Limits to Arbitrage when Market Participation Is Restricted

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November 28, 2003

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Abstract

There is an extensive literature claiming that it is often difficult to make use of arbitrage opportunities in financial markets. This paper provides a new reason why existing arbitrage opportunities might not be seized. We consider a world with short-lived securities, no short-selling constraints and no transaction costs. We show that to exploit all existing arbitrage opportunities, traders should pay attention to all financial markets simultaneously. It gives a general result stating that failure to do so will leave some arbitrage opportunies unexploited with probability one.

Key words: Arbitrage, Bounded rationality.

JEL codes: D52, G12.

1 Introduction

One of the fundamental concepts in finance is arbitrage, defined as the simultaneous purchase and sale of the same, or essentially similar, security in two different markets for advantageously different prices, see Sharpe and Alexander (1990). The efficient market hypothesis relies to a large extent on the assumption that, whenever present, arbitrage opportunities will be exploited quickly. The behavioral finance literature as in Shleifer (2000), p. 2, questions this hypothesis:

> The key forces by which markets are supposed to attain efficiency, such as arbitrage, are likely to be much weaker and more limited than the efficient markets theorists have supposed.

In reality arbitrage opportunities are limited by a number of factors like the existence of transactions costs, short-selling contraints, or mispricing of securities deepening in the short run.

This paper claims that even under close to ideal circumstances, i.e. the case where transactions costs are zero, short-selling constraints do not exist and securities are short-lived so deepening of mispricing is impossible, existing arbitrage opportunities might not be seized. We show that this is generally the case whenever traders restrict their attention to a subset of the securities traded at a certain point in time. At the heart of our argument is therefore the observation that attention is only available in limited amounts, following Radner and Rothschild (1975).

Van Zandt (1999) argues that individuals are bounded not so much by the total amount of information processing they can handle, as by the amount they can perform in a given amount of time. This leads to parallel or distributed processing, where information processing tasks are broken down into steps that are shared among the members of the organization and where each of these steps takes time.

Limits to the capability of information processing are the main reasons for traders to specialize to subsets of securities. In investment firms, for instance, analysts typically concentrate on the stocks within a particular industry sector. In Vayanos (2003), this feature is modeled by a processing constraint. Agents are assumed to analyze portfolio's of at most a fixed number of inputs, where an input can either be an asset examined directly, or a subordinate's portfolio.

In this paper we consider the finance version of a two-period general equilibrium model with restricted market participation. The model is a special case of the restricted market participation models of Siconolfi (1988) and Polemarchakis and Siconolfi (1997). As a consequence of the two-period time horizon, all traded assets are short-lived. This is the most favorable case for arbitrage, as it makes deepening of mispricing impossible. Investors can buy and sell assets in period 0 without being subject to short-selling constraints or transactions costs. They are however subject to information processing constraints. An investor is assumed to be unable to be active in the markets of all traded assets simultaneously.

Assets have payoffs in period 1, depending on the realization of the state of nature. Asset payoffs are real, i.e. denominated in terms of the consumption good. Investors consume in both periods. In this context, the usual definition of no-arbitrage is both the absence of a costless portfolio with non-negative returns in each future state of nature and strictly positive returns in at least one state, and of a portfolio yielding income in period 0 and with non-negative returns in each future state of nature.

Since investors restrict their attention to certain subsets of assets, they might not be able to make use of certain arbitrage opportunities. One might expect, however, that, under suitable assumptions, they are able to do so collectively. In particular, one might expect that this is the case as long as the subsets of assets to which investors pay attention overlap. This paper makes the point that this intuition is wrong. For almost all asset structures, as soon as each investor is limited in his trading opportunities to some extent, some arbitrage opportunities will be left unexploited, even at the collective level. Recall from Geanakoplos and Mas-Colell (1989), for example, that for every asset market model with real assets there is a corresponding model with nominal payoffs. Hence changes in real payoffs can be seen resulting from changes of commodity prices.

Section 2 outlines our model and derives the appropriate no-arbitrage conditions. Section 3 shows a first example that the no-arbitrage conditions in a restricted market participation model may differ from the usual no-arbitrage conditions. Section 4 derives the main result: this is typically the case, no matter how small the restriction in market participation.

2 Arbitrage

We consider the case that is most favorable to arbitrage. All traded assets are short-lived, which prohibits deepening of mispricing, transactions costs are absent, and short-sales are not restricted, apart from restrictions that prevent bankruptcy. In particular, we consider a model with two time periods, $t = 0, 1$, and one state of nature s out of S possible states of nature realizing at $t = 1$. There is a finite number of investors $i = 1, \ldots, I$ and a single good, called income, in each state.

At $t = 0$ investors allocate their money between consumption and investment in one of the available assets $j = 1, \ldots, J$. Throughout we restrict attention to the case $J \leq S$. The symbols $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{S}$ denote the sets $\{1, \ldots, I\}$, $\{1, \ldots, J\}$, and $\{1, \ldots, S\}$, respectively.

Assets pay off in period 1. The payoff of asset j in state s is given by A_s^j. Investor i has a utility function U^i and an initial income stream $\omega^i \in \mathbb{R}_+^{S+1}$. The set of possible income streams is given by $X^i = \mathbb{R}_+^{S+1}$.

Investor i has only access to a limited set of asset markets. Cognitive restrictions require him to restrict attention to the set $\mathcal{J}^i \subset \mathcal{J}$ of assets.

Let $q \in \mathbb{R}^J$ denote the asset prices and $\theta^i \in \mathbb{R}^J$ the net asset portfolio of agent i, i.e. negative components of θ^i denote sales of the corresponding assets and positive components denote purchases.

The optimization problem of investor i is given by

$$\max_{\theta^i \in \mathbb{R}^J,\ x^i \in \mathbb{R}_+^{S+1}} U^i(x^i)$$

subject to

$$\begin{array}{rcll} x^i - \omega^i & \leq & \begin{pmatrix} -q \\ A \end{pmatrix} \theta^i, & \\ \theta^i_j & = & 0, & j \in \mathcal{J} \setminus \mathcal{J}^i. \end{array}$$

Investor i has arbitrage opportunities if he can purchase a portfolio at no cost today, with non-negative payoffs at each state s and a strictly positive payoff in at least one state, or a portfolio yielding postive income in period 0 and non-negative payoffs at each future state. For investor i this leads to the following no-arbitrage condition, which is labelled NAC^i,

$$\nexists \theta^i \in \mathbb{R}^J \text{ such that } \theta^i_j = 0,\ j \in \mathcal{J} \setminus \mathcal{J}^i, \text{ and } \begin{pmatrix} -q \\ A \end{pmatrix} \theta^i > 0. \qquad (\text{NAC}^i)$$

It is well-known that NAC^i is satisfied if and only if $q \in Q^i$, where

$$Q^i = \left\{ q \in \mathbb{R}^J \mid \exists \pi \in \mathbb{R}^S_{++} \text{ such that for every } j \in \mathcal{J}^i,\ q_j = \textstyle\sum_{s \in \mathcal{S}} \pi_s A^j_s \right\}.$$

The following result follows immediately:

$$\mathcal{J}^{i^1} \subset \mathcal{J}^{i^2} \Rightarrow Q^{i^2} \subset Q^{i^1}.$$

Asset prices are said to satisfy the no-arbitrage condition NAC if the no-arbitrage condition is satisfied for all investors. So asset prices q satisfy NAC if and only if NAC^i is satisfied for every $i \in \mathcal{I}$. This is easily seen to be equivalent to the statement that $q \in Q = \cap_{i \in \mathcal{I}} Q^i$.

3 Networks of Agents

Another interesting no-arbitrage condition is the one which follows if some omniscient investor could oversee all the possibilities offered in the market. This leads to the market no-arbitrage condition NAC$^{\mathrm{m}}$,

$$\nexists \theta \in \mathbb{R}^J \text{ such that } \begin{pmatrix} -q \\ A \end{pmatrix} \theta > 0. \qquad (\mathrm{NAC}^{\mathrm{m}})$$

It is well-known that NAC$^{\mathrm{m}}$ is satisfied if and only if $q \in Q^{\mathrm{m}}$, where

$$Q^{\mathrm{m}} = \left\{ q \in \mathbb{R}^J \mid \exists \pi \in \mathbb{R}^S_{++} \text{ such that } q = \textstyle\sum_{s \in \mathcal{S}} \pi_s A_s \right\}.$$

In this section it is examined under what circumstances NAC and NAC$^{\mathrm{m}}$ are the same, i.e. under what conditions partly informed investors exploit all arbitrage possibilities present.

The following lemma is easily shown.

Proposition 3.1 *It holds that $Q^m \subset Q$.*

Proposition 3.1 states that NAC$^{\mathrm{m}}$ implies NAC. Of course, if some agent can trade in all markets then the concepts of NAC and NAC^m coincide.

Proposition 3.2 *If for some investor $i \in \mathcal{I}$ it holds that $\mathcal{J}^i = \mathcal{J}$, then $Q^m = Q$.*

Investor i in the proposition is omniscient, so the result follows trivially.

A first intuition would be that if the $\mathcal{J}^i$ overlap, then NAC implies NAC$^{\mathrm{m}}$. The market of asset j' is said to be related to the market of asset j'' if for some agent $i \in \mathcal{I}$ it holds that $j', j'' \in \mathcal{J}^i$. The market of asset j' is said to be indirectly related to the market of asset j'' if there is a sequence of markets $j^1, \ldots, j^n$ such that $j^1 = j'$ and $j^n = j''$ and j^k and j^{k+1} are directly related for all $k \in \{1, \ldots, n-1\}$.

Neither direct relatedness nor direct relatedness of all markets is sufficient for the sets Q and Q^{m} to coincide. In fact, consider the following example where it holds that all markets are directly related. Notice that one needs at least three assets for an interesting example.

Example 3.3 Consider three investors all having strictly monotonic utility functions. Suppose that

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

We will consider the case where investor i is assumed not to trade in asset i. So, $\mathcal{J}^1 = \{2, 3\}$, $\mathcal{J}^2 = \{1, 3\}$ and $\mathcal{J}^3 = \{1, 2\}$. Consider the asset price system $q = (5, 3, 5)$. We claim that

$q \notin Q^{\mathrm{m}}$. Indeed, $\theta = (-1,-3,-1)$ is an arbitrage portfolio. However q belongs to Q^{m} because for $\mathcal{J}^1 = \{2,3\}$, $\pi = (1/2, 1/6, 13/6)$, for $\mathcal{J}^2 = \{1,3\}$, $\pi = (1,2,1)$ and for $\mathcal{J}^3 = \{1,2\}$, $\pi = (13/6, 1/6, 1/2)$ are state price vectors demonstrating the absence of arbitrage opportunities.

The example is the strongest example possible in the sense that adding one market to one agent gives equivalence between NAC and NAC$^{\mathrm{m}}$ by Proposition 3.2.

4 Limits to Arbitrage

This section shows that Example 3.3 is not an exceptional case. It makes the striking observation that in finance economies with restricted market participation, forgone arbitrage opportunities are the rule rather than the exception. To make this statement more precise, we use the following notation. Let $\mathcal{A}$ denote the set of $(S \times J)$–matrices and

$$\mathcal{A}_+ = \left\{ A \in \mathcal{A} \mid \exists \theta \in \mathbb{R}^J \setminus \{0\},\ A\theta \in \mathbb{R}^S_+ \right\},$$

i.e. $\mathcal{A}_+$ is the set of asset return matrices for which there is a non-trivial asset portfolio giving non-negative returns in each state. We will restrict attention to the set of asset return matrices $\mathcal{A}_+$. Asset return matrices outside $\mathcal{A}_+$ are hardly interesting, as the next result shows that there are no limits on asset prices imposed by arbitrage in that case.

Proposition 4.1 *It holds that $A \in \mathcal{A} \setminus \mathcal{A}_+$ if and only if $Q^m = \mathbb{R}^J$.*
Proof: Let $A \in \mathcal{A}_+$ and choose $\theta \in \mathbb{R}^J \setminus \{0\}$ and $q \in \mathbb{R}^J$ so that $A\theta \in \mathbb{R}^S_+$ and $q\theta < 0$. Then there exists no $\pi \in \mathbb{R}^S_{++}$ with $q = \pi A$. Hence, $Q^{\mathrm{m}} \neq \mathbb{R}^J$.

To prove the converse inclusion, let $A \in \mathcal{A} \setminus \mathcal{A}_+$. Suppose there exists $q \in \mathbb{R}^J$ such that $\{\lambda q \mid \lambda > 0\} \cap \{\pi A \mid \pi \in \mathbb{R}^S_{++}\} = \emptyset$. By the separating hyperplane theorem, there exists $\theta \in \mathbb{R}^J \setminus \{0\}$ such that $\lambda q\theta \leq \pi A\theta$ for all $\lambda > 0$ and $\pi \in \mathbb{R}^S_{++}$. Taking the limit for $\lambda \downarrow 0$, we obtain $0 \leq \pi A\theta$ for all $\pi \in \mathbb{R}^S_{++}$. Taking the limit for $\pi_s \downarrow 0$ for all $s \in \{1, \ldots, S\} \setminus \{\hat{s}\}$ and $\pi_{\hat{s}} \to 1$, we get $0 \leq A_{\hat{s}}\theta$. Hence, $A\theta \in \mathbb{R}^S_+$, so $A \in \mathcal{A}_+$, a contradiction. Q.E.D.

For all $A \in \mathcal{A} \setminus \mathcal{A}_+$ it holds that $Q^{\mathrm{m}} = \mathbb{R}^J$. Therefore, irrespective of the participation structure, $Q^{\mathrm{m}} = \cap_{i \in \mathcal{I}} Q^i$.

The next result claims that in general arbitrage opportunities are left unexploited for asset return matrices in $\mathcal{A}_+$.

Theorem 4.2 *Suppose that for all investors $i \in \mathcal{I}$ it holds that $\mathcal{J}^i \neq \mathcal{J}$. Then there exists an open subset $\mathcal{O}$ of the set $\mathcal{A}_+$ with $\mathcal{A}_+ \setminus \mathcal{O}$ having Lebesgue measure zero such that $Q^{\mathrm{m}} \neq \cap_{i \in \mathcal{I}} Q^i$ for all $A \in \mathcal{O}$.*

Before proving Theorem 4.2, we introduce some extra notation. Let Θ_+ and Θ^i_+, $i \in \mathcal{I}$, be closed convex cones defined by

$$\Theta_+ = \left\{\theta \in \mathbb{R}^J \mid A\theta \in \mathbb{R}^S_+\right\},$$
$$\Theta^i_+ = \left\{\theta \in \Theta_+ \mid \theta_j = 0 \ \text{ if } j \in \mathcal{J}\backslash\mathcal{J}^i\right\}.$$

Let C be a closed convex cone. A half–line L emanating from the origin is called an extreme ray of C if $L \subseteq C$ and every closed line segment in C with a relative interior point in L has both endpoints in L. Let K be an arbitrary subset of $\mathbb{R}^J$. Then the convex cone generated by K is a subset of $\mathbb{R}^J$ containing zero and all those vectors which can be represented as a linear combination with positive weights of finitely many points in K. The convex cone generated by the empty set consists of the zero vector alone.

Let T denote the set of all those vectors $\theta \in \Theta_+$ with $\|\theta\| = 1$ such that the half–line emanating from the origin and passing through θ is an extreme ray of the cone Θ_+. Lemma 4.3 is an immediate implication of Corollary 18.5.2 of Rockafellar (1997).

Lemma 4.3 *Let $A \in \mathcal{A}_+$ have rank J. Then Θ_+ is the convex cone generated by T.*

Lemma 4.4 *Let $A \in \mathcal{A}_+$ have rank J. Then the following conditions are equivalent:*

(C1) $\cap_{i\in\mathcal{I}} Q^i \subseteq Q$,

(C2) $\Theta_+ \subseteq \sum_{i\in\mathcal{I}} \Theta^i_+$,

(C3) $T \subseteq \cup_{i\in\mathcal{I}} \Theta^i_+$.

Proof:
$(C1) \Rightarrow (C2)$ Consider $\overline{\theta} \in \Theta_+$. If $A\overline{\theta} = 0$, then $\overline{\theta} = 0$, and $\overline{\theta} \in \sum_{i\in\mathcal{I}} \Theta^i_+$. Suppose that $A\overline{\theta} \in \mathbb{R}^S_+\backslash\{0\}$. By condition $(C1)$, $0 < q\overline{\theta}$ for all $q \in \cap_{i\in\mathcal{I}} Q^i$. Therefore, $0 \leq q\overline{\theta}$ for all $q \in \mathrm{cl}(\cap_{i\in\mathcal{I}} Q^i)$. Since the non–empty open set Q is contained in each of the sets Q^i,

$$\mathrm{cl}\left(\cap_{i\in\mathcal{I}} Q^i\right) = \cap_{i\in\mathcal{I}} \mathrm{cl}\,(Q^i),$$

see Rockafellar (1997), Theorem 6.5. Observe that

$$\left\{q \in \mathbb{R}^J \mid \exists \pi \in \mathbb{R}^S_+ \text{ such that for every } j \in \mathcal{J}^i,\ q_j = \textstyle\sum_{s\in\mathcal{S}} \pi_s A^j_s\right\} \subseteq \mathrm{cl}\,(Q^i).$$

In fact, equality holds as well, but the inclusion $\subseteq$ is sufficient for our purposes. Thus, the inequality $0 \leq q\overline{\theta}$ holds for all $(q, \pi) \in \mathbb{R}^J \times \mathbb{R}^{SI}$ satisfying

$$\begin{array}{rcll} q_j &=& \sum_{s\in\mathcal{S}} \pi^i_s A^j_s, & \text{for all } i \in \mathcal{I},\ j \in \mathcal{J}^i, \\ 0 &\leq& \pi^i_s, & \text{for all } i \in \mathcal{I},\ s \in \mathcal{S}. \end{array}$$

Farkas' Lemma, see Rockafellar (1997), Corollary 22.3.1, implies that for all $i \in \mathcal{I}$, $j \in \mathcal{J}^i$, and $s \in \mathcal{S}$ there exist numbers θ^i_j and $\mu^i_s \geq 0$ such that

$$\sum_{j \in \mathcal{J}^i} A^j_s \theta^i_j - \mu^i_s \quad = 0, \quad \text{for all } i \in \mathcal{I},\ s \in \mathcal{S}, \tag{1}$$

$$\sum_{\{i \in \mathcal{I} | j \in \mathcal{J}^i\}} \theta^i_j \quad = \overline{\theta}_j, \quad \text{for all } j \in \mathcal{J}. \tag{2}$$

Define θ^i_j to be zero for all $i \in \mathcal{I}$ and $j \in \mathcal{J} \backslash \mathcal{J}^i$, and let $\theta^i = (\theta^i_1, \ldots, \theta^i_J)$. Then $\theta^i \in \Theta^i_+$ and $\sum_{i \in \mathcal{I}} \theta^i = \overline{\theta}$.

$(C2) \Rightarrow (C1)$ Let $q \in \cap_{i \in \mathcal{I}} Q^i$, and let $\theta \in \mathbb{R}^J$ be such that $A\theta \in \mathbb{R}^S_+ \backslash \{0\}$. We must show that $0 < q\theta$. Indeed, using C2, for $i \in \mathcal{I}$ we can choose $\theta^i \in \Theta^i_+$ such that $0 \leq q\theta^i$ and $\sum_{i \in \mathcal{I}} \theta^i = \theta$. Moreover, there is some $i_0 \in \mathcal{I}$ with $A\theta^{i_0} \in \mathbb{R}^S_+ \backslash \{0\}$, so $0 < q\theta^{i_0}$, and therefore $0 < q\theta$.

$(C2) \Rightarrow (C3)$ Let $\theta \in T$. As θ is an element of Θ_+, condition (C2) implies that there are $\theta^i \in \Theta^i_+$ such that $\sum_{i \in \mathcal{I}} \theta^i = \theta$. As θ is a non–zero vector, there is $i_0 \in \mathcal{I}$ such that θ^{i_0} is a non–zero vector. Observe that the line segment with endpoints $2\theta^{i_0}$ and $2 \sum_{i \in \mathcal{I} \backslash \{i_0\}} \theta^i$ contains the vector θ in its relative interior. Therefore, there exists a positive number t such that $2\theta^{i_0} = t\theta$. This implies that θ is an element of $\Theta^{i_0}_+$.

$(C3) \Rightarrow (C2)$ By Lemma 4.3, Θ_+ is the convex cone generated by T. By Condition $(C3)$, it is contained in the convex cone generated by $\cup_{i \in \mathcal{I}} \Theta^i_+$. Clearly, the latter is equal to $\sum_{i \in \mathcal{I}} \Theta^i_+$. Q.E.D.

For each $\theta \in \Theta_+$ with $\|\theta\| = 1$, let $\mathcal{S}(\theta) = \{s \in \mathcal{S} \mid A_s \theta = 0\}$. Denote by $\operatorname{codim}(\theta)$ the codimension of the linear subspace of $\mathbb{R}^J$ spanned by the vectors A_s, $s \in \mathcal{S}(\theta)$. Observe that $\operatorname{codim}(\theta) \geq 1$. Moreover, the codimension of the linear subspace spanned by the vectors A_s, $s \in \mathcal{S}(\theta)$ together with vector θ equals $\operatorname{codim}(\theta) - 1$.

Lemma 4.5 *Let $A \in \mathcal{A}$ with rank J and $\theta \in \Theta_+$ with $\|\theta\| = 1$ be given. Then $\theta \in T$ if and only if* $\operatorname{codim}(\theta) = 1$.

Proof: Let $A \in \mathcal{A}$ with rank J and $\theta \in \Theta_+$ with $\|\theta\| = 1$ be given. Let L denote a half–line emanating from the origin and passing through the point θ.

Suppose that $\operatorname{codim}(\theta) > 1$. Then the codimension of the linear space spanned by the vectors A_s, $s \in \mathcal{S}(\theta)$, together with vector θ is non–zero. Hence, there exists a vector $\xi \in \mathbb{R}^J \backslash \{0\}$ such that $A_s \xi = 0$ for all $s \in \mathcal{S}(\theta)$ and $\theta\xi = 0$. As $A_s\theta > 0$ for all $s \in \mathcal{S} \backslash \mathcal{S}(\theta)$, there is an $\varepsilon > 0$ such that $A_s(\theta + t\xi) > 0$ for all $t \in [-\varepsilon, \varepsilon]$ and $s \in \mathcal{S} \backslash \mathcal{S}(\theta)$. Thus, the closed line segment with endpoints $(\theta - \varepsilon\xi)$ and $(\theta + \varepsilon\xi)$ lies entirely in Θ_+ and contains vector θ in its relative interior. However, neither of its endpoints belongs to L. Therefore, L is not an extreme ray of Θ_+, and θ is not an element of the set T.

Suppose that codim $(\theta) = 1$. Let θ' and θ'' be two points in Θ_+ such that $\lambda\theta' + (1-\lambda)\theta'' = t\theta$ for some $\lambda \in (0,1)$ and $t \geq 0$. We must show that θ' and θ'' both belong to L. Indeed, $A_s\theta' \geq 0$ and $A_s\theta'' \geq 0$ for all $s \in \mathcal{S}$. If $s \in \mathcal{S}(\theta)$, then $A_s\theta = 0$, and therefore $A_s\theta' = A_s\theta'' = 0$. Thus, all three vectors θ, θ', and θ'' belong to a linear subspace orthogonal to the span of the vectors A_s, $s \in \mathcal{S}(\theta)$. As the dimension of this linear subspace is equal to 1, there are real numbers t' and t'' such that $\theta' = t'\theta$ and $\theta'' = t''\theta$. If the number t' were negative, θ' would be a non–zero vector such that $0 \leq A\theta' = t'A\theta \leq 0$. This would contradict the choice of A in the set of matrices $\mathcal{A}_+$ with rank J. Therefore, it follows that $t' \geq 0$, and so $\theta' \in L$. It follows similarly that $\theta'' \in L$. Q.E.D.

The following example illustrates the set T.

Example 4.6 Suppose that $I = 3$, $S = 4$, $J = 3$, $\mathcal{J}^i = \mathcal{J}\backslash\{i\}$ for $i \in \mathcal{I}$, and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}.$$

Observe that A is an element of the set $\mathcal{A}_+$ with rank J. Moreover, the matrix A is in general position. Each 3×3 submatrix of A is non–singular. The set T consists of the four elements reported in the Table 1.

Table 1: The elements of the set T

	Element 1	Element 2	Element 3	Element 4
$j = 1$	0	1	-1	0
$j = 2$	1	0	2	-1
$j = 3$	-1	-1	0	2

Elements 1 and 4 of the set T belong to Θ^1_+, element 2 belongs to Θ^2_+, and element 3 belongs to Θ^3_+. By Lemma 4.4 the sets Q and $\cap_{i \in \mathcal{I}} Q^i$ coincide.

Corollary 4.7 *Let $A \in \mathcal{A}$ with rank J and $\theta \in T$ be given. Then the set $\mathcal{S}(\theta)$ consists of at least $J - 1$ distinct elements.*

Lemma 4.8 *Suppose that $\mathcal{J}^i \neq \mathcal{J}$ for all $i \in \mathcal{I}$. Then there exists an open subset $\mathcal{A}'$ of $\mathcal{A}$ with $\mathcal{A}\backslash\mathcal{A}'$ having Lebesgue measure zero such that $T \cap \Theta^i_+ = \emptyset$ for all $i \in \mathcal{I}$ and for all $A \in \mathcal{A}'$.*

Proof: For every $j \in \mathcal{J}$ and for every subset M of $\mathcal{S}$ with cardinality $J-1$, define the function $F_{jM} : \mathcal{A} \times \mathbb{R}^J \to \mathbb{R}^{J+1}$ as follows,

$$F_{jM}(A, \theta) = \begin{bmatrix} A_s\theta, \ s \in M \\ \theta \cdot \theta - 1 \\ \theta_j \end{bmatrix}.$$

Define the sets $\mathcal{A}_{jM}$ as

$$\mathcal{A}_{jM} = \left\{ A \in \mathcal{A} \mid \text{there is no } \theta \in \mathbb{R}^J \text{ such that } F_{jM}(A, \theta) = 0 \right\}.$$

To see that $\mathcal{A}_{jM}$ is open, let $A_{(n)}$ be the sequence of matrices in $\mathcal{A} \backslash \mathcal{A}_{jM}$ converging to some $A \in \mathcal{A}$. Then there exists a sequence $\theta_{(n)}$ in $\mathbb{R}^J$ such that $F_{jM}(A_{(n)}, \theta_{(n)}) = 0$ for all n. Since the sequence $\theta_{(n)}$ is bounded, it has a convergent subsequence converging to some $\theta \in \mathbb{R}^J$. Hence, $F_{jM}(A, \theta) = 0$, and the matrix A belongs to the complement of the set $\mathcal{A}_{jM}$.

The partial derivatives of the function F_{jM} with respect to θ and A_s, $s \in M$, are represented in Table 2. For simplicity we take M equal to $\{1, \ldots, J-1\}$. It is easy to see that for all $(A, \theta) \in F_{jM}^{-1}(0)$ the matrix of the partial derivatives has full row rank. That is, F_{jM} is transversal to zero. The Transversality Theorem implies that the complement of the set $\mathcal{A}_{jM}$ has Lebesgue measure zero.

Table 2: Partial derivatives of the function F_{jM}, $M = \{1, \ldots, J-1\}$.

	θ	A_1	A_2	$\ldots$	A_{J-1}
$A_1\theta$	A_1	θ	0	$\ldots$	0
$A_2\theta$	A_2	0	θ	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$A_{J-1}\theta$	A_{J-1}	0	0	$\ldots$	θ
$\theta \cdot \theta - 1$	2θ	0	0	$\ldots$	0
θ_j	e	0	0	$\ldots$	0

The symbol e is a J–dimensional row–vector such that $\mathrm{e}_l = 0$ for all $l \in \mathcal{J} \backslash \{j\}$ and $\mathrm{e}_j = 1$.

Finally, define $\mathcal{A}'$ as the set of matrices with rank J in the intersection of all sets $\mathcal{A}_{jM}$. Then $\mathcal{A}'$ is open and its complement has Lebesgue measure zero.

Let $A \in \mathcal{A}'$ and $\theta \in T$. Suppose that $\theta \in \Theta_+^i$ for some $i \in \mathcal{I}$. Then $\theta_j = 0$ for every $j \in \mathcal{J} \backslash \mathcal{J}^i$. Corollary 4.7 implies that there is a subset M of the set $\mathcal{S}$ with cardinality $J-1$ such that $A_s\theta = 0$ for all $s \in M$. Therefore, $F_{jM}(A, \theta) = 0$ for every $j \in \mathcal{J} \backslash \mathcal{J}^i$, a contradiction to $A \in \mathcal{A}'$. Thus, we have proved that $T \cap \Theta_+^i = \emptyset$ for all $i \in \mathcal{I}$ and $A \in \mathcal{A}'$.

Q.E.D.

Proof of Theorem 4.2: Define the set $\mathcal{O}$ as the matrices with rank J in the intersection of the sets $\mathcal{A}_+$ and $\mathcal{A}'$. As both the set of matrices in $\mathcal{A}$ with rank J and the set $\mathcal{A}'$ is open in $\mathcal{A}$, $\mathcal{O}$ is open in $\mathcal{A}_+$. Since both the set of matrices in $\mathcal{A}$ with rank J and the set $\mathcal{A}'$ have full Lebesgue measure, the set $\mathcal{A}_+ \backslash \mathcal{O}$ has Lebesgue measure zero. For all $A \in \mathcal{O}$ the set T is non–empty, whereas its intersection with the collection of cones Θ^i_+ is empty. By Lemma 4.4, $\cap_{i \in \mathcal{I}} Q^i \neq Q$. Q.E.D.

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