Massey operations and the Poincaré series of certain local rings

by

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Introduction. Throughout this paper R denotes a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m}$. For a finitely generated R-module M we let $P_R(M)$ be the power series

$$P_R(M) = \sum_{p=0}^{\infty} \dim_k \operatorname{Tor}_p^R(k,M) Z^p$$

The Poincaré series of R is the power series $P_R = P_R(k)$.

The conjecture due to Kaplansky and Serre that P_R be a rational function is still far from being solved, although the rationality of P_R has been established for several classes of rings: For complete intersections by Zariski and Tate [7], for rings of the form $R = k[X_1,\ldots,X_n]/(X_1,\ldots,X_n)^r$ by Golod [2], for rings of imbedding dimension equal to 2 by Scheja [5], for rings with "two relations" by Shamash [6], and lately for Gorenstein rings of imbedding dimension 3 by Wiebe [8].

The main results in this paper are the following:

(i) If R is a local complete intersection with socle $0 : \mathfrak{m}$, and we put $\bar{R} = R/0 : \mathfrak{m}$, then $P_{\bar{R}}$ is a rational function.

(ii) If $R(M)$ is the extension of R by a finitely generated R-module M (see below for the definition of $R(M)$) then

$$P_{R(M)} = P_R \circ (1 - ZP_R(M))^{-1}$$

Let Φ_R be the power series

$$\sum_{p=0}^{\infty} \dim_k \mathrm{Ext}_R{}^p(k,R)Z^p$$

We find a relationship between the rationality problems of P_R and Φ_R for rings R of dimension zero. Cf. Wiebe [8].

The method of proof is based on the use of Massey operations on differential graded algebras, a technique exploited by Golod [2] and Shamash [6] in computing P_R for certain rings R.

Notations and definitions.

The term "R-algebra" will be used in the sense of Tate [7]. By an augmented R-algebra F we will mean an R-algebra F with a surjective augmentation map $F \longrightarrow R/\mathfrak{m}$. $\tilde{H}(F)$ will denote the kernel of the induced map $H(F) \longrightarrow R/\mathfrak{m}$. deg x will denote the degree of a homogeneous element x.

Let M be a finitely generated R-module. We shall let R(M) denote the algebra over R whose underlying R-module is the direct sum $R \oplus M$ and whose ring structure is given by $(r,m)(r',m') = (rr', rm'+r'm)$. Note that R is a local noetherian ring; it will be referred to as the extension of R by M.

1. Massey operations on R-algebras.

Definition. Let F be an augmented R-algebra and let n be a positive integer. Let I be a set consisting of n successive integers and consider an indexed set

$$M = \{ \gamma_{i,j} \}_{\substack{i,j \in I \\ i \leq j}}$$

of homogeneous elements (of non-negative degree) in the augmentation ideal of F. We will call M an n-ary trivialized Massey operation on F provided that

$$d\gamma_{i,j} = \sum_{k=i}^{j-1} (-1)^{[i,k]} \gamma_{i,k}\gamma_{k+1,j} \quad \text{for } i < j$$

$$d\gamma_{i,j} = 0 \quad \text{for } i = j$$

where $[i,k] = \sum_{t=i}^{k} (1+\deg \gamma_{t,t})$.

Observe that if $\{\gamma_{i,j}\}_{i_0 \leq i \leq j \leq j_0}$ is a $(j_0 - i_0 + 1)$-ary trivialized Massey operation and if $i_0 \leq i_1 \leq j_1 \leq j_0$, then $\{\gamma_{i,j}\}_{i_1 \leq i \leq j \leq j_1}$ is a $(j_1 - i_1 + 1)$-ary trivialized Massey operation.

We will say that F has a trivial Massey operation γ if there exists a set S of homogeneous cycles in F, representing a minimal set of generators for $\tilde{H}(F)$, and γ is a function with values in F, defined on the set of finite sequences of elements in S (with repetitions), such that

(i) $\gamma(z) = z$ for all $z \in S$

and

(ii) for any sequence $z_1, \ldots, z_n$, $\{\gamma_{i,j}\}_{1 \leq i \leq j \leq n}$ is a trivialized Massey operation, $\gamma_{i,j}$ being defined by $\gamma_{i,j} = \gamma(z_i, \ldots, z_j)$.

S will be called the set of cycles belonging to γ.

The following proposition is a slight generalization of a result essencially due to Golod [2]. See also Shamash [6].

Proposition 1. Let F be an augmented R-algebra with a trivial Massey operation γ. Assume that F is R-free as a module, and let N be a graded R-module whose homogeneous components N_p are free R-modules of rank equal to $\dim \tilde{H}_{p-1}(F) \otimes_R k$, for all p.

Let T be the tensor algebra generated over R by N. Then the differential on F may be extended to a differential on the graded R-module $X = F \otimes_R T$, turning X into an R-free resolution of $R/\mathfrak{m}$.

Moreover if $dF \subset \mathfrak{m}F$ and $\operatorname{im} \gamma \subset \mathfrak{m}F$ then X is a minimal resolution of $R/\mathfrak{m}$.

Proof. Let S be the set of cycles belonging to γ. Choose a homogeneous basis U for N and a bijective map $U \longrightarrow S$ of degree -1. The image of an element $u_i \in U$ will be denoted by z_i.

Every element of X is uniquely expressible as a sum of tensors $f \otimes u_1 \otimes \dots \otimes u_n$ where $n \geq 0$, f denotes a homogeneous element of F and $u_i \in U$ for $1 \leq i \leq n$, (for $n = 0$ the symbol combination $f \otimes u_1 \otimes \dots \otimes u_n$ shall denote the element $f \otimes 1$). Now define the differential on these selected generators inductively by

$$d(f \otimes 1) = (df) \otimes 1$$

and for $n \geq 1$

(1) $$d(f \otimes u_1 \otimes \dots \otimes u_n) = d(f \otimes u_1 \otimes \dots \otimes u_{n-1}) \otimes u_n + (-1)^{\deg f} f \cdot \gamma(z_1, \dots, z_n)$$

d can now be extended uniquely to all of X by linearity. Using induction on n one easily verifies the following formula

(2) $$d(f \otimes u_1 \otimes \dots \otimes u_n) = (df) \otimes u_1 \otimes \dots \otimes u_n + \sum_{t=1}^{n} \left[(-1)^{\deg f} f \cdot \gamma(z_1, \dots, z_t) \otimes u_{t+1} \otimes \dots \otimes u_n \right]$$

It follows from the definition of γ that

(3) $$d\gamma(z_1, \dots, z_n) = \sum_{k=1}^{n-1} (-1)^{[1,k]} \gamma(z_1, \dots, z_k) \gamma(z_{k+1}, \dots, z_n)$$

where $[1,k] = \sum_{t=1}^{k} (1 + \deg z_t)$. Using (2) and (3) it is a matter of straightforward computation to show that $d^2 = 0$. We omit the details.

We furnish X with the augmentation map $X \to k$ induced by the augmentation map $F \to k$. We shall indicate a proof of the fact that X is acyclic.

Let x be a homogeneous cycle in the augmentation ideal of X. There exist unique homogeneous elements $f_{u_1, \dots, u_n} \in F$ such that

(4) $$x = \sum_{\substack{u_1, \dots, u_n \\ n \geq 0}} f_{u_1, \dots, u_n} \otimes u_1 \otimes \dots \otimes u_n$$

where $f_{u_1, \dots, u_n} = 0$ for all but a finite number of indices. The integer $\sup\{n \mid f_{u_1, \dots, u_n} \neq 0\}$ will be called the wight of x and will be denoted by $w(x)$.

For $w(x) = 0$ we obviously have $x \in B(X)$. Now assume that $w(x) = w \geqslant 1$. Let the elements $f_{u_1,\dots,u_w}$ in (4) which are coeffisients of terms of weight equal to w be denoted simply by $f_1,\dots,f_k$. In differentiating (4) using the formula (2) and looking at the terms of wight w and $w-1$, one can see that $f_1,\dots,f_k$ are cycles in the augmentation ideal of F. Here one has to use the fact that the "selected" cycles z are linearly independent modulo $B(F)$. Hence to each $f \in \{f_1,\dots,f_k\}$ there exist elements $r_1,\dots,r_m$ in R and cycles $z_1,\dots,z_m$ in S and an element g in F such that

$$f = dg + \sum_{i=1}^{m} r_i z_i \tag{5}$$

Let $f \otimes u_1 \otimes \dots \otimes u_w$ be one of the terms in (4) of weight equal to w. Using (5) we obtain

$$d(g \otimes u_1 \otimes \dots \otimes u_w + 1 \otimes \sum_{i=1}^{m} r_i z_i \otimes u_1 \otimes \dots \otimes u_w)$$
$$= f \otimes u_1 \otimes \dots \otimes u_w + (\text{terms of weight} < w).$$

It follows that there exist elements y and x' in X such that

$$x = dy + x'$$

where $w(x') \leqslant w(x)-1$. By induction on $w(x)$ it now follows that $x \in B(X)$.

The last statement in the proposition is trivial in view of (2). ∎

The following lemma is easily verified and we omit the proof.

Lemma. Let X be an R-algebra. Let $n \geqslant 2$ and let $\{\gamma_{i,j}\}_{1 \leq i \leq j \leq n}$ be a trivialized n-ary Massey opperation on X. Suppose that there exists a $y \in X$ such that $dy = \gamma_{1,1}$. Put $\gamma'_{2,j} = \gamma_{1,j} + (-1)^{\deg \gamma_{1,1}} \cdot y\gamma_{2,j}$

and $\gamma'_{i,j} = \gamma_{i,j}$ for $i > 2$

Then $\{\gamma'_{i,j}\}_{2 \leq i \leq j \leq n}$ is a trivialized (n-1)-ary Massey operation on X. ∎

Proposition 2. Let X be a minimal R-algebra resolution of $R/\mathfrak{m}$ and let F be a sub-R-algebra of X such that F, as an R-module, is a direct summand of X. Let $\{\gamma_{i,j}\}_{1 \leq i \leq j \leq n}$ be a trivialized n-ary Massey operation on F with $\gamma_{i,i} \in \mathfrak{m}F$ for all i. Then $\gamma_{i,j} \in \mathfrak{m}F$ for all $i \leq j$.

Proof. Since $\{\gamma_{i,j}\}_{1 \leq i \leq j \leq n}$ is also a trivialized Massey operation on X and moreover $\mathfrak{m}X \cap F = \mathfrak{m}F$, it is no loss of generality assuming that $F = X$.

We will prove the proposition by induction on n. For $n = 1$ it is trivial. Let $t \geqslant 2$. Assume that it has been proved for $n < t$. Now let $n = t$. By the induction hypothesis and the observation made in the definition of Massey operations it suffices to show that $\gamma_{1,n} \in \mathfrak{m}X$.

Since X is acyclic and $\gamma_{1,1}$ is a cycle in $\mathfrak{m}X$ there exists a $y \in X$ such that $dy = \gamma_{1,1}$. Using the lemma we can construct an (n-1)-ary Massey operation $\{\gamma'_{i,j}\}_{2 \leq i \leq j \leq n}$ on X such that

$$\gamma_{1,n} = \gamma'_{2,n} - (-1)^{\deg \gamma_{1,1}} y\gamma_{2,n}$$

and such that $\gamma'_{2,2}$ is a cycle of positive degree. The minimality of the resolution X gives $\gamma_{2,2} \in \mathfrak{m}X$. Moreover we have $\gamma'_{i,i} = \gamma_{i,i} \in \mathfrak{m}X$ for $n \geq i > 2$. Hence the induction hypothesis gives $\gamma'_{2,n} \in \mathfrak{m}X$ and $\gamma_{2,n} \in \mathfrak{m}X$. Hence $\gamma_{1,n} \in \mathfrak{m}X$. ∎

2. The Poincaré series of local complete intersections reduced modulo the socle.

Theorem 1. Let R be a local complete intersection with maximal ideal $\mathfrak{m}$. Let $n = \dim \mathfrak{m}/\mathfrak{m}^2$ and put $\bar{R} = R/0 : \mathfrak{m}$. Then R is either a local complete intersection, or else the Poincaré series of $\bar{R}$ is given by

$$P_{\bar{R}} = ((1-Z)^n - Z^2)^{-1}$$

Proof. Since complete intersections are Cohen-Macaulay rings we will have $0 : \mathfrak{m} \neq 0$ if and only if $\dim R = 0$. Hence we may assume, without loss of generality, that $\dim R = 0$. By the Cohen structure theory there exists a regular local ring $\tilde{R}, \tilde{\mathfrak{m}}$ and a surjective ringhomomorphism $\varphi : \tilde{R} \to R$ such that $\ker \varphi$ is generated by a maximal R-sequence $v_1, \ldots, v_n$ contained in $\tilde{\mathfrak{m}}^2$. Let $t_1, \ldots, t_n$ be a minimal set of generators for $\tilde{\mathfrak{m}}$. Then there exist elements r_{ij} in $\tilde{\mathfrak{m}}$ such that

$$v_i = \sum_{j=1}^{n} r_{ij} t_j, \quad i = 1, \ldots, n$$

Let $\tilde{u}$ be the determinant of the matrix (r_{ij}) and observe that $u \in \tilde{\mathfrak{m}}^n$. Put $u = \varphi(u)$. Proposition 1 in [4] shows that $0 \neq u \in 0 : \mathfrak{m}$. On the other hand $0 : \mathfrak{m}$ is a simple submodule of R since R is a zerodimensional Gorenstein ring, cf. [1]. Hence u generates $0 : \mathfrak{m}$. Thus letting $\mathfrak{a} = (v_1, \ldots, v_n, \tilde{u})$ we can conclude that $\bar{R} \approx R/\mathfrak{a}$.

Let F be the minimal R-algebra resolution of $R/\mathfrak{m}$ obtained from the R-algebra R by the adjunction of n variables of degree 1 and n variables of degree 2. Cf. Tate [7]. Let $\bar{F}$ be the $\bar{R}$-algebra F/uF.

If $\mathfrak{a}$ can be generated by n elements then these elements form an $\tilde{R}$-sequence, in which case $\bar{R}$ is a complete intersection. Let us assume that this is not the case, i.e. $\mathfrak{a}$ is minimally generated by $\{v_1, \ldots, v_n, \tilde{u}\}$ and $n \geq 2$. In this case we have $\mathfrak{a} \subset \tilde{\mathfrak{m}}^2$ and $\bar{F}$ can be extended to a minimal $\bar{R}$-algebra resolution of $\bar{R}/\bar{\mathfrak{m}}$ by adjoining one variable of degree 2 (corresponding to the new relation u) and variables of degree ≥ 3. Cf. [3]. The existence of such an extension enables us to apply proposition 2 later.

We will now show that $\bar{F}$ has a trivial Massey operation. First choose a set S of cycles representing a basis for $\tilde{H}(\bar{F})$. Let $\psi : F \longrightarrow \bar{F}$ be the canonical map. To each cycle z in S, select an element Z in $\psi^{-1}(z)$. We obviously have

$$dZ \in uF \tag{1}$$

We shall define inductively a function Γ assigning to each finite sequence of "selected" elements $Z_1,\ldots,Z_m$ in $\psi^{-1}(S)$ an element $\Gamma(Z_1,\ldots,Z_m)$ in F satisfying

(i) $\Gamma(Z) = Z$

(ii) $d\Gamma(Z_1,\ldots,Z_m) = \sum_{k=1}^{m-1}(-1)^{[1,k]}\Gamma(Z_1,\ldots,Z_k)\Gamma(Z_{k+1},\ldots,Z_m)$

where $[1,k] = \sum_{t=1}^{k}(1+\deg Z_t)$.

Let $\psi(\Gamma(Z_1,\ldots,Z_m))$ be denoted by $\gamma(z_1,\ldots,z_m)$. Observe that if $\Gamma(Z_1,\ldots,Z_m)$ has been defined then $\gamma(z_1,\ldots,z_m)$ is one of the components of a trivialized n-ary Massey operation, since ψ is an R-algebra homomorphism. It follows from proposition 2 that $\gamma(z_1,\ldots,z_m)\in \bar{\mathfrak{m}}\bar{F}$ hence

(2) $$\Gamma(Z_1,\ldots,Z_m)\in \mathfrak{m}F$$

Let $m \geqslant 2$ and suppose that Γ has been defined on sequences of length $< m$. Now let $Z_1,\ldots,Z_m$ be an arbitrary sequence of length m.

Put $Y = \sum_{k=1}^{m-1}(-1)^{[1,k]}\Gamma(Z_1,\ldots,Z_k)\Gamma(Z_{k+1},\ldots,Z_m)$

Using the induction hypotesis and relations of type (i) and (ii) one easily shows that

$$dY = (-1)^{[1,1]}d(Z_1)\Gamma(Z_2,\ldots,Z_m)-\Gamma(Z_1,\ldots,Z_{m-1})d(Z_m)$$

It follows from (1) and (2) that

$$dY \in (uF)(\mathfrak{m}F) = 0$$

F is acyclic. We choose an element in $d^{-1}(Y)$ and denote it by $\Gamma(Z_1,\ldots,Z_m)$. The construction is now complete, and the function γ sending the sequence $z_1,\ldots,z_m$ to $\gamma(z_1,\ldots,z_m) = \psi(\Gamma(Z_1,\ldots,Z_m))$ is obviously a trivial Massey operation on $\bar{F}$. In view of proposition 1 we can find a minimal resolution X of the $\bar{R}$-module $\bar{R}/\bar{m}$ of the form

$$X = \bar{F} \underset{\bar{R}}{\otimes} T$$

where T is the tensoralgebra generated by a free graded R-module N with

$$\operatorname{rank} N_p = \dim \tilde{H}_{p-1}(\bar{F}) \quad \text{for all } p \quad .$$

Hence letting $\chi(\ )$ denote the Poincaré series of graded modules, i.e. the formal power series

$$\chi(\ ) = \sum_{p=0}^{\infty} \operatorname{rank}((\ )_p) Z^p$$

we have

$$P_{\bar{R}} = \chi(X) = \chi(\bar{F}) \cdot \chi(T) \tag{3}$$

We have $\chi(\bar{F}) = (1-Z)^{-n}$ cf. [7].

Moreover

$$\chi(T) = (1-\chi(N))^{-1} \tag{4}$$

From the exact homology sequence associated to the exact sequence of complexes

$$0 \rightarrow F/mF \xrightarrow{u\circ} F \rightarrow \bar{F} \rightarrow 0$$

one obtains isomorphisms

$$H_q(F) \approx H_{q-1}(F/mF) \quad \text{for} \quad q \geq 1$$

It follows that

(5) $$\chi(N) = Z^2\chi(F/\mathfrak{m}F) = Z^2(1-Z)^{-n}$$

The desired formula for P_R now follows from (3), (4) and (5). ∎

<u>Remark.</u> It might be possible to use similar methods to compute P_R for the following rings R :

Let $\tilde{R}$ be a regular ring of dimension n. Let $v_1,\ldots,v_n$ be a maximal $\tilde{R}$-sequence and let $u_1,\ldots,u_n$ be elements in R such that $v_i = \sum_{j=1}^{n} r_{ij}u_j$. Put $\tilde{u} = \det(r_{ij})$, $\mathfrak{a} = (v_1,\ldots,v_n,u)$ and $R = \tilde{R}/\mathfrak{a}$.

A homological investigation of ideals of the type $\mathfrak{a}$ has been carried out by Northcott [4].

3. <u>A change of ring theorem and a relationship between P_R and Φ_R.</u>

<u>Theorem 2.</u> Let $R(M)$ be the extension of R by a finitely generated R-module M. Then

$$P_{R(M)} = P_R\cdot(1-ZP_R(M))^{-1}$$

<u>Proof.</u> We will identify the residue field of R and that of $R(M)$ and denote it by k. Let X be a minimal R-algebra resolution of k. Consider the $R(M)$-algebra $X^* = X \otimes_R R(M)$, furnished with the canonical augmentation map $X^* \to k$. We will show that X has a trivial Massey operation γ.

From the exact sequence of R-modules

$$0 \to M \to R(M) \to R \to 0,$$

which may be regarded as a sequence of $R(M)$-modules, we obtain the exact sequence of complexes over $R(M)$

$$0 \to X \otimes_R M \to X^* \to X \to 0$$

We will identify $X \otimes_R M$ with its image in X^*. Since X is acyclic, the exact homology sequence shows that the inclusion $X \otimes_R M \hookrightarrow X^*$ induces an isomorphism of $R(M)$-modules $H(X \otimes_R M) \approx \tilde{H}(X^*)$. Hence we can find a set S of homogeneous cycles in $X \otimes_R M$ representing a minimal set of generators for $\tilde{H}(X^*)$. Since $M^2 = 0$ we have $(X \otimes M)^2 = 0$ in X^*. It is now simple to construct a trivial Massey operation γ on X^*. Simply put $\gamma(z) = z$ for all $z \in S$ and for each sequence $z_1, \ldots, z_n$ of elements in S, of length $n \geq 2$, put $\gamma(z_1, \ldots, z_n) = 0$.

Using proposition 1 we can construct a minimal resolution of the $R(M)$-module k of the form $X \otimes_{R(M)} T$ where T is the tensoralgebra of a free graded $R(M)$-module N with

$$\operatorname{rank} N_p = \dim \tilde{H}_{p-1}(X^*) \otimes_{R(M)} k \quad \text{for all } p$$

We have isomorphisms of degree zero :

$$\tilde{H}(X^*) \otimes_{R(M)} k \approx H(X \otimes_R M) \otimes_{R(M)} k \approx H(X \otimes_R M) \approx \operatorname{Tor}^R(k,M)$$

Thus

$$\chi(N) = ZP_R(M)$$

Since $\chi(T) = (1-\chi(N))^{-1}$ and $P_R = \chi(X^*)$ we have

$$P_{R(M)} = \chi(X^* \otimes T) = P_R \cdot (1-ZP_R(M))^{-1}$$

Lemma. Let R be a local ring of dimension zero. Let E be the injective envellope of the R-module $R/\mathfrak{m}$. Then $R(E)$ is a local Gorenstein ring of dimension zero.

Proof. Since $R(E)$ is finitely generated over R, it has dimension zero. The anihilator of E is zero; moreover the socle of E is generated by one element, say e_o, since $\mathrm{Hom}_R(R/\mathfrak{m}, E) \approx R/\mathfrak{m}$. One easily shows that the socle of the ring $R(E)$ is generated by the element $(0, e_o)$. It follows that $R(E)$ is a Gorenstein ring; Cf. Bass [1]. Note that E is finitely generated since $\dim R = 0$. ∎

Theorem 3. Let R denote a local ring of dimension zero. Then the following statements are equivalent :

(i) P_R is rational for all R.

(ii) $P_R(M)$ is rational for all R and all finitely generated modules M.

(iii) Φ_R is rational for all R and P_R is rational for all Gorenstein rings R.

(iv) $\dfrac{P_R}{\Phi_R}$ is rational for all R.

Proof. Let M be a finitely generated R-module. The rationality of P_R and $P_{R(M)}$ implies the rationality of

$P_R(M)$, because of the preceding theorem. Hence (i) implies (ii). In the following let E be the injective envellope of the R-module $R/\mathfrak{m}$. We have

$$\mathrm{Tor}^R(R/\mathfrak{m}, E) \approx \mathrm{Ext}_R(R/\mathfrak{m}, R)$$

hence $\Phi_R = P_R(E)$, showing that (ii) implies (iii). It follows from the preceding theorem that

$$(1) \qquad P_{R(E)} = P_R \circ (1 - Z\Phi_R)^{-1}$$

Assume that (iii) is true. We will show that $\frac{P_R}{\Phi_R}$ is rational for an arbitrary R. By the lemma $R(E)$ is a Gorenstein ring. Hence $P_{R(E)}$, as well as Φ_R, is rational by the assumption. It follows from (1) that P_R is rational. Hence so is $\frac{P_R}{\Phi_R}$.

From (1) we obtain

$$(2) \qquad P_R = P_{R(E)} \circ (1 + ZP_{R(E)} \frac{\Phi_R}{P_R})^{-1}$$

Since $R(E)$ is a Gorenstein ring, we have $\Phi_{R(E)} = 1$. It follows from (iv) that $P_{R(E)}$ is rational. Hence it follows from (iv) and (2) that P_R is rational, showing that (iv) implies (i). ∎

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