$\left(\frac{K_f}{\phi_f} + \frac{K_s}{\phi_s} + \frac{K_\phi}{\phi_s}\right)\frac{d\phi_s}{dt} = \frac{\Delta V}{V}(K_f - K_s) \ .$$

Proof

Summing the three equations gives

$$0 = K_f\frac{1}{v_f}\frac{dv_f}{dt} - K_s\frac{1}{v_s}\frac{dv_s}{dt} - K_\phi\frac{1}{\phi_s}\frac{d\phi_s}{dt}$$

and after substituting the continuity equations into this equation, lemma 2 follows.

The derivation

After rewriting the defintion of K as

$$K = -\frac{V}{\Delta V}\Delta P$$

we see that if we can express ΔP as

$$-\Delta P = \frac{\Delta V}{V}F$$

for some function F then $K = F$ and we will be done.

First, recall that a variation in total pressure ΔP in poromechanics splits as

$$-\Delta P = -\Delta P_f - \Delta P_{\text{eff}} \ .$$

Inserting the equation of state $\Delta P_{\text{eff}} = K_\phi\Delta\phi_s$ gives

$$-\Delta P = -\Delta P_f - K_\phi\Delta\phi_s$$

and after exchanging Δ's with time derivatives on the right hand side we have

$$-\Delta P = -\frac{dP_f}{dt} - K_\phi \frac{d\phi_s}{dt} \ .$$

Now use Lemma 1 to get

$$-\Delta P = K_f \frac{\Delta V}{V} - (\frac{K_f}{\phi_f} + K_\phi)\frac{d\phi_s}{dt}$$

and finally Lemma 2 to get

$$-\Delta P = \frac{\Delta V}{V}\left(K_f - (\frac{K_f}{\phi_f} + K_\phi)(K_f - K_s)\left(\frac{K_f}{\phi_f} + \frac{K_s}{\phi_s} + \frac{K_\phi}{\phi_s}\right)^{-1}\right)$$

which has the form

$$-\Delta P = \frac{\Delta V}{V}F \ .$$

Remembering that $K_\phi = (\phi_s/K_d - 1/K_s)^{-1}$ we see that the expression for F only involves the quantities K_s, K_d, ϕ_f and K_f and we have proven that

$$K = K(K_s, K_d, \phi_f, K_f) \ .$$

The author has checked that this expression for K is equal to Gassmann's equation using Maple.

Finally, here is how to find the three equations:

Equation 1

We have that

$$\frac{\Delta V_f}{V_f} = -\frac{\Delta P_f}{K_f}$$

is equivalent to

$$-\frac{dP_f}{dt} = \frac{K_f}{v_f}\frac{dv_f}{dt}$$

because

$$\frac{\Delta V_f}{V_f} = \frac{\Delta(V_f/N_f)}{V_f/N_f} = \frac{\Delta v_f}{v_f}$$

where N_f is the amount of matter in the volume V_f.

Equation 2

By definition

$$\frac{dP_s}{dt} = -\frac{\partial}{\partial v_s}\left(\frac{\partial u_s}{\partial v_s}\right)\frac{dv_s}{dt} - \frac{\partial}{\partial \phi_s}\left(\frac{\partial u_s}{\partial v_s}\right)\frac{d\phi_s}{dt} \; .$$

Also by definition

$$\frac{\partial u_s}{\partial v_s} = -P_s \; ,$$

and we have proven in the section about the poroelastic potential that

$$-\frac{\partial}{\partial v_s}\left(\frac{\partial u_s}{\partial \phi_s}\right) = \frac{\tau}{\phi_s} \; .$$

Since the order of partial differentiation doesent matter

$$-\frac{\partial}{\partial \phi_s}\left(\frac{\partial u_s}{\partial v_s}\right) = -\frac{\partial}{\partial v_s}\left(\frac{\partial u_s}{\partial \phi_s}\right) \; ,$$

and we get that

$$\frac{dP_s}{dt} = \frac{\partial P_s}{\partial v_s}\frac{dv_s}{dt} + \frac{\tau}{\phi_s}\frac{d\phi_s}{dt} \; .$$

Using now

$$\tau = P_f - P_s$$

and

$$\frac{\partial P_s}{\partial v_s} = -\frac{K_s}{v_s} \; ,$$

we get equation 2:

$$\frac{dP_s}{dt} = -\frac{K_s}{v_s}\frac{dv_s}{dt} + \frac{P_f - P_s}{\phi_s}\frac{d\phi_s}{dt} \; .$$

Equation 3

Starting with the equation of state

$$\Delta P_{\text{eff}} = K_\phi \Delta \phi_s$$

and using $P_{\text{eff}} = -\phi_s \tau$ we get

$$\frac{d(\phi_s \tau)}{dt} = -K_\phi \frac{d\phi_s}{dt} \; ,$$

33

which is equivalent to

$$\frac{d\tau}{dt} = -\frac{K_\phi + \tau}{\phi_s}\frac{d\phi_s}{dt} \ .$$

Using again $\tau = P_f - P_s$ we get equation 3:

$$\frac{d(P_f - P_s)}{dt} = -\frac{K_\phi + P_f - P_s}{\phi_s}\frac{d\phi_s}{dt} \ .$$

6   Conclusion and Outlook

I think that the thermodynamic approach, although it contains a lot of long calculations, provide a clear insigt into the entropy production of the studied models. The models that are derived this way are by construction thermodynamically admissible.

It is interesting to see that we get very much by just extending the set of internal variables by one.

I suggest that further extensions should proceed in small steps. Results are needed before we embark on an even more complicated model.

Perhaps the poroelastic potential could set ground for more research on reflection and refraction of elastic waves in porous media?

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