# A short introduction to mathematical finance

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**Abstract**

We give a brief survey of some fundamental concepts, methods and results in the mathematics of finance. The survey covers the 3 topics

**Chapter 1: Markets and arbitrages.**
The one-period model. The multi-period model.
The continuous time model.

**Chapter 2: Contingent claims and completeness.**
Hedging. Complete markets.

**Chapter 3: Pricing of contingent claims.**
The Black and Scholes formula.

## Introduction

The role of mathematics in economics has increased steadily during the last decades and this trend has been extra strong in finance. In 1997 Myron Scholes and Robert Merton were awarded the Nobel Prize in Economics, mainly for their work related to the celebrated Black and Scholes option pricing formula. (Fischer Black died in 1995). This formula is a spectacular example of how the advanced mathematical theory of stochastic analysis can be useful in economics.

The purpose of this paper is to give a first introduction to the mathematical modelling of finance. For more information we refer to [D], [K], [KS2], [LL] and [Ø2] and the references therein.

# 1 Markets and arbitrages

Although the main emphasis of this survey is on the continuous time models, we will first discuss some simple discrete time models, because they provide good motivation for the more advanced and technically complicated time-continuous case.

**The one-period model**

In this model there are just two instants of time $t$: $t = 0$ (initial time) and $t = T > 0$ (terminal time).

**Definition 1.1** A *market* in the one-period model consists of two $(n + 1)$-dimensional vectors

(1.1) $X(0) = (X_0(0), X_1(0), \ldots, X_n(0))$ and $X(T) = (X_0(T), X_1(T), \ldots, X_n(T))$

representing the prices $X_0(t), \ldots, X_n(t)$ of $n+1$ securities/assets at times $t = 0$ and $t = T$, respectively. The first component $X_0(t)$ represents the price of a safe investment, say a bank account, while the other components $X_1(t), \ldots, X_n(t)$ represent the prices of $n$ risky investments, say of stocks, where $n$ is a natural number. We assume that the value of $X(0)$ is deterministic and known, while the value of the price $X(T)$ at the future time $T$ is random and unknown. Thus we regard $X(T)$ as a *random variable* on a given probability space $(\Omega, \mathcal{F}, P)$, where $\Omega$ is a set, $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$ and $P : \mathcal{F} \to [0, 1]$ is a probability measure.

The price $X_0(t)$ of the safe investment is often called the *numeraire*. We assume from now on that $X_0(t) > 0$ for $t \in \{0, T\}$. Then if we take $X_0(t)$ as the unit (*numeraire*), the price vector becomes

$$\overline{X}(t) = (1, X_0^{-1}(t)X_1(t), \ldots, X_0^{-1}(t)X_n(t)), \qquad t = 0, T \; . \tag{1.2}$$

This market $\{\overline{X}(t)\}_{t=0,T}$ is called the *normalization* of the market $\{X(t)\}_{t\in\{0,T\}}$. A market $\{Y(t)\}_{t\in\{0,T\}}$ is called *normalized* if $Y_0(t) = 1$ for $t = 0$ and $t = T$.

**Definition 1.2 a)** A *portfolio* in the one-period model is an $(n + 1)$-dimensional deterministic vector

$$\theta = (\theta_0, \theta_1, \ldots, \theta_n) \tag{1.3}$$

Here $\theta_i$ represents the number of units of security number $i$ which are held at time $t = 0$; $i = 0, 1, \ldots, n$.

**b)** *The value* at time $t$ of the portfolio $\theta$ is given by

$$V^\theta(t) = \theta \cdot X(t) = \sum_{i=0}^{n} \theta_i X_i(t) \; , \tag{1.4}$$

where $\cdot$ denotes the dot product in $\mathbf{R}^{n+1}$.

**c)** The portfolio $\theta$ is called an *arbitrage* if

$$V^\theta(0) \le 0 \le V^\theta(T) \quad \text{a.s. and } P[V^\theta(T) > 0] > 0 \tag{1.5}$$

where a.s. means 'almost surely' or 'with probability 1' (with respect to $P$).

In other words, $\theta$ is an artibrage if it can generate a positive fortune with positive probability starting with a non-positive fortune, without any risk of a loss. Intuitively, this cannot be possible in a market in equilibrium. Therefore the absence of arbitrage is often used as an equilibrium criterion of a market.

**Remark.** If the market $\{X(t)\}_{t\in\{0,T\}}$ is *normalized* then it has an arbitrage $\theta$ in the sense of (1.5) if and only if there exists a portfolio $\widetilde{\theta}$ satisfying the weaker condition

$$V^{\widetilde{\theta}}(T) \ge V^{\widetilde{\theta}}(0) \quad \text{a.s. and } P[V^{\widetilde{\theta}}(T) > V^{\widetilde{\theta}}(0)] > 0 \tag{1.6}$$

To see this, assume $\widetilde{\theta}$ satisfies (1.6). Define

$$\theta = \left(-\sum_{i=1}^n \widetilde{\theta}_i X_i(0), \widetilde{\theta}_1, \ldots, \widetilde{\theta}_n\right)$$

Then

$$V^\theta(0) = -\sum_{i=1}^n \widetilde{\theta}_i X_i(0) + \sum_{i=1}^n \widetilde{\theta}_i X_i(0) = 0$$

and

$$V^\theta(T) = -\sum_{i=1}^n \widetilde{\theta}_i X_i(0) + \sum_{i=1}^n \widetilde{\theta}_i X_i(T)$$

Hence

$$\begin{aligned} V^\theta(T) - V^\theta(0) &= \sum_{i=1}^n \widetilde{\theta}_i X_i(T) - \sum_{i=1}^n \widetilde{\theta}_i X_i(0) \\ &= \sum_{i=1}^n \widetilde{\theta}_i X_i(T) - \sum_{i=1}^n \widetilde{\theta}_i X_i(0) = V^{\widetilde{\theta}}(T) - V^{\widetilde{\theta}}(0)\ , \end{aligned}$$

so $\theta$ satisfies (1.5) since $\widetilde{\theta}$ satisfies (1.6).

However, if $\{X(t)\}_{t\in\{0,T\}}$ is *not* normalized, then the existence of a portfolio $\theta$ satisfying (1.6) need not imply the existence of a portfolio satisfying (1.5). For example, let

$$X(0) = (1,1) \qquad \text{and} \qquad X(T) = (1+Y, 3-2Y),$$

where $Y \geq 0$ is a random variable assuming arbitrary small and arbitrary large values with positive probability. Then $\widetilde{\theta} = (2, 1)$ gives $V^{\widetilde{\theta}} = 3$ and $V^{\widetilde{\theta}}(T) = 5$, so (1.6) holds. On the other hand, if we try to find $\theta = (a, b)$ such that (1.5) holds, we get $V^\theta(0) = a + b \leq 0$, and hence $V^\theta(T) = a(1 + Y) + b(3 - 2Y) \leq 2b - bY$, which cannot satisfy (1.5) for any $b \in \mathbf{R}$.

**Example 1.3** Suppose $Y(\omega)$ is a random variable, $a \in \mathbf{R}$ a constant and suppose that

(1.7) $$X(0) = (1, a) \; , \qquad X(T) = (1, Y)$$

Choose $\theta = (\theta_0, \theta_1)$. Then

$$V^\theta(0) = \theta_0 + \theta_1 a \qquad \text{and} \qquad V^\theta(T) = \theta_0 + \theta_1 Y \; .$$

So $(\theta_0, \theta_1)$ is an arbitrage if

(1.8) $$\theta_0 + \theta_1 a \leq 0 \leq \theta_0 + \theta_1 Y \qquad \text{a.s. } P \text{ and } P[\theta_0 + \theta_1 Y > 0] > 0$$

**(i)** Suppose

$$Y \geq a \quad \text{a.s. } P \text{ and } P[Y > a] > 0.$$

Then $(\theta_0, \theta_1) = (-a, 1)$ (i.e. borrow the amount $a$ in the bank and use it to buy one stock) is an arbitrage by (1.8).

**(ii)** Similarly, if

$$Y \leq a \quad \text{a.s. } P \text{ and } P[Y < a] > 0$$

then $(\theta_0, \theta_1) = (a, -1)$ is an arbitrage by (1.8). The remaining case is when

**(iii)** $\qquad P[Y > a] > 0 \qquad \text{and} \qquad P[Y < a] > 0.$

In this case no $\theta = (\theta_0, \theta_1)$ can be an arbitrage, because if $\theta_1 > 0$ then

$$P[\theta_0 + \theta_1 Y < \theta_0 + \theta_1 a] = P[Y < a] > 0$$

and if $\theta_1 < 0$ then

$$P[\theta_0 + \theta_1 Y < \theta_0 + a_1 a] = P[Y > a] > 0 \; .$$

Moreover, if $\theta_1 = 0$ then $\theta = (\theta_0, \theta_1)$ is not an arbitrage either.

We conclude that *the market (1.7) has no arbitrage if and only if (iii) holds.*

This example actually gives a complete characterization for $n = 2$ of markets which do not have arbitrages, because of the following result:

**Lemma 1.4** Let $\{X(t)\}_{t\in\{0,T\}}$ be a one-period market with $n \geq 2$ arbitrary. The following are equivalent:

(i) $\{X(t)\}_{t\in\{0,T\}}$ has no arbitrage

(ii) $\{\overline{X}(t)\}_{t\in\{0,T\}}$ has no arbitrage.

The proof is simple and is left to the reader.

Combining Example 1.3 with Lemma 1.4 we get

**Corollary 1.5** A on-period market $\{X(t)\}_{t\in\{0,T\}}$ with $n = 2$ securities has no arbitrage if and only if

$$P[\overline{X}_1(T) > \overline{X}_1(0)] > 0 \quad \text{and} \quad P[\overline{X}_1(T) < \overline{X}_1(0)] > 0 \tag{1.9}$$

We now seek a similar criterion for non-existence of arbitrage for arbitrary $n \geq 2$. In this connection the following concept is fundamental:

**Definition 1.6 a)** A probability measure $Q$ on $\mathcal{F}$ is called a *martingale measure* for the normalized market $\{\overline{X}(t)\}_{t\in\{0,T\}}$ if

$$E_Q[\overline{X}(T)] = \overline{X}(0)\ , \tag{1.10}$$

where $E_Q$ denotes expectation with respect to $Q$.

**b)** If – in addition to (1.10) – the measure $Q$ is *equivalent* to $P$, written $Q \sim P$ (in the sense that $P$ and $Q$ hav the same null sets), then we say that $Q$ is an *equivalent martingale measure.*

One reason for the importance of this concept is the following:

**Theorem 1.7 a)** Suppose there exists an equivalent martingale measure $Q$ for the normalized market $\{\overline{X}(t)\}_{t\in\{0,T\}}$. Then the market $\{X(t)\}_{t\in\{0,T\}}$ has no arbitrage.

**b)** Conversely, suppose the market $\{X(t)\}_{t\in\{0,T\}}$ has no arbitrage. Then there exists an equivalent martingale measure $Q$ for $\{\overline{X}(t)\}_{t\in\{0,T\}}$.

*Proof of a).* Suppose an arbitrage $\theta(t)$ for $\{\overline{X}(t)\}_{t\in\{0,T\}}$ exists. Let $\overline{V}^\theta(t)$ be the corresponding value process for $\{\overline{X}(t)\}_{t\in\{0,T\}}$. Then, since $\overline{V}^\theta(T) \geq \overline{V}^\theta(0)$ a.s. $P$ we have

$$\overline{V}^\theta(T) \geq \overline{V}^\theta(0) \quad \text{a.s. } Q, \tag{1.11}$$

since $Q$ is equivalent to $P$. Similarly, since $P[\overline{V}^\theta(T) > \overline{V}^\theta(0)] > 0$ we have

$$Q[\overline{V}^\theta(T) > \overline{V}^\theta(0)] > 0 \ . \tag{1.12}$$

On the other hand, since $Q$ is a martingale measure for $\{\overline{X}(t)\}_{t\in\{0,T\}}$ we have

$$E_Q[\overline{V}^\theta(T)] = E_Q[\theta \cdot \overline{X}(T)] = \theta \cdot E_Q[\overline{X}(T)] = \theta \cdot \overline{X}(0) = \overline{V}^\theta(0) \ . \tag{1.13}$$

Clearly (1.13) contradicts (1.11) combined with (1.12). We conclude that $\{\overline{X}(t)\}_{t\in\{0,T\}}$, and hence $\{X(t)\}_{t\in\{0,T\}}$, cannot have an arbitrage. We refer to [LL, Theorem 1.2.7] for a proof of b). □

**Remark.** The reader can easily verify that when $n=2$ then (1.9) is equivalent to the existence of an equivalent martingale measure for $\{\overline{X}(t)\}_{t\in\{0,T\}}$.

### The multi-period (discrete time) model

We now introduce a more elaborate model, where trading and price changes can take place in $k$ instants of time $t = t_j$, where $t_0 = 0 < t_1 < t_2 < \cdots < t_{k-1}$. In addition we have a terminal time $T = t_k > t_{k-1}$. Put $\mathcal{T} = \{t_0, t_1, \ldots, t_k\}$.

The *market* is now represented by an $(n+1)$-dimensional *stochastic process* $\{X(t)\}_{t\in\mathcal{T}}$ on a probability space $(\Omega, \mathcal{F}, P)$, where $X_i(t)$ is the price of security $i$ at time $t$. As before we assume that $X(0)$ is deterministic and known, while $X(t)$ may be random for $t \neq 0$. Similarly, a *portfolio* is now an $(n+1)$-dimensional stochastic process $\{\theta(t)\}_{t\in\mathcal{T}}$ (on the same probability space), where $\theta_i(t)$ is the number of units of security number $i$ held at time $t$.

In this model it is necessary to emphasize that when an agent makes a decision about her portfolio at time $t$, she only has knowledge about the price process up to that time and not about future prices (except their probability distributions). To express this mathematically we let $\mathcal{G}_t$ denote the *$\sigma$-algebra generated by the random variables* $\{X_i(s);\ s \leq t,\ i = 0,, \ldots, n\}$. Heuristically $\mathcal{G}_t$ represents the history of the process $\{X(s)\}$ up to time $t$. Then we require that $\theta(t)$ should be measurable with respect to $\mathcal{G}_t$, for all $t \in \mathcal{T}$. If this is the case, we say that $\theta(t)$ is *adapted* (to $\mathcal{G}_t$). From now on we will assume that all our portfolios are adapted.

If $\theta(t)$ is a portfolio we can as before define the corresponding *value process* $V^\theta(t)$ by

$$V^\theta(t) = \theta(t) \cdot X(t) \ ; \qquad t \in \mathcal{T} \ .$$

We say that the portfolio $\theta(t)$ is *self-financing* if for each $j = 0, \ldots, k-1$ we have

$$V^\theta(t_{j+1}) = V^\theta(t_j) + \theta(t_j) \cdot (X(t_{j+1}) - X(t_j)) \tag{1.14}$$

or

(1.15) $$\Delta V^\theta(t_j) = \theta(t_j) \cdot \Delta X(t_j) \ ,$$

where $\Delta V^\theta(t_j) = V^\theta(t_{j+1}) - V^\theta(t_j)$ and $\Delta X(t_j) = X(t_{j+1}) - X(t_j)$.

In other words, when the agent decides about her portfolio $\theta(t_{j+1})$ at time $t_{j+1}$, she can use only the money $V^\theta(t_j)$ available at time $t_j$ plus the profit $\theta(t_j) \cdot \Delta X(t_j)$ coming from the change in prices from $t_j$ to $t_{j+1}$ and the portfolio choice she made at time $t_j$.

We now proceed as in the one-period case:

**Definition 1.8** A self-financing (and adapted) portfolio is an *arbitrage* if

(1.16) $$V^\theta(0) \le 0 \le V^\theta(T) \quad \text{a.s. } P \text{ and } \quad P[V^\theta(T) > 0] > 0 \ .$$

If we assume that $X_0(t) \neq 0$ for all $t \in \mathcal{T}$ we can define the normalized process $\overline{X}(t)$ by

$$\overline{X}(t) = (1, X_0^{-1}(t)X_1(t), \ldots, X_0^{-1}(t)X_n(t)) \ ; \qquad t \in \mathcal{T}$$

As before we have

**Lemma 1.9** The following are equivalent:

(i) $\{X(t)\}_{t\in\mathcal{T}}$ has no arbitrage

(ii) $\{\overline{X}(t)\}_{t\in\mathcal{T}}$ has no arbitrage

**Definition 1.10 a)** A probability measure $Q$ on $\mathcal{F}$ is called a *martingale measure* for the normalized market $\{\overline{X}(t)\}_{t\in\mathcal{T}}$ if

(1.17) $$E_Q[\overline{X}(t_{j+1}) \mid \mathcal{G}_{t_j}] = \overline{X}(t_j) \ ; \qquad j = 0, 1, \ldots, k \ .$$

**b)** If – in addition to (1.14) – the measure $Q$ is equaivalent to $P$, then we say that $Q$ is an *equivalent martingale measure.*

We can now state the multi-period version of Theorem 1.7:

**Theorem 1.11 a)** Suppose there exists an equivalent martingale measure $Q$ for the normalized market $\{\overline{X}(t)\}_{t\in\mathcal{T}}$. Then the market $\{X(t)\}_{t\in\mathcal{T}}$ has no arbitrage.

**b)** Conversely, if the market $\{X(t)\}_{t\in\mathcal{T}}$ has no arbitrage, then there exists an equivalent martingale measure $Q$ for $\{\overline{X}(t)\}_{t\in\mathcal{T}}$.

*Proof of a).* Suppose an arbitrage $\theta$ for $\{\overline{X}(t)\}_{t\in\mathcal{T}}$ exists. Let $\overline{V}^\theta(t)$ be the corresponding value process for $\{\overline{X}(t)\}_{t\in\mathcal{T}}$. Then since $\overline{V}^\theta(T) \ge 0$ a.s. $P$ we have

(1.18) $$\overline{V}^\theta(T) \ge 0 \quad \text{a.s. } Q$$

because $Q$ is equivalent to $P$.

Similarly, since $P[\overline{V}^\theta(T) > 0] > 0$, we have

$$Q[\overline{V}^\theta(T) > 0 \ . \tag{1.19}$$

On the other hand, since $Q$ is a martingale measure for $\{\overline{X}(t)\}_{t\in\mathcal{T}}$ it follows that

$$\begin{aligned}
E_Q[\overline{V}^\theta(T)] &= E_Q[\overline{V}^\theta(0) + \sum_{j=0}^{k-1} \Delta\overline{V}^\theta(t_j)] \\
&\stackrel{*}{=} \overline{V}^\theta(0) + \sum_{j=0}^{k-1} E_Q[\theta(t_j)\cdot\Delta X^\theta(t_j)] \\
&= \overline{V}^\theta(0) + \sum_{j=0}^{k-1} E_Q[E_Q[\theta(t_j)\cdot\Delta X(t_j) \mid \mathcal{G}_{t_j}]] \\
&= \overline{V}^\theta(0) + \sum_{j=0}^{k-1} E_Q[\theta(t_j)E_Q[\Delta X(t_j) \mid \mathcal{G}_{t_j}]] \\
&= \overline{V}^\theta(0) \ ,
\end{aligned}$$

where we at $(*)$ have used the self-financing property of $\theta$. This is impossible in view of (1.18) and (1.19). We conclude that $\{\overline{X}(t)\}_{t\in\mathcal{T}}$, and hence $\{X(t)\}_{t\in\mathcal{T}}$, cannot have an arbitrage. $\square$

We refer to [LL, Theorem 1.2.7] for a proof of b).

**The continuous time model**

We now assume that trading and price changes can take place at any time $t \in [0,T]$, where $T > 0$ is fixed. Hence the *market* is now represented by a *continuous time* $(n+1)$-dimensional stochastic process $X(t) = (X_0(t), X_1(t), \ldots, X_n(t))$; $t \in [0,T]$ on a given probability space $(\Omega, \mathcal{F}, P)$. More precisely, we will assume that $X(t)$ is an *Ito process* of the form

$$dX_0(t) = \rho(t,\omega)X_0(t)dt \ ; \qquad X_0(0) = 1 \tag{1.20}$$

$$dX_i(t) = \mu_i(t,\omega)dt + \sigma_i(t,\omega)dB(t) \ ; \qquad X_i(0) = x_i \ ; \quad 1 \le i \le n \tag{1.21}$$

where $B(t) = (B_1(t), \ldots, B_m(t))$ is $m$-dimensional Brownian motion with filtration $\{\mathcal{F}_t\}_{t\ge 0} = \{\mathcal{F}_t^{(m)}\}_{t\ge 0}$, $\rho(t,\omega) \in \mathbf{R}$, $\mu_i(t,\omega) \in \mathbf{R}$ and $\sigma_i(t,\omega)$ is row number $i$ of an $n \times m$ matrix $\sigma(t,\omega) = [\sigma_{ij}(t,\omega)]_{\substack{1\le i\le n \\ 1\le j\le m}}$, so that $\sigma_i dB$ means $\sum_{j=1}^{m} \sigma_{ij} dB_j$. We assume that all these coefficients $\rho$, $\mu_i$, $\sigma_{ij}$ are $(t,\omega)$-measurable and $\{\mathcal{F}_t^{(m)}\}$-adapted, that $\rho(t,\omega)$ is bounded and

$$(1.22)\qquad P\Big[\int_0^T \{|\mu_i(t,\omega)| + \sum_{j=1}^m \sigma_{ij}^2(t,\omega)\}dt < \infty\Big] = 1 \qquad \text{for all } i\ .$$

Under these conditions (1.20)–(1.21) can be interpreted in the *Ito integral* sense

$$(1.23)\qquad X_0(t) = 1 + \int_0^t \rho(s,\omega)X_0(s)ds$$

$$(1.24)\qquad X_i(t) = x_i + \int_0^t \mu_i(s,\omega)ds + \int_0^t \sum_{j=1}^m \sigma_{ij}(s,\omega)dB_j(s)\ ; \qquad 1 \le i \le n$$

Note that the solution of (1.20) is

$$(1.25)\qquad X_0(t) = \exp\Big(\int_0^t \rho(s,\omega)ds\Big)$$

and hence that

$$(1.26)\qquad \xi(t) := X_0^{-1}(t) = \exp\Big(-\int_0^t \rho(s,\omega)ds\Big) > 0$$

Also note that

$$(1.27)\qquad d\xi(t) = -\rho(t,\omega)\xi(t)dt\ ; \qquad \xi(0) = 1\ .$$

For more information about Ito integrals we refer to e.g. [KS1] and [Ø2].

A *portfolio* in this market is an $(n+1)$-dimensional $\mathcal{F}_t$-adapted and $(t,\omega)$-measurable process $\theta(t) = (\theta_0(t), \theta_1(t), \dots, \theta_n(t))$. As before $\theta_i(t)$ gives the number of units of security $i$ held at time $t$. The *value process* $V^\theta(t)$ of a portfolio is, as before,

$$V^\theta(t) = \theta(t)\cdot X(t)\ ; \qquad t \in [0,T]$$

We say that the portfolio $\theta(t)$ is *self-financing* if

$$(1.28)\qquad dV^\theta(t) = \theta(t)\cdot dX(t)$$

i.e., if

$$(1.29)\qquad V^\theta(t) = V^\theta(0) + \int_0^t \theta(s)\cdot dX(s)\ ; \qquad t \in [0,T]$$

where the integral on the right is the Ito integral obtained by substituting (1.20), (1.21) for $X(t)$.

**Remarks.** **1)** Note that the self-financing condition (1.28) is just the continuous time analogue of (1.15).

**2)** The reader can easily verify that all *constant* portfolios $\theta = (\theta_0, \dots, \theta_n)$ are self-financing.

In the continuous time model it is necessary to add one more condition on the portfolios allowed:

**Definition 1.12** A self-financing portfolio $\theta$ is called *admissible* if there exists $K = K(\theta) < \infty$ such that

$$V^\theta(t,\omega) \geq -K \qquad \text{for a.a. } (t,\omega) \in [0,T] \times \Omega \tag{1.30}$$

(here, and in the following, "almost all $t \in [0,T]$" means with respect to Lebesgue measure on $[0,T]$.)

The condition (1.30) is natural from a modelling point of view: There must be a bound on the size of the debt that an agent can have during her portfolio. The condition is also mathematically convenient: It excludes the so-called *doubling strategies*. See [Ø2, Chapter 12] for more details.

We now proceed as in the multi-period case:

**Definition 1.13** An admissible portfolio $\theta(t)$ is called an *arbitrage* if

$$V^\theta(0) \leq 0 \leq V^\theta(T) \qquad \text{a.s. } P \text{ and} \qquad P[V^\theta(T) > 0] > 0\ . \tag{1.31}$$

**Example 1.14** Suppose $n = 2$ and

$$\begin{aligned} dX_0(t) &= 0 & X_0(0) &= 1 \\ dX_1(t) &= dt + dB_1(t) + dB_2(t)\ ; & X_1(0) &= 1 \\ dX_2(t) &= 3dt - 2dB_1(t) - 2dB_2(t)\ ; & X_2(0) &= 1 \end{aligned}$$

Then $\theta(t) = (-3, 2, 1)$ (constant) is an arbitrage, because

$$V^\theta(t) = -3 + 2(1 + t + B_1(t) + B_2(t)) + 1 \cdot (1 + 3t - 2B_1(t) - 2B_2(t)) = 5t\ .$$

As in the discrete time case we can define the *normalized price process* $\overline{X}(t)$ by

$$\overline{X}(t) = \xi(t)X(t) = (1, \xi(t)X_1(t), \dots, \xi(t)X_n(t)) \tag{1.32}$$

where $\xi(t) = X_0^{-1}(t)$ as in (1.26). Note that by Ito's formula we have

$$d\overline{X}(t) = \xi(t)dX(t) + X(t)d\xi(t) = \xi(t)[dX(t) - \rho X(t)dt]\ . \tag{1.33}$$

We say that $\{X(t)\}_{t \in [0,T]}$ is *normalized* if $\overline{X}(t) = X(t)$, i.e. if $X_0(t) = 1$ for all $t$.

**Lemma 1.15 a)** The following are equivalent:

(i) $\{X(t)\}_{t\in[0,T]}$ has no arbitrage

(ii) $\{\overline{X}(t)\}_{t\in[0,T]}$ has no arbitrage

**b)** Suppose $\{X(t)\}_{t\in[0,T]}$ is normalized. Then $\{X(t)\}_{t\in[0,T]}$ has an arbitrage if and only if there exists an admissible portfolio $\theta$ such that

$$V^\theta(0) \le V^\theta(T) \quad \text{a.s. and} \quad P[V^\theta(T) > V^\theta(0)] > 0\,. \tag{1.34}$$

*Proof.* a) Suppose $\theta$ is an arbitrage for $\{X(t)\}_{t\in[0,T]}$. Let

$$\overline{V}^\theta(t) = \theta(t)\cdot\overline{X}(t) = \xi(t)V^\theta(t) \tag{1.35}$$

be the corresponding value process for the normalized market. Then

$$\begin{aligned} d\overline{V}^\theta &= d(\xi(t)V^\theta(t)) = \xi(t)dV^\theta(t) + V^\theta(t)d\xi(t) \\ &= \xi(t)\theta(t)dX(t) - \rho(t)\xi(t)V^\theta(t)dt \\ &= \xi(t)\theta(t)[dX(t) - \rho(t)X(t)dt] = \theta(t)d\overline{X}(t) \end{aligned}$$

Hence

$$\overline{V}^\theta(t) = V(0) + \int_0^t \theta(t)d\overline{X}(t)\,. \tag{1.36}$$

In particular, $\theta$ is admissible for $\{\overline{X}(t)\}_{t\in[0,T]}$. Moreover, since

$$V^\theta(0) \le 0 \le V^\theta(T) \quad \text{a.s. and} \qquad P[V^\theta(T) > 0] > 0$$

we have by (1.34)

$$\overline{V}^\theta(0) \le 0 \le \overline{V}^\theta(T) \quad \text{a.s. and} \qquad P[\overline{V}^\theta(T) > 0] > 0\,.$$

Hence $\theta$ is an arbitrage for $\{\overline{X}(t)\}_{t\in[0,T]}$.

The argument goes both ways and hence a) is proved.

b) Suppose $\{X(t)\}_{t\in[0,T]}$ is normalized and let $\theta$ be an admissible portfolio satisfying (1.33).

Define $\check{\theta}(t) = (\check{\theta}_0(t), \check{\theta}_1(t), \ldots, \check{\theta}_n(t))$ by

$$\check{\theta}_i(t) = \theta_i(t) \qquad \text{for } i = 1, 2, \ldots, n$$

and put

$$\check{\theta}_0(0) = -\sum_{i=1}^n \theta_i(0)X_i(0)$$

and

$$\check{\theta}_0(t) = \sum_{i=1}^{n} \Big( \int_0^t \theta_i(s) dX_i(s) - \theta_i(t) X_i(t) \Big)$$

Then $V^{\check{\theta}}(0) = 0$ and because $dX_0(t) = 0$ we have

$$V^{\check{\theta}}(t) = \check{\theta}(t) X(t) = \check{\theta}_0(t) + \sum_{i=1}^{n} \theta_i(t) X_i(t) = \int_0^t \check{\theta}(s) dX(s) \ .$$

So $\check{\theta}$ is admissible and $\theta$ satisfies (1.31) since

$$V^{\check{\theta}}(t) = V^{\theta}(t) - V^{\theta}(0) \ .$$

□

Just as in the discrete time case there is a striking relation between markets with no arbitrage and equivalent martingale measures. However, in this case the relation is more complicated:

**Definition 1.16 a)** A probability measure $Q$ on $\mathcal{F}_T^{(m)}$ is called a *martingale measure* for the normalized market $\{\overline{X}(t)\}_{t\in[0,T]}$ if

(1.37) $$E_Q[\overline{X}(s) \mid \mathcal{F}_t^{(m)}] = \overline{X}(t) \qquad \text{for all } s > t$$

**b)** If – in addition to (1.37) – the measure $Q$ is equivalent to $P$, then we say that $Q$ is an *equivalent martingale measure* for $\{\overline{X}(t)\}_{t\in[0,T]}$.

We now state without proof the continuous time analogue of Theorem 1.11:

**Theorem 1.17 [DS]**

**a)** Suppose there exists an equivalent martingale measure for $\{\overline{X}(t)\}_{t\in[0,T]}$. Then the market $\{X(t)\}_{t\in[0,T]}$ satisfies the "*no free lunch with vanishing risk*" (NFLVR)-condition.

**b)** Conversely, if the market $\{X(t)\}_{t\in[0,T]}$ satisfies the NFLVR-condition, then there is an equivalent martingale measure for $\{\overline{X}(t)\}_{t\in[0,T]}$.

**Remark.** We will not define the NFLVR-condition here, but simply point out that it is slightly *stronger* than the "no arbitrage"-condition. We refer to [DS] for details and for the proof of Theorem 1.17. Hence we have

**Corollary 1.18** Suppose there exists an equivalent martingale measure for $\{\overline{X}(t)\}_{t\in[0,T]}$. Then the market $\{X(t)\}_{t\in[0,T]}$ has no arbitrage.

When do equivalent martingale measures exist? Consider the following situation

**Example 1.19** Suppose $X(t) = (1, Y(t))$, where

$$Y(t) = t + B(t) \; ; \qquad t \in [0, T] \; ,$$

$B(t)$ being 1-dimensional Brownian motion. Does the market $\{X(t)\}_{t\in[0,T]}$ have an arbitrage? Since $Y(t)$ can assume both positive and negative values with positive probability our intuition tells us that the answer is *no*. Let us try to verify this by constructing a measure $Q \sim P$ such that $X(t)$ is a $Q$-martingale.

To this end put

$$M_t(\omega) = \exp(-B(t,\omega) - \tfrac{1}{2}t) \; ; \qquad t \in [0, T]$$

and define the measure $Q$ on $\mathcal{F}_T$ by

$$dQ(\omega) = M_T(\omega) dP(\omega) \; .$$

Since $M_T(\omega) > 0$ we see that $Q \sim P$. Moreover,

$$E_Q[1] := \int_\Omega 1 \, dQ(\omega) = \int_\Omega M_T(\omega) dP(\omega) = M_0 = 1 \; ,$$

since $M_t$ is a $P$-martingale. (The reader can easily check this by using the Ito formula.) Hence $Q$ is a probability measure and it remains to show that $Y(t)$ is a $Q$-martingale. To do this we apply the following well-known result about conditional expectation (see e.g. [Ø2, Lemma 8.6.2] for a proof).

**Lemma 1.20** Suppose $Q$ is a probability measure on $\mathcal{F}$ of the form

$$dQ(\omega) = f(\omega) dP(\omega)$$

for some $f(\omega) \geq 0$. Let $Y$ be an $\mathcal{F}$-measurable random variable such that $E_Q[|Y|] < \infty$ and let $\mathcal{G} \subset \mathcal{F}$ be a $\sigma$-algebra. Then if $E_P[f \mid \mathcal{G}] \neq 0$ we have

$$E_Q[Y \mid \mathcal{G}] = \frac{E[fY \mid \mathcal{G}]}{E[f \mid \mathcal{G}]}$$

where $E[\cdot] = E_P[\cdot]$ means expectation with respect to $P$.

Applied to our situation this gives, for $s < t < T$,

$$E_Q[Y(t) \mid \mathcal{F}_s] = \frac{E[M_T Y(t) \mid \mathcal{F}_s]}{E[M_T \mid \mathcal{F}_s]} = \frac{E[E[M_T Y(t) \mid \mathcal{F}_t] \mid \mathcal{F}_s]}{M_s}$$
$$= \frac{E[Y(t) E[M_T \mid \mathcal{F}_s]}{M_s} = \frac{E[Y(t) M_t \mid \mathcal{F}_s]}{M_s} \; .$$

Now note that by the Ito formula

$$\begin{aligned} d(M_tY(t)) &= M_t dY(t) + Y(t)dM_t + dM_t dY(t) \\ &= M_t(dt + dB(t)) + Y(t)(-M_t dB(t)) + (-M_t)dt \\ &= M_t(1 - Y(t))dB(t) \ . \end{aligned}$$

Hence $M_tY(t)$ is a $P$-martingale and therefore by the above we get

$$E_Q[Y(t) \mid \mathcal{F}_s] = \frac{E[M_tY(t) \mid \mathcal{F}_s]}{M_s} = \frac{M_sY(s)}{M_s} = Y(s) \ ,$$

which proves that $Y(t)$ is a $Q$-martingale.

We conclude that $Q$ is an equivalent martingale measure for $X(t)$. Hence the market cannot have an arbitrage.

This example is a special case of the following important result, which gives a general method of constructing equivalent martingale measures:

**Theorem 1.21 (The Girsanov theorem)**
Suppose $Y(t)$ is an Ito process in $\mathbf{R}^n$ of the form

$$dY(t) = \beta(t,\omega)dt + \sigma(t,\omega)dB(t)$$

where $B(t) \in \mathbf{R}^m$, $\beta(t,\omega) \in \mathbf{R}^n$ and $\sigma(t,\omega) \in \mathbf{R}^{n\times m}$. Suppose there exist processes $u(t,\omega) \in \mathbf{R}^m$, $\alpha(t,\omega) \in \mathbf{R}^n$ such that

$$\sigma(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega)$$

and such that

$$E\Big[\exp\Big(\tfrac{1}{2}\int_0^t u^2(s,\omega)ds\Big)\Big] < \infty$$

Put

(1.38) $$M_t(\omega) = \exp\Big(-\int_0^t u(s,\omega)dB(s) - \tfrac{1}{2}\int_0^t u^2(s,\omega)ds\Big); \qquad 0 \le t \le T$$

and define the measure $Q$ on $\mathcal{F}_T$ by

(1.39) $$dQ(\omega) = M_T(\omega)dP(\omega) \ .$$

Then $Q$ is a probability measure on $\mathcal{F}_T$, $Q \sim P$ and

$$\widetilde{B}(t) := \int_0^t u(s,\omega)ds + B(t) \ ; \qquad 0 \le t \le T$$

is a Brownian motion with respect to $Q$. Moreover, in terms of $\widetilde{B}(t)$ the process $Y(t)$ has the stochastic integral representation

$$dY(t) = \alpha(t,\omega)dt + \sigma(t,\omega)d\widetilde{B}(t) \ .$$

In particular, if $\alpha(t,\omega) = 0$ we obtain that $Q$ is an equivalent martingale measure for $Y(t)$.

**Remark.** Note that the filtration $\widetilde{\mathcal{F}}_t$ generated by $\widetilde{B}(s)$; $s \le T$ need not be the same as the filtration $\mathcal{F}_t$ generated by $B(s)$; $s \le t$. It is easy to see that in general we have

$$\widetilde{\mathcal{F}}_t \subseteq \mathcal{F}_t \ .$$

But there are cases where $\widetilde{\mathcal{F}}_t \neq \mathcal{F}_t$. See [RY, Remark 2°), p. 306].

However, if $u(s,\omega) = u(s)$ is *deterministic* it is clear that

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_t \ .$$

This applies, for example, to the generalized Black & Scholes model in Section 4.

In view of the explicit construction in the Girsanov theorem and Corollary 1.18, it is natural to expect that one can give conditions for the non-existence of arbitrage directly in terms of the coefficients $\rho$, $\mu$ and $\sigma$ in the equations (1.20)–(1.21) defining the market $\{X(t)\}_{t\in[0,T]}$. This is indeed the case. For example, one can prove the following result:

**Theorem 1.22 a)** Suppose there exists an $\mathcal{F}_t^{(m)}$-adapted process $u(t,\omega) \in \mathbf{R}^m$ such that

(1.40) $\sigma_i(t,\omega)u(t,\omega) = \mu_i(t,\omega) - \rho(t,\omega)X_i(t,\omega)$ ; for $1 \le i \le n$, for a.a. $(t,\omega)$

and

(1.41) $$E\Big[\exp\Big(\tfrac{1}{2}\int_0^T u^2(t,\omega)dt\Big)\Big] < \infty$$

Then the market $\{X(t)\}_{t\in[0,T]}$ has no arbitrage.

**b) [K, Th. 0.2.4]** Conversely, if the market $\{X(t)\}_{t\in[0,T]}$ has no arbitrage, then there exists an $\mathcal{F}_t^{(m)}$-adapted process $u(t,\omega)$ such that (1.40) holds.

We refer to [K, Th. 0.2.4] or [Ø2, Th. 12.1.8] for a proof.

We illustrate Theorem 1.21 by looking at some special cases:

**Example 1.23**

**(i)** Suppose $n = m$ and that $\sigma(t,\omega) \in \mathbf{R}^{n\times n}$ is an invertible matrix for a.a. $(t,\omega)$. Then clearly the system of equations (1.40) has the unique solution

(1.42) $$u(t,\omega) = \sigma^{-1}(t,\omega)[\mu(t,\omega) - \rho(t,\omega)\widehat{X}(t,\omega)]$$

where $\widehat{X}(t,\omega)$ is (the transposed of) the vector $(X_1(t,\omega),\dots,X_n(t,\omega))$, obtained by removing the 0'th component $X_0(t)$ from $X(t)$. So if this $u(t,\omega)$ given by (1.42) satisfies (1.41) then we know that the market has no arbitrage.

**(ii)** Even when $n = m$ and $\sigma$ is not invertible there may be solutions $u(t,\omega)$ of (1.40). Consider the market with $n = m = 2$ and

(1.43) $$\begin{cases} dX_0(t) = 0 \\ dX_1(t) = 2dt + dB_1(t) + dB_2(t) \\ dX_2(t) = -2dt - dB_1(t) - dB_2(t) \end{cases}$$

Then (1.40) gets the form

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

which has (for example) the solution $u_1 = 2$, $u_2 = 0$. Since this gives

$$E\Big[\exp\Big(\tfrac{1}{2}\int_0^T u^2(t,\omega)dt\Big)\Big] = E[\exp(2B(T,\omega))] = \exp(2T) < \infty ,$$

we conclude that the market (1.43) has no arbitrage.

**(iii)** If we modify the market above to

(1.44) $$\begin{cases} dX_0(t) = 0 \\ dX_1(t) = 1dt + dB_1(t) + dB_2(t) \\ dX_2(t) = -2dt - dB_1(t) - dB_2(t) \end{cases}$$

then the corresponding system (1.40) gets the form

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

which has no solutions. We conclude by Theorem 1.22 b) that this market has an arbitrage. Indeed, if we choose

$$\theta(t) = (0, -1, -1)$$

then $\theta$ is self-financing (since it is constant) and

$$V^\theta(t) = t \qquad \text{for } t \geq 0,$$

so $\theta$ is an arbitrage for the market (1.44).

# 2 Contingent claims and completeness

From now on we assume we are given a continuous time market $\{X(t)\}_{t\in[0,T]}$ given by (1.20) and (1.21). We start this section by recalling the definition of a *European call option*:

**Definition 2.1** A *European call option* is the right – but not the obligation – to buy one stock of a specified type at a specified price $K$ (the *exercise* price) and at a specified future time $T$ (the *time of maturity*).

If $S(t,\omega)$ denotes the market price of the stock at time $t$, then there are two possibilities at the time $T$ of maturity:

(i) If $S(T,\omega) > K$ then the owner of this option will buy the stock for the price $K$ and immediately sell it on the open market for the price $S(T,\omega)$, thereby obtaining the payoff $S(T,\omega) - K$.

(ii) If $S(T,\omega) \leq K$ then the owner will not exercise the option and the payoff is 0.

Thus we can express the payoff $F(\omega)$ at time $T$ of a European call option by

$$F(\omega) = (S(T,\omega) - K)^+ = \begin{cases} S(T,\omega) - K & \text{if } S(T,\omega) > K \\ 0 & \text{if } S(T,\omega) \leq K \end{cases} \tag{2.1}$$

More generally, we introduce the following concepts:

**Definition 2.2 a)** A *European contingent* $T$*-claim* (or just a $T$*-claim*) is a lower bounded $\mathcal{F}_T^{(m)}$-measurable random variable $F(\omega)$.

**b)** We say that the $T$-claim $F(\omega)$ is *attainable* in the market $\{X(t)\}_{t\in[0,T]}$ if there exists an admissible portfolio $\theta(t) \leq \mathbf{R}^{n+1}$ and a real number $z$ such that

$$F(\omega) = V_z^\theta(T) := z + \int_0^T \theta(t) \cdot dX(t) \quad \text{a.s.,} \tag{2.2}$$

i.e. such that the value process equals $F$ a.s. at the terminal time $T$. If such a $\theta(t)$ exists, we call it a *replicating* or *hedging* portfolio for $F$.

**c)** The market $\{X(t)\}_{t\in[0,T]}$ is called *complete* if every *bounded* $T$-claim is attainable.

Some important questions are:

(2.3) Which claims are attainable in a given market $\{X(t)\}_{t\in[0,T]}$?

(2.4) Which markets $\{X(t)\}_{t\in[0,T]}$ are complete?

(2.5) If a $T$-claim $F$ is attainable, how do we find the corresponding initial value $z$ and the replicating portfolio $\theta(t)$? Are they unique?

Let us illustrate the situation in a simple case:

**Example 2.3** Suppose the market is given by

$$X(t) = (1, B(t)) \in \mathbf{R}^2 \; ; \qquad t \in [0,T] \; .$$

Is the claim

$$F(\omega) = B^2(T,\omega)$$

attainable? We seek an admissible portfolio $\theta(t) = (\theta_0(t), \theta_1(t))$ and a real number $z$ such that

$$F(\omega) = B^2(T,\omega) = z + \int_0^T \theta(t) \cdot dX(t) = z + \int_0^T \theta_1(t) dB(t) \tag{2.6}$$

By the Ito formula we see that

$$B^2(T,\omega) = T + \int_0^T 2B(t)dB(t) \; .$$

We conclude that

$$z = T, \quad \theta_1(t) = 2B(t)$$

do the job (2.6). Then we choose $\theta_0(t)$ to make the portfolio $\theta(t)$ self-financing. For this we need that

$$V_z^\theta(t) = z + \int_0^t \theta(s) \cdot dX(s) = \theta(t) \cdot X(t)$$

i.e.

$$T + \int_0^t 2B(s)dB(s) = \theta_0(t) + 2B^2(t) \; .$$

So we choose $\theta_0(t) = T + \int_0^t 2B(s)dB(s) - 2B^2(t) = T - t - B^2(t)$. Then $\theta(t) = (\theta_0(t), \theta_1(t))$ is an admissible portfolio which replicates $F$ and hence $F$ is attainable.

There is a striking characterization of completeness of a market $\{X(t)\}_{t \in [0,T]}$ in terms of equivalent martingale measures, due to Harrison and Pliska [HP] and Jacod [J]:

**Theorem 2.4** A market $\{X(t)\}_{t\in[0,T]}$ is complete if and only if there is *one and only one* equivalent martingale measure $Q$ for $\{\overline{X}(t)\}_{t\in[0,T]}$.

(Compare with the equivalent martingale measure condition for non-arbitrage in Theorem 1.17!)

Again one may ask if there is a more direct criterion for completeness in terms of the coefficients $\rho$, $\mu$ and $\sigma$ in the equations (1.20)–(1.21).

Here is a partial answer:

**Theorem 2.5** Let $\{X(t)\}_{t\in[0.T]}$ be the market given by (1.20)–(1.21) and let $\widehat{X}(t)$ be (the transposed of) the vector $(X_1(t),\ldots,X_n(t))$. Suppose there exists an $\mathcal{F}_t^{(m)}$-adapted process $u(t,\omega)\in\mathbf{R}^m$ such that

$$\sigma(t,\omega)u(t,\omega)=\mu(t,\omega)-\rho(t,\omega)\widehat{X}(t,\omega)\qquad\text{for a.a. }(t,\omega)\tag{2.7}$$

and

$$E\Big[\exp\Big(\tfrac{1}{2}\int_0^T u^2(s,\omega)ds\Big)\Big]<\infty\,.\tag{2.8}$$

Then the market $\{X(t)\}_{t\in[0,T]}$ is complete if and only if $\sigma(t,\omega)$ has a *left inverse* $\Lambda(t,\omega)\in\mathbf{R}^{m\times n}$, i.e. if and only if

$$\operatorname{rank}\sigma(t,\omega)=m\qquad\text{for a.a. }(t,\omega)\tag{2.9}$$

For a proof we refer to [K, Th. 0.3.5] or [Ø2, Th. 12.2.5].

**Remark.** Note that when (2.7) and (2.8) hold, then the corresponding (unique) equivalent martingale measure $Q$ for $\{X(t)\}_{t\in[0,T]}$ is given by (1.38) and (1.39) in the Girsanov theorem (Theorem 1.21), with $u(t,\omega)$ as in (2.7).

**Example 2.6** Consider the market defined by

$$\begin{cases} dX_0(t)= & 0 & ; & X_0(0)=1\\ dX_1(t)= & dB_1(t)+3dB_2(t) & ; & X_1(0)=x_1\\ dX_2(t)= & dt-dB_1(t)-2dB_2(t) & ; & X_2(0)=x_2\end{cases}\tag{2.10}$$

Here equation (2.7) gets the form

$$\begin{bmatrix}1 & 3\\ -1 & -2\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

which has the unique solution

$$\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\begin{bmatrix}-3\\ 1\end{bmatrix}$$

Clearly

$$E\Big[\exp\Big(\tfrac{1}{2}\int_0^T u^2(s,u)ds\Big)\Big]=E\Big[\exp\Big(\tfrac{1}{2}\int_0^T 10dt\Big)\Big]=\exp(5T)<\infty\,,$$

so by Theorem 2.5 this market is complete.

**Example 2.7** Suppose the market is given by

$$X(t) = \begin{bmatrix} 1 \\ Z(t) \end{bmatrix} \in \mathbf{R}^2 , \tag{2.11}$$

where

$$dZ(t) = \sigma_1 dB_1(t) + \sigma_2 dB_2(t) \in \mathbf{R}$$

with $\sigma_1$ and $\sigma_2$ constants.

Here equation (2.7) becomes

$$[\sigma_1 \quad \sigma_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

which has infinitely many solutions $\left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in \mathbf{R}^2$. We also see that

$$\text{rank } \sigma \leq 1 < 2 = m$$

in this case, so (2.9) does not hold. We conclude from Theorem 2.5 that the market (2.11) is *not* complete.

Hence there exist claims which are not attainable in this market. Here is one of them: Define $F(\omega) = B_1^2(T)$. Then

$$F(\omega) = T + \int_0^T 2B_1(s) dB_1(s) . \tag{2.12}$$

On the other hand, if $F$ is attainable, there exist $\mathcal{F}_t$-adapted $\theta(t, \omega) \in \mathbf{R}$ and $z \in \mathbf{R}$ such that

$$F(\omega) = V_z^\theta(T) \tag{2.13}$$

where

$$V_z^\theta(t) = z + \int_0^t \theta(s, \omega)[\sigma_1 dB_1(s) + \sigma_2 dB_2(s)] . \tag{2.14}$$

Moreover, $V_z^\theta(t)$ is lower bounded. This implies that $V_z^\theta(t)$ is a *supermartingale* and then it follows by the *Doob-Meyer decomposition* [KS1] that $\theta(t, \omega)$ and $z$ must be unique. Hence, by comparing (2.12) and (2.14) we get $T = z$ and

$$2B_1(s) = \sigma_1 \theta(s) \qquad \text{and} \qquad 0 = \sigma_2 \theta(s) \tag{2.15}$$

This contradiction shows that $F$ cannot be attainable.

# 3 Pricing of contingent claims

We motivate this section by again referring to the European call option in Definition 2.1. Now we ask the question: How much would a buyer be willing to pay at time $t = 0$ to become the owner of such an option? And what amount would the *seller* of such an option be willing to accept as a payment?

Again we generalize to a situation where a person – the *buyer* – is being offered a guaranteed (stochastic) payment $F(\omega)$, a given $T$-claim, at time $T$ by a *seller*. The *buyer* can now argue as follows: If I – the buyer – pay the price $y$ for this guarantee, then I start out with an initial fortune $-y$ in my investment strategy. With this initial fortune (debt) it must be possible for me to hedge to time $T$ a value $V_{-y}^{\theta}(T)$ which, when the guaranteed payment is added, gives me a nonnegative result:

$$V_{-y}^{\theta}(T,\omega) + F(\omega) \geq 0 \quad \text{a.s.}$$

By this point of view the maximal price $p = p(F)$ that the buyer is willing to pay is given by

(*Buyer's price of the contingent claim F*)

$$p(F) = \sup\{y \; ; \; \text{There exists an admissible portfolio } \theta \text{ such that}$$

$$V_{-y}^{\theta}(T) := -y + \int_0^T \theta(s)dX(s) \geq -F(\omega) \quad \text{a.s.}\} \tag{3.1}$$

On the other hand, the *seller* can adopt a similar non-risk attitude: If I – the seller – accept a price $z$ for this guarantee, then I can use this as an initial fortune in an investment strategy. With this initial value it must be possible to hedge to time $T$ a value $V_z^{\theta}(T)$, which is no less than the amount $F(\omega)$ that I have promised to pay to the buyer:

$$V_z^{\theta}(T,\omega) \geq F(\omega) \quad \text{a.s.}$$

Thus the minimal price $q = q(F)$ that the seller is willing to accept is given by

(*Seller's price of the contingent claim F*)

$$q(F) = \inf\{z \; ; \; \text{There exists an admissible portfolio } \theta \text{ such that}$$

$$V_z^{\theta}(T) := z + \int_0^T \theta(s)dX(s) \geq F(\omega) \quad \text{a.s.}\} \tag{3.2}$$

**Definition 3.1** If $p(F) = q(F)$ we call this common value the *price* (at $t = 0$) of the contingent claim $F(\omega)$.

In general we have (see e.g. [K, Prop. 0.4.1] or [Ø2, Th. 12.3.2])

**Theorem 3.2 a)** Suppose (2.7) and (2.8) hold and let $Q$ be as in (1.39). Let $F(\omega)$ be a $T$-claim such that $E_Q[\xi(T)F] < \infty$. Then

$$p(F) \le E_Q[\xi(T)F] \le q(F) \le \infty \tag{3.3}$$

**b)** Suppose, in addition to the conditions in a), that the market is *complete*. Then the price of the $T$-claim is given by

$$p(F) = E_Q[\xi(T)F] = q(F)\ . \tag{3.4}$$

Therefore, for *complete* markets (satisfying (2.7) and (2.8)) there is a unique, canonical price of a $T$-claim. For incomplete markets, however, we can only give an interval $[p(F), q(F)]$ within which the price should be. Unfortunately, this interval may in many cases be large and therefore (3.3) does not give a satisfactory answer in this case. We refer to the paper by P. Leukert [L] in this volume for more information about pricing in incomplete markets.

**Example 3.3** Consider again the market $\{X(t)\}_{t\in[0,T]}$ given by (2.10) in Example 2.6. Suppose the claim has the form

$$F(\omega) = X_1^2(T,\omega)$$

Since this market is complete, we know that the price of this $T$-claim is given by (3.4), i.e.

$$\begin{aligned}
p(F) &= q(F) = E_Q[X_1^2(T,\omega)] = \\
&= E\Big[\exp\Big(\int_0^T u_1 dB_1(t) + \int_0^T u_2 dB_2(t) - \tfrac{1}{2}\int_0^T (u_1^2+u_2^2)dt\Big)X_1^2(T,\omega)\Big] \\
&= E[\exp(-3B_1(T) + B_2(T) - 5T)(x_1 + B_1(T) + 3B_2(T))^2] \\
&= \frac{1}{2\pi T}\int_{\mathbf{R}^2} \exp(-3y_1 + y_2 - 5T)(x_1 + y_1 + 3y_2)\exp\Big(-\frac{y_1^2+y_2^2}{4T}\Big)dy_1 dy_2\ ,
\end{aligned}$$

by using the known distribution of the 2-dimensional Brownian motion $(B_1(T), B_2(T))$. (This expression can be simplified further.)

**Remark.** In this paper we only consider the pricing of *European* claims. For a survey on the pricing of *American* options see the paper by K. Aase [A] in this volume.

# 4 The generalized Black & Scholes model

We now discuss in detail the following special case of the general model (1.20)–(1.21):

Suppose the market $X(t) = (X_0(t), X_1(t)) = (A(t), S(t))$ is given by

$$dX_0(t) = dA(t) = \rho(t)A(t)dt \; ; \qquad A(0) = 1 \tag{4.1}$$

and

$$dX_1(t) = dS(t) = \alpha(t)S(t)dt + \beta(t)S(t)dB(t) \; . \tag{4.2}$$

Here $\rho(t)$, $\alpha(t)$ and $\beta(t)$ are *deterministic* functions (i.e. they do not depend on $\omega$) satisfying the conditions

$$\int_0^T (|\rho(t)| + |\alpha(t)| + |\beta^2(t)|)dt < \infty \tag{4.3}$$

and

$$\int_0^T \frac{(\alpha(t) - \rho(t))^2}{\beta^2(t)} dt < \infty \; . \tag{4.4}$$

This is a generalization of the classical Black & Scholes market, where $\rho, \alpha$ and $\beta \neq 0$ are constants [BS]. So we will just refer to it as the *generalized Black & Scholes market*. This market consists of only two assets: The bank account, with price dynamics given by (4.1) and the stock, with price dynamics given by (4.2).

First of all, let us apply the general theory outlined above to check the properties of this market:

**Non-existence of arbitrage:**

The equation (1.40) gets the form

$$\beta(t)S(t,\omega)u(t,\omega) = \alpha(t)S(t,\omega) - \rho(t)S(t,\omega)$$

which has the solution

$$u(t,\omega) = u(t) = \frac{\alpha(t) - \rho(t)}{\beta(t)} \; . \tag{4.5}$$

By (4.4) we see that (1.41) holds and we conclude by Theorem 1.22 that this market has *no arbitrage*.

**Completeness:** Again by (1.40) we see that the conditions (2.7) and (2.8) of Theorem 2.5 hold. Moreover, since $\sigma(t) \neq 0$ for a.a. $t$, we have rank $\sigma(t) = 1 = m$ for a.a. $t$. So by Theorem 2.5 we conclude that this market is *complete*.

The (unique) equivalent martingale measure $Q$ is given by (see the Remark following Theorem 2.5)

$$dQ(\omega) = \exp\Big( - \int_0^T u(t)dB(t) - \tfrac{1}{2} \int_0^T u^2(t)dt \Big) dP(\omega) \qquad \text{on } \mathcal{F}_T \tag{4.6}$$

with $u(t)$ given by (4.5).

So by Theorem 3.2 b) the price of the European call option (Definition 2.1)

$$F(\omega) = (S(T,\omega) - K)^+$$

is given by, with $\xi(T) = \exp\Big( - \int_0^T \rho(t)dt \Big)$,

$$\begin{aligned} p(F) = q(F) &= E_Q[\xi(T)(S(T,\omega) - K)^+] \\ &= \xi(T) E_Q\Big[\Big( x_1 \exp\Big\{ \int_0^T \beta(t)dB(t) + \int_0^T (\alpha(t) - \tfrac{1}{2}\beta^2(t))dt \Big\} - K \Big)^+\Big] \\ &= \xi(T) E_Q\Big[\Big( x_1 \exp\Big\{ \int_0^T \beta(t)d\widetilde{B}(t) + \int_0^T (\rho(t) - \tfrac{1}{2}\beta^2(t))dt \Big\} - K \Big)^+\Big] \qquad (4.7) \end{aligned}$$

where

$$\widetilde{B}(t) = \int_0^t u(s)ds + B(t) = \int_0^t \frac{\alpha(s) - \rho(s)}{\beta(s)} ds + B(t) \tag{4.8}$$

is a Brownian motion with respect to $Q$.

Using the known distribution of Brownian motion the expectation in (4.7) can be written

$$\begin{aligned} &p(F) = q(F) = \\ &\xi(T) \int_{\mathbf{R}} \Big( x_1 \exp\Big\{ y + \int_0^T (\rho(t) - \tfrac{1}{2}\beta^2(t))dt \Big\} - K^+ \Big) \frac{1}{\sqrt{2\pi}\,\Lambda} \exp\Big( - \frac{y^2}{2\Lambda^2} \Big) dy \qquad (4.9) \end{aligned}$$

where

$$\Lambda = E_Q\Big[\Big( \int_0^T \beta(t)d\widetilde{B}(t) \Big)^2\Big] = \int_0^T \beta^2(t)dt\,. \tag{4.10}$$

This is the (*generalized*) *Black & Scholes pricing formula* for European call options.

Note that – surprisingly – the coefficient $\alpha(t)$ does not appear in this formula!

In the (classical) special case when $\rho(t) = \rho$, $\alpha(t) = \alpha$ and $\beta(t) = \beta > 0$ are *constants*, we get the *classical Black & Scholes formula*

$$p(F) = q(F) = x_1\Phi(z) - e^{-\rho T} K\Phi(z - \beta\sqrt{T}\,) \ , \tag{4.11}$$

where

$$z = \frac{\ln(\frac{x_1}{K}) + (\rho + \frac{1}{2}\beta^2)T}{\beta\sqrt{T}} \tag{4.12}$$

and

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{y^2}{2}} dy$$

is the standard normal distribution function.

Finally, we turn to the question (2.5):

How do we find the initial value $z$ and the replicating portfolio $\theta$ for a given $T$-claim in the generalized Black & Scholes market?

To discuss this, let us investigate more closely by direct computation the relation between an admissible portfolio $\theta(t) = (\theta_0(t), \theta_1(t)) = (\xi(t), \eta(t))$ and the corresponding value process

$$V^\theta(t) = \theta(t) \cdot X(t) = \xi(t)A(t) + \eta(t)S(t) \tag{4.13}$$

Since $\theta$ is self-financing we have

$$dV^\theta(t) = \xi(t)dA(t) + \eta(t)dS(t) \tag{4.14}$$

From (4.13) we get

$$\xi(t) = \frac{V^\theta(t) - \eta(t)S(t)}{A(t)} \tag{4.15}$$

which substituted in (4.14) gives, using (4.1),

$$dV^\theta(t) = (V^\theta(t) - \eta(t)S(t))\rho(t)dt + \eta(t)dS(t)$$

Combining this with (4.2) we get

$$dV^\theta(t) = \rho(t)V^\theta(t)dt + \eta(t)S(t)[(\mu(t) - \rho(t))dt + \sigma(t)dB(t)] \tag{4.16}$$

This can be written

$$dV^\theta(t) - \rho(t)V^\theta(t)dt = \sigma(t)\eta(t)S(t)d\widetilde{B}(t) \ ,$$

where

$$\widetilde{B}(t) = \int_0^t \frac{\mu(s) - \rho(s)}{\sigma(s)} ds + B(t) \ ; \qquad 0 \le t \le T \tag{4.17}$$

Multiplying by the integrating factor $e^{-\int_0^t \rho(s)ds}$ we get

$$d\Big(e^{-\int_0^t \rho(s)ds}V^\theta(t)\Big) = e^{-\int_0^t \rho(s)ds}\sigma(t)\eta(t)S(t)d\widetilde{B}(t)$$

or

$$e^{-\int_0^t \rho(s)ds}V^\theta(t) = V^\theta(0) + \int_0^t e^{-\int_0^s \rho(r)dr}\sigma(s)\eta(s)S(s)d\widetilde{B}(s) \tag{4.18}$$

From this we see that if we want to replicate a given $T$-claim $F(\omega)$, this amounts to finding $\eta(s)$ such that

$$e^{-\int_0^T \rho(s)ds}F(\omega) = V^\theta(0) + \int_0^T e^{-\int_0^s \rho(r)dr}\sigma(s)\eta(s)S(s)d\widetilde{B}(s)\ . \tag{4.19}$$

If such an $\eta$ is found, then we let $\xi$ be given by (4.15) and this makes the portfolio $\theta = (\xi,\eta)$ self-financing. To prove that this $\theta$ will also be admissible, we must verify that $\{V^\theta(t)\}_{t\in[0,T]}$ is lower bounded.

To this end, let as before $Q$ be the measure on $\mathcal{F}_T$ defined by (4.6), (4.5). Then by taking the conditional expectation of (4.19) with respect to $Q$ and $\mathcal{F}_t$ we get

$$\begin{aligned} &E_Q\Big[e^{-\int_0^t \rho(s)ds}F(\omega)\mid\mathcal{F}_t\Big] \\ &= V^\theta(0) + \int_0^t e^{-\int_0^s \rho(r)dr}\sigma(s)\eta(s)S(s)d\widetilde{B}(s) = V^\theta(t)\ ; \qquad 0\le t\le T \end{aligned} \tag{4.20}$$

From this we conclude that $V^\theta(t)$ is lower bounded since $F$ is. Hence $\theta = (\xi,\eta)$ is admissible.

Next, note that if we take the $Q$-expectation of (4.19) we get

$$V^\theta(0) = e^{-\int_0^t \rho(r)dr}E_Q[F] \tag{4.21}$$

We proceed to show that there does indeed exist an $\eta$ satisfying (4.19), if $V^\theta(0)$ is given by (4.21). To this end define

$$G(\omega) = e^{-\int_0^T \rho(r)dr}F(\omega)$$

Assume for simplicity that

$$E_Q[G^2] < \infty \tag{4.22}$$

Then we will show that there exists a unique $\mathcal{F}_t$-adapted $w(s,\omega)$ such that

$$G(\omega) = E_Q[G] + \int_0^T w(s,\omega)d\widetilde{B}(s) \tag{4.23}$$

and

$$(4.24)\quad E\Big[\int_0^T w^2(s,\omega)ds\Big] < \infty \quad \text{and} \quad \Big\{\int_0^t w(s,\omega)d\widetilde{B}(s)\Big\}_{t\in[0,T]} \quad \text{is lower bounded.}$$

We mention some of the methods which can be used to achieve this:

**Method 1 The Ito representation theorem**

Since the two filtrations $\widetilde{\mathcal{F}}_t$ and $\mathcal{F}_t$ of $\widetilde{B}$ and $B$ coincide in this case (see the Remark following Theorem 1.21), we can apply the Ito representation theorem to the $\widetilde{\mathcal{F}}_T$-measurable $G(\omega) \in L^2(Q)$ and this gives the existence and uniqueness of a $w(s,\omega)$ satisfying (4.23) and (4.24). The disadvantage with this method is that it says nothing about how to find $w$ explicitly.

**Method 2 The PDE method**

In the *Markovian case*, i.e. when the payoff has the form

$$(4.25)\qquad G(\omega) = f(S(T,\omega))$$

for some (deterministic) function $f\colon \mathbf{R} \to \mathbf{R}$, then one tries to find a solution $V^\theta(t,\omega)$, $\eta(t,\omega)$ of (4.16) of the form

$$(4.26)\qquad V^\theta(t,\omega) = \Psi(t, S(t,\omega))\,, \qquad \eta(t,\omega) = \Big[\frac{\partial}{\partial x}\psi(t,x)\Big]_{x=S(t,\omega)}$$

for some function $\Psi(t,x)\colon \mathbf{R}^2 \to \mathbf{R}$.

This substitution leads, by the Ito formula, to a partial differential equation in $\Psi(t,x)$ with boundary values

$$\Psi(T,x) = f(x)\ .$$

This equation can be solved using the Feynman-Kac formula.

This method was originally used by Black & Scholes [BS] for the special case of the European call option

$$F(\omega) = f(S(T,\omega))$$

with

$$(4.27)\qquad f(x) := (x-K)^+ \qquad \text{(see (2.1))}.$$

It has the advantage of giving explicit formulas for both $V^\theta(0) = E_Q[G]$ and $\eta(t,\omega)$. The disadvantage is that it does not apply to non-Markovian payoffs, like the so-called *knock-out option*

$$(4.28)\qquad F(\omega) = \begin{cases} 0 & \text{if } \max\limits_{0\le t\le T} S(t,\omega) > K \\ 1 & \text{otherwise} \end{cases}$$

**Method 3 Backward stochastic differential equations**

Put

(4.29) $$Y(t) = \sigma(t)\eta(t)S(t)$$

Then (4.16) can be written

(4.30) $$dV^\theta(t) = \rho(t)V^\theta(t)dt + u(t)Y(t)dt + Y(t)dB(t)\ ; \qquad 0 \le t \le T$$

In addition we have the *terminal* condition

(4.31) $$V^\theta(T) = F(\omega) \quad \text{a.s.}$$

Equations (4.30)–(4.31) constitute what is called a *backward stochastic differential equation* (BSDE):

We seek *two* $\mathcal{F}_t$-adapted stochastic process $V^\theta(t), Y(t)$ satisfying (4.30) for $0 \le t \le T$ and such that the terminal value of $V^\theta$ is given by (4.31). The existence (and uniqueness) of a solution $V^\theta(t), Y(t)$ of (4.30)–(4.31) follows from the general theory of backward stochstic differential equations (see e.g. [PP] and [P]). The theory of BSDE has many applications. Unfortunately it is only in special cases that explicit solution formulas can be found.

**Method 4 The generalized Clark-Ocone formula**

This formula states that if $G \in \mathbf{D}_{1,2}$ and satisfies some additional conditions then

(4.32) $$G(\omega) = E_Q[G] + \int_0^T E_Q[D_tG \mid \mathcal{F}_t]d\widetilde{B}(t)\ ,$$

where $\widetilde{B}$ and $Q$ are as before (see (4.6) and (4.8)) and $D_tG$ is *the Malliavin derivative* of $G$ at $t$. See [KO].

The disadvantage here is that the space $\mathbf{D}_{1,2}$ (which we do not define here) does not contain all of $L^2(Q)$. In particular, the formula does not apply to discontinuous payoffs like (4.28). On the other hand, since $\mathbf{D}_{1,2}$ is dense in $L^2(Q)$ we can for a given $G \in L^2(Q)$ choose $G_n \in \mathbf{D}_{1,2}$ such that $G_n \to G$ in $L^2(Q)$. Then by the Ito isometry we get that, with $w$ as in (4.23),

$$E_Q[D_tG_n \mid \mathcal{F}_t] \to w(t,\omega) \qquad \text{in } L^2[\lambda \times Q],$$

where $\lambda$ is Lebesgue measure on $[0,T]$. So we may use $E[D_tG_n \mid \mathcal{F}_t]$ as an approximation of the $w$ we seek. In some cases this limiting procedure can be used to obtain an exact expression for $w$. The calculation by this method is carried out in the European call option case in [Ø1]. The result in the classical Black & Scholes market, with $\rho(t) = \rho$, $\alpha(t) = \alpha$

and $\beta(t) = \beta$ constants, is that the portfolio $\theta(t) = (\xi(t), \eta(t))$ needed to replicate the European call

$$F(\omega) = (S(T, \omega) - K)^+$$

is given by

$$\eta(t) = e^{\rho(t-T)} S^{-1}(t) E[Y_y(T-t) \mathcal{X}_{[K,\infty)}(Y_y(T-t))]_{y=S(t)} , \tag{4.33}$$

where

$$Y_y(s) = y \exp(\beta B(s) + (\rho - \tfrac{1}{2}\beta^2)s) ; \qquad 0 \leq s \leq T , \tag{4.34}$$

and $\xi(t)$ is determined by (4.14).

Note that since the distribution of $B(T)$ is known the expectation (4.33) can be expressed more explicitly as an integral with respect to the normal distribution.

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