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A reduction theorem for capacity of positive maps

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Abstract

We prove a reduction theorem for capacity of positive maps of finite dimensional C^*−algebras, thus reducing the computation of capacity to the case when the image of a nonscalar projection is never a projection.

Introduction

In quantum information theory there has been a great deal of interest in the concept of capacity of completely positive maps. A drawback with capacity is that it is usually quite difficult to compute, hence there is a need for developing computational techniques. In the present paper we shall prove a reduction theorem for capacity which reduces its computation to the ergodic case. As a consequence we get a partial result towards the additivity of capacity for tensor products.

If P is a finite dimensional C^*−algebra we denote by Tr_P the trace on P which takes the value 1 at each minimal projection. Let η denote the real function $\eta(t) = -t \log t$ for $t > 0$, and $\eta(0) = 0$. Then the entropy $S(a)$ of a positive operator a in P is defined by $S(a) = \mathrm{Tr}_P(\eta(a))$. If M is another finite dimensional C^*−algebra let $\Phi\colon M \to P$ be a positive unital linear trace preserving map, i.e. $\mathrm{Tr}_P(\Phi(x)) = \mathrm{Tr}_M(x)$ for all $x \in M$. Note that we only assume Φ is positive and not completely positive, since the latter stronger assumption is in most cases unnecessary. Let C denote the positive operators in M with trace 1. If $a \in C$ let

$$C(\Phi, a) = \sup S(\Phi(a)) - \sum_i \lambda_i S(\Phi(a_i)),$$

where the sup is over all convex combinations of operators $a_i \in C$ with $\sum_i \lambda_i a_i = a$. The *capacity* $C(\Phi)$ of Φ is defined by

$$C(\Phi) = \sup_{a \in C} C(\Phi, a).$$

For a discussion of capacity see e.g. [2].

1 The reduction theorem

If P is a finite dimensional C^*-algebra and ω is a state on P let Q_ω denote its density operator in P. Then the entropy of ω (with respect to P) is $S(\omega) = S(Q_\omega)$. We shall need three properties of entropy, namely: it is subadditive, i.e. $S(\omega_1+\omega_2) \leq S(\omega_1)+S(\omega_2)$; it is concave, i.e. $S(\lambda\omega_1+(1-\lambda)\omega_2) \geq \lambda S(\omega_1)+(1-\lambda)S(\omega_2)$, and if $N \subseteq M \subseteq P$ are C^*-subalgebras then $S(\omega \mid N) \geq S(\omega \mid M)$. Our first result is taken from the book [3] and is an inequality in the opposite direction.

Lemma 1 *Let $M \subseteq P$ be finite dimensional C^*-algebras, and let $e_1, \ldots, e_n$ be projections in M with sum 1. Let $N = \bigoplus_{i=1}^n N_i$, where $N_i = e_i M e_i$. Let ω be a state on P. Then*

$$\sum_i \omega(e_i) S(\frac{\omega|N_i}{\omega(e_i)}) = S(\omega|N) - \sum_i \eta(\omega(e_i)) \leq S(\omega).$$

Proof. Let $s_i = \omega(e_i)$. Then

$$\begin{aligned} S(\omega|N) &= \sum_i S(\omega(e_i.e_i)) \\ &= \sum_i S(\frac{\omega(e_i.e_i)}{s_i} s_i) \\ &= \sum_i s_i S(\frac{\omega(e_i.e_i)}{s_i}) + \eta(s_i) \end{aligned}$$

which proves the equality in the lemma.

In order to prove the inequality let f_k be minimal projections in P and $\alpha_k > 0$ such that the density operator Q_ω for ω is of the form $Q_\omega = \sum_k \alpha_k f_k$, so in particular $\sum_k \alpha_k = 1$. Thus $S(\omega) = S(Q_\omega) = \sum_k \eta(\alpha_k)$. By the first part of the proof we have

$$\begin{aligned} S(\omega|N) &= \sum_i S(\omega(e_i.e_i)) \\ &= \sum_i S(\sum_k \alpha_k e_i f_k e_i) \\ &\leq \sum_{i,k} S(\alpha_k e_i f_k e_i) \\ &= \sum_{i,k} \alpha_k S(e_i f_k e_i) + \eta(\alpha_k)\mathrm{Tr}_P(e_i f_k e_i) \\ &= \sum_{i,k} \alpha_k \eta(\mathrm{Tr}_P(e_i f_k e_i)) + \eta(\alpha_k)\mathrm{Tr}_P(e_i f_k e_i) \\ &\leq \sum_i \eta(\sum_k \alpha_k \mathrm{Tr}_P(e_i f_k e_i)) + \sum_k \eta(\alpha_k) \end{aligned}$$

$$
\begin{aligned}
&= \sum_i \eta(\mathrm{Tr}_P(e_i Q_\omega e_i)) + S(\omega) \\
&= \sum_i \eta(\omega(e_i)) + S(\omega),
\end{aligned}
$$

where the first inequality follows from subadditivity of S and second from concavity. We also used that $e_i f_k e_i = Tr_P(e_i f_k e_i)p$, where p is a minimal projection. The proof is complete.

From the definition of capacity it is clear that if $\Phi\colon M \to P$ is as before, and $N \subseteq M$, then $C(\Phi|N) \le C(\Phi)$. Our next result describes a situation when we have equality. We shall use a result of Broise, see [5] , that if a is a self-adjoint operator in M such that $\Phi(a^2) = \Phi(a)^2$ then $\Phi(aba) = \Phi(a)\Phi(b)\Phi(a)$ for all $b \in M$. In particular, if e is a projection in M such that $\Phi(e)$ is a projection, then the above identity holds for a replaced by e. The ergodic case alluded to in the introduction is the case when the only operators a which satisfy $\Phi(a^2) = \Phi(a)^2$ are the scalar operators.

Theorem 2 *Let M, P be finite dimensional C^*-algebras. Let $\Phi\colon M \to P$ be a positive unital trace preserving map. Suppose $e_1, \ldots, e_n$ are projections in M with sum 1 such that $\Phi(e_i)$ is a projection for all i. Let $N = \bigoplus e_i M e_i$. Then $C(\Phi) = C(\Phi|N)$.*

Proof. Clearly $C(\Phi) \ge C(\Phi|N)$. For the opposite inequality let $a, a_m \in C$ such that $a = \sum_m \lambda_m a_m$. Let $Q = \bigoplus \Phi(e_i) P \Phi(e_i)$. Since $\Phi(e_i x e_i) = \Phi(e_i)\Phi(x)\Phi(e_i)$ for all $x \in M, \Phi(E_N(x)) = E_Q(\Phi(x))$, where E_N and E_Q denote the conditional expectations on N and Q respectively. Thus

$$
S(\Phi(a)) \le S(E_Q(\Phi(a))) = S(\Phi(E_N(a))).
$$

Therefore by Lemma 1 applied to the states ω_m defined by $Q_{\omega_m} = \Phi(a_m)$ and $e_1, \ldots, e_n$ yields the following inequality.

$$
\begin{aligned}
&S(\Phi(a)) - \sum_m \lambda_m S(\Phi(a_m)) \\
&\le S(\Phi(E_N(a))) - \sum_m \lambda_m \sum_i \mathrm{Tr}_P(\Phi(e_i)\Phi(a_m)\Phi(e_i)) S(\frac{\Phi(e_i)\Phi(a_m)\Phi(e_i)}{\mathrm{Tr}_P(\Phi(e_i)\Phi(a_m)\Phi(e_i))}) \\
&= S(\Phi(E_N(a))) - \sum_m \lambda_m \sum_i \mathrm{Tr}_P(\Phi(e_i a_m e_i)) S(\frac{\Phi(e_i a_m e_i)}{\mathrm{Tr}_P(\Phi(e_i a_m e_i))}) \\
&= S(\Phi(E_N(a))) - \sum_{m,i} \lambda_m \mathrm{Tr}_M(e_i a_m e_i) S(\frac{\Phi(e_i a_m e_i)}{\mathrm{Tr}_M(e_i a_m e_i)}) \\
&= S(\Phi(E_N(a))) - \sum_{m,i} \mu_{m,i} S(\frac{\Phi(e_i a_m e_i)}{\mathrm{Tr}_M(e_i a_m e_i)}),
\end{aligned}
$$

where $\sum_{m,i} \mu_{m,i} = 1$, and $\frac{e_i a_m e_i}{\mathrm{Tr}_M(e_i a_m e_i)} = E_N(\frac{e_i a_m e_i}{\mathrm{Tr}_M(e_i a_m e_i)}) \in N$ with trace 1. Since the above inequality holds for all families (a_m) as above

$$
C(\Phi, a) \le C(\Phi|N, E_N(a)).
$$

Since this holds for all $a \in M$

$$C(\Phi) = \sup_a C(\Phi, a) \leq \sup_a C(\Phi|N, E_N(a)) = C(\Phi|N),$$

proving the theorem.

We can now state our main reduction theorem. Note that if the projections e_i are minimal with the property that $\Phi(e_i)$ is a projection, then $\Phi|e_iMe_i$ is ergodic in the sense defined above, so the theorem is a reduction to the ergodic case.

Theorem 3 *Let M, P be finite dimensional C^*-algebras and $\Phi: M \to P$ a positive unital trace preserving map. Let $e_1, \ldots, e_n$ be projections in M with sum 1 such that $\Phi(e_i)$ is a projection for each i. Let $M_i = e_iMe_i$ and $\Phi_i = \Phi|M_i: M_i \to \Phi(e_i)P\Phi(e_i)$ be the restriction map to M_i. Then*

$$C(\Phi) = \log \sum_{i=1}^{n} e^{C(\Phi_i)}.$$

Proof. By Theorem 2 it suffices to consider $a = \sum_i a_i \in M, a_i = ae_i \in M_i$, where $a_i = \sum_j \lambda_{ji} a_{ji}$ with $\mathrm{Tr}_M(a_{ji}) = 1$, $a_{ji} \in M_i^+$, $\sum_{ji} \lambda_{ji} = 1$. Let $s_i = \mathrm{Tr}_M(e_i a) = \mathrm{Tr}_M(a_i) = \mathrm{Tr}_P(\Phi(e_i)\Phi(a))$. Then we have

$$\begin{aligned}
S(\Phi(a)) \quad - & \quad \sum_{ji} \lambda_{ji} S(\Phi(a_{ji})) \\
= & \quad \sum_i [S(\Phi(e_i)\Phi(a)) - \sum_j \lambda_{ji} S(\Phi(a_{ji}))] \\
= & \quad \sum_i [S(s_i(\frac{1}{s_i}\Phi(e_i)\Phi(a))) - s_i \sum_j \frac{\lambda_{ji}}{s_i} S(\Phi(a_{ji}))] \\
= & \quad -\sum_i s_i \log s_i + \sum_i s_i [S(\frac{1}{s_i}\Phi(e_i)\Phi(a)) - \sum_j \frac{\lambda_{ji}}{s_i} S(\Phi(a_{ji}))]
\end{aligned}$$

We have

$$S(\frac{1}{s_i}\Phi(e_i)\Phi(a)) - \sum_j \frac{\lambda_{ji}}{s_i} S(\Phi(a_{ji})) \leq C(\Phi|M_i).$$

Therefore

$$\begin{aligned}
S(\Phi(a)) \quad - & \quad \sum_{ji} \lambda_{ji} S(\Phi(a_{ji})) \\
\leq & \quad -\sum_i s_i (\log s_i - C(\Phi|M_i)) \\
= & \quad -\sum_i s_i (\log s_i - \log \frac{C(\Phi|M_i)}{\sum_k e^{C(\Phi|M_k)}}) + \log \sum_i e^{C(\Phi|M_i)}
\end{aligned}$$

Since the sum $\sum_i s_i(\log s_i - \log \frac{e^{C(\Phi|M_i)}}{\sum_k e^{C(\Phi|M_k)}})$ is a relative entropy, it is nonnegative, see Lemma 4.5 in [4]. Hence we have

$$S(\Phi(a)) - \sum_{ji} \lambda_{ji} S(\Phi(a_{ji})) \leq \log \sum_i e^{C(\Phi|M_i)},$$

Since this holds for all a we conclude that $C(\Phi) \leq \log \sum_i e^{C(\Phi|M_i)}$.

For the converse inequality let $\varepsilon > 0$, and choose $b_i \in M_i^+$ with $\mathrm{Tr}_M(b_i) = 1$, $\mu_{ji} \geq 0$ with $\sum_j \mu_{ji} = 1$ and $a_{ji} \in M_i^+$ with trace 1 such that $\sum_j \mu_{ji} a_{ji} = b_i$, and

$$S(\Phi(b_i)) - \sum_j \mu_{ji} S(\Phi(a_{ji})) \geq C(\Phi|M_i) - \varepsilon.$$

Let now $s_i \geq 0$ have sum 1, and let $a_i = s_i b_i$, $\lambda_{ji} = s_i \mu_{ji}$. Put $a = \sum_i a_i = \sum_{ji} \lambda_{ji} a_{ji}$. Then by the above inequality we have

$$S(\frac{1}{s_i}\Phi(e_i)\Phi(a_i)) - \sum_j \frac{\lambda_{ji}}{s_i} S(\Phi(a_{ji})) \geq C(\Phi|M_i) - \varepsilon.$$

Thus by the computations in the beginning of the proof we have

$$S(\Phi(a)) - \sum_{ji} \lambda_{ji} S(\Phi(a_{ji})) \geq -\sum_i s_i(\log s_i - C(\Phi|M_i)) - \varepsilon.$$

Hence by the same computation we did above we obtain

$$\begin{aligned} S(\Phi(a)) \quad &- \quad \sum_{ji} \lambda_{ji} S(\Phi(a_{ji})) \\ &\geq \quad -\sum_i s_i(\log s_i - \log \frac{C(\Phi|M_i)}{\sum_k e^{C(\Phi|M_k)}}) + \log \sum_k e^{C(\Phi|M_k)} - \varepsilon. \end{aligned}$$

For the value $s_i = \frac{C(\Phi|M_i)}{\sum_k C(\Phi|M_k)}$ the value of the relative entropy is 0, hence

$$C(\Phi) \geq S(\Phi(a)) - \sum_{ji} \lambda_{ji} S(\Phi(a_{ji})) \geq \log \sum_k e^{C(\Phi|M_k)} - \varepsilon.$$

Since ε is arbitrary the proof is complete.

A good illustration of an application of the theorem is the case when Φ is a trace preserving projection map of M into itself, i.e.$\Phi(x) = \Phi(\Phi(x))$ for all $x \in M$. Then the image $N = \Phi(M)$ is a Jordan subalgebra of M, and if Φ is completely positive then Φ is a conditional expectation, and $\Phi(M)$ is a C^*-algebra, see [1]. The rank of N -*rankN*- is the maximal number of minimal projections in N with sum 1.

Corollary 4 *Let $\Phi: M \to M$ be a trace preserving projection map. Then*

$$C(\Phi) = \log \operatorname{rank} \Phi(M).$$

Proof. Let $n = \text{rank}\, N$ and $e_1, \dots, e_n$ be minimal projections in $\Phi(M)$ with sum 1. Then $e_k M e_k = \mathbb{C} e_k$ for all k, hence $C(\Phi|e_k M e_k) = 0$, so by the theorem

$$C(\Phi) = \log \sum_i^n e^0 = \log n.$$

The proof is complete.

The main problem concerning capacity is whether it is additive under tensor products, i.e. whether $C(\Phi \otimes \Psi) = C(\Phi) + C(\Psi)$ when $\Phi \otimes \Psi$ is positive, in particular when they are both completely positive. Our next result reduces the problem to the case when both maps are ergodic.

Corollary 5 *Let M, N, P, Q be finite dimensional C^*-algebras and $\Phi\colon M \to P$ and $\Psi\colon N \to Q$ be positive unital trace preserving maps such that $\Phi \otimes \Psi\colon M \otimes N \to P \otimes Q$ is positive. Let $e_i \in M$ and $f_j \in N$ be projections with sum 1 such that $\Phi(e_i)$ and $\Psi(f_j)$ are projections. Let*

$$\Phi_i = \Phi|e_i M e_i\colon e_i M e_i \to \Phi(e_i) P \Phi(e_i),$$

$$\Psi_j = \Psi|f_j N f_j\colon f_j N f_j \to \Psi(f_j) Q \Psi(f_j).$$

Suppose $C(\Phi_i \otimes \Psi_j) = C(\Phi_i) + C(\Psi_j)$ for all i, j. Then

$$C(\Phi \otimes \Psi) = C(\Phi) + C(\Psi).$$

Proof. We apply Theorem 3 to the projections $e_i \otimes f_j$ and the corresponding maps $\Phi_i \otimes \Psi_j$. Thus we have

$$\begin{aligned} C(\Phi \otimes \Psi) &= = \log \sum_{ij} e^{C(\Phi_i \otimes \Psi_j)} = \log \sum_{ij} e^{C(\Phi_i) + C(\Psi_j)} \\ &= \log \sum_{ij} e^{C(\Phi_i)} e^{C(\Psi_j)} = \log \sum_i e^{C(\Phi_i)} \sum_j e^{C(\Psi_j)} \\ &= C(\Phi) + C(\Psi). \end{aligned}$$

The proof is complete.

If Φ is completely positive and *id* is the identity map of N let f_j be a minimal projection for each j. Then the assumptions of the above corollary hold for the projections $1 \otimes f_j$. Hence we have

Corollary 6 *Let M and N be finite dimensional C^*-algebras as before with Φ completely positive. Then $C(\Phi \otimes \text{id}) = C(\Phi) + \log rankN$.*

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