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The 2-hop spanning tree problem.

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Abstract

Given a graph G with a specified root node r. A spanning tree in G where each node has distance at most 2 from r is called a 2-*hop spanning tree*. For given edge weights the 2-*hop spanning tree problem* is to find a minimum weight 2-hop spanning tree. The problem is *NP*-hard and has some interesting applications. We study a polytope associated with a directed model of the problem give a completeness result for wheels and a vertex description of a linear relaxation. Some classes of valid inequalities for the convex hull of incidence vectors of 2-hop spanning trees are derived by projection techniques.
Keywords: Integer programming, hop-constrained spanning tree, polyhedra.

1 Introduction

Optimization problems in connection with trees are of major importance in combinatorial optimization. Tree problems arise in many applications as in telecommunication network design, computer networking and facility location. For a thorough treatment of trees, applications, theoretical and algorithmic issues, see Magnanti and Wolsey [4].

In some applications one is interested in trees with additional properties, like diameter or degree constraints, or that subtrees (off a root node) satisfy a cardinality constraint, see [4]. Recently Gouveia [3], studied the problem of finding a minimum weight spanning tree (in a given graph) satisfying *hop constraints*. The situation may described as follows. Let r be a given (fixed) node in a graph G. For a spanning tree T we define, for each $v \in T$, $\text{dist}(v)$ as the number of edges in the (unique) rv-path in T (in particular, $\text{dist}(r) = 0$). Let h be a positive integer. If $\text{dist}(v) \leq h$ for all $v \in T$ we say that T is a h-*hop tree*. Thus in a h-hop tree all nodes are "close" to the root, and the maximum distance is no larger than h. The h-*hop (constrained) spanning tree problem* is to find, for a given weight function defined on the edges of the graph, a minimum weight h-hop tree. This problem is *NP*-hard in general. Different models for

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this problem (as well as a Steiner version), relations between these models and some algorithms are presented in [3].

The purpose of this paper is to study the h-hop spanning tree problem for the special case $h = 2$; we call this the 2-*hop spanning tree problem* (**2HST**). We point out some interesting applications and study a directed model for this problem. In particular, we study the problem when the underlying graph is a wheel, and give a complete linear description of the convex hull of (directed) 2-hop trees in this case. Some consequences of this result for the convex hull of 2-hop trees are also discussed.

For graph theory and polyhedral theory used in the paper, see Schrijver [7] and Nemhauser and Wolsey [5]. For a polytope P we let $\mathrm{vert}(P)$ denote its set of vertices. If D is a directed graph and S and T are disjoint subsets of nodes in D the (S,T) denotes the set of arcs with initial endnode in S and terminal endnode in T. Similarly, $[S,T]$ denotes the edges between two node sets S and T in an undirected graph.

2 Some applications

Let $G = (V, E)$ be a graph (undirected, with no parallel edges or loops) and let $r \in V$ be a given root node. Also let $\mathbf{c} \in \mathbb{R}^E$ be a given weight function, so c_{ij} denotes the weight of edge $[i,j] \in E$. The 2-hop spanning tree problem is to find a 2-hop tree with $\mathbf{c}(T) := \sum_{[i,j]\in T} c_{ij}$ smallest possible. The structure of a 2-hop spanning tree T is simple; it is a spanning tree such that each nonroot node v is either adjacent to r (i.e., $[r,v] \in T$) or r and v have a common neighbor (i.e., $[r,u], [u,v] \in T$ for some $u \neq r, v$). Equivalently, T is a union of stars covering V such that these stars are pairwise disjoint except that they all contain the root r. We mention some application areas for this problem.

Telecommunications. Consider a local computer network or a telecommunication network where a number of sites (computers or switches) are to be connected to some central (switching) unit with connection to the "rest of the world". The problem of designing such a local network is often modelled as a spanning tree problem with root node being the central unit. A hop constraint on the tree is of interest in order to meet a specified delay constraint, as the total delay is proportional to the number of intermediate nodes on the communication path. The hop constraint may also represent a reliability constraint or simply the required network hierarchy.

Transportation. A problem in freight transportation is to transport goods from origin to destination points using containers. Each container either goes directly between the two destinations or it goes via a depot where all goods are unloaded and thereafter sent by some other container to the final destination (unless the depot was the destination). A special case is when all goods have the same destination node r. Assume that the capacity of each container is large compared to the size and number of goods. The problem is to decide where to send containers so that the goods from each of the nodes are transported to r with minimum total cost. The selected containers then correspond to a 2-hop spanning tree and the transportation problem becomes a 2-hop spanning tree problem.

Statistics. An important problem in cluster analysis in applied statistics is the *cluster median problem*, see e.g. [1]. A given set $v_1, \ldots, v_n$ of n elements or objects is to be partitioned into a number of clusters (subsets) such that "each cluster contains rather equal elements". For each pair of elements one has given a distance $d_{ij} \geq 0$ measuring how unequal elements i and j are (and $d_{i,i} = 0$). The problem is to partition the elements into subsets or clusters and choose one element in each subset, the *cluster median*, such that the total sum of distances from each node to its median is smallest possible. The cluster median problem corresponds to the 2-hop spanning tree problem in the graph with node set $\{r, v_1, \ldots, v_n\}$ and weight function $c_{v_i,v_j} = d_{ij}$ for $i \neq j$ and $c_{r,v_i} = d_{i,i}$ for $i \leq n$.

Plant location. The 2HST problem is closely related to other well-known combinatorial optimization problems. Consider the simple plant location problem (see e.g., [5]), which may be seen as the integer linear programming problem to minimize $\sum_{i \in I} d_i y_i + \sum_{i \in I, j \in J} f_{i,j} x_{i,j}$ subject to the constraints (i) $x_{i,j} \leq y_i$ for $i \in I$ and $j \in J$, (ii) $\sum_{i \in I} x_{i,j} = 1$ for $j \in J$, and (iii) all variables are 0–1. Here I represents the set of possible plant locations and J the set of customers. This problem is obtained as a special case of the 2-hop spanning tree problem when we let the node set be $\{r\} \cup I \cup J$ and define edges and weights by $c_{r,i} = d_i$ for each $i \in I$, $c_{i,i'} = 0$ for all $i, i' \in I$, $i \neq i'$ and $c_{i,j} = f_{i,j}$ for $i \in I$, $j \in J$. As the plant location problem is *NP*-hard, this construction shows that also the 2-hop spanning tree problem is *NP*-hard. The 2-hop spanning tree problem could be viewed as a variant of the simple plant location problem where the distinction between locations and customers have been removed.

3 A directed model

There are several possible integer linear programming formulations of the 2HST problem. Different models may be derived from similar ones for the spanning tree problem; a thorough discussion of different such models and relations may be found in [4]. We present a model based on variables associated with arcs in the corresponding directed graph. We assume (for technical reasons) that for each nonroot node v G contains $[r, v]$ and the edges $[r, u]$ and $[u, v]$ for some $u \neq r, v$.

Let $D = (V, A)$ be the directed graph obtained from the graph G when we replace the edge $[r, v]$ by the arc (r, v) for each $[r, v] \in E$ and furthermore replace the edge $[u, v] \in E$ by the two (distinct) arcs uv and vu whenever $u, v \neq r$. We introduce a vector $\mathbf{y} \in \mathbb{R}^A$ with one component y_{uv} associated with the arc $uv \in A$. Let $\mathbf{c} \in \mathbb{R}^E$ be an objective function in 2HST and define $\mathbf{d} \in \mathbb{R}^A$ by $d_{uv} = d_{vu} = c_{uv}$. We denote the set of ingoing arcs to a node v by $\delta^-(v)$. Similarly, $\delta^-(S)$ is the set of ingoing arcs to a set S of nodes.

We say that $F \subseteq A$ is a *directed 2-hop spanning tree* if each node $v \neq r$ has exactly one ingoing arc and for this arc, say uv, either $u = r$ or F contains the arc ru. Thus a directed 2-hop spanning tree is an r-arborescence where the distance from the root to each node is either 1 or 2. Consider the integer linear program

$$\begin{array}{lll} \text{minimize} & \mathbf{d}^T\mathbf{y} & \\ \text{subject to} & & \\ \text{(i)} & \mathbf{y}(\delta^-(v)) = 1 & \text{for } v \in V \setminus \{r\}; \\ \text{(ii)} & y_{uv} \leq y_{ru} & \text{for } uv \in A,\ u, v \neq r; \\ \text{(iii)} & y_{uv} \in \{0,1\} & \text{for } uv \in A. \end{array} \tag{1}$$

It is easy to check that the feasible (integer) solutions in (1) are the incidence vectors of directed 2-hop spanning trees. We call constraints (ii) the *2-hop constraints*. Note that the *subset constraints*

$$\mathbf{y}(\delta^-(S)) \geq 1 \quad \text{for } S \subset V,\ r \notin S \tag{2}$$

are implied by the constraints (1)(i) and (ii). To see this, choose a node $v \in S$ and consider the equation $\mathbf{y}(\delta^-(v)) = 1$. Then $1 = \mathbf{y}(\delta^-(v)) = \mathbf{y}((V \setminus S, v)) + \mathbf{y}(S \setminus \{v\}, v)) \leq \mathbf{y}((V \setminus S, v)) + \mathbf{y}((r, S \setminus \{v\})) \leq \mathbf{y}(\delta^-(S))$ due to the 2-hop constraints $y_{uv} \leq y_{ru}$ for $u \in S \setminus \{v\}$. The subset constraints are essential in formulations of the directed spanning tree problem (r-arborescence problem), but in connection with 2-hop directed spanning trees they are redundant even for the LP relaxation of (1).

Let $P \subset \mathbb{R}^A$ be the integer polytope with vertices being the incidence vectors of directed 2-hop spanning trees, i.e., P is the convex hull of the feasible solutions in (1). (P depends on the graph G, but we omit indicating this dependence in our notation). We call P the *directed 2-hop spanning tree polytope*. It is easy to see that $\dim(P) = |A| - |V| + 1$ meaning that the affine hull of P is described by the equations $\mathbf{y}(\delta(v)) = 1$ for $v \neq r$. Note that each directed 2-hop spanning tree F contains at most one of the arcs uv and vu for every pair of distinct nonroot nodes u and v. This means that the inequality $y_{uv} + y_{vu} \leq 1$ is a valid inequality for P. Let P^{lp} denote the linear relaxation of P, i.e. the polytope consisting of the points satisfying (1)(i), (ii) and $0 \leq y_{uv} \leq 1$ for each $uv \in A$. One sees that the inequality $y_{uv} + y_{vu} \leq 1$ is also valid for P.

Then $P \subseteq P^{lp}$ and an important question for optimization is how well P^{lp} approximates P. We give a result in this direction in the next section.

Due to our construction of D and the weight function $\mathbf{d}$ we get a correspondence between the 2HST problem and problem (1). Let T be the linear transformation which maps each vector $\mathbf{y} \in \mathbb{R}^A$ to the vector $\mathbf{x} \in \mathbb{R}^E$ as follows: $x_{ru} = y_{ru}$ for each $[r, u] \in E$ and $x_{uv} = y_{uv} + y_{vu}$ for each (undirected) edge $[u, v] \in E$ with $u, v \neq r$. Let $Q \subset \mathbb{R}^E$ be the convex hull of incidence vectors of 2-hop spanning trees in G. One can show (based on our assumption on G given in the beginning of this section) that $\dim(Q) = |E| - 1$ so the only equation satisfied by all points in Q is the cardinality constraint $\mathbf{x}(E) = |V| - 1$.

Proposition 3.1 *For every graph G we have*

$$Q = T(P) \subseteq T(P^{lp}). \tag{3}$$

Moreover, if $\mathbf{y}$ *is an optimal solution of the integer program (1), then* $\mathbf{x} = T(\mathbf{y})$ *is the incidence vector of an optimal* 2*-hop spanning tree in the 2HST problem.*

Proof. Let $\mathbf{y}$ be a vertex of P. Thus, $\mathbf{y}$ is a feasible solution of (1) and $\mathbf{y} = \chi^S$ for some directed 2-hop spanning tree S in D. As noted above, for each edge $[u,v]$ joining two nonroot nodes u and v, S does not contain both the arcs uv and vu. Therefore $\mathbf{x} = T(\mathbf{y})$ must be the incidence vector of some subset F of E. In fact, it follows from the properties of a directed 2-hop spanning tree that F must be a 2-hop spanning tree in G. This proves that each vertex of P is mapped via T to a vertex of Q. Let $\mathbf{x} = \chi^F$ be a vertex of Q, so F is a 2-hop spanning tree. We can find a directed 2-hop spanning tree S with $\mathbf{x} = T(\chi^S)$ as follows. First, let S contain all arcs rv for which $[r,v] \in F$. Moreover, if $[u,v] \in F$ where u and v are nonroot nodes, then S contains exactly one of the two edges $[r,u]$ and $[r,v]$ (as F is a 2-hop spanning tree), say that it contains $[r,u]$; we then let S contain the arc uv. Then S is a directed 2-hop spanning tree and $\mathbf{x} = T(\chi^S)$. This shows that T maps the vertex set of P *onto* the vertex set of Q, i.e., $\mathrm{vert}(Q) = T(\mathrm{vert}(P))$. Therefore $T(P) = T(\mathrm{conv}(\mathrm{vert}(P))) = \mathrm{conv}(T(\mathrm{vert}(P))) = \mathrm{conv}(\mathrm{vert}(Q)) = Q$. We note that the last inclusion in (3) follows directly from the fact that $P \subseteq P^{lp}$. Finally, if $\mathbf{y}$ is an optimal solution of (1), let $\mathbf{x} = T(\mathbf{y})$ and note that $\mathbf{c}^T\mathbf{x} = \mathbf{d}^T\mathbf{y}$. The optimality of $\mathbf{x}$ follows from this. □

From this result it is clear that one can solve the 2HST problem by solving problem (1). Although the number of variables in (1) is almost twice the number of edges in G, this can be useful as the linear relaxation of (1) is strong.

4 Complete description for wheels

In this section we study the polytope P in the case when the underlying graph G is a wheel. We give a complete description of the vertices of the linear relaxation P^{lp} and determine all the additional inequalities needed to define P. These results are derived from a recent result for set packing polytopes.

Assume that G is an *n-wheel*, i.e. a cycle with n nodes augmented with a node, the root node r, and an edge between this node and all the other nodes. To fix the notation, we assume that $V = \{r, v_1, \ldots, v_n\}$ is the node set of G and its edges are $[r, v_i]$ and $[v_i, v_{i+1}]$ for $i \le n$ where we identify v_{n+1} and v_1. Let $D = (V, A)$ be the directed graph associated with G (as described in section 3) and let $\mathbf{y} \in \mathbb{R}^A$. To simplify notation we write y_i, $y_{i,i+1}$ and $y_{i+1,i}$ in stead of y_{rv_i}, $y_{v_iv_{i+1}}$ and $y_{v_{i+1}v_i}$, respectively. The linear system defining P^{lp} (see (1)) then becomes

$$\begin{array}{lll} \text{(i)} & y_i + y_{i-1,i} + y_{i+1,i} = 1 & \text{for } i \le n; \\ \text{(ii)} & y_i \ge y_{i,i+1} & \text{for } i \le n; \\ \text{(iii)} & y_i \ge y_{i,i-1} & \text{for } i \le n; \\ \text{(iv)} & \mathbf{y} \ge \mathbf{0}. & \end{array} \tag{4}$$

P^{lp} is a $2n$-dimensional polytope in $\mathbb{R}^{3n}$. We next project this polytope into the $2n$-dimensional space of the variables $y_{i,i-1}$ and $y_{i,i+1}$ for $i \le n$. This is done by eliminating the variables y_i using the equations in (4)(i) and Fourier-Motzkin elimination. Note also that each inequality $y_i \geq 0$ is redundant. We obtain $y_i = 1 - y_{i-1,i} - y_{i+1,i}$ for each $i \leq n$ and the linear system defining the projection $P^{lp}_{\triangle}$ of P^{lp} is

$$\begin{array}{lll} \text{(i)} & y_{i-1,i} + y_{i+1,i} + y_{i,i+1} \leq 1 & \text{for } i \leq n; \\ \text{(ii)} & y_{i-1,i} + y_{i+1,i} + y_{i,i-1} \leq 1 & \text{for } i \leq n; \\ \text{(iii)} & y_{i,i-1}, y_{i,i+1} \geq 0 & \text{for } i \leq n. \end{array} \tag{5}$$

By a reordering of the variables this system may be written in a more convenient form. Let $\mathbf{z} \in \mathbb{R}^{2n}$ be defined via $\mathbf{y}$ by ordering the components of $\mathbf{y}$ according to the following *cyclic ordering* of all the arcs in D that are not incident to the root: $(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),\ldots,(n,1),(1,n)$. Then the linear system in (5) becomes

$$\begin{array}{lll} \text{(i)} & z_i + z_{i+1} + z_{i+2} \leq 1 & \text{for } i \leq n; \\ \text{(ii)} & z_i \geq 0 & \text{for } i \leq n. \end{array} \tag{6}$$

Thus, if we let $\mathcal{S}$ denote the solution set of (6) we have that $P^{lp}_{\triangle} = \mathcal{S}$ (with proper correspondence between variables, as described above). The polytope $\mathcal{S}$ was studied in Dahl [2] in connection with the stable set problem in a circulant graph. We shall below apply the results of [2] to get information about P^{lp} and P.

Let G_m denote the *circulant graph* of order m: it has nodes $1,\ldots,m$ and edges $[i,i+1]$ and $[i,i+2]$ for $i \leq m$, where node numbers are calculated modulo m (so e.g. node 1 and node $m+1$ are equal). It is useful to imagine the nodes of V placed consecutively along a circle so node 1 and m are adjacent. The graph G_m is linked to the linear system (6) in the following way. The integral solutions of (6) are all 0–1 and correspond to the stable sets in G_m, i.e. a subset S of $\{1,\ldots,m\}$ such that $|i-j| \geq 3$ for all $i,j \in S$ being distinct. Thus the integer hull $\mathcal{S}_I$ of $\mathcal{S}$ is the stable set polytope in the circulant graph G_m and $\mathcal{S}$ is the linear relaxation corresponding to nonnegativity and clique constraints.

Consider a point $\mathbf{y} \in \mathbb{R}^A$ with each components being either 0 or 1/2 and such that for each $i \leq n$ two of the variables y_i, $y_{i-1,i}$ and $y_{i+1,i}$ are 1/2 and the remaining variable is 0. The point $\mathbf{y}$ lies in P^{lp} if and only if there is no $i \leq n$ with $y_i = y_{i+1} = 0$. If this condition holds (i.e., $\mathbf{y} \in P^{lp}$) we call $\mathbf{y}$ a 1/2-tree solution, and if, furthermore, the number of variables y_i that are 1/2 is odd, we call $\mathbf{y}$ and *odd 1/2-tree solution.*

Proposition 4.1 *Assume that G is the n-wheel as described above. Then the vertices of P^{lp} are (i) the incidence vectors of all 2-hop spanning trees, (ii) the point $\mathbf{y}$ with all components being 1/3 (provided that n is not a multiple of 3), and (iii) all odd 1/2-tree solutions.*

Proof. A complete description of the vertices of $\mathcal{S}$ was given in [2]. If this is combined with the fact that $\mathcal{S}$ is a projection of P^{lp} (as given above) it is rather easy to derive the desired result. $\square$

Thus the polytope P^{lp} is essentially 1/2-integral (all components of a vertex is a integral multiple of 1/2); the only exception is the point with all variables being 1/3.

Example. Let $n = 5$, so $m = 2n = 10$. The vertices of $\mathcal{S}$ (the relaxation of the stable set polytope in the circulant graph) are the incidence vectors of stable sets and the vectors $\mathbf{z}^1 = (1/3, \ldots, 1/3)$, $\mathbf{z}^2 = (1/2)\chi^{\{1,3,5,7,9\}}$ and $\mathbf{z}^3 = (1/2)\chi^{\{2,4,6,8,10\}}$. The vertices of P^{lp} are the incidence vectors of directed 2-hop spanning trees, $(1/3, \ldots, 1/3)$ and (i) the vector given by $y_{ri} = y_{i,i+1} = 1/2$ and $y_{i+1,i} = 0$ for $i \leq 5$, (ii) the vector given by $y_{ri} = y_{i+1,i} = 1/2$ and $y_{i,i+1} = 0$ for $i \leq 5$.

We now turn to the directed 2-hop spanning tree polytope P. We know that $P = P_I^{lp}$ and this implies that $P = \{(\mathbf{y}^1, \mathbf{y}^2) : \mathbf{y}^1 \in \mathcal{S}_I,\ \mathbf{y}^2 = \mathbf{W}\mathbf{y}^1\}$ where $\mathbf{y}^1 \in \mathbb{R}^{2n}$ contains the variables $y_{i,i+1}$ and $y_{i,i-1}$ for all i, $\mathbf{y}^2 \in \mathbb{R}^{2n}$ contains the variables y_i for all i and the equation $\mathbf{y}^2 = \mathbf{W}\mathbf{y}^1$ represents the equations $y_i = 1 - y_{i-1,i} - y_{i+1,i}$ for $i \leq n$.

We define certain subsets of the arc set A. Recall the cyclic ordering given above. Let B be the set of "boundary arcs" $(i, i+1)$ and $(i+1, i)$ for $i \leq n$. Let B' be a subset of B with no pair of consecutive elements in the cyclic ordering (so, e.g., B' does not contain both $(2,3)$ and $(3,2)$). Then $I = B \setminus B'$ is called a 1-*interval set* as it consists of consecutive "intervals" $I_1, \ldots, I_t$ separated by just one element (arc) in the cyclic ordering. Here an interval is a set of consecutive arcs in the cyclic ordering.

Associated with each 1-interval set is a rank (or canonical) inequality

$$\mathbf{y}(I) \leq r(I) \tag{7}$$

where $r(I) := \max\{|S \cap I| : S \text{ is a directed 2-hop spanning tree in } G\}$. Due to the definition of $r(I)$ the inequality is valid for the directed 2-hop spanning tree polytope P. Note that the coefficient of each variable y_i for $i \leq n$ is zero in this inequality. In [2] it was shown that if $|I_s| = 3k_s + 1$ for some nonnegative integer k_s (so $|I_s| \equiv 1 \pmod 3$) then

$$r(I) = \sum_{s=1}^{t} k_s + \lfloor t/2 \rfloor. \tag{8}$$

Theorem 4.2 *When G is the n-wheel a complete linear description of P consists of the inequalities in (4), the inequality $\mathbf{y}(B) \leq \lfloor 2n/3 \rfloor$ and the 1-interval inequalities $\mathbf{y}(I) \leq r(I)$ for which $|I_s| \equiv 1 \pmod 3$ for $s = 1, \ldots, t$ and with $t \geq 3$ odd.*

Proof. Again the result may be derived from a corresponding result for the stable set polytope given in [2] and we omit the details. □

We conclude this section with some algorithmic remarks. In general we can solve the 2HST problem (in G) by solving the corresponding directed 2-hop spanning tree problem, confer Proposition 3.1. Furthermore, this directed problem may be reduced whenever G is a wheel, to the stable set problem in the circulant graph described above. Note that this transformation changes the arc weights for the boundary arcs (as the weight of arcs $(i-1,i)$ and $(i+1,i)$ are decreased by $c_{r,i}$) so that the weights of arcs $(i,i+1)$ and $(i+1,i)$ may no longer be equal. The stable set problem in circulant graphs is polynomially solvable (for arbitrary weights). In fact it may be solved by linear programming as follows. Observe that if variables for two consecutive nodes in the circulant graph are fixed, say x_1 and x_2, such that at most one is 1 then the remaining variables are found by solving a linear programming problem with a coefficient matrix which is an interval matrix (i.e., a $(0,1)$-matrix where the ones occur consecutively in each row). Such matrices are known to be totally unimodular, so an optimal vertex solution must be integral and it therefore corresponds to an optimal stable set for the fixed values of x_1 and x_2. By comparing the three possible ways of fixing x_1 and x_2 one gets an optimal stable set. This solution may be transformed into an optimal solution of the 2-hop problem in question.

Example, continued. In the example above a complete description of the stable set polytope $\mathcal{S}_I$ consists of the inequalities defining $\mathcal{S}$ (clique inequalities and nonnegativity constraints), the (anti-wheel) inequality $\sum_{j=1}^{10} z_j \leq 3$ and the two 1-interval inequalities $z_1+z_3+z_5+z_7+z_9 \leq 2$ and $z_2+z_4+z_6+z_8+z_{10} \leq 2$. A complete linear description of P consists of the system (4), the inequality $\sum_{i=1}^{5}(y_{i,i+1} + y_{i+1,i}) \leq 3$ and the two 1-interval inequalities $\sum_{i=1}^{5} y_{i,i+1} \leq 2$ and $\sum_{i=1}^{5} y_{i+1,i} \leq 2$.

5 Projections and the undirected model

We return to the case when G is a general graph satisfying the conditions given in the beginning of section 3. An interesting general technique in polyhedral combinatorics is to use extended formulations and projections to get strong linear relaxations of hard combinatorial optimization problems (see [6]). We recall Proposition 3.1 which describes a relation between "directed and undirected spanning tree polytopes". We examine this relation further.

Consider the linear transformation $T : \mathbb{R}^A \rightarrow \mathbb{R}^E$ defined in section 3. The following projection technique may be used to find relaxations of Q (the convex hull of all 2-hop spanning trees). Assume that $\mathbf{a}^T\mathbf{y} \leq \alpha$ is a valid inequality for P. This means that $P \subseteq H$ where H is the halfspace in $\mathbb{R}^A$ consisting of the points satisfying the inequality $\mathbf{a}^T\mathbf{y} \leq \alpha$. From this inclusion and Proposition 3.1 we obtain $Q = T(P) \subseteq T(H)$. Here $T(H)$ is a polyhedron and any valid valid inequality for $T(H)$ is therefore also valid for Q. In particular, it may happen that $\mathbf{a}$ is such that $T(H)$ is a halfspace in $\mathbb{R}^E$, say induced by the inequality $\mathbf{b}^T\mathbf{x} \leq \beta$, and then this inequality is valid for Q.

We next establish different classes of valid inequalities for Q. In each case one may prove the validity by direct arguments, but we prove the stronger fact that all of the inequalities are obtained by projection from the directed model. For a node v we let $\Gamma(v)$ denote its set of adjacent nodes. If $S \subset V$ and $F \subseteq \delta(S)$ is such that no pair of edges in F has a common endnode in S, we call F a *subboundary of* S. Note that then $|F| \leq |S|$. Let d_v denote the degree of v (number of incident edges).

Proposition 5.1 *The following set of inequalities are valid for Q (the convex hull of incidence vectors of 2-hop spanning trees) and they may be obtained by projection from the directed model:*

$$
\begin{array}{lll}
\text{(i)} & \mathbf{x}(E[S]) + \mathbf{x}(F) \leq |S| & \text{for } S \subset V \text{ and } F \text{ a subboundary of } S; \\
\text{(ii)} & \sum_{e \in E} x_e = |V| - 1; & \\
\text{(iii)} & x_{uv} \leq x_{ur} + x_{vr} & \text{for } uv \in E,\ u, v \neq r; \\
\text{(iv)} & \mathbf{x}(\delta(v) \setminus \{[r,v]\}) + (2 - d_v)x_{rv} \leq 1 & \text{for } v \neq r; \\
\text{(v)} & x_{rv} + \mathbf{x}([r, S_1]) + \mathbf{x}([S_2, v]) \geq 1 & \text{for each partition } S_1, S_2 \text{ of } \Gamma(v) \setminus \{r\}; \\
\text{(vi)} & 0 \leq x_e \leq 1 & \text{for } e \in E.
\end{array}
\tag{9}
$$

Proof. (i) Adding $\mathbf{y}(\delta^-(v)) = 1$ for $v \in S$ gives $\mathbf{y}(A[S]) + \mathbf{y}(\delta^-(S)) = |S|$ where $A[S]$ denotes the set of arcs with both endnodes in S. Add to this inequality the inequalities $y_{vu} - y_{rv} \leq 0$ for each $e = [v, u] \in F$ and suitable $-y_e \leq 0$. Inserting $\mathbf{x}$ in the resulting inequality gives (i).

(ii). This follows by adding all the equations $\mathbf{y}(\delta^-(v)) = 1$ for $v \neq r$.

(iii). Adding the inequalities $y_{uv} \leq y_{ru}$ and $y_{vu} \leq y_{rv}$ gives $y_{uv} + y_{vu} \leq y_{ru} + y_{rv}$ and inserting $\mathbf{x}$ gives (iii).

(iv). Add $\mathbf{y}(\delta^-(v)) = 1$ and $y_{vu} - y_{rv} \leq 0$ for each $u \in \Gamma(v) \setminus \{r\}$ and insert $\mathbf{x}$.

(v) Add $\mathbf{y}(\delta^-(v)) = 1$, $y_{ru} - y_{uv} \geq 0$ for $u \in S_1$ and suitable nonnegativity constraints and insert $\mathbf{x}$.

(vi) The nonnegativity is trivial and the upper bounds has been shown before. $\square$

We call each inequality in (9)(i) a *generalized subtour* inequality. A special case is obtained by choosing, for some $u \in V \setminus S$, $F = [S, u]$ which gives the subtour inequality $\mathbf{x}(E[S']) \leq |S'| - 1$ where $S' = S \cup \{u\}$. This inequality is nonredundant if and only if $E[S']$ is a clique and each node of S' is adjacent to the root. Another special case is the 3-*path* inequality

$$\mathbf{x}(T) \leq 2$$

where T is a path in G with three edges such that either all its nodes are different from r or T contains r as an endnode. This inequality is the generalized subtour inequality where S is the set of the two internal nodes of the path and F consists of the two edges incident to the endnodes of T.

The inequalities (9)(iii) are the undirected counterpart to the 2-hop constraints in (1).

The *connectivity* inequalities in (9)(v) contains as special case the *degree* inequalities

$$\mathbf{x}(\delta(v)) \geq 1 \text{ for each } v \neq r.$$

We mention two other special cases of the connectivity inequalities and, for simplicity, we concentrate on the n-wheel. First, if we let, for some $i \leq n$, $v = v_i$,

$S_1 = \{v_{i+1}\}$ and $S_2 = \{v_{i-1}\}$ we get the valid inequality $x_{i-1,i}+x_{r,i}+x_{r,i+1} \geq 1$. Next, if we let $v = v_i$, $S_1 = \{v_{i-1}, v_{i+1}\}$ and $S_2 = \emptyset$ the connectivity inequality becomes $x_{r,i-1} + x_{r,i} + x_{r,i+1} \geq 1$.

We remark that a valid *integer* linear programming model for 2HST consist of the degree inequalities, the constraints (9)(iii) and (vi) plus integrality constraints. Thus, all the other inequalities described in Proposition 5.1 give rise to tighter formulations of the problem. Note that all the inequalities in the proposition above are also valid for $T(P^{lp})$ which is a relaxation of Q.

Example, continued. Recall our 5-wheel example. Then all the facets of the undirected 2-hop spanning tree polytope Q are induced by inequalities among the types given in Proposition 5.1 (plus the anti-wheel inequality $\mathbf{x}(B) \leq 3$). There are 81 facets and and among these 63 are generalized subtour or connectivity inequalities. This illustrates that the "undirected polytope" Q is much more complex than the "directed polytope " P. However, in this example, all the facets of Q are obtained from simple inequalities using projection applied to a linear relaxation of P.

6 Concluding remarks

We have studied the 2-hop spanning tree problem and relations between polytopes associated with two different formulations of this problem. It seems that a directed formulation may be very tight for the problem.

We leave open two questions concerning the relation between the polytopes P and Q in the case when G is the n-wheel. First, from low-dimensional test examples it seems that the valid inequalities given in Proposition 5.1 give a complete linear description of Q, but we have no proof that this holds in general (for wheels). Another (weaker) question is if $Q = T(P^{lp})$ for wheels. In fact, we did not use the interval inequalities in order to derive the valid inequalities described in Proposition 5.1.

References

[1] T.S. Arthanari and Y. Dodge. *Mathematical programming in statistics*. Wiley, 1981.

[2] G. Dahl. Stable set polytopes for a class of circulant graphs. Technical Report 249, University of Oslo, Institute of Informatics, Oslo, Norway, May 1997.

[3] L. Gouveia. Using variable redefinition for computing minimum spanning and steiner trees with hop constraints. Technical Report 2, Faculdade de Ciências da Universidade de Lisboa, Centro de investigação operacional, Lisboa, Portugal, 1996.

[4] T.L. Magnanti and L.A. Wolsey. *Optimal trees*, volume 7 of *Handbooks in Operations Research and Management Science*, chapter 9, pages 503–615. North-Holland, 1995.

[5] G. Nemhauser and L.A. Wolsey. *Integer and combinatorial optimization.* Wiley, 1988.

[6] W.R. Pulleyblank. Polyhedral combinatorics. In Nemhauser et al., editor, *Optimization*, volume 1 of *Handbooks in Operations Research and Management Science*, chapter 5, pages 371–446. North-Holland, 1989.

[7] A. Schrijver. *Theory of linear and integer programming.* Wiley, Chichester, 1986.