Post-Newtonian methods and the gravito-electromagnetic analogy

Mikjel Thorsrud

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Department of Physics
University of Oslo

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Contents

Introduction

I would say that the starting point of this thesis was the course ‘Classical mechanics and electrodynamics’ taught by my supervisor Jon Magne Leinaas the final semester of my Bachelor studies. This was the first time I was exposed to electrodynamics in a relativistic framework, and I was very fascinated by the transformation properties of the electromagnetic field under a change of reference frame. To develop a better understanding for the interplay between electrodynamics and special relativity, I considered a special case with a static charge distribution and two observers O and O'. In my example O was at rest relative to the charge distribution, while O' was moving. Accordingly only observer O' experiences a magnetic field since there is no current in the reference frame of O. Since the observers “see” different electromagnetic fields, they will not agree on the (measured) acceleration of a charged particle in the field. At the pioneering days of electrodynamics this apparent paradox was explained by introducing a so called “aether”. In the shed of special relativity however, we all know today that electrodynamics makes perfectly sense without any aether. In the simple example I considered I was able to calculate that the electromagnetic field transforms exactly like it should in order to secure a relativistic correct transformation of the path of the particle. This simple calculation taught me to appreciate the view of magnetism as a relativistic (second order) effect. Later that semester I found a paper claiming that all of electrodynamics can be derived from Coulomb’s law (which describes the electric force between two charged particles at rest) and special relativity alone.

At that time I had no good understanding of general relativity and curved spacetimes. To me, given the obvious analogy between Coulomb’s law and Newton’s gravitational law, it should be possible to apply exactly the same idea to the phenomenon of gravitation. Accordingly I was convinced that there had to be a gravitational counterpart to magnetism. A search in Google soon verified that I was right, gravito-magnetism was a theoretically undisputed part of general relativity and even in the final stage of being experimentally verified! Today however, two years later, I realize that I was right for the wrong reason. Gravitation, as described by general relativity, is a manifestation of curved spacetime, and special relativity is only valid locally. My reasoning starting with Newton’s law of gravitation and special relativity was certainly not compatible with the view of gravitation as a geometric phenomenon. Nevertheless, it turns out that general relativity predicts effects which qualitatively resembles that of magnetism.

In theoretical physics there is a well known analogy between general relativity and electrodynamics which is based on a linearization of Einstein’s field equation. The linearization of general relativity is reviewed in chapter 2, while the gravito-electromagnetic analogy is spelled out in chapter 3. Such a kind of analogy is interesting since Einstein’s equation (on component form) is extremely complicated mathematically and hard to gain physical insight into. It turns out though, that a lot of papers in the rich literature on the topic are in lack of a systematic method. My idea was therefore to study the analogy in a more consistent way using state of the art perturbative methods. This led me to the PPN-formalism and so-called post-Newtonian methods which provides a systematic way to expand any metric theory of gravity, and which also takes account for non-linear effects. After having learned the methods, I introduced suitable variables and gauge (coordinate) conditions, and reformulated the post-Newtonian limit of general relativity in a way which was appropriate for my discussion. I also applied the same kind of systematic expansion to electrodynamics. This enabled me to compare the theories in a consistent way beyond their lowest order approximations. This work, described in chapter 4, is basically just a comparison of the mathematical structure of the considered approximations of the theories. In the following chapter I extend the perspective by exploring the huge conceptual difference between the theories. Based on calculations I investigate the geometric significance of curvature in the post-Newtonian approximation of general relativity.

I will not come up with any excuses for having written such a voluminous thesis. Rather I will tell you how to come through it in a reasonable time. Chapter 1 bears the title 'A brief introduction to gravitational theory'. Allthough it is a brief introduction compared to a text-book treatment, I do not consider it as particularly brief in the context of a thesis. This chapter became a bit longer than first anticipated as it turned out being difficult for me just to write down a lot of equations without explaining how they hang together. In the process I learned a lot though (as I usually took another approach to the material than when I first learned it), and for completeness I have chosen to include it all. This makes my thesis pretty self-contained, and I frequently refer back to the introduction chapter. With a satisfactory basic understanding of general relativity chapter 1 can be dropped all together (but be aware that the material is really obligatory for the following chapters). The following two chapters is about linearization and the gravito-electromagnetic analogy respectively. I think these chapters should be read quickly through even by a reader familiar with the material, at least in order to see my approach. Chapter 4 and 5 constitute the original part of my thesis. They cover the part of my work over the last year which I actually accomplished to complete (for every idea that worked I had several which failed). In the conclusion, chapter 6, I give a brief summary of my main results, which are discussed more thoroughly in the relevant chapters.

Have an enjoyable reading!

Part I

Preliminaries

Chapter 1

Brief introduction to gravitational theory

In this chapter I will introduce the necessary maths and concepts for my thesis. Even though I have tried to make this chapter fairly self-contained, it is by no means meant as a complete introduction to (the basic concepts of) general relativity. Rather, I focus on those parts of the theory which will be important for my sub sequent work. The moral has been to introduce the formalism needed (in a fairly self contained way), neither less nor more. This means that several topics which are usually included in good courses of general relativity will not be discussed here. Examples are Cartan's formalism, differential forms, and strategies for finding exact solutions of the field equations. On the other hand, concepts which are really important for my thesis will be treated in greater detail than it is usually made time for in a course. Much effort is put to explain for example the arbitrariness of coordinates in general relativity, which is often referred to as the "gauge" invariance of the theory. Allthough these ideas are really elementary parts of the theory, they are conseptually demanding for students who are primarily trained for physics in flat spacetime. We also discuss tensor fields (and their bases) in the general context of an arbitrary manifold before specializing to spacetime. These basic consepts lays the foundation for understanding gravity as Einstein saw it.

In this chapter originality is not a goal in itself. I am not the inventor of general relativity(!) and this chapter is to a large extent a synthesis of things I have learned other places. In particular the textbooks [1], [2], [3], [4], [5], [6], [7], [8] and [9] has been important sources of knowledge and inspiration for me in my strives to learn general relativity.

1.1 Spacetime and coordinates

As conscious beings we know that spacetime has four dimensions, one time-dimension and three space dimensions. Physicists likes to characterize points in spacetime by what happens there, and therefore usually refer to them as *events*. An event can be uniquely labeled by four numbers, a four-tuple. More formally we can designate a function $\phi : S \to R^4$, which for each point $\mathcal{P}$ in spacetime S gives a unique four-tuple $\phi(\mathcal{P}) = (x^0, x^1, x^2, x^3)$. Such a function is said to provide a one-to-one map from S to R^4. If ϕ is a continues function with a continues inverse, such four-tuples are called *coordinates*. In spacetime coordinates have four components x^μ, although we will often write x as an abbreviation for x^μ when it cannot be misunderstood. We will use the standard convention that Greek letters, like μ, run from 0 to 3, while Latin letters, like i, run from 1 to 3. The component $\mu = 0$ is the time component, while i are the spatial components, ie:

$$x^\mu = (ct, x, y, z). \tag{1.1}$$

To understand general relativity it is important to realize the arbitrary nature of the coordinates. The smooth function ϕ is not a unique function. Another smooth function $\phi_2 : S \to R^4$ will give another set of coordinates $x^{\mu'}$, and the new coordinates will be some (smooth) function of the old ones:

$$x^\mu \to x^{\mu'} = x^{\mu'}(x). \tag{1.2}$$

This transformation is called a *coordinate transformation*. Notice from (1.2) that we denote the new set of coordinates by using a primed *index*, ie. we write $x^{\mu'}$ rather than x'^μ. This convention is known as *kernel index notation*, and has, as we shall see, the main advantage that transformation laws of tensors become particularly easy to memorize.

Allthough coordinates are completely arbitrary, there exists spacetimes with special symmetries where some certain coordinates are preferred because they make things simple. A particular simple and important special case is the flat spacetime, where the preferred coordinates, which we shall call *Lorentz coordinates*, are related by the well known Lorentz transformations[1]

$$x^{\mu'} = L^{\mu'}_{\nu}(\mathbf{v})x^\nu. \tag{1.3}$$

What is it that makes Lorentz coordinates preferred in flat spacetime? Consider two events $\mathcal{P}_A$ and $\mathcal{P}_B$ in spacetime, and an inertial observer O equipped with a meter-stick and a clock. According to special relativity there exists coordinates x^μ which coincide with the physical distances measured by O. More precisely we can say that if $\mathcal{P}_A$ and $\mathcal{P}_B$, according to O's measurements are separated by the time interval Δt, and the space interval Δr, then the preferred coordinates (the Lorentz coordinates) satisfy $x^0_B - x^0_A = c\Delta t$, and $\sqrt{(x^1_B - x^1_A)^2 + (x^2_B - x^2_A)^2 + (x^3_B - x^3_A)^2} = \Delta r$. The preferred coordinates $x^{\mu'}$ of another inertial observer O' with velocity $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ relative to O, are given by the Lorentz transformation (1.3). Allthough Lorentz coordinates are not unique, they are preferred because they have an immediate physical interpretation (as proper time and proper distances). Moreover, the significance of the coordinates as a measure of distances is not restricted to specific areas of spacetime, but have a *global* character. In a spacetime of arbitrary curvature however, it is never possible to introduce coordinates with this property globally. As we will see later, the best we can do is to introduce coordinates which locally, around a given point P, has significance as a measure of distances. The point P is totally arbitrary, and everywhere else the coordinates looses their physical significance. In a spacetime of arbitrary curvature, it is therefore best to accept that coordinates have no immediate physical significance, and choose coordinates which takes advantage of the present spacetime symmetries (if any). As an example we can mention the spacetime outside a black whole, which is called the Schwartzschild spacetime. Such a spacetime has rotational symmetry. A popular choice of coordinates which takes advantage of this symmetry, are the so called Schwartzschild-coordinates. They are used because the geometry of spacetime can be written compactly in terms of these coordinates, and not because they have any special physical significance.

The important message from this section is of course that coordinates are only artifices used in physics and does not exist a priori in the nature. This simple idea applies equally well to flat spacetime as curved of course; it is simply a matter of convenience that we usually limit ourselves to the Poincare group in special relativity. Because of this arbitrariness, all laws in general relativity are written on a form which does not depend on the choice of coordinates. The invariance of the *form* of physical laws under coordinate transformations is called *general covariance*.

[1]When we also include translations (change of origin) the transformation group is called the *Poincare* group.

1.2 The equivalence principle

In this section we shall briefly discuss the *equivalence principle*. This principle led Einstein to realize that gravitation was a geometric phenomenon and put him on track towards his theory of general relativity.

An early version of the equivalence principle was used already by Isac Newton in his 1686 work Principia. Newton argued that the mass m_g in his gravitational law $\mathbf{F} = m_g\mathbf{g}$, was the same as the inertial mass m_i in his second law of mechanics $\mathbf{F} = m_i\mathbf{a}$, ie. that $m_g = m_i$ for *any* body. A consequence of this is that all bodies fall in a gravitational field with the same acceleration regardless of their mass or internal structure. Today this principle is generally referred to as the *weak equivalence principle*. Einstein realized that a consequence of this principle was that for an observer in a freely falling elevator in a gravitational field, the laws of mechanics are just the same as for an inertial observer far from gravitational fields. To Einstein this was not a coincidence, and he added a key-element to the principle. According to Einstein should not only the laws of mechanics behave in the usual way in such an elevator, but *any* law of nature, including for example the laws of electromagnetism. Today this extension of the weak equivalence principle is generally referred to as the *Einstein equivalence principle*. For simplicity however, and since the Einstein equivalence principle is the only one needed in this thesis, we shall just refer to it as the equivalence principle.

There is also a long tradition for formulating the equivalence principle in an “opposite” way. A physicist in an elevator experiencing a fictious force cannot decide whether it is due to a gravitational field or not. For example, experiments performed inside an elevator at rest on the earth gives similar results as if the elevator was accelerated in a region where gravity is absent. This formulation of the equivalence principle will be useful in chapter 5.

It should be stressed that in the above discussion, it is assumed that inhomogeneities in the gravitational field can be neglected, ie. that the elevator is sufficiently small to make detection of inhomogeneities impossible. The above mentioned experiments are thus local of character.

1.3 Vectors, dual-vectors and tensors

In this section we shall discuss geometrical objects like vectors and dual-vectors in a general context. For convenience we will use Greek letters to denote components of such objects. Greek letters are usually supposed to run from 0 to three 3, but the discussions in this section is neither limited to spacetime nor four dimensions. Therefore the reader is encouraged to read the Greek letters as “any symbol” in this section. In the next section we will generalize the discussion to fields of such objects living on a curved manifold.

A *vector* is a geometric object which can be represented graphically as a quantity with magnitude and direction. We will denote vectors with boldface, for example $\mathbf{v}$. A *vector space* V is a collection of vectors. The rigor definition and axioms of a vector space is a subject of linear algebra and is not necessary here, but we should remember that if $\mathbf{a}$ and $\mathbf{b}$ are vectors in V, then any linear combination $c_1\mathbf{a}+c_2\mathbf{b}$ is also a vector in V. Let us consider a set of vectors $\{\mathbf{v_k}\}$ in a vector space V. $\{\mathbf{v_k}\}$ is said to be a *linearly independent* set if the only solution of the equation $a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_k\mathbf{v}_k = 0$ is the trivial solution where all the coefficients are zero, ie. $a_k = 0$ for all k. A vector *basis* for V is defined as a linearly independent set of vectors, denoted $\{\mathbf{e}_\mu\}$, which span the space V. The dimension of the space V equals the number of basis-vectors. The definition of a basis implies that there exists coefficients a^μ such that an arbitrary vector $\mathbf{a}$ in V can be expressed as a linear combination of the basis vectors: $\mathbf{a} = a^\mu\mathbf{e}_\mu$. The vector itself is an abstract geometric object, while the coefficients a^μ

are the components of $\mathbf{a}$ relative to the basis $\{\mathbf{e}_\mu\}$. As a matter of convenience, we will often loosely refer to the "vector a^μ", although we know it is just the components relative to a given basis.

Since there exist an infinite number of possible bases for a vector space, we should consider change of basis transformations. Allthough these relations follows naturally from what is stated above, it is rewarding to formalize and summarize them, as it turns out that we will make use of them quite often. If $\{\mathbf{e}_\mu\}$ is a basis for a vector space V, than a new basis $\{\mathbf{e}_{\mu'}\}$ must be some linear-combination of the vectors $\mathbf{e}_\mu$:

$$\mathbf{e}_{\mu'} = \mathbf{e}_\nu M^\nu_{\ \mu'}. \tag{1.4}$$

The coeffisiens $M^\nu_{\ \mu'}$ can be thought of as elements of a transformation matrix. The inverse transformation matrix is simply written $M^{\mu'}_{\ \mu}$ and defined by:

$$M^\mu_{\ \mu'} M^{\mu'}_{\ \nu} = \delta^\mu_\nu. \tag{1.5}$$

The inverse transformation is then written:

$$\mathbf{e}_\mu = \mathbf{e}_{\nu'} M^{\nu'}_{\ \mu}. \tag{1.6}$$

Since the vector $\mathbf{a} = a^\mu \mathbf{e}_\mu$ is invariant under a change of basis, it follows that the components transforms inversely:

$$a^{\mu'} = M^{\mu'}_{\ \nu} a^\nu. \tag{1.7}$$

With these definitions the invariance of a vector under a basis transformation follows naturally:

$$\mathbf{a} = a^{\mu'} \mathbf{e}_{\mu'} = M^{\mu'}_{\ \mu} a^\mu \mathbf{e}_\nu M^\nu_{\ \mu'} = \delta^\nu_{\ \mu} a^\mu \mathbf{e}_\nu = a^\nu \mathbf{e}_\nu. \tag{1.8}$$

We will now introduce the concept of dual vectors which are entities in a dual vector space V^* which is associated with and has the same dimension as the ordinary vector space V. This idea is also used in quantum mechanics where a quantum state can be represented either as bra or a ket. In quantum mechanics a bra $\langle\psi|$ acts on a ket $|\phi\rangle$ to produce a complex number $\langle\psi|\phi\rangle \in \mathbf{C}$. In the context useful for relativity, a dual vector is defined as a linear function which maps vectors to *real* numbers. Such dual vectors are also called *one-forms*. We will denote dual vectors with boldface and underline, for example $\underline{\mathbf{a}}$. A one-form $\underline{\mathbf{a}}$ acting on a vector $\mathbf{v}$ to give a number is written $\underline{\mathbf{a}}(\mathbf{v}) = r \in R$. Let us now formalize the mentioned properties of a dual vector. The fact that the dual vector is a *linear* function means that they map linear combinations of vectors in the following way:

$$\underline{\alpha}(k_1\mathbf{v}_1 + k_2\mathbf{v}_2) = k_1\underline{\alpha}(\mathbf{v}_1) + k_2\underline{\alpha}(\mathbf{v}_2). \tag{1.9}$$

If $\underline{\alpha}$ and $\underline{\beta}$ are dual-vectors in a space V^*, it follows that the linear combination $k_1\underline{\alpha} + k_2\underline{\beta}$ is also a dual vector, defined by:

$$(k_1\underline{\alpha} + k_2\underline{\beta})(\mathbf{v}) = k_1\underline{\alpha}(\mathbf{v}) + k_2\underline{\beta}(\mathbf{v}). \tag{1.10}$$

The basis one-forms are written $\underline{\mathbf{w}}^\mu$ and defined by

$$\underline{\mathbf{w}}^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu, \tag{1.11}$$

where the Kronecker symbol is defined by

$$\delta^\mu_\nu = \begin{cases} 1 & \text{, if } \mu = \nu \\ 0 & \text{, if } \mu \neq \nu \end{cases} \tag{1.12}$$

An arbitrary one-form $\underline{\mathbf{a}}$ can be written as a linear combination of the basis one-forms: $a_\mu \underline{\mathbf{w}}^\mu$. The coefficients a_μ are the components of the one-form $\underline{\mathbf{a}}$ relative to the basis $\underline{\mathbf{w}}^\mu$. From (1.11) we see that the components a_μ are given by letting $\underline{\mathbf{a}}$ act on the vector basis $\mathbf{e}_\mu$:

$$\underline{\mathbf{a}}(\mathbf{e}_\mu) = a_\nu \underline{\mathbf{w}}^\nu(\mathbf{e}_\mu) = a_\nu \delta^\nu_\mu = a_\mu. \tag{1.13}$$

Change of basis transformations for dual vectors are written:

$$\begin{aligned} \underline{\mathbf{w}}^{\mu'} &= M^{\mu'}_{\ \nu} \underline{\mathbf{w}}^\nu, \\ \underline{\mathbf{w}}^{\mu} &= M^{\mu}_{\ \nu'} \underline{\mathbf{w}}^{\nu'}, \end{aligned} \tag{1.14}$$

while the components a_μ must transform inversely:

$$\begin{aligned} a_{\mu'} &= M^{\nu}_{\ \mu'} a_\nu, \\ a_{\mu} &= M^{\nu'}_{\ \mu} a_{\nu'}. \end{aligned} \tag{1.15}$$

Notice that transformation is inverse compared to the case with ordinary basis vectors. This is natural from the point of view of notation and placement of indices, but it is also mathematically required such that definition (1.11) holds in an arbitrary basis:

$$\begin{aligned} \underline{\mathbf{w}}^{\mu'}(\mathbf{e}_{\nu'}) &= M^{\mu'}_{\ \nu} \underline{\mathbf{w}}^\nu(\mathbf{e}_\alpha M^\alpha_{\ \nu'}) = M^{\mu'}_{\ \nu} M^\alpha_{\ \nu'} \underline{\mathbf{w}}^\nu(\mathbf{e}_\alpha) \\ &= M^{\mu'}_{\ \nu} M^\alpha_{\ \nu'} \delta^\nu_\alpha = M^{\mu'}_{\ \nu} M^\nu_{\ \nu'} = \delta^{\mu'}_{\nu'}. \end{aligned} \tag{1.16}$$

Finally we note that the action of a dual vector acting on vector can be written in a simple and useful way in terms of the components:

$$\underline{\alpha}(\mathbf{v}) = \alpha_\mu \underline{\mathbf{w}}^\mu(v^\nu \mathbf{e}_\nu) = \alpha_\mu v^\nu \underline{\mathbf{w}}^\mu(\mathbf{e}_\nu) = \alpha_\mu v^\nu \delta^\mu_\nu = \alpha_\mu v^\mu. \tag{1.17}$$

We shall now see that vectors and dual-vectors are special cases of a more general geometric object, called a tensor. A function f of several variables are called a *multi-linear* function if it is linear in all its arguments. A tensor of rank $\binom{m}{n}$ is defined as a multi-linear function which maps m dual vectors and n vectors to R. A tensor of rank $\binom{0}{n}$ is called a *covariant* tensor of rank n, while a tensor of rank $\binom{m}{0}$ is called a *contravariant* tensor of rank m. Hence we can recognize a vector as a contravariant tensor of rank 1 and a dual vector as a covariant tensor of rank 1. A tensor $\binom{m}{n}$ which maps both dual vectors ($m \neq 0$) and vectors ($n \neq 0$) is called a *mixed* tensor. To define a basis for an arbitrary tensor we must introduce the tensor product denoted $\otimes$. If T is a tensor of rank $\binom{m}{n}$ and U is a tensor of rank $\binom{p}{q}$, then $T \otimes U$ is a tensor of rank $\binom{m+p}{n+q}$ defined by

$$\begin{aligned} &T \otimes U(\underline{\mathbf{a}}^1, \underline{\mathbf{a}}^2, \ldots, \underline{\mathbf{a}}^{m+p}, \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{n+q}) \\ &= T(\underline{\mathbf{a}}^1, \underline{\mathbf{a}}^2, \ldots, \underline{\mathbf{a}}^{m}, \mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{n}) U(\underline{\mathbf{a}}^{m+1}, \underline{\mathbf{a}}^{m+2}, \ldots, \underline{\mathbf{a}}^{m+p}, \mathbf{v}^{n+1}, \mathbf{v}^{n+2}, \ldots, \mathbf{v}^{n+q}). \end{aligned} \tag{1.18}$$

This definition tells us how the $\binom{m+p}{n+q}$ tensor $T \otimes U$ maps $m+p$ dual vectors and $n+q$ ordinary vectors to $\mathbb{R}$. The definition give a three point algorithm for calculating the map: 1) calculate the number given by putting the first m dual vectors and the first n vectors of the argument into T, 2) calculate another number by putting the last p dual vectors and the last q vectors of the argument into U, 3) multiply these two numbers. Note that $T \otimes U \neq U \otimes T$ since the map is sensitive to the order of the arguments. We can now use the definition to construct a basis for an arbitrary tensor of rank $\binom{m}{n}$:

$$\mathbf{e}_{\mu_1} \otimes \mathbf{e}_{\mu_2} \otimes \cdots \otimes \mathbf{e}_{\mu_m} \otimes \underline{\mathbf{w}}^{\nu_1} \otimes \underline{\mathbf{w}}^{\nu_2} \otimes \cdots \otimes \underline{\mathbf{w}}^{\nu_n}. \tag{1.19}$$

In a space of dimension d the basis consists of d^{m+n} basis tensors. For example in spacetime $\binom{1}{1}$ tensors lives in a space with $4^{1+1} = 16$ basis tensors:

$$\mathbf{e_1} \otimes \underline{\mathbf{w}}^1, \quad \mathbf{e_1} \otimes \underline{\mathbf{w}}^2, \quad \mathbf{e_1} \otimes \underline{\mathbf{w}}^3, \quad \mathbf{e_1} \otimes \underline{\mathbf{w}}^4, \quad \mathbf{e_2} \otimes \underline{\mathbf{w}}^1, \quad \dots \tag{1.20}$$

Thus the tensors will also have d^{m+n} components $T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}$, one for each basis tensor. In a given basis an arbitrary $\binom{m}{n}$ tensor T can be written:

$$T = T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \mathbf{e}_{\mu_1} \otimes \cdots \otimes \mathbf{e}_{\mu_\mathbf{m}} \otimes \underline{\mathbf{w}}^{\nu_1} \otimes \cdots \otimes \underline{\mathbf{w}}^{\nu_n}. \tag{1.21}$$

The components of the tensor are found by using the basis for the vectors and the dual vectors as argument for the map:

$$T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = T(\underline{\mathbf{w}}^{\mu_1},\ \dots, \underline{\mathbf{w}}^{\mu_m},\ \mathbf{e}_{\nu_1},\ \dots \mathbf{e}_{\nu_n}). \tag{1.22}$$

This can be viewed as a generalization of (1.13) and can be verified using definitions (1.11) and (1.18). Such a calculation is cumbersome for a tensor of arbitrary rank, but, to provide an example, let us consider the special case of a $\binom{1}{1}$ tensor T:

$$\begin{aligned} T(\underline{\mathbf{w}}^\alpha,\ \mathbf{e}_\beta) &= T^\mu_\nu \mathbf{e}_\mu \otimes \underline{\mathbf{w}}^\nu (\underline{\mathbf{w}}^\alpha,\ \mathbf{e}_\beta) \\ &= T^\mu_\nu \mathbf{e}_\mu (\underline{\mathbf{w}}^\alpha) \underline{\mathbf{w}}^\nu (\mathbf{e}_\beta) \\ &= T^\mu_\nu \delta^\alpha_\mu \delta^\nu_\beta \\ &= T^\alpha_\beta. \end{aligned} \tag{1.23}$$

The transformation properties for tensor components under a change of basis is a straight forward generalization of the relations for vectors and dual vectors:

$$T^{\mu'_1 \dots \mu'_m}{}_{\nu'_1 \dots \nu'_n} = (M^{\mu'_1}_{\mu_1} \dots M^{\nu_1}_{\nu'_1} \dots) T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n}. \tag{1.24}$$

This definition ensures that the tensor T is invariant under a change of basis:

$$\begin{aligned} \mathbf{e}_{\mu_1} \otimes \cdots \otimes \underline{\mathbf{w}}^{\nu_1} \otimes \dots \quad \to \quad & \mathbf{e}_{\mu'_1} \otimes \cdots \otimes \underline{\mathbf{w}}^{\nu'_1} \otimes \dots \\ &= (\mathbf{e}_{\mu_1} M^{\mu_1}_{\mu'_1}) \otimes \cdots \otimes (M^{\nu'_1}_{\nu_1} \underline{\mathbf{w}}^{\nu_1}) \otimes \dots \end{aligned} \tag{1.25}$$

A tensor of particular importance for us, is of course the *metric* tensor, which is a symmetric covariant tensor of rank 2. The *inner product*, is per definition a symmetric map from two vectors to $\mathbb{R}$: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. The metric tensor is defined:

$$g(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}. \tag{1.26}$$

Thus the components of the metric tensor becomes

$$g_{\mu\nu} = g(\mathbf{e}_\mu, \mathbf{e}_\nu) = \mathbf{e}_\mu \cdot \mathbf{e}_\nu. \tag{1.27}$$

In an n dimensional space the metric tensor has n^2 components, but since $g_{\mu\nu} = g_{\nu\mu}$, only $(n^2+n)/2$ of them are independent. The inverse metric operator has components defined by

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu. \tag{1.28}$$

Notice that although $g_{\mu\nu}$ are the components of a covariant tensor while $g^{\mu\nu}$ are components of a contravariant tensor, we have chosen to give them the same name, namely the symbol 'g'. We will often call $g_{\mu\nu}$ the covariant components of the metric tensor, and $g^{\mu\nu}$ the contravariant components, although, strictly speaking, they are components of different (but accociated) tensors[2].

Allthoug a tensor T of rank $\binom{m}{n}$ is defined as a map from from m one-forms and n vectors to $\mathbb{R}$, it is no problem to leave some of the arguments of T open (unused) by letting T act on fewer than m one-forms and n vectors. If T act on m' one-forms and n' vectors (assuming $m' < m$ and $n' < n$), then it follows from definition (1.18) that the resulting map is a new tensor of rank $\binom{m-m'}{n-n'}$. Such an operation is called a *contraction* . As an example consider the operation where T is a $\binom{2}{1}$ tensor mapping a vector $\mathbf{u}$ and a one-form $\underline{\mathbf{a}}$. The operation is not well-defined before we choose which argument of T to be un-used/open. Choosing the second argument, we can write this operation $T(\underline{\mathbf{a}}, \quad, \mathbf{u})$. It is easy to show from definition (1.18) that the resulting map is a $\binom{1}{0}$ tensor (vector):

$$\begin{aligned} T(\underline{\mathbf{a}}, \quad, \mathbf{u}) &= T^{\mu_1\mu_2}{}_{\nu_1}\mathbf{e}_{\mu_1} \otimes \mathbf{e}_{\mu_2} \otimes \underline{\mathbf{w}}^{\nu_1}(a_\alpha \underline{\mathbf{w}}^\alpha, \quad, u^\beta \mathbf{e}_\beta) \\ &= T^{\mu_1\mu_2}{}_{\nu_1} a_\alpha u^\beta \mathbf{e}_{\mu_1}(\underline{\mathbf{w}}^\alpha)\mathbf{e}_{\mu_2}(\quad)\underline{\mathbf{w}}^{\nu_1}(\mathbf{e}_\beta) \\ &= T^{\mu_1\mu_2}{}_{\nu_1} a_\alpha u^\beta \delta^\alpha_{\mu_1}\delta^{\nu_1}_\beta \mathbf{e}_{\mu_2}(\quad) \\ &= T^{\mu_1\mu_2}{}_{\nu_1} a_{\mu_1} u^{\nu_1}\mathbf{e}_{\mu_2}(\quad) \\ &\equiv S^{\mu_2}\mathbf{e}_{\mu_2}(\quad). \end{aligned} \tag{1.29}$$

Hence the new tensor, named S, has components

$$S^{\mu_2} = T^{\mu_1\mu_2}{}_{\nu_1} a_{\mu_1} u^{\nu_1}. \tag{1.30}$$

The generalization to arbitrary tensors is obvious. The metric tensor for example, maps a vector $\mathbf{v}$ to a one-form $\underline{\mathbf{v}} \equiv g(\mathbf{v}, \quad)$, while the inverse metric maps the one form back to vector, ie. $\mathbf{v} = g(\underline{\mathbf{v}}, \quad)$. Notice that we have given the one-form the same name as the vector, namely the symbol v. This notation is due to the fact that, since they are related by the metric tensor, $\mathbf{v}$ and $\underline{\mathbf{v}}$ are just different representations of the same physical content. In terms of components the map is written in the following way:

$$g_{\mu\nu}u^\mu = u_\nu. \tag{1.31}$$

Using u_ν as argument for the inverse metric we get:

$$g^{\alpha\nu}u_\nu = g^{\alpha\nu}g_{\mu\nu}u^\mu = \delta^\alpha_\mu u^\mu = u^\alpha. \tag{1.32}$$

Observe from (1.31) and (1.32) that the covariant components of the metric tensor acts as a *lowering* operator, while the contravariant vectors act as a *raising* operator.

This section contains a lot of equations with the bases written out explicitly. The purpose was to show various tensor properties from first principle. From now on however, when a basis is chosen, we will rarely care to write it out, and instead we will write tensors and tensor equations in terms of their components. We will also usually refer to the "tensor" $T^{\mu_1\mu_2}{}_{\nu_1}$, although it is just the components of the tensor T.

1.4 Tensor fields on manifolds

In the previous section we introduced the mathematics of tensors. In this section we will broaden the perspective to *fields* of tensors. This is necessary since, in spacetime, the physics is always represented by some kind of tensor field. To get there we will need to introduce some new ideas, such as the concept of a manifold.

[2] As we shall see in a moment, this is not a special convention for the metric tensor, but holds for any kind of tensor.

1.4.1 Differentiable manifolds

A *differentiable manifold* is a manifold on which calculus can be used. For applications in physics manifolds are, almostly without exception, always differentiable. We shall therefore follow the tradition to just call it a *manifold* (instead of a differentiable manifold). Before giving the rigor definition of a manifold, let us mention a few well known examples. The Euclidean plane $\mathbb{R}^2$ and the four-dimensional Minkowski spacetime are examples of manifolds without curvature. The surface S^2 of a three dimensional sphere is an example of a curved manifold. Informally, a manifold M is a space of arbitrary dimension and curvature which can be *coordinatised*. Accordingly, for a space to be a manifold, it must be possible to introduce coordinate-systems to every part of M. With mathematical rigor a manifold can be defined in the following way:

Manifold: *The n dimensional space M is a manifold if it is possible to divide M into overlapping and open regions M_i such that for each region there exist a continuous one-to-one function f_i (with a continuous inverse f_i^{-1}) which maps points $P \in M_i$ to $\mathbb{R}^n$, $f_i : M_i \to \mathbb{R}^n$.*

In this definition it is assumed that the union of M_i covers the entire manifold ($\bigcup_i M_i = M$), and that in regions with overlapping functions there exist smooth functions (coordinate transformations) which relates the coordinates to each other. The actual maps $f_i(P)$ of the region M_i is of course what we usually refer to as coordinates (or the coordinate system). The reason why the definition consider different regions M_i with its own coordinates $f_i(P)$, is simply that it is often impossible to describe the entire manifold with a single coordinate system.

Consider for example the sphere S^2 in the usual (θ, ϕ) coordinates. These coordinates are not well-behaved at the north-pole (and the south-pole) where the ϕ coordinate is undefined and there exist an infinite number of coordinates $(0, \phi)$, which all represent the same point. Therefore the function $f : M \to \mathbb{R}^n$ is not a one-to-one function. This problem can be overcome by introducing *two* sets of coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2). On the earth we could for example choose $\theta_1 = 0$ at the north-pole and $\theta_2 = 0$ at Blindern. The sphere can therefore be coordinatised by two functions f_1 and f_2. At the north-pole and at Blindern one of the coordinate systems are well-behaved, and at all other places there exist a smooth coordinate transformation between the coordinate systems. Thus the sphere S^2 is a manifold.

Intrinsic and extrinsic properties of a manifold To introduce the ideas of *intrinsic* and *extrinsic* properties of a manifold we shall discuss the notion of curvature. The mathematical definition of curvature, ie. the Riemann curvature tensor, will be saved for section 1.6.5. It is convenient being able to use the term curvature in a more vague way before we come to section 1.6.5 though, so here we will introduce it in a more loose way. There are two different concepts of curvature which are important to keep distinct, namely *intrinsic* and *extrinsic* curvature. As an example of the latter consider the two dimensional surface of a cylinder. Embedded in the three dimensional euclidean space $\mathbb{R}^3$ the cylinder looks like an object with curvature. This kind of curvature however, is concerned with the actual embedding in $\mathbb{R}^3$ and is not an intrinsic property of the manifold itself. To see this, simply unroll the cylinder and observe that the geometry is the same as the two dimensional Euclidean plane. All intrinsic properties of the manifold, such as distances measured along curves on the manifold, remain unchanged by the action of unrolling. An observer living on the surface of the cylinder can therefore not distinguish the geometry of the surface from the geometry of the flat plane. The surface of a cylinder is therefore an example of a manifold with extrinsic curvature, but with no intrinsic curvature. The surface of a sphere S^2 however, is a manifold with intrinsic curvature. An inhabitant of S^2 will for example find the circumference of a circle divided by its diameter to be less than π. This is a characteristic of curved spaces, and the inhabitant can thus conclude that he lives in a world with a non-Euclidean geometry.

The curvature of spacetime as described by general relativity is intrinsic of nature. Therefore it is important to define entities (such as tensor fields) on curved manifolds M in terms of things that are intrinsic to M. For example, the vector space S^2 must be defined without reference to the basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ of the Cartesian coordinate system $\mathbb{R}^3$. In the next section we shall see how to define vectors from the intrinsic point of view.

1.4.2 The coordinate basis

Going from the notion of a vector in itself, as discussed in chapter 1.3, to a *field* of vectors on a manifold M, requires some new ideas. First of all, to each point x on M we must define a vector space associated with exactly that point. The vector space at a given point is called the *tangent space* T_p. Vectors at different places belong to different tangent spaces. Thus it is not possible to define operations between tensors defined at different places on the manifold. This is natural since on a curved manifold, vectors cannot be moved carelessly around like in Euclidean spaces. We know that *parallel transporting*[3] a vector around in a curved space will change the vector. The change when moving from a point to another is not well defined though, but will depend on the chosen path. This is not a controversy, but perhaps the clearest characteristic of curvature, and gives a clear motivation why each point on a manifold is associated with its own vector space.

Next we must define the tangent spaces T_p in terms of something that is intrinsic to M. The new idea is to associate vectors with derivative operators, but first we need some new definitions. A *path* on M is defined as a series of connected points. A *curve* is defined as a path parametrized by a scalar which varies smoothly along the path. Thus there is a unique number $\in \mathbb{R}$ associated with every point on the curve, and a curve on M is simply a continuous one-to-one map from $\mathbb{R}$ to M. In spacetime the curve is defined by the four functions $x^\mu(\lambda)$, where λ is some scalar quantity. The scalar is by definition an invariant quantity which do not depend on the coordinates. For every path then, there exist an infinite number of curves, one for each choice of parameter.

We are now ready to give a definition of vectors in terms of things which are intrinsic to the manifold. We claim that any vector $\mathbf{v}$ at a point P can be represented by the directional derivative $\frac{d}{d\lambda}$ associated with some curve $x^\mu(\lambda)$ passing through P. The tangent space T_p is then simply defined as the space of directional derivatives associated with all possible curves going through P. A very natural basis for T_p suggests itself from the chain rule for derivatives:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}. \tag{1.33}$$

We see that an arbitrary directional derivative $\frac{d}{d\lambda}$ can be written as a linear combination of the partial derivatives ∂_μ with coefficients $u^\mu = \frac{dx^\mu}{d\lambda}$. Thus the partial derivatives constitute a natural basis for T_p. This basis is so important that it is given its own name, coordinate basis:

Coordinate basis for the tangent space: *The coordinate basis for T_p is the set of partial derivatives $\{\partial_\mu\}$.*

[3]The reader not familiar with the concept of parallel transport must be patient and wait until section 1.6.3.

For convenience we will often refer to the coordinate basis vectors as $\mathbf{e}_\mu$, but this should only be interpreted as short notation for ∂_μ. Usually we like to think of a vector as a quantity with a magnitude and a direction. The new definition of vectors as directional derivatives formalizes this view in a neat way. To the operator $\frac{d}{d\lambda}$ the associated direction is simply the direction of the curve at the point p. The magnitude, on the other hand, is determined by the choice of parameter. For example if we want a vector which point in the same direction as $\frac{d}{d\lambda}$, but with three times the magnitude, we choose a new parameter $\lambda_2 \equiv (1/3)\lambda$ such that $\frac{d}{d\lambda_2} = \frac{d\lambda}{d\lambda 2}\frac{d}{d\lambda} = 3\frac{d}{d\lambda}$. Also notice that the differential operator $\frac{d}{d\lambda}$ can be interpreted as the tangent vector to the accociated curve $x^\mu(\lambda)$. This can be seen by comparing (1.33) to the usual definition of the tangent vector $\mathbf{u}$ of the curve $x^\mu(\lambda)$:

$$\mathbf{u} = \frac{dx^\mu}{d\lambda}\mathbf{e}_\mu. \tag{1.34}$$

The vector space T_p of a point p is then spanned by the infinite set of tangent vectors associated with all possible curves going through p, hence the name 'tangent space'.

In chapter 1.3 we saw that bases for tensors of arbitrary rank could be constructed from the vector bases and the dual vector bases. Thus, to be able to construct the coordinate basis for arbitrary ranked tensors, we will also need to identify the coordinate basis for one-forms (like we identified partial derivatives as the coordinate basis for vectors). This is found by demanding that the usual relation $\underline{\mathbf{w}}^\mu(\partial_\nu) = \delta^\mu_\nu$ also holds in the coordinate basis. Let us rewrite this relation slightly:

$$\underline{\mathbf{w}}^\mu(\partial_\nu) = \delta^\mu_\nu = \frac{\partial x^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\alpha}\delta^\alpha_\nu = \frac{\partial x^\mu}{\partial x^\alpha}\underline{\mathbf{w}}^\alpha(\partial_\nu). \tag{1.35}$$

Remember that the gradient $\underline{d}f$ of a scalar field $f(x)$ is defined by[4]:

$$\underline{d}f = \frac{\partial f}{\partial x^\mu}\underline{\mathbf{w}}^\mu. \tag{1.36}$$

The '$\underline{d}$' is called the exterior derivative operator, and is used to differentiate anti-symmetric covariant tensors of arbitrary rank. Since the only anti-symmetric covariant tensor[5] we will need to differentiate are scalar fields, we will take (1.36) as our definition of $\underline{d}$. Since coordinates are scalar fields, (1.35) can now be rewritten

$$\underline{\mathbf{w}}^\mu(\partial_\nu) = \underline{d}x^\mu(\partial_\nu). \tag{1.37}$$

The coordinate basis for the dual space T_p^* of one-forms is therefore represented by the set of exterior derivatives of coordinates:

Coordinate basis for the dual space: *The coordinate basis for T_p^* is the set of exterior derivatives of the coordinates: $\{\underline{d}x^\mu\}$.*

We have now identified the coordinate basis of vectors and dual vectors, and can construct bases for tensors of arbitrary rank.

The coordinate basis is a natural and very important kind of basis in general relativity. It can be thought of as the formal way to set up the basis vectors to point along the coordinate axes, which

[4]Note that the gradient of a scalar field actually is a one-form field, not a vector field, as often taught when restricted to Euclidean spaces. In Euclidean spaces the components of the metric tensor are just the unity matrix, which means that the components of a one-form is equal to the components of the corresponding vector.

[5]A covariant tensor of rank p which is anti-symmetric in all its indices is called a *p-form*. Scalars and one-forms have too few indices to interchange the order of indices, and are just defined as 0-forms and 1-forms respectively.

is the usual way to think of a coordinate system. Also note that, given a coordinate system, the coordinate basis is a *unique* basis. Therefore a coordinate transformation necessarily induce a change of the coordinate basis. The transformation law is given directly by the chain rule. Hence, under a coordinate transformation $x^\mu \to x^{\mu'}$, the coordinate basis vectors transform in the following way:

$$\partial_\mu \to \partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu. \tag{1.38}$$

In chapter 1.3 we summarized the transformation properties of tensors under arbitrary basis transformations in terms of a transformation matrix $M^\mu_{\ \mu'}$ defined in (1.4). It is convenient to use the same notation also for transformations between coordinate bases. The transformation matrix for coordinate transformations is defined

$$M^\mu_{\ \mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}, \tag{1.39}$$

such that the transformation of the coordinate basis vectors can be written on the generic form $\mathbf{e}_{\mu'} = M^\mu_{\ \mu'}\mathbf{e}_\mu$. Since the matrix

$$M^{\mu'}_{\ \mu} = \frac{\partial x^{\mu'}}{\partial x^\mu} \tag{1.40}$$

is the inverse of $M^\mu_{\ \mu'}$, the *components* of vectors transforms as

$$u^{\mu'} = M^{\mu'}_{\ \mu} u^\mu. \tag{1.41}$$

For one-forms, and arbitrary tensors, the transformation properties are also just like the formulas given in chapter 1.3, with $M^\mu_{\ \mu'}$ and $M^{\mu'}_{\ \mu}$ defined by (1.39) and (1.40). In particular, the transformation of the metric tensor under a coordinate transformation becomes:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}. \tag{1.42}$$

Also note that for non-linear coordinate transformations, the transformation matrices will depend on the position, $M^\mu_{\ \mu'} = M^\mu_{\ \mu'}(x)$.

Allthough it can be argued that the coordinate basis is the most natural kind of basis on a manifold, there is nothing that prevents us from choosing another kind of bases. In general relativity all calculations are usually done in coordinate basis (because that is the simplest way). When the calculations are done, however, it might be convenient to change basis. For example, we will see that orthonormal bases has a special physical significance and are associated with observers and their measurements. It is therefore often useful to change basis without changing the coordinates (the new basis is then certainly not a coordinate basis). An arbitrary basis can be written as a linear combination of the coordinate basis vectors, $\mathbf{e}_{\mu'} = M^\mu_{\mu'}\partial_\mu$, where the coefficients $M^\mu_{\mu'} = M^\mu_{\mu'}(x)$ may depend on the position.

1.4.3 The metric and path lengths

As explained, there can be various reasons for choosing another kind of bases than the coordinate basis, when studying tensor fields on manifolds. Each basis has an associated metric where the components are defined in the usual way $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$, as stated in chapter 1.3 (where we discussed tensors in a given space). The only difference here, where we discuss the metric on a manifold (where tensor spaces are defined on each point), is of course that the metric is a function of the position, a tensor *field*. However, when we simply refer to the metric without referring to a particular basis, we always mean the metric associated with the coordinate basis. We will see in this section that a given coordinate system together with the associated metric defines the geometry of the manifold, in the sense that path lengths on the manifold are well defined.

Associated with the metric there is a *line element* defined by

$$ds^2 = (\mathbf{u} \cdot \mathbf{u}) d\lambda^2 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda^2, \tag{1.43}$$

where $u^\mu = \frac{dx^\mu}{d\lambda}$ is the tangent vector of a curve $x^\mu(\lambda)$. The line element is usually written on the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.44}$$

and has an interpretation as the square of the infinitesimal path length between two neighboring points λ and $\lambda + d\lambda$ along the curve $x^\mu(\lambda)$. The form (1.44) shows explicitly that the path length of a line segment does not depend on the particular choice of parametrization. Personally, when I see a line element on the form (1.44) I prefer to read it on the equivalent form

$$ds^2 = (\mathbf{e}_\mu \cdot \mathbf{e}_\nu) dx^\mu dx^\nu. \tag{1.45}$$

Roughly speaking, I think of the line-element in the following way: "the (physical) distance between two neighboring points is given by the length of the basis vector times the coordinate separation". This view is of course only correct for orthogonal spacetimes ($\mathbf{e}_\mu \cdot \mathbf{e}_\nu = 0$ when $\mu \neq \nu$), but most spacetimes are orthogonal or at least very close too. Mathematicians however, deny to use infinitesimal entities like the coordinate separation dx^μ. Instead they read it as the one-form $\underline{d}x^\mu$. So when a mathematician see a line-element written on the standard form (1.44) they read it as

$$ds^2 = g_{\mu\nu} \underline{d}x^\mu \otimes \underline{d}x^\nu. \tag{1.46}$$

The right hand side of this equation is exactly the metric tensor written out in the basis $\underline{d}x^\mu \otimes \underline{d}x^\nu$. Thus, mathematicians treat the symbol ds^2 simply as a name for the metric tensor, and does not read anything out of the power of two. According to this view, the line element and the metric tensor is exactly the same mathematical structure, and the words 'line-element' and 'metric' are often used interchangeably. The words are also often used interchangeably in papers written by physicists, although physicists most often I suppose, think of the metric as a tensor and associate the line-element with distances.

We can now define the path-length between two points of finite separation $\Delta\lambda$ along the curve $x^\mu(\lambda)$. For space-like curves it is defined

$$\Delta s = \int_{\lambda_0}^{\lambda_0 + \Delta\lambda} d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}, \tag{1.47}$$

while for time-like curves we have

$$\Delta s \equiv \Delta\tau = \int_{\lambda_0}^{\lambda_0 + \Delta\lambda} d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}. \tag{1.48}$$

The definition of time-like, light-like and space-like curves is just like in special relativity[6]. For time-like paths the path-length Δs is related to the *proper time* interval $\Delta\tau$ by $\Delta s = c\Delta\tau$, where the

[6]Time-like curves have tangent vectors satisfying $\mathbf{u} \cdot \mathbf{u} < 0$, while for light-like and space-like curves they satisfy $\mathbf{u} \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{u} > 0$ respectively.

proper time is defined as the time interval measured on a comoving *standard clock*[7]. Accordingly, for time-like curves we can relate the line-element to the infinitesimal proper time interval $d\tau$:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2 d\tau^2. \tag{1.49}$$

We will reserve the name *four-velocity* for tangent vectors which are parametrized by the proper time τ, ie. $\frac{dx^\mu}{d\lambda}$ is called the four-velocity only if $\lambda = \tau$. From (1.49) we get the useful *four-velocity identities*

$$\mathbf{u}\cdot\mathbf{u} = \begin{cases} -c^2 & \text{, for time-like vectors} \\ 0 & \text{, for light-like vectors.} \end{cases} \tag{1.50}$$

Let us finally look at the metric and the line element of a few manifolds. The line element of the Euclidean volume in Cartesian coordinates reads

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{1.51}$$

hence the metric is $g_{ij} = diag(1,1,1)$. A coordinate transformation from the Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$ to spherical coordinates $(x^{1'}, x^{2'}, x^{3'}) = (r, \theta, \phi)$ defined by

$$x = r\ sin(\theta)\ cos(\phi), \quad y = r\ sin(\phi)sin(\theta), \quad z = r\ cos(\theta), \tag{1.52}$$

gives the new metric

$$g_{\mu'\nu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}}g_{\alpha\beta} = diag(1, r^2, r^2 sin^2(\theta)), \tag{1.53}$$

and hence the line element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta\ d\phi^2. \tag{1.54}$$

Another familiar manifold is of course the flat spacetime, called the *Minkowski spacetime*, which in Cartesian coordinates has line element

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1.55}$$

The associated metric $\eta_{\mu\nu} = diag(-1,1,1,1)$ is called the *Minkowski metric*. Note that the name ‘Minkowski metric’ and the associated symbol $\eta_{\mu\nu}$, is reserved for a particular choice of coordinates (Cartesian). Thus, if we choose to describe the Minkowski spacetime in spherical coordinates, we will neither use the symbol $\eta_{\mu\nu}$ nor call the metric $g_{\mu\nu} = diag(-1, 1, r^2, r^2 sin\theta)$ the ‘Minkowski metric’.

1.4.4 Example: coordinates and path lengths on a two dimensional curved manifold

In this example, we will study the geometry of a two dimensional curved manifold which can be embedded in the Euclidean space $\mathbb{R}^3$. We will analyze the geometry both from the extrinsic and intrinsic point of view and imagine a physicist living in the two dimensional world of the plane. This makes it possible to illustrate how the notion of coordinates are equally meaningful from the intrinsic point of view. In particular we will see that our friend living inside the manifold will need what we shall call an *operational* definition of the coordinates in order to be able to associate them with actual points in his world. The main purpose of this section is to illustrate some ideas which are important to gain insight into the notion of curved spacetime, where the intrinsic observers are *us* (unless you are reading my thesis from a higher dimensional space...).

[7] A standard clock is what the name suggests, ie. a standard clock like the one hanging on the wall.

The three dimensional Euclidean manifold ($\mathbb{R}^3$) in Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$ has line element

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{1.56}$$

By introducing cylindrical coordinates $(x^{1'}, x^{2'}, x^{3'}) = (r, \phi, z)$ defined by

$$x = r\ cos(\phi), \quad y = r\ sin(\phi), \quad x^{3'} = x^3 = z, \tag{1.57}$$

the line element takes the form

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2. \tag{1.58}$$

The coordinate transformation of the metric from (1.56) to (1.58) is geometrically obvious, but can also be verified by the standard transformation rule of course. The curved plane we shall discuss has rotational symmetry around the z axis and is defined by the equation[8]

$$z = -r^{\frac{3}{2}}. \tag{1.59}$$

The line element (1.58) can be rewritten

$$ds^2 = \left(1 + (\frac{dz}{dr})^2\right) dr^2 + r^2 d\phi^2, \tag{1.60}$$

which for the curved plane gives

$$ds^2 = \left(1 + \frac{9}{4} r\right) dr^2 + r^2 d\phi^2. \tag{1.61}$$

This is the line element of the curved two-dimensional world which our friend lives in. What is the significance of the coordinates (r, ϕ) in the line element? From the extrinsic point of view, where the plane is embedded in $\mathbb{R}^3$, r is of course still the distance from the z axis, while ϕ is still the angle between the x axis and the projection down to the $x-y$ plane. More formally we can say that r and ϕ are the polar coordinates of the projection plane $z = 0$. From the intrinsic point of view however, we cannot refer to the embedding in $\mathbb{R}^3$, and the significance of the coordinates must be found directly from the line element (1.61), which works as a "definition" of the coordinates. First we observe that the r coordinate has no immediate significance as distance. Along a path of constant ϕ ($d\phi = 0$) the infinitesimal distance ds between two points with coordinates r and $r + dr$ is $dr\sqrt{1 + \frac{9}{4} r}$. Integrating we find that the length of a (radial) path from $r = 0$ to r is

$$\Delta s = \frac{8}{27} \left[(\frac{9r}{4} + 1)^{3/2} - 1 \right]. \tag{1.62}$$

The inverse expression is:

$$r = \frac{4}{9} \left[(\frac{27}{8} \Delta s + 1)^{2/3} - 1 \right]. \tag{1.63}$$

This equation can be taken as a definition of the r coordinate for the inhabitant of the plane. If he wonder about the r coordinate of a given point, he can just measure the distance Δs along a radial path to the origin of the coordinate system, and then insert Δs into (1.63). Thus we have found

[8] I chose this condition because it gives simple calculations. I have avoided the much more discussed two-sphere S^2 (which also give simple calculation) because there are no preferred origin (on the surface of a sphere). Simply too much symmetry to illustrate a few points!

an *operational* definition of the r coordinate from the intrinsic point of view. This is perhaps not the simplest way to define the r coordinate. Along a path with constant r coordinate the line element reads $ds^2 = r^2 d\phi^2$. The path length of circles with origin at $r = 0$ is therefore $2\pi r$, just like in Euclidean space. This suggests that our friend may use another operational definition for the r coordinate. If he wonder about the radial coordinate of a given point P, he can just simply measure the circumference Δs of the circle with origin in $r = 0$ passing through the point P. The r coordinate is then given by the formula $r = \Delta s/2\pi$.

We have seen that in the chosen coordinates, the r coordinate is not a measure of the distance from the origin, but is defined such that the circumference of circles around the origin satisfy the same expression as in Euclidean space. These coordinates are of course just one example out of the unlimited space of possibilities. The chosen coordinates are very convenient though, since they take advantage of the symmetry around the z axis and since there is a simple operational definition of the radial coordinate. Another natural choice is to introduce a new radial coordinate ρ which has significance as the distance from the origin measured along a radial path ($d\phi = 0$). Equation (1.62) then suggests that we define:

$$\rho = \frac{8}{27}\left[(\frac{9r}{4} + 1)^{3/2} - 1\right]. \tag{1.64}$$

The inverse expression is found by replacing Δs with ρ in (1.63). An immediate consequence of the new coordinate is that $g_{\rho\rho} = 1$. This can be stated without calculations since the new radial coordinate is a direct measure for path lengths along radial line segments. The metric element $g_{\phi\phi}$ is unchanged since we use the same ϕ coordinate. We therefore still have $g_{\phi\phi} = r^2$, but r must be rewritten in terms of ρ. The off-diagonal elements of the metric tensor is still zero. All this reasoning can be verified, of course, by the formal transformation rule (1.42) with $x^\mu = (r, \phi)$ and $x^{\mu'} = (\rho, \phi)$. The line element associated with the new coordinates is therefore

$$ds^2 = d\rho^2 + \frac{4^2}{9^2}\left((\frac{27}{8}\rho + 1)^{2/3} - 1\right)^2 d\phi^2. \tag{1.65}$$

We have now seen two different sets of coordinates which might be useful for our friend living on the curved manifold. Both coordinate systems have some similarities to the polar coordinates $\hat{r}, \hat{\phi}$ of the Euclidean plane with line element

$$ds^2 = d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2. \tag{1.66}$$

I have added the "hats" in order to avoid confusing the flat spacetime coordinates with those of the curved manifold. Comparing (1.61) with (1.66), we see that the line element for a path with $dr = 0$ is on exactly the same form as in the Euclidean case. Comparing (1.65) with (1.66) though, we see that it is instead the line element of a path with constant ϕ that is on the same form. Thus we have showed that it is possible to introduce coordinates on the curved manifold such that, along a special line, distances are related to the coordinates in the same way as in Euclidean space. It is however not possible to choose coordinates such that the line element takes the same form as (1.66) for a general path. On curved manifolds, we must therefore accept that coordinates are nothing more than numbers, ie. maps from the manifold to $\mathbb{R}^n$, which can be related to distances and path lengths only after calculations (which involves the metric).

1.5 Spacetime as a differentiable manifold

So far we have focused on the notion of coordinates, tensor fields and their transformation properties in the general context of an arbitrary manifold. The next logical step is to discuss the relevance for spacetime physics. We have not yet established the kinematics of general relativity, and have therefore not yet demonstrated how the tensors are computed in the first place. A reader not familiar with general relativity should therefore, at this point, only accept that physical quantities are represented by tensors of various sorts (just like in special relativity), which can be calculated using strategies to be introduced later.

A physical theory is a mathematical model which is supposed to describe processes in the nature. The model can be thought of as a mathematical representation for (some aspect of) the physical system under consideration. In some kind of theories, and especially in gauge theories, a single physical state may have several different (but somehow related) representations. Since a successful physical theory predicts the outcome of experiments, the formalism must establish a well defined relation between the representation and the associated physical measurable quantity. In spacetime physical quantities are usually represented in terms of tensor fields. For example the path of a particle may be represented by the field of tangent vectors. The physical measurable quantities must then be the components of the tensor relative to some basis. But what basis? Obviously not the coordinate basis, since this one depends on the chosen coordinates. Measurable quantities can obviously not rely on the arbitrary choice of coordinates! One of the fundamental axioms of general relativity, is that the physical quantities are given by the tensor components relative to orthonormal bases which are oriented in a direction defined by the motion of the observer. For this reason, orthonormal frames are also sometimes called *physical bases*. The details are explained in the next section.

Why then care about the coordinate basis at all, if orthonormal bases are so important? We have seen that the geometry of a manifold, in the sense of distances measured along paths on it, is well defined first when it is equipped with a metric[9]. In general relativity, the metric is the solution of the field equation. All calculations are usually simplest when done in the coordinate basis. Changing to an orthonormal basis is therefore usually done after all calculations are finished, and when the only remaining question is how the result relates to measurements.

1.5.1 Orthonormal bases

Let us start with some definitions. An *orthogonal* basis satisfies:

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = g_{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu. \tag{1.67}$$

An *orthonormal basis* is an orthogonal basis which is *normalized*, such that $g_{\mu\nu} = (-1, 1, 1, 1)$. The Minkowski metric is therefore per definition an orthormal basis. As the only two kinds of bases we will work with turns out to be coordinate bases and orthonormal bases, we shall use the convention that $\mathbf{e}_{\hat{\mu}}$ denotes basis vectors of an orthonormal frame, while $\mathbf{e}_\mu$ denotes coordinate basis vectors. We use the same notation for components of vectors, such that $u^{\hat{\mu}}$ and u^μ denotes components of the vector $\mathbf{u}$ relative to an orthonormal basis and a coordinate basis respectively. In the special case of Minkowski spacetime, where the coordinate basis is orthonormal, we shall use the coordinate basis notation. We are then ready to define the physical basis associated with a particular observer:

Physical basis: *The physical basis* $\{\mathbf{e}_{\hat{\mu}}\}$ *for a particular observer with four velocity* $\mathbf{u}$ *satisfy* $\mathbf{e}_{\hat{0}} = \frac{\mathbf{u}}{c}$ *and* $\mathbf{e}_{\hat{\mu}} \cdot \mathbf{e}_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$.

[9]Remember that when only saying “a metric”, we for sure mean the metric associated with a coordinate basis. If a particular coordinate system is already established we say “the metric”, which then for sure mean the metric associated with *the* coordinate basis of the chosen coordinates.

The observer can be thought of as carrying a laboratory, whose time direction and spatial directions are defined by the orthogonal unit vectors $\mathbf{e}_{\hat{\mu}}$, along the world line. There are many possibilities for the spatial basis vectors. This ambiguity corresponds to the observer's freedom to orient the spatial axes of his laboratory. The time basis vector $\mathbf{e}_{\hat{0}}$ however, is fixed. Since a particle with the same velocity as the observer will move only in the time direction according to the laboratory frame, $\mathbf{e}_{\hat{0}}$ must be parallel to the observer's four velocity $\mathbf{u}$. The factor of $1/c$ in $\mathbf{e}_{\hat{0}} = \frac{\mathbf{u}}{c}$ is a normalization factor ensuring $\mathbf{e}_{\hat{0}} \cdot \mathbf{e}_{\hat{0}} = -1$, see (1.50). Consider now a particle with the same four-velocity $\mathbf{u}$ as the observer. We can find the components relative to the laboratory frame by writing the vector in both bases:

$$\mathbf{u} = u^{\mu}\mathbf{e}_{\mu} = u^{\hat{\mu}}\mathbf{e}_{\hat{\mu}} = u^{\hat{0}}\frac{\mathbf{u}}{c}, \tag{1.68}$$

which implies

$$u^{\hat{\mu}} = (c, 0, 0, 0). \tag{1.69}$$

Notice that $u^{\hat{\mu}}$ has exactly the same components as it would if spacetime was flat. This is just what we should expect of a physical basis, since in a sufficient small region spacetime always look flat. For two observers with different velocity, but at the same location (at one particular moment) the components of tensors relative to their physical bases are related by the Lorentz transformations. In other words, special relativity is valid *locally* also in curved spacetime.

The orthonormal basis can be constructed as a linear combination of the coordinate basis vectors in the usual way: $\mathbf{e}_{\hat{\mu}} = M^{\mu}_{\hat{\mu}}\mathbf{e}_{\mu} \equiv M^{\mu}_{\hat{\mu}}\partial_{\mu}$, where $M^{\mu}_{\hat{\mu}}$ is the coordinate basis component μ of the orthonormal basis vector $\mathbf{e}_{\hat{\mu}}$, ie. $M^{\mu}_{\hat{\mu}} = (\mathbf{e}_{\hat{\mu}})^{\mu}$. When the basis transformation is written on the generic form in terms of the matrix $M^{\mu}_{\hat{\mu}}$, the standard transformation law for (the components of) arbitrary tensors, see (1.24), can be employed.

1.5.2 Local Lorentz frames

In the previous section we discussed the transformation from a coordinate basis to a (globally) orthonormal basis. This is an example of a basis transformation which is not associated with a coordinate transformation. However, recall that there is always an unique coordinate basis associated with a given coordinate system, ie. $\mathbf{e}_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$. Thus, associated with a coordinate transformation there is always a corresponding transformation of the coordinate basis. In this section we shall see that it is always possible to change to a coordinate system such that locally, at a point P, the coordinate basis is orthonormal, ie. $g_{\mu\nu}(P) = \eta_{\mu\nu}$. This assertion is part of a larger theorem which is sometimes called the *local flatness theorem*. It states that the coordinates can be chosen such that in addition to being orthonormal locally, the partial derivatives of the metric vanish:

Local flatness theorem: *For any spacetime, there exists coordinates such that at an arbitrary point P, the coordinate basis satisfies: $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $\partial_{\alpha}g_{\mu\nu}(P) = 0$.*

The local flatness theorem can be proved by Taylor expanding the standard transformation law for the metric, see for example [8, ch.6.2]. The proof also shows that in general it is *not* possible to find coordinates where also all the second order derivatives $\partial_{\alpha}\partial_{\beta}g_{\mu\nu}$ vanish. From special relativity we know that a *global Lorentz frame* is associated with a coordinate system where $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere. The local flatness theorem shows that this is not possible in a curved spacetime. In curved spacetime the closest we can get to a global Lorentz frame around a point P is therefore to introduce a coordinate system where $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $\partial_{\alpha}g_{\mu\nu}(P) = 0$. In this thesis I will refer to the coordinates of such a coordinate systems as *geodesic coordinates*[10]. This coordinate system is said to define a *local Lorentz frame*. The local Lorentz frame represents the closest one can come to the global Lorentz frame of special relativity (where $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere). Later we will show that test particles only affected by gravity has no acceleration in a local Lorentz frame. Thus, in general relativity, where gravity is *not* interpreted as a force, the local Lorentz frame actually is a local *inertial* reference frame. This means that the local Lorentz frame is the reference frame associated with freely falling (inertial) obserwerers. The local flatness theorem is conceptually interesting and important for the understanding of spacetime geometry. It shows us that locally, in the infinitesimal neighborhood of a given point, spacetime appear to be flat. This should not be a big surprise, as the equivalence principle tells us that it is not possible for an un-accelerated observer by means of local experiments, to decide whether he is in free fall in a gravitational field or in an inertial rest frame in flat spacetime.

It should be stressed that there is no *unique* local Lorentz frame associated with a given point. Once a local Lorentz frame is chosen there is still freedom left to perform (local) Lorentz transformations. This ensures that the principle of relativity holds, ie. that physical laws are equal in all inertial frames (which is local Lorentz frames in curved spacetime) and that the speed of light is invariant[11].

We have seen that there are two ways to transform to orthonormal bases: pure basis transformations (section 1.5.1) and coordinate transformations. If you change to an orthonormal basis in order to see the physical content of a *tensorial* quantity, it does not matter what method you choose, but the former is often preferable as they can be done globally (while the latter only locally). If you want to see the physical content of a *non-tensorial* quantity however, the latter method *must* be chosen. The physical content of non-tensorial quantities can be found by changing to a special kind of coordinate system associated with the *proper reference frame* of the observer. The local Lorentz frame is the proper reference frame of an inertial observer. In chapter 5 we will see examples of more general proper reference frames, where the observer may be accelerated or even rotating. It should be stressed that for a *tensorial* quantity it does not matter whether the measurement is performed by an inertial observer or not. The accelerated observer will measure the same as an inertial observer with the same (instantaneous) velocity.

As a concluding remark, let us comment on the role of coordinates again. In special relativity the terms 'observer', 'coordinates' and 'reference frame' are often used interchangeably. In flat spacetime they, roughly speaking, describe the same thing. This is at least correct when we restrict ourselves to the Poincare group (which we, practically speaking, always do in special relativity). Given an observer there is automatically an associated coordinate system. This coordinate system defines the global Lorentz/inertial frame of the observer. In curved spacetime however, coordinates are in general not associated with observers and their reference frames, but the local flatness theorem tells us that it is at least always possible to introduce a coordinate system which locally defines an inertial frame of reference (ie. a local Lorentz frame).

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[10] Some authors call it *Riemannian normal coordinates*.

[11] Please note that the principle of relativity is not more general in general relativity than in special relativity. There is only one principle of relativity! The only difference in curved spacetime is that reference frames can only be defined locally. The principle of relativity is therefore a local principle in curved spacetime, while it is a global principle in flat spacetime. The only thing that is more 'general' in general relativity is that it accounts for more general geometries than the Minkowski geometry. Several relativists actually do not like the name 'general relativity', and argue that *geometrodynamics* would be a much better name. This name reflects the essence of Einteins theory: the dynamic interaction between of matter fields and spacetime geometry.

1.5.3 Coordinate time versus proper time

We have already defined the proper time as the time measured on a comoving standard clock. The proper time should not be confused with the time coordinate $x^0 = ct$ in the line element. The "clock" related to the x^0 coordinate is often called the *coordinate clock*. The coordinate clock can be adjusted such that it ticks at another rate of course, this is simply a coordinate transformation. Thus the time measured on a coordinate clock has no significance as a measure of proper time intervals. Standard clocks however, can by definition never be adjusted, their fate is to give the proper time along a particular path. It is still possible to give the coordinate time a physical interpretation though. Very often coordinates are chosen such that $\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu}$. Spacetimes where there exist such coordinates are called *asymptotic Minkowskian spacetimes*. In such spacetimes we therefore have the limit

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2d\tau^2 \tag{1.70}$$

far away from the gravitational system. For standard clocks at rest in the coordinate system we have $dx = dy = dz = 0$ and the line-element gives $dt = d\tau$. Thus, for asymptotic Minkowskian spacetimes (such as the Schwarzschild spacetime), the time coordinate clock can be interpreted as the time measured on a standard clock at rest at infinity in the coordinate system.

1.6 Curvature

As soon as the manifold was defined, we where able to define functions, tensors, and parametrized paths on it. The geometry of the manifold, in the sense of well defined path lengths, needed additional structure, namely the metric tensor, to be well defined. Now the time has come to introduce more mathematical structure, the curvature tensor. First we need to introduce the *connection*, a mathematical structure which depends on the metric, but is not a tensor.

1.6.1 Covariant differentiation

Consider the partial derivative $\partial_\nu V^\mu$, where V^μ is a $\binom{1}{0}$ tensor. This looks like a mixed tensor of rank $\binom{1}{1}$ since there is one upper and one lower index. It actually fails to be a tensor though, since it does not transform as a tensor:

$$\begin{aligned}\partial_{\nu'} V^{\mu'} &= \frac{\partial x^\nu}{\partial x^{\nu'}}\partial_\nu\left(\frac{\partial x^{\mu'}}{\partial x^\mu}V^\mu\right)\\ &= \frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^\mu}\partial_\nu V^\mu + \frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\mu}V^\mu.\end{aligned} \tag{1.71}$$

The last term (in the last line) would not have appeared if $\partial_\nu V^\mu$ was a tensor. When formulating physical laws in curved spacetime we can therefore not use the partial derivative in equations. This is forbidden by the principle of general covariance, which states that the formulation of a physical law shall not depend on the choice of coordinates. What we need is the *covariant derivative*, an operator which transforms as a tensor an reduces to the usual derivative operator in Cartesian coordinates in flat spacetime. The *covariant derivative* operator D maps a $\binom{k}{l}$ tensor to a $\binom{k}{l+1}$ tensor. Thus $D\mathbf{V}$ is a $\binom{1}{1}$ tensor with components $(D\mathbf{V})^\mu_\nu \equiv D_\nu V^\mu$, one contravariant index specifying which component of the vector, and one covariant index specifying the direction of the derivative. The components of the covariant derivative of the vector field $\mathbf{V}$ is defined

$$D_\nu V^\mu \equiv \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda}V^\lambda, \tag{1.72}$$

where $\Gamma^{\mu}_{\nu\lambda}V^{\lambda}$ is called the *connection coefficients*. Since the covariant derivative transform according to the standard transformation law for a tensor

$$D_{\nu'}V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}D_{\nu}V^{\mu}, \tag{1.73}$$

the connection coefficients must transform as

$$\Gamma^{\mu'}_{\nu'\lambda'} = \frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\Gamma^{\mu}_{\nu\lambda} - \frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\mu'}}{\partial x^{\nu}\partial x^{\lambda}}, \tag{1.74}$$

which is easy to show by inserting (1.71) and (1.72) into (1.73) and solving the equation for $\Gamma^{\mu'}_{\nu'\lambda'}$. This is clearly not the transformation law for a tensor, due to the appearance of the last term on the right hand side of (1.74), which shows that the connection coefficients are *not* the components of a tensor object. It is the combination of the partial derivative and the connection coefficients, ie. the covariant derivative defined by (1.72), which becomes a true tensor. The transformation law for the connection coefficients can be reformulated

$$\Gamma^{\mu'}_{\nu'\lambda'} = \frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\Gamma^{\mu}_{\nu\lambda} + \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial x^{\nu'}\partial x^{\lambda'}}, \tag{1.75}$$

by solving (1.74) for $\Gamma^{\mu}_{\nu\lambda}$ and then interchanging the convention for which coordinates to be written with primed indices[12].

The (components of the) covariant derivative of a tensor of rank $\binom{k}{l}$ is defined

$$\begin{aligned} D_{\alpha}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_k} &= \partial_{\alpha}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_k} \\ &+ \Gamma^{\mu_1}_{\alpha\lambda}T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_k} + \Gamma^{\mu_2}_{\alpha\lambda}T^{\mu_1\lambda\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_k} + \ldots \\ &- \Gamma^{\lambda}_{\alpha\nu_1}T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_k} - \Gamma^{\lambda}_{\alpha\nu_2}T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\lambda\ldots\nu_k} - \ldots \end{aligned} \tag{1.76}$$

Hence the covariant derivative equals the partial derivative plus one term for each free index. Notice the minus sign in the terms associated with the covariant indices of T. For example, the covariant derivative of a one form has components

$$D_{\nu}u_{\mu} = \partial_{\nu}u_{\mu} - \Gamma^{\lambda}_{\nu\mu}u_{\lambda}. \tag{1.77}$$

The covariant derivative of a scalar field, which is a object with no free indices, equals the partial derivative. All this follows naturally from the transformation law of tensors, see for example [5, ch.3] for more details. In the following we will usually consider the special case of a vector field, but the generalization to a tensor field of arbitrary rank is given by proper index placement together with (1.76).

So far we have only considered the covariant derivative along the coordinate axes. We will now generalize to covariant derivatives along an arbitrary curve $x^{\mu}(\lambda)$. Just like the ordinary directional derivative along $x^{\mu}(\lambda)$ is $\frac{dV^{\mu}}{d\lambda} \equiv u^{\nu}\partial_{\nu}V^{\mu}$, we define the *covariant directional derivative*

$$\frac{DV^{\mu}}{d\lambda} \equiv u^{\nu}D_{\nu}V^{\mu}, \tag{1.78}$$

where $u^{\nu} \equiv \frac{dx^{\mu}}{d\lambda}$ is the tangent vector field to the curve. Note that the covariant directional derivative of a *vector* defines another *vector* field along the curve $x^{\mu}(\lambda)$. In general we have that the covariant

[12] It is totally arbitrary, of course, what we choose as primed indices. The essential point is that the primed and the unprimed indices refer to different coordinate systems.

directional derivative of a tensor T of rank $\binom{k}{l}$ is a new tensor of the same rank. This is contrary to the covariant derivative of T, which is a $\binom{k}{l+1}$ tensor. What we have called the covariant derivative of T, denoted DT, actually is the *gradient* of T. The covariant derivative, like any gradient, can be thought of as a derivative in an unspecified direction. A particular curve needs to be specified before the gradient becomes a derivative[13]. We see that the relation between the covariant derivative (1.72) and the covariant directional derivative (1.78) simply is that the latter is the result when the former is contracted with respect to the tangent vector u^μ. It may be rewarding to introduce abstract notation also for the covariant directional derivative. We simply state that that $\frac{D}{d\lambda}T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l}$ are the components of the tensor $D_{\mathbf{u}}T$, just like $D_\mu T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l}$ are the components of DT. The relation between DT and $D_{\mathbf{u}}T$ is then defined

$$D_{\mathbf{u}}T \equiv DT(\ ,\ \ ,\ldots,\mathbf{u}), \tag{1.79}$$

ie. DT is the tensor which take the tangent vector $\mathbf{u}$ as input and has the corresponding covariant directional derivative $D_{\mathbf{u}}T$ as output. Note that (1.78) can be rewritten

$$D_{\mathbf{u}}\mathbf{V} \equiv (u^\nu D_\nu V^\mu)\,\mathbf{e}_\mu. \tag{1.80}$$

1.6.2 The Christoffel connection

So far we have just analyzed the transformation properties of the connection coefficients without giving an explicit definition of them in terms of the metric. The transformation law in itself is not enough to give an unique definition of the connection coefficients. It is possible to give several different definitions of the connection coefficients which all transform according to (1.74). What we would like though, is an *unique* definition. To get rid of the ambiguity we must impose a new condition. In flat spaces, the partial derivatives transform as tensors if we choose Cartesian coordinates. Thus the connection coefficients are all zero in Cartesian coordinates in flat spaces. We now impose the condition that in curved spacetime the same should hold in a local Lorentz frame, ie. that $\Gamma^\mu_{\nu\lambda}(P) = 0$ in a coordinate system where $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $\partial_\sigma g_{\mu\nu}(P) = 0$. This is a natural condition from the point of view that the local Lorentz frame is the closest we can get to the global Lorentz frames of flat spacetime. The condition is implemented by demanding

$$D\mathbf{g} = 0, \tag{1.81}$$

which is called the condition of *metric compability*. The metric tensor is therefore sometimes called 'covariant constant'. Equation (1.81) is an example of a *coordinate independent* statement, and the notation is sometimes called *abstract notation*. In a coordinate basis, the statement (1.81) takes the form of a component equation:

$$D_\alpha g_{\mu\nu} = 0. \tag{1.82}$$

From this condition we will now derive an unique expression for the connection coefficients, and verify that they all vanish in a local Lorentz frame. It turns out that the expression for the connection coefficients simplifies much in a coordinate basis. In my subsequent work I will not make use of the most general expression, therefore I will restrict the derivation to coordinate bases. Actually, what I will derive is the *Christoffel connection*, which is the name for the connection in a coordinate basis.

[13] Some texts, for example [1], consequently use the name 'gradient' on the object we call covariant derivative, and the name 'covariant derivative' on the object we call a covariant directional derivative. More often than not though, authors just tend to call it 'covariant derivative' altogether, and the context determines what is meant.

It turns out that the Christoffel connection, in contrast to the general case, is symmetric in the lower indices, ie. $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$. This follows straight away from the condition that the Christoffel symbol vanish in a local Lorentz frame. Transforming from geodesic coordinates (where $\Gamma^{\lambda}_{\mu\nu}(P) = 0$) to arbitrary coordinates we get

$$\Gamma^{\mu'}_{\nu'\lambda'}(P) = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x^{\nu'} \partial x^{\lambda'}}, \tag{1.83}$$

according to (1.75). Since partial derivatives commute, ie. $\partial_{\nu'}\partial_{\lambda'} = \partial_{\lambda'}\partial_{\nu'}$, and the point P is arbitrary, the symmetry is shown explicitly. We will make use of this symmetry property in the derivation of the Christoffel symbol. The symmetry should therefore be regarded as an axiom of the derivation[14].

By using (1.76) equation (1.82) can be rewritten

$$D_{\alpha} g_{\mu\nu} = \partial_{\alpha} g_{\mu\nu} - \Gamma^{\lambda}_{\alpha\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\alpha\nu} g_{\mu\lambda} = 0. \tag{1.84}$$

Solving this equation for the Christoffel connection we get the unique expression we have sought after. By permuting indices (1.84) can also be written:

$$\begin{aligned} D_{\mu} g_{\nu\alpha} &= \partial_{\mu} g_{\nu\alpha} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\alpha} - \Gamma^{\lambda}_{\mu\alpha} g_{\nu\lambda} = 0, \\ D_{\nu} g_{\alpha\mu} &= \partial_{\nu} g_{\alpha\mu} - \Gamma^{\lambda}_{\nu\alpha} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\alpha\lambda} = 0. \end{aligned} \tag{1.85}$$

Subtracting the second and third of these permutations from (1.84) and using that the metric is symmetric (always), and that the connection coefficients are symmetric in the lower indices (only in coordinate basis) we get

$$2\Gamma^{\lambda}_{\mu\nu} g_{\alpha\lambda} = \partial_{\mu} g_{\nu\alpha} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu}. \tag{1.86}$$

By multiplying the equation with $g^{\alpha\sigma}$ and using $g_{\alpha\lambda} g^{\alpha\sigma} = \delta^{\sigma}_{\lambda}$ we finally get our sought after expression

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_{\mu} g_{\alpha\nu} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu} \right). \tag{1.87}$$

In a local Lorentz frame all partial derivatives of the metric vanish. Hence also (all components of) the Christoffel connection vanish in a local Lorentz frame according to (1.87). Thus we have shown that metric compability is equivalent to the desired condition of vanishing Christoffel connection in a local Lorentz frame. The desired condition have given us a unique definition of the (Christoffel) connection which is the one that is always used in general relativity (when working in a coordinate basis).

1.6.3 Parallel transport

As pointed out in section 1.4.2, tensors at different locations belongs to different tangent spaces. It is therefore not possible to define vector operations, such addition or the scalar product, between vectors at different locations. What we need is *parallel transport*, which describes the change in a vector when it is moved along a curve $x^{\mu}(\lambda)$. Vectors at different locations can be compared first when they

[14] Our desired condition is of course that the Christoffel symbols vanish in a local Lorentz frame. Mathematically though, I derive this as a consequence of metric compability. Since I also need to use the symmetry property in this derivation, it should be taken as an axiom, and not as a consequence of what I am deriving! This axiom is related to the choice of a *torsion free* connection in general relativity, a property often discussed in more detailed texts. Some authors prefer to pick the condition of vanishing Christoffel symbols as the fundamental axiom. The point though, is that all this hang together, metric compability (and a torsion free connection) is equivalent to vanishing Christoffel symbols in local Lorentz frames.

are parallel transported to a common event in spacetime. With the Christoffel connection at hand, it turns out that simple reasoning will lead us to the mathematical definition of parallel transport.

In flat spacetime with Cartesian coordinates, parallel transport is trivial. Just keep the components of the vector constant as you move the vector around. The local Lorentz frames have showed us that curved spacetime looks flat locally. Therefore we should expect that also in a local Lorentz frame, parallel transport is given by keeping the components of the vector constant. Consider now parallel transport of a vector $\mathbf{V}$ along a curve $x^\nu(\lambda)$, and let $u^\nu \equiv \frac{dx^\nu}{d\lambda}$ be the field of tangent vectors to the curve. We choose a point P along the curve with coordinates $x^\nu(\lambda_P)$, and pick a coordinate system which defines a local Lorentz frame at P, ie. $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu}(P) = 0$. The condition that V^μ is constant for an infinitesimal movement of the vector around P along the curve is satisfied if

$$\left.\frac{dV^\mu}{d\lambda}\right|_{\lambda=\lambda_P} = \partial_\nu V^\mu u^\nu|_{\lambda=\lambda_P} = 0. \tag{1.88}$$

This is not a tensor expression and is only valid in this very special coordinate system. The coordinate independent version of this expression is given by replacing the ordinary derivative with the covariant derivative:

$$\left.\frac{DV^\mu}{d\lambda}\right|_{\lambda=\lambda_P} = D_\nu V^\mu u^\nu|_{\lambda=\lambda_P} = 0. \tag{1.89}$$

In the considered coordinate system this expression is just the same as (1.88) as all the Christoffel coefficients vanish at P. The big difference between (1.88) and (1.89) though, is that the latter is valid in any coordinate system. Since the point P along the curve was arbitrary we get the following definition of parallel transport:

$$\frac{DV^\mu}{d\lambda} = u^\nu D_\nu V^\mu = 0. \tag{1.90}$$

This definition is the unique covariant expression which follows when demanding that parallel transport have the same meaning in a local Lorentz frame as in flat spaces[15]. The procedure we followed is actually given its own name, *principle of minimal coupling*. It states that a covariant equation valid in flat spacetime, can be generalized to curved spacetime by the substitutions $\partial_\mu \to D_\mu$ and $\eta_{\mu\nu} \to g_{\mu\nu}$.

Parallel transport defines a unique continuation of a vector along a given curve. Starting with a vector V^μ at a given point P this continuation along a curve $x^\mu(\lambda)$ (with tangent vector field u^ν) is defined by (1.90). The change of the vector when moving from a given point to another is not well defined, but will depend on the chosen curve between the events. If a vector is parallel transported along a closed curve back to the starting point, it may actually have changed depending on the path. This is not a controversy, but as we shall see, perhaps the clearest characteristic of curvature.

An important consequence of choosing a metric compatible connection, is that the metric is always parallel transported:

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\alpha}{d\lambda} D_\alpha g_{\mu\nu} = 0. \tag{1.91}$$

The nice thing about this, is that inner products is always preserved when two vectors are parallel

[15] We motivated the definition of parallel transport from the pictorial idea of moving a vector around on the manifold. From a mathematical point of view this might sound fuzzy. Some authors prefer to start with the definition itself rather than the physical ideas behind. Parallel transport along $x^\mu(\lambda)$ of a vector $\mathbf{V}$ at a point P is then just defined as the continuation of $\mathbf{V}$ given by equation (1.90).

transported along a common curve[16]:

$$\begin{aligned}\frac{D}{d\lambda}(\mathbf{U}\cdot\mathbf{V}) &= \frac{D}{d\lambda}(g_{\mu\nu}U^\mu V^\nu)\\ &= \left(\frac{D}{d\lambda}g_{\mu\nu}\right)U^\mu V^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}U^\mu\right)V^\nu + g_{\mu\nu}U^\mu\left(\frac{D}{d\lambda}V^\nu\right)\\ &= 0.\end{aligned} \tag{1.92}$$

On the second line, the first term vanish since the Christoffel connection is metric compatible, while the second and the third term vanish since we consider parallel transport of the vectors. This means that the norm of a vector ($\sqrt{\mathbf{U}\cdot\mathbf{U}}$), and the sense of orthogonality ($\mathbf{U}\cdot\mathbf{V} = 0$) is preserved under parallel transport.

Equipped with the concept of parallel transport, we can now make a geometric interpretation of the covariant directional derivative which is due to the mathematician Levi-Civita. Consider a vector field $\mathbf{V}$ and a curve $x^\mu(\lambda)$. Let $\mathbf{V}(\lambda)$ be short hand notation for $\mathbf{V}(x^\mu(\lambda))$. Let $\mathbf{V}_\parallel(\lambda+\Delta\lambda)$ be the vector $\mathbf{V}(\lambda+\Delta\lambda)$ parallel transported to $x^\mu(\lambda)$ along the curve. The covariant derivative can then be interpreted geometrically as

$$\frac{D}{d\lambda}\mathbf{V}(\lambda) = \lim_{\Delta\lambda\to 0}\frac{\mathbf{V}_\parallel(\lambda+\Delta\lambda)-\mathbf{V}(\lambda)}{\Delta\lambda}. \tag{1.93}$$

The only difference from the definition of the ordinary derivative, is that the two vectors $\mathbf{V}(\lambda)$ and $\mathbf{V}(\lambda+\Delta\lambda)$, which belong to different tangent planes, are parallel transported to the same event before subtracted. In a local Lorentz frame the components of the vector remains constant under parallel transport, and the definition becomes equal to the definition of the ordinary directional derivative.

With the concept of parallel transport at hand we can also give a geometric interpretation of the connection coefficients. So far we have discussed them from a rather formal point of view, defining them as the contribution which makes the covariant derivative transform as a tensor. Some authors prefer to introduce the connection coefficients in terms of the following definition:

$$D_\nu\mathbf{e}_\sigma = \Gamma^\mu_{\nu\sigma}\mathbf{e}_\mu. \tag{1.94}$$

This shows that the connection coefficient is a measure of the change (ie. the twisting, turning, expansion and contraction) of the basis vector field relative to parallel transport[17]. More precisely we see from (1.94) that "$\Gamma^\mu_{\nu\sigma}$ is the $\mathbf{e}_\mu$-direction-component of the change of the basis vector $\mathbf{e}_\sigma$ in the $\mathbf{e}_\nu$-direction". The geometric interpretation of the connection also have consequences for our view of covariant differentiation of course. Written out explicitly (1.80) becomes

$$D_\mathbf{u}\mathbf{V} = u^\nu\partial_\nu V^\mu\mathbf{e}_\mu + u^\nu V^\lambda\Gamma^\mu_{\nu\lambda}\mathbf{e}_\mu. \tag{1.95}$$

Inserting (1.94) this can be rewritten

$$D_\mathbf{u}\mathbf{V} = u^\nu\partial_\nu V^\mu\mathbf{e}_\mu + V^\lambda u^\nu D_\nu\mathbf{e}_\lambda. \tag{1.96}$$

This shows explicitly that the covariant derivative not only derivate the components of the vector, but also takes account for the change of the basis vectors relative to parallel transport. The covariant derivative may therefore be viewed as a generalized derivative operator which derivates *tensors* rather than components.

[16] I have not showed that the usual product rule of differentiation applies to the covariant derivative of a contraction. But I can ensure you it actually does, see for example [5, ch.3].

[17] Remember that if the covariant derivative of the basis vectors where zero they would be related by parallel transport (along an arbitrary curve). The covariant derivative is thus a measure of the change relative to parallel transport.

1.6.4 The geodesic equation

A *test particle* is a point particle not contributing to the gravitational field (spacetime curvature). A (mass) distribution can be idealized as a test particle if its spatial extension and mass is small enough to be neglected. According to general relativity, a test-particle only affected by gravity is a free moving particle. Its motion is called *geodesic motion*. The observed three acceleration of such a particle is just a manifestation of spacetime curvature.

A particle not influenced by any force follows a "straight line". In Euclidean spaces a path $x^i(s)$ defines a straight line if

$$\frac{d^2x^i}{ds^2} = 0, \tag{1.97}$$

where the parameter s is the path length of the curve. Consider a parameter λ related to the path length by $\lambda = c_1 s + c_2$, where c_1 and c_2 are constants. Equation (1.97) then implies

$$\frac{d}{d\lambda}\frac{dx^i}{d\lambda} = 0. \tag{1.98}$$

Thus we can say that in Euclidean space, a curve $x^i(\lambda)$ defines a straight line if the tangent vectors $\frac{dx^i}{d\lambda}$ are constant (as long as the parameter λ is related to the path length as explained above). Generalizing this equation to curved spacetime means that the ordinary derivative must be replaced by the covariant (directional) derivative. Thus a straight line is a path whose tangent vectors are related by parallel transport:

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0. \tag{1.99}$$

This can be rewritten

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}, \tag{1.100}$$

which is called the *geodesic equation*. The invariant parameter λ in this equation must be related to the proper time τ by $\lambda = c_1\tau + c_2$. A parameter related to the proper time in this way, is called an *affine parameter*. These are the only parameters suited to solve the geodesic equation on the standard form of (1.100). Note that the derivation of the geodesic equation is not restricted to a time-like path (although these are of particular interest since they define the paths of free moving test particles). The geodesic equation can therefore also be used to calculate space-like geodesics as long as the parameter λ relates to s, which is a measure of the path length, in the same way as it relates to the proper time τ for a time-like curve[18].

Note that if all the connection coefficients vanish the geodesic equation gives $\frac{d^2x^\mu}{d\lambda^2} = 0$, ie. no coordinate acceleration. Thus, in a local Lorentz frame, where all the components of the Christoffel connection is zero, there is no observed acceleration. In general relativity, an inertial frame is a reference frame where the three-acceleration of a test particle only influenced by gravity is zero. An inertial frame is therefore a freely falling frame. Metric compatibility is therefore equivalent to demanding that the local inertial frames of curved spacetime actually is the local Lorentz frame. The equivalence principle suggests that there should be a local inertial frame also in curved spacetime. Now we have identified the local inertial frame as the local Lorentz frame. It is quite pleasant that the local Lorentz frame, which is the closest thing we can get to the global Lorentz frame of flat spacetime, actually is the same thing as the local inertial frame! Metric compability is therefore the natural condition when searching for a unique definition for the connection in general relativity!

[18] It should be noted that there also exist a family of parameters such that the geodesic equation solves light-like geodesics. These parameters are not related to the distance though, since the path length of a light-like curve is vanishing by definition.

In special relativity the vector $\frac{d^2x^\mu}{d\tau^2}$ is called the four acceleration. It should be stressed that in curved spacetime $\frac{d^2x^\mu}{d\lambda^2}$ is not a vector at all, since it is a linear combination of the tangent vector $u^\mu = \frac{dx^\mu}{d\lambda}$ (which is a vector) and the Christoffel connection (which is not a vector). In curved spacetime the vector quantity associated with the acceleration is $A^\mu = \frac{D}{d\lambda}\frac{dx^\mu}{d\lambda}$, or in abstract notation $D_{\mathbf{u}}\mathbf{u}$. In curved spacetime it is the vector A^μ which is called the four-acceleration, not $\frac{d^2x^\mu}{d\lambda^2}$[19]. The tangent vector $u^\mu = \frac{dx^\mu}{d\lambda}$ though, is a vector both in curved and flat spacetime. The reason for this is just that the coordinates are scalars, and from (1.76) we see that the ordinary derivative of scalars coincide with the covariant derivative (since a scalar has no free indices)[20]. It should also be noted that the deeper reason for $\frac{d^2x^\mu}{d\lambda^2}$ being a tensor in special relativity, is that the transformations are restricted to the Poincare group, ie. transformations between inertial frames. It then follows from first principle that $\frac{d^2x^\mu}{d\lambda^2}$ transforms as a tensor:

$$\frac{d^2x^{\mu'}}{d\lambda^2} = \frac{d}{d\lambda}\frac{dx^{\mu'}}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{dx^\mu}{d\lambda}\right) = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{d^2x^\mu}{d\lambda^2}, \tag{1.101}$$

where in the last step has used the fact that the Poincare transformation is constant in spacetime, ie. $\frac{d}{d\lambda}\frac{\partial x^{\mu'}}{\partial x^\mu} = 0$. Sometimes more general transformations is studied even in flat spacetime[21]. In that case the transformation matrix $\frac{\partial x^{\mu'}}{\partial x^\mu}$ may not be constant. This means that also in flat spacetime, $\frac{d^2x^\mu}{d\lambda^2}$ is only a vector under the transformations of the Poincare group. If one chooses to study more general coordinate transformations, the four-acceleration must be generalized to $A^\mu = \frac{D}{d\lambda}\frac{dx^\mu}{d\lambda}$, just like in curved spacetime[22].

Lagrangian formalism There is also another route to the kinematics of curved spacetime. As shown in any textbook of general relativity, the geodesic equation can also be derived from the variational principle (ie. $\delta s' = 0$) by varying the action

$$s' = \int_{\lambda_1}^{\lambda_2} d\lambda\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{1.102}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$. Note that s' is the path length of the curve $x^\mu(\lambda)$ from λ_1 to λ_2. Varying the action s' gives the same result as varying the action

$$s = \int_{\lambda_1}^{\lambda_2} L, \tag{1.103}$$

[19]Note that according to (1.99) a free particle has vanishing four acceleration, ie. $A^\mu = 0$.

[20]To avoid confusion it should be stressed that even though the coordinates x^μ is written with an upper index, it is by no means any $\binom{1}{0}$ tensor. It is a scalar, a $\binom{0}{0}$ tensor. x^μ should be thought of as four different functions of the spacetime position, not as the components of a position vector. Remember that the notion of a position vector is only meaningful in flat spaces, not in curved, where all vectors live in tangent spaces (actually they live in tangent spaces in flat spaces to, but the concept of a position vector is nevertheless a useful concept in flat spaces)). In the language of differential forms it is a 0-form.

[21]In the literature an often studied example are so called hyperbolically accelerated reference frames

[22]There is no general agreement whether general coordinate transformations in Minkowski spacetime belongs to general relativity or special relativity. I prefer to classify it as special relativity, since general relativity really is about how matter fields creates spacetime curvature, and how the matter fields in the next turn respond to the curvature. Allthough it is correct that a general coordinate transformation demands the introduction of some tools which characterize the framework of general relativity (like a non vanishing connection), the spacetime remains flat, and I call it special relativity.

where

$$L = L(x^\mu, \dot{x}^\mu) = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{1.104}$$

We prefer the latter convention since it simplifies calculations. Varying the action s leads to the covariant *Euler-Lagrange equations*:

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = 0. \tag{1.105}$$

Solving this equation gives back the geodesic equations. This means that geodesics are paths of extremal length, ie. that the particle follow the path between two events which give (local) maxima for the proper time. Covariant Lagrange dynamics give us another strategy to calculate particle trajectories. Instead of solving the geodesic equation, which is a second order non-linear differential equation, we can use standard strategies for solving the covariant Lagrange equations. A specific coordinate x^μ is called a *cyclic coordinate* if it satisfy $\frac{\partial L}{\partial x^\mu} = 0$. If x^μ is a cyclic coordinate it follows from the Euler-Lagrange equation that the *conjugate momentum*

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} \tag{1.106}$$

is a constant of motion. Covariant Lagrange dynamics is therefore well suited for situations with symmetries in the Lagrangian. When solving the equations, the four-velocity identities (1.50) may be useful. Finally note that the Lagrangian (1.104) can be written down almost directly from the line element:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu d\lambda^2 = 2Ld\lambda^2. \tag{1.107}$$

Note that we have only considered material particles so far. For completeness let me briefly comment on the situation for a photon. The geodesic equation is also valid for a photon as far as we choose a suitable invariant parameter λ (we cannot choose the proper time in the case of a photon, since the path length of a light like geodesic is zero). Regarding the Euler-Lagrange equations, the Lagrangian 1.104 must be replaced by

$$L = \frac{1}{2}\mathbf{P}\cdot\mathbf{P} = \frac{1}{2}g_{\mu\nu}P^\mu P^\nu, \tag{1.108}$$

where $\mathbf{P}$ is the four momentum of the photon which in an orthonormal basis have components

$$P^{\hat{\mu}} = (\frac{E}{c}, \mathbf{p}), \tag{1.109}$$

where E is the energy and $\mathbf{p}$ is the three momentum of the photon in that frame.

1.6.5 The Riemann curvature tensor

With the Christoffel connection defined, we are now ready to introduce a new piece of mathematical structure to the manifold, the *Riemann curvature tensor*. Curvature is characterized by this tensor. This section is not very detailed, and instead of writing out all derivations, I will refer to a few textbook references. As with the connection, everything here is restricted to a coordinate basis.

In a coordinate basis the Riemann curvature tensor is defined by

$$R(\quad, \mathbf{A}, \mathbf{u}, \mathbf{v}) = (D_\mathbf{u}D_\mathbf{v} - D_\mathbf{v}D_\mathbf{u})\,\mathbf{A}, \tag{1.110}$$

From this equation we see that the Riemann tensor is a map from three vectors to a new vector, ie. a $\binom{1}{3}$ tensor. We find the components of the Riemann tensor by the usual method. Writing the the tensor out on the generic form

$$R = R^{\mu}_{\ \nu\alpha\beta}\mathbf{e}_{\mu} \otimes \underline{\mathbf{w}}^{\nu} \otimes \underline{\mathbf{w}}^{\alpha} \otimes \underline{\mathbf{w}}^{\beta}, \tag{1.111}$$

and letting it act on *basis* vectors, we find

$$R^{\mu}_{\ \nu\alpha\beta}\mathbf{e}_{\mu} = \left(D_{\alpha}D_{\beta} - D_{\beta}D_{\alpha}\right)\mathbf{e}_{\nu}. \tag{1.112}$$

Carrying out the differentiation[23] one gets

$$R^{\mu}_{\ \nu\alpha\beta}\mathbf{e}_{\mu} = \left(\Gamma^{\rho}_{\ \nu\beta}\Gamma^{\mu}_{\ \rho\alpha} - \Gamma^{\rho}_{\ \nu\alpha}\Gamma^{\mu}_{\ \rho\beta} + \partial_{\alpha}\Gamma^{\mu}_{\ \nu\beta} - \partial_{\beta}\Gamma^{\mu}_{\ \nu\alpha}\right)\mathbf{e}_{\mu}. \tag{1.113}$$

Thus the components of the Riemann tensor is

$$R^{\mu}_{\ \nu\alpha\beta} = \Gamma^{\rho}_{\ \nu\beta}\Gamma^{\mu}_{\ \rho\alpha} - \Gamma^{\rho}_{\ \nu\alpha}\Gamma^{\mu}_{\ \rho\beta} + \partial_{\alpha}\Gamma^{\mu}_{\ \nu\beta} - \partial_{\beta}\Gamma^{\mu}_{\ \nu\alpha}. \tag{1.114}$$

Equipped with the Riemann tensor the previously mentioned characteristics of curvature can be expressed quantitatively. We have for example mentioned that a vector parallel transported around a closed path on a curved manifold will change. It is a standard textbook example to consider a vector V^{μ} at P parallel transported along a closed loop of infinitesimal size. It can then be shown that the change in V^{μ} is characterized by the Riemann tensor at P:

$$\Delta V^{\kappa} = \frac{1}{2}\Delta A^{\alpha\beta}R^{\kappa}_{\ \mu\beta\alpha}V^{\mu}, \tag{1.115}$$

where $\Delta A^{\alpha\beta}$ is the area of the projection of the loop on the plane spanned by the coordinate lines x^{α} and x^{β}.

Another significance of the Riemann tensor, is that it is accociated with tidal forces arising from inhomogeneities in the gravitational field. Consider two particles with four velocity $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ which are separated by the infinitesimal separation vector $\mathbf{s}^{\mu}$. It can then be showed, see for example [4, ch.21], that the acceleration four vector is given by

$$\frac{D^2}{d\tau^2}s^{\mu} = -R^{\mu}_{\nu\sigma\rho}u^{\nu}s^{\sigma}u^{\rho}. \tag{1.116}$$

The Riemann tensor, like any tensor, cannot be transformed away. If the tensor has no-vanishing components in one coordinate system, it will have so in *all* coordinate systems. Similarly, if all components of the tensor are zero in one coordinate system, it will be so in every coordinate system. This follows directly from the fact that the new components are linear combinations of the old components according to the transformation law for tensors. Tidal forces can therefore not be transformed away. Gravitational (three) acceleration, however, which is associated with the Christoffel connection, can be transformed away due to the fact that the connection is not a tensor. This corresponds to going into a freely falling frame, ie. a local Lorentz frame.

[23] The first derivative is simply given by (1.94). The next derivative goes much like (1.96). See for example [7] for more details.

1.7 The Einstein field equation

Einstein's insight was that gravitation is not a force, in the conventional sense, but a manifestation of curved spacetime. In general relativity the source of spacetime curvature is matter and radiation fields. The part of the theory describing the production of spacetime curvature we refer to as the *dynamics* of general relativity. Matter and radiation fields in the next turn responds back to the curvature of spacetime. The part of the theory defining the motion of fields (and particles) in a curved spacetime background, we refer to as the *kinematics* of the theory. General relativity is founded on a single equation, the *Einstein field equation*, which defines both the dynamics and the kinematics of the theory. By geometric methods we have already found the kinematics of a *test particle*, ie. the geodesic equation. The Einstein field equation however, is not restricted to the special case of a test particle, but defines the evolution of an arbitrary system which might even be massive enough to be self gravitating. We will see that in the special case of a test particle, Einstein's field equation can be used to derive the geodesic equation.

The previous chapters have been quite detailed. The purpose was to give a good introduction to the conceptual and mathematical foundation of the theory. These chapters are actually not only the prerequisites for general relativity, but in fact for any theory explaining gravity from the geometric point of view. The best known alternative to general relativity is the Brans-Dicke theory. The similarities between these competing theories, is that neither of them introduce a conventional force to explain the kinematics, but explain it as a manifestation of spacetime curvature. The difference lays in the dynamics, ie. the theories do not agree on the curvature produced by a given field. Among these theories, general relativity has become the standard theory, not because of uniqueness, but because of its simplicity. It is by far the simplest theory consistent with observations so far. We can say that the previous chapters outline the basic prerequisites for understanding gravity as a geometric phenomenon, while this chapter introduce the simplest realization of these ideas. From the conceptual point of view, the previous chapters can therefore be regarded as more important than this one. For this reason we will take a much more pragmatic approach here, focusing on what will be really necessary for my subsequent work.

1.7.1 The energy-momentum tensor

Like the gravitational field couples to the scalar quantity mass in Newton's theory, it couples to a symmetric tensor of rank 2 which is called the *energy-momentum tensor* in Einstein's theory. The energy-momentum tensor describes material characteristics as energy density, energy flux, shear forces, pressure and stress. In a local reference frame (with basis vectors $\mathbf{e}_{\hat{\mu}}$) the components of the energy-momentum tensor

$$T^{\hat{\mu}\hat{\nu}} \equiv T(\mathbf{e}_{\hat{\mu}}, \mathbf{e}_{\hat{\nu}}) \tag{1.117}$$

have the following physical significance:

$$\begin{aligned} T^{\hat{0}\hat{0}} &= \text{energy density.} \\ T^{\hat{0}\hat{i}} &= \text{energy flux in direction i.} \\ T^{ij} &= \text{the } i\text{th component of the force per unit area exerted} \\ &\quad\ \text{across a surface with normal in the } j \text{ direction.} \end{aligned} \tag{1.118}$$

Notice that the spatial components T^{ij} is just the stress tensor of classical mechanics. The diagonal elements of T^{ij} are called *normal stress*, while the off-diagonal elements *shear stress*. The normal stress is called *pressure* if it is independent of direction, ie. if all diagonal components are equal in the rest-frame of the fluid. Allthough material fields are usually build up by particles, we idealize them as *fluids*, ie. a continuous matter distribution. We will now take a look at the energy-momentum tensor of some fluids which will be important in my later work. A *perfect fluid* is a fluid which can be characterized completely by its pressure and energy content. *Dust* is defined as a pressure-less perfect fluid.

Dust Since there are no internal forces in dust, it is characterized solely by its energy content. We define ρ to be the mass density, and hence ρc^2 the energy density, of the fluid as measured (locally) in the rest-frame of the fluid. In this frame the only non-vanishing component of the energy-momentum tensor is $T^{\hat{0}\hat{0}} = \rho c^2$. We need a coordinate independent expression though. Since the four velocity has components $u^{\hat{\mu}} = (c, 0, 0, 0)$ in this special Lorentz frame (see (1.69)), we have $T^{\hat{\mu}\hat{\nu}} = \rho u^{\hat{\mu}} u^{\hat{\nu}}$ and hence the covariant expression for the energy-momentum tensor for dust becomes

$$T^{\mu\nu} = \rho u^\mu u^\nu. \tag{1.119}$$

Let us consider a local reference frame not comoving in the fluid, and verify that the physical significance of the components still is as defined by (1.118). If the fluid has velocity v in the x direction according to this local reference frame, the four velocity will have components[24] $u^{\hat{\mu}} = (\gamma c, \gamma v, 0, 0)$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. In this frame the energy-momentum tensor will have components

$$T^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \gamma^2 \rho c^2 & \gamma^2 \rho c v^1 & 0 & 0 \\ \gamma^2 \rho c v^1 & \gamma^2 \rho v^1 v^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.120}$$

This is just the components we are expecting. Consider for example the component $T^{\hat{0}\hat{0}}$. In a comoving Lorentz frame the energy density is ρc^2. In this non-comoving frame however, there must be an additional factor of γ^2. One of them is due to the relativistic energy formula ($E = \gamma m c^2$), while the other one comes from length contraction of the volume (remember we are considering energy *density*, energy content divided by volume).

Perfect fluid A perfect fluid can be characterized completely by its energy content and pressure. Such a fluid looks isotropic in its rest frame, ie. in a comoving local reference frame. It can be shown that a perfect fluid has no heat transport or viscosity[25]. Defining p to be the pressure measured in the rest frame of the fluid, the energy-momentum tensor have components

$$T^{\hat{\mu}\hat{\nu}} = diag(\rho c^2, p, p, p) \tag{1.121}$$

in this frame. The covariant expression is

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu}. \tag{1.122}$$

[24] In a local reference frame, expressions from special relativity hold. This follows directly from the fact that $g_{\mu\nu} = \eta_{\mu\nu}$ locally in this coordinate system. Consider for example the four velocity $u^\mu \equiv \frac{dx^\mu}{d\tau}$. In an arbitrary coordinate system it holds that $u^\mu = \frac{dx^\mu}{dt}\frac{dt}{d\tau}$. Assuming $\frac{dy}{dt} = \frac{dz}{dt} = 0$, we get (from the line element) $\frac{dt}{d\tau} = \frac{1}{\sqrt{-g_{00} - g_{11}\frac{v^2}{c^2}}}$, where $v = \frac{dx}{dt}$. In a local reference frame we have $g_{00} = -1$ and $g_{11} = 1$, which gives the familiar expression from special relativity, ie. $u^{\hat{\mu}} = (\gamma c, \gamma v, 0, 0)$.

[25] Viscosity characterize the "thickness" of a fluid, ie. a measure of its resistance.

In the following work I will use the perfect fluid as model for matter. This is a good model for fluids with negligible shear stress, heat transport and viscosity. The model is an average of the considered properties (ρ and p) over scales that are large compared to atomic scales.

Energy and momentum conservation In flat spacetime energy and momentum conservation is defined by the four equations

$$\partial_\mu T^{\mu\nu} = 0, \tag{1.123}$$

where the zero component express energy conservation and the other momentum. This can be proved for each kind of field individually by applying the relevant dynamic equation. We can generalize this law to curved spacetime, by the standard procedure. First make use of the equivalence principle to claim that the conservation law should take the same form in a local Lorentz frame as in the global Lorentz frame of flat spacetime. Then use the fact that the Christoffel connection vanish locally in this frame, to rewrite from partial derivative to covariant derivative. This gives an expression valid in an arbitrary coordinate system. As already explained, all these steps are implemented simply by replacing partial derivatives with covariant derivatives, and the Minkowski metric with a general metric. Hence in curved spacetime we have the covariant expression

$$D_\mu T^{\mu\nu} = 0. \tag{1.124}$$

It should be commented that this only expresses a *local* conservation law for energy and momentum (since it reduces to (1.123) only in a local Lorentz frame). In flat spacetime the equation can be integrated up to give a global conservation law. Not so in curved spacetime where there is in general no conservation law for energy and momentum. The energy and momentum will change in response to a dynamic spacetime curvature, and this evolution is defined by (1.124).

1.7.2 Einstein's field equations

Einstein's field equation read:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.125}$$

where $R_{\mu\nu}$ and R are called the *Ricci tensor* and *Ricci scalar* respectively. These tensors are formed out of contraction of the Riemann curvature tensor, and defined by

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} \tag{1.126}$$

and

$$R \equiv g^{\mu\nu} R_{\mu\nu}. \tag{1.127}$$

The sum of tensors on the left-hand side of (1.125) is often referred to as *Einstein's curvature tensor* or only *Einstein's tensor*:

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.128}$$

The field equation can also be re-formulated on an equivalent form which is often convenient. Taking the trace of (1.125) we get

$$R = -\frac{8\pi G}{c^4} T. \tag{1.129}$$

Inserting this back into (1.125) we get

$$R_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T). \tag{1.130}$$

I will make use of this formulation of the field equation in my sub-sequent work.

Einstein's field equation relates the curvature of spacetime, represented by the left-hand side of (1.125), to the energy-momentum tensor. All kinds of matter and radiation contributes to the energy-momentum tensor. The energy associated with the gravitational field itself however, does not contribute. Indeed, the common perception today, is that it is impossible to set up a covariant expression for the energy-momentum tensor of the gravitational field. Finding a covariant expression for the energy-momentum tensor of the gravitational field was the subject of intense research for several decades. This expression was supposed to be included on the right-hand side of the field equation together with -and on equal footing as- the energy-momentum tensors for matter and radiation fields. The (possible) non-existence of such an expression does not mean that there is no contribution to the gravitational field from the energy of the gravitational field itself though. The common view today is that the field equation itself automatically takes account of such non-linear self coupling.

The Einstein equation is ten non-linear partial differential equations for the symmetric metric tensor $g_{\mu\nu}$. Non-linear differential equations are usually extremely hard to solve analytically. Einstein's field equation is no exception, and there exist no (known) general solution. Analytic solutions only exist in simple special cases with much symmetry. This is the reason why perturbation theory is so important in general relativity.

The ten equations are not all independent, but related by the four equations:

$$D_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0. \tag{1.131}$$

These equations follows from an important relation in differential geometry called Bianchi's second identity[26]. This means that Einstein's equation's are really only $10 - 4 = 6$ independent equations for the 10 independent components of the symmetric metric tensor. The four degrees of ambiguity, corresponds exactly to the freedom in choice of coordinates[27]. Note that (1.131) can be written $D_\mu E^{\mu\nu} = 0$. A direct consequence of this is that the local law for energy and momentum conservation -equation (1.124)- follow as an consequence of the field equation (1.125). Einstein demanded that this law should follow as an consequence of the field equations, and (1.125) is by far the simplest[28] -and therefore most natural- alternative.

1.7.3 The geodesic equation derived from the field equation.

In the previous section we noticed that $D_\mu T^{\mu\nu} = 0$ follows as a consequence of Einstein's field equation. This equation do not only imply a local conservation law for energy and momentum, but also defines the kinematics of general relativity. In this section I will show that in the special case of a test-particle, modeled as dust, this equation is reduced to the geodesic equation. At this point the reader may wonder why I choose to do this in full detail when the rest of section 1.7 is so "sketchy". The reason is that I will make use of it in a later chapter, to show that linearized gravity cannot be regarded a self-consistent theory of gravitation.

Inserting the energy-momentum tensor for dust (1.119) into (1.124) we get

$$u^\nu D_\mu \left(\rho u^\mu \right) + \rho u^\mu D_\mu u^\nu = 0. \tag{1.132}$$

[26] The Bianchi identity is included in every textbook of differential geometry or general relativity. I will not bother with it here though, since I make no use of it in my following work.

[27] Full freedom in choice of coordinates demands four degrees of freedom corresponding to the four coordinate transformations $x^\mu \to x^{\mu'}(x)$ possible in spacetime.

[28] Einstein initially tried $R_{\mu\nu} \propto T_{\mu\nu}$ as a possible candidate for a field equation, but soon rejected it since it was not consistent with the local law for energy and momentum conservation.

This begins to look like the geodesic equation on the form (1.99). What we need to show is that the first term on the left-hand side is zero. Multiplying this equation with u_ν and using the four identity $u^\nu u_\nu = -c^2$ we get

$$-c^2 D_\mu \left(\rho u^\mu\right) + \rho u^\mu u_\nu D_\mu u^\nu = 0. \tag{1.133}$$

Covariant differentiation of the four velocity identity gives

$$u_\mu D_\nu u^\mu + u^\mu D_\nu u_\mu = 0. \tag{1.134}$$

It is easy to show that both terms on the left hand side are equal:

$$u^\mu D_\nu u_\mu = u^\mu D_\nu \left(g_{\alpha\mu} u^\alpha\right) = u_\alpha D_\nu u^\alpha = u_\mu D_\nu u^\mu, \tag{1.135}$$

where in the second term we have used that the metric tensor is covariant constant, ie. $D_\nu g_{\alpha\mu} = 0$. Equation (1.134) therefore implies

$$u_\mu D_\nu u^\mu = 0. \tag{1.136}$$

Inserting this into (1.133) we get

$$D_\mu \left(\rho u^\mu\right) = 0. \tag{1.137}$$

Inserting this back into (1.132) and dividing by ρ we get

$$u^\mu D_\mu u^\nu = 0 \tag{1.138}$$

which is equivalent to (1.99) -the geodesic equation!

1.7.4 The Schwartzschild solution

Most of my following work will make use of perturbative methods. It will be very convenient to test my calculations (for sloppy errors) against known exact solutions. Here I will present, with no derivation, such an solution.

In 1916, only a few months after Einstein's field equation where published, Karl Schwarzschild found the solution outside a non-rotating black hole. Today, this solution is known as the Schwarzschild spacetime. In the so called *Schwartzschild-coordinates*, the line element takes the form

$$ds^2 = -(1 - \frac{R_s}{r})c^2 dt^2 + \frac{dr^2}{1 - \frac{R_s}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.139}$$

where $R_s \equiv \frac{2GM}{c^2}$ is the so called Schwarzschild radius, and where M is the mass of the black hole. Notice the coordinate singularity at the sphere $r = R_s$[29], which is called event-horizon of the black whole. The Schwarzschild spacetime does not only describe the solution for a black whole, but is the *exterior solution*, ie. the solution outside the mass distribution, of any spherical symmetric non-rotating mass distribution. Let us comment on the significance of the coordinates, ie. the Schwartzschild coordinates. Note that the Scwarzschild solution is asymptotic Minkowskian, which means that -see chapter 1.5.3- the time coordinate ticks at the same rate as a standard clock at infinity which is at rest in the coordinate system (ie. $r \to \infty$ and $dx = dy = dz = 0$). Also note that the radial coordinate r is not a measure of the proper distance from the origin, but rather is defined such that the proper circumference of a circle at r is simply $2\pi r$.

[29]This is a coordinate singularity, and not a physical singularity, which can be transformed away by a coordinate transformation.

The line element (1.139) is the Schwarzschild spacetime described on the standard form. It turns out that these coordinates are not suited for comparison against my approximate solutions which are isotropic in the spatial coordinates[30]. Therefore we will need to re-express the Schwarzschild spacetime in isotropic coordinates. This can be implemented by a coordinate transformation of the radial coordinate:

$$r \to \rho = \frac{1}{2}(r - \frac{R_s}{2} + \sqrt{r^2 - R_s r}). \tag{1.140}$$

Inverting this equation we get

$$r = \rho \left(1 + \frac{R_s}{4\rho}\right)^2, \tag{1.141}$$

and hence

$$1 - \frac{R_s}{r} = \frac{\left(1 - \frac{R_s}{4\rho}\right)^2}{\left(1 + \frac{R_s}{4\rho}\right)^2}. \tag{1.142}$$

From (1.141) and (1.142) we get:

$$\frac{dr^2}{1 - \frac{R_s}{r}} = \left(1 + \frac{R_s}{4\rho}\right)^4 d\rho^2. \tag{1.143}$$

Accordingly the line-element takes the form:

$$ds^2 = -\frac{\left(1 - \frac{R_s}{4\rho}\right)^2}{\left(1 + \frac{R_s}{4\rho}\right)^2} c^2 dt^2 + \left(1 + \frac{R_s}{4\rho}\right)^4 (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2). \tag{1.144}$$

In coordinates $x = \rho \sin\theta \cos\phi$, $y = \rho \sin\theta \sin\phi$, and $z = \rho \cos\theta$ this is written

$$ds^2 = -\frac{\left(1 - \frac{R_s}{4\rho}\right)^2}{\left(1 + \frac{R_s}{4\rho}\right)^2} c^2 dt^2 + \left(1 + \frac{R_s}{4\rho}\right)^4 (dx^2 + dy^2 + dz^2), \tag{1.145}$$

where $\rho^2 = x^2 + y^2 + z^2$.

So far, there is no general solution the Einstein equation, but solutions can be obtained under special circumstances, involving high degree of symmetry. Fortunately, a great part of the spacetimes we want to describe, actually involves the needed symmetry, like the field outside a symmetric object like a planet. However, a lot of interesting phenomenas can not be solved analytically and need to be solved numerically. In the next chapter, we will discuss an approximation to the field equations, valid in the weak-field regime (like in our solar system), which indeed *have* a general analytic solution. This is the linear approximation.

[30]Isotropic spatial coordinates means coordinates where the spatial metric components are all equal.

Chapter 2

Linearized gravity

As exact analytic solutions to Einstein's field equations are known only in simple special cases, perturbative methods are very important in gravitational theory. In this chapter we shall introduce the very simplest strategy for obtaining approximate solutions, namely linearization. By neglecting the non-linear terms in the field equation, we will obtain the so called *linearized field equation*. By a suitable choice of coordinates this equation takes the form of an inhomogeneous wave equation, the Helmholtz equation, which has a general analytic solution. It turns out that this equation is precisely what follows from a (quantum) field theoretical approach to gravity, if one demands a linear field equation and a massless spin-2 particle (the graviton). This perspective was investigated by Feynmann, Gupta and others in the early 60's, see for example [10]. The resulting formalism is therefore often called *the linearized theory of gravity*. It is not regarded as a serious candidate for a quantum theory of gravity though, since such a theory is believed to be of non-linear nature[1]. After the formalism is derived, we will explicitly show that linearized theory cannot be regarded as a self-consistent theory of gravity (even not on the classical level). The name 'linearized theory of gravity' is therefore, at best, misleading. In our approach however, linearized theory is only an approximation to general relativity which gives good accuracy in the case of weak gravitational fields. Nevertheless, we will adopt the standard terminology of linearized theory (which is also often used in the full theory) which is clearly inspired by the field-theoretical approach to gravity. As an example of such terminology we can mention the standard (coordinate) transformation law for the metric which is called the 'gauge symmetry' of the theory.

In my subsequent work, we will go beyond the accuracy provided by the linearized field equation, but it will still play an important role as a tool for calculations.

2.1 Expansion around flat spacetime

The starting point of linearized theory is to expand the Einstein equation around the flat spacetime metric. We therefore write:

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x). \tag{2.1}$$

Notice that all the space and time dependence belongs to the function $h(x)$, since the Minkowski metric is constant. The basic idea is now to rewrite the Einstein equation in terms of $h_{\alpha\beta}$ and neglect

[1]From a quantum field perspective the linear nature of electrodynamics comes from the fact that the mediating boson, the photon, does not couple back to the electromagnetic field since it is charge-less. In the gravitational case however, the gravitational field bears the gravitational charge, which is energy. This non-linear nature of gravity is the field theoretical reason for the Einstein equation being non-linear.

all terms that are non-linear. The result is the so called linearized field equation. In general we can say that the solutions to the linearized field equation are accurate to first order in $h_{\alpha\beta}$. They are therefore good approximations only in regions where $h_{\alpha\beta} \ll 1$. It should be stressed that the linearized field equation can be used in regions of the spacetime where the gravitational field is weak regardless of whether the gravitational field is globally weak or not. Indeed, it is not possible, by measuring the distant spacetime geometry of a given source, to discover whether that source has strong internal gravity, or weak. See [1, ch.19.3] for details and further discussions.

Basically we treat $h_{\alpha\beta}$ as a perturbation of the Minkowski metric, and assume that $|h_{\alpha\beta}| \ll 1$ (and $|\partial_\nu h_{\alpha\beta}| \ll 1$). This assumption clearly restricts both the physical situation (ie. the gravitational field must be weak enough to ensure that there exist some coordinate frame where $h_{\alpha\beta} \ll 1$) *and* the choice of coordinates (ie. we must, if it exists, choose a coordinate frame where $h_{\alpha\beta} << 1$).

As we will show later, the perturbation $h_{\alpha\beta}$ does actually not transform according to the standard transformation law for a covariant tensor of rank 2. It is therefore not a real tensor. Hence we must clarify how to raise and lower the indices of $h_{\alpha\beta}$. If the perturbation was a real tensor we would raise and lower it with the metric, for example:

$$h^{\alpha}_{\nu} = g^{\alpha\mu} h_{\mu\nu} = \eta^{\alpha\mu} h_{\mu\nu} + h^{\alpha\mu} h_{\mu\nu}. \tag{2.2}$$

In linearized theory however, we would neglect the last term which is quadratic in the perturbation, and the background metric $\eta^{\mu\nu}$ ($\eta_{\mu\nu}$) would act as a raising (lowering) operator. Even though $h_{\mu\nu}$ is not a real tensor, we follow the textbook tradition and *define* the background metric to act as a raising/lowering operator:

$$h^{\mu}_{\nu} \equiv \eta_{\alpha\nu} h^{\alpha\mu} = \eta^{\alpha\mu} h_{\alpha\nu}. \tag{2.3}$$

A consequence of this definition is that we cannot simply write the contravariant components of the metric as $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$. To find out how to expand the contravariant components of the metric around the flat background, we write

$$g^{\mu\nu} = \eta^{\mu\nu} + X^{\mu\nu}, \tag{2.4}$$

where $X^{\mu\nu}$ is a not yet determined function (of the same smallness as $h_{\mu\nu}$ of course). We then determine $X^{\mu\nu}$ from the relation which defines the contravariant components of the metric:

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}. \tag{2.5}$$

Inserting (2.4) into (2.5) we get

$$\begin{aligned} \delta^{\mu}_{\nu} + X^{\mu\alpha}\eta_{\alpha\nu} + \eta^{\mu\alpha} h_{\alpha\nu} + O(h^2) &= \delta^{\mu}_{\nu} \\ X^{\mu\alpha}\eta_{\alpha\nu} + \eta^{\mu\alpha} h_{\alpha\nu} &= 0 \\ X^{\mu}_{\nu} + h^{\mu}_{\nu} &= 0. \end{aligned} \tag{2.6}$$

In the first step we have used that quadratic terms in the perturbation ($O(h^2)$) are neglected in linearized theory, and in the second step we have used the background to raise and lower the perturbation (as defined above). Raising the indices again we get $X^{\mu\nu} = -h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}$ according to our definition. We can therefore (see (2.4)) conclude that the contravariant expansion must be defined by:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{2.7}$$

This definition is consistent with the tradition to raise and lower the perturbation with the Minkowski metric.

2.2 Symmetries of linearized theory

In this section we will review the symmetries of linearized theory, and we will see that there is a clear analogy to electrodynamics. As a reminder, we know that the full theory is invariant[2] under arbitrary coordinate transformations

$$x^\mu \to x^{\mu'}(x), \tag{2.8}$$

where $x^{\mu'}(x)$ is an invertible and differentiable function of the old coordinates, and with a differentiable inverse. Also remember the transformation law for the metric:

$$g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta}. \tag{2.9}$$

Inspired by standard field theory, some authors calls this the *gauge symmetry* of general relativity. This terminology suggests that there should be an associated conserved quantity. However, far as I know, the only invariance under the transformations (2.9) is the *form* of tensors (and therefore covariant equations), ie. their functional dependence is preserved, while the numerical values of their components will change. As far as I can see, there is therefore really no good reason at all to call (2.9) a *gauge* symmetry. In the linearized theory however, there is a corresponding invariance, as we shall soon see.

Let us now work out the relations in linearized theory corresponding to (2.8) and (2.9). We have already imposed some condition on the coordinates by assuming a coordinate-frame where the perturbation is small. This condition does however not uniquely fix the coordinates, and we can still make small changes in the coordinates that leave $\eta_{\mu\nu}$ unchanged, but makes small changes in $h_{\mu\nu}$. Linearized theory is invariant under so called *infinitesimal coordinate transformations*:

$$x^\mu \to x^{\mu'}(x) = x^\mu + \epsilon^\mu(x), \tag{2.10}$$

where $\epsilon^\mu(x)$ are four arbitrary functions with partial derivatives $\partial_\alpha \epsilon^\mu$ small enough to leave $|h_{\mu'\nu'}| \ll 1$ and $|\partial_{\alpha'} h_{\mu'\nu'}| \ll 1$. From (2.9) we can now find the gauge symmetry of linearized gravity. First note that

$$\frac{\partial x^\alpha}{\partial x^{\mu'}} = \frac{\partial}{\partial x^{\mu'}}(x^{\alpha'} - \epsilon^\alpha(x)) = \delta^\alpha_\mu - \frac{\partial}{\partial x^{\mu'}}\epsilon^\alpha(x) = \delta^\alpha_\mu - \frac{\partial x^\beta}{\partial x^{\mu'}}\frac{\partial}{\partial x^\beta}\epsilon^\alpha(x) = \delta^\alpha_\mu - \frac{\partial \epsilon^\alpha}{\partial x^\mu}, \tag{2.11}$$

where the last step is correct to linear accuracy in ϵ^μ[3]. Inserting this into (2.9) we find, to linear order in ϵ^μ:

$$g_{\mu'\nu'} = g_{\mu\nu} - \frac{\partial \epsilon^\alpha}{\partial x^\mu}\delta^\beta_\nu g_{\alpha\beta} - \frac{\partial \epsilon^\beta}{\partial x^\nu}\delta^\alpha_\mu g_{\alpha\beta} = g_{\mu\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu} - \frac{\partial \epsilon_\mu}{\partial x^\nu}. \tag{2.12}$$

Thus the transformation of $h_{\mu\nu}$ under infinitesimal coordinate transformations is given by

$$h_{\mu\nu} \to h_{\mu'\nu'} = h_{\mu\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu} - \frac{\partial \epsilon_\mu}{\partial x^\nu}. \tag{2.13}$$

This is called the *gauge transformation of linearized theory*. Notice that $h_{\mu\nu}$ is *not* a real tensor, ie. it does not transform according to the transformation law for a covariant tensor of rank two. Indeed, if you insert $h_{\mu\nu}$ into the standard transformation law for a tensor, what you find, after neglecting all

[2] Here 'invariant' refers to the invariance of the *form* of all equations in general relativity, ie. it is a *covariant* theory.

[3] Notice that, in linearized theory, we can replace $\partial_\mu f$ and $\partial_{\mu'} f$ by each other whenever f is a function of the same smallness as $h_{\mu\nu}$ and the coordinates x^μ and $x^{\mu'}$ are related by infinitesimal coordinate transformations.

higher order terms in the small quantities $h_{\mu\nu}$ and ϵ^α, is that $h_{\mu\nu}$ is invariant, ie. $h_{\mu\nu} \to h_{\mu'\nu'} = h_{\mu\nu}$. This is clearly not consistent with the result in equation (2.13), and $h_{\mu\nu}$ is not a real tensor.

The invariant quantity corresponding to the gauge transformation (2.13) is the linearized Riemann tensor. This tensor has components

$$R^{(1)}_{\alpha\beta\sigma\rho} = \frac{1}{2}\left(\partial_\sigma\partial_\beta h_{\alpha\rho} + \partial_\alpha\partial_\rho h_{\beta\sigma} - \partial_\sigma\partial_\alpha h_{\beta\rho} - \partial_\beta\partial_\rho h_{\alpha\sigma}\right), \tag{2.14}$$

see for example [1, eq.18.9]. It is a straightforward exercise to show that the components of this tensor are unaffected by gauge transformations of the form (2.13). Since the exercise contains considerably (but simple) algebra, let me just briefly give the recipe. First substitute $\partial_\mu \to \partial_{\mu'}$ and $h_{\mu\nu} \to h_{\mu'\nu'}$ into (2.14). There will be 8 terms on the form $\partial_{\sigma'}\partial_{\beta'}\partial_\alpha\epsilon_\rho$ (notice that *one* of the partial derivatives are with respect to the old coordinates). To linearized accuracy this term can be replaced by $\partial_\sigma\partial_\beta\partial_\alpha\epsilon_\rho$ (notice that now *every* partial derivatives are with respect to the old coordinates), which is accurate to linearized order[4]. When this is done you can interchange the order of the partial derivatives, and as it happens, the eight terms cancel exactly. Thus we have

$$R^{(1)}_{\alpha\beta\sigma\rho}(x) = R^{(1)}_{\alpha'\beta'\sigma'\rho'}(x'). \tag{2.15}$$

To linearized accuracy there is a *real* invariance[5]; the numerical values of the components of the Riemann tensor are invariant! This justifies the terminology used in linearized theory, ie. calling (2.13) a gauge transformation. In section (2.6) we shall see that there is a clear analogy to electrodynamics.

In the full theory, Poincare transformations have no special significance since there is (in general) no global Minkowski background. In the linearized theory of gravity however, the coordinate system is *almost* Minkowskian. It is therefore interesting to discuss the transformation properties of $h_{\mu\nu}$ under Poincare transformations. Linearized theory is clearly not invariant under arbitrary Lorentz transformations, but must be restricted to infinitesimal coordinate transformations, ie. those that does not spoil the condition $h_{\mu\nu} \ll 1$. Rotations never spoil this condition. Boosts however, can have arbitrary large gamma-factors, and must be limited to those that preserves the smallness of $h_{\mu\nu}$. Also note that nothing in the derivation is changed if we add a translation (a constant coordinate transformation), which means that linearized theory is invariant under the Poincare group (but with a restriction on the boost). Having clarified these restrictions, let us consider the transformation properties of the perturbation under Lorentz transformations:

$$\begin{aligned} g_{\alpha\beta}(x) = \eta_{\alpha\beta} + h_{\alpha\beta}(x) \to g_{\alpha'\beta'}(x') &= L^\alpha_{\ \alpha'} L^\beta_{\ \beta'} g_{\alpha\beta}(x) \\ &= L^\alpha_{\ \alpha'} L^\beta_{\ \beta'} \eta_{\alpha\beta} + L^\alpha_{\ \alpha'} L^\beta_{\ \beta'} h_{\alpha\beta}(x) \\ &= \eta_{\alpha'\beta'} + L^\alpha_{\ \alpha'} L^\beta_{\ \beta'} h_{\alpha\beta}(x). \end{aligned} \tag{2.16}$$

Accordingly the perturbation transforms as an ordinary tensor under Lorentz transformations:

$$h_{\alpha\beta} \to h_{\alpha'\beta'} = L^\alpha_{\ \alpha'} L^\beta_{\ \beta'} h_{\alpha\beta}. \tag{2.17}$$

It turns out that the perturbation $h_{\mu\nu}$, which in general is not a tensor, transform as a tensor in the special case where the infinitesimal coordinate transformation is a Lorentz transformation.

[4] See footnote 3.

[5] In the full theory there is only an invariance of the form of the Riemann tensor, ie. its functional dependence on the metric remains unchanged, but here, in linearized theory, there is a real invariance in the sense of numerical invariance of the components.

2.3 The linearized field equation

In this section we will derive the linearized field equation. First we introduce the notation

$$\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma(1)}_{\alpha\beta} + \Gamma^{\gamma(2)}_{\alpha\beta}, \tag{2.18}$$

where $\Gamma^{\gamma(1)}_{\alpha\beta}$ is linear in $h_{\alpha\beta}$ and $\Gamma^{\gamma(2)}_{\alpha\beta}$ is quadratic. According to the definition of the Christoffel connection (1.87), we see that

$$\Gamma^{\gamma(1)}_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left[\partial_\beta h_{\delta\alpha}(x) + \partial_\alpha h_{\delta\beta}(x) - \partial_\delta h_{\alpha\beta}(x)\right], \tag{2.19}$$

since the Minkowski metric is constant. We will of course not need $\Gamma^{\gamma(2)}_{\alpha\beta}$ in linearized theory, but for later use we note that

$$\Gamma^{\gamma(2)}_{\alpha\beta} = -\frac{1}{2}h^{\gamma\delta}\left[\partial_\beta h_{\delta\alpha}(x) + \partial_\alpha h_{\delta\beta}(x) - \partial_\delta h_{\alpha\beta}(x)\right], \tag{2.20}$$

where the minus sign comes from the definition (2.7). Using the same kind of notation the linearized Einstein equation can be written

$$E^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.21}$$

What we need is therefore to linearize the Ricci tensor and the Ricci scalar. We start with the former. From (1.114) and (1.126) we see that the Ricci tensor has components

$$R_{\alpha\beta} = \partial_\gamma \Gamma^{\gamma}_{\alpha\beta} - \partial_\beta \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\delta}_{\gamma\delta} - \Gamma^{\gamma}_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}. \tag{2.22}$$

The last two terms do not contribute to $R^{(1)}_{\alpha\beta}$ since $\Gamma^{\gamma(1)}_{\alpha\delta}$ multiplied by itself is quadratic in $h_{\alpha\beta}$. Hence the linearized Ricci tensor has components

$$\begin{aligned} R^{(1)}_{\alpha\beta} &= \partial_\gamma \Gamma^{\gamma(1)}_{\alpha\beta} - \partial_\beta \Gamma^{\gamma(1)}_{\alpha\gamma} \\ &= -\frac{1}{2}\eta^{\gamma\delta}\partial_\gamma\partial_\delta h_{\alpha\beta} + \frac{1}{2}\partial_\gamma\partial_\alpha \eta^{\gamma\delta}h_{\delta\beta} + \frac{1}{2}\partial_\beta\partial_\delta\eta^{\gamma\delta}h_{\alpha\gamma} - \frac{1}{2}\eta^{\gamma\delta}\partial_\beta\partial_\alpha h_{\delta\gamma}. \end{aligned} \tag{2.23}$$

In the first term we recognize the d'Alembertian operator, $\eta^{\gamma\delta}\partial_\gamma\partial_\delta = -\frac{1}{c^2}\partial_t^2 + \nabla^2 \equiv \Box$. We also introduce the notation $h \equiv h^{\gamma}_{\gamma} = \eta^{\gamma\delta}h_{\gamma\delta}$, and refer to h as the 'trace' of $h_{\mu\nu}$. Inserting the d'Alembertian and raising indices with the background metric gives

$$R^{(1)}_{\alpha\beta} = -\frac{1}{2}\Box h_{\alpha\beta} + \frac{1}{2}\partial^\gamma\partial_\alpha h_{\gamma\beta} + \frac{1}{2}\partial_\beta\partial^\gamma h_{\alpha\gamma} - \frac{1}{2}\partial_\alpha\partial_\beta h. \tag{2.24}$$

The linearized Ricci scalar then becomes

$$R^{(1)} \equiv \eta^{\alpha\beta}R^{(1)}_{\alpha\beta} = -\Box h + \partial^\mu\partial^\nu h_{\mu\nu}, \tag{2.25}$$

and according to (2.21) the linearized Einstein tensor becomes:

$$E^{(1)}_{\alpha\beta} = -\frac{1}{2}\Box h_{\alpha\beta} + \frac{1}{2}\partial^\gamma\partial_\alpha h_{\gamma\beta} + \frac{1}{2}\partial_\beta\partial^\gamma h_{\alpha\gamma} - \frac{1}{2}\partial_\alpha\partial_\beta h + \frac{1}{2}\eta_{\alpha\beta}\Box h - \frac{1}{2}\eta_{\alpha\beta}\partial^\mu\partial^\nu h_{\mu\nu}. \tag{2.26}$$

We now introduce the so called "trace-reversed tensor"

$$\overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h. \tag{2.27}$$

Notice that the inverse relation is similar, ie.

$$h_{\alpha\beta} = \overline{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\overline{h}, \tag{2.28}$$

which is easy to show by taking the trace of (2.27) and inserting the result back into the equation. In this notation the linearized Einstein tensor simplifies to:

$$E^{(1)}_{\alpha\beta} = -\frac{1}{2}\Box\overline{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\partial^\rho\partial^\sigma\overline{h}_{\rho\sigma} + \frac{1}{2}\partial^\rho\partial_\beta\overline{h}_{\alpha\rho} + \frac{1}{2}\partial^\rho\partial_\alpha\overline{h}_{\beta\rho}, \tag{2.29}$$

which is straight forward to confirm by inserting (2.27). Finally we insert this into (2.21) and get

$$\Box\overline{h}_{\alpha\beta} + \eta_{\alpha\beta}\partial^\rho\partial^\sigma\overline{h}_{\rho\sigma} - \partial^\rho\partial_\beta\overline{h}_{\alpha\rho} - \partial^\rho\partial_\alpha\overline{h}_{\beta\rho} = -\frac{16\pi G}{c^4}T_{\alpha\beta}, \tag{2.30}$$

which is the linearized field equation.

2.4 Lorentz gauge

In section 2.2 we figured out the gauge symmetry of linearized theory, see (2.13). As we will soon proof, this gauge freedom can be used to choose the condition

$$\partial^\nu\overline{h}_{\mu\nu} = 0. \tag{2.31}$$

We will follow the tradition to call this condition the *Lorentz gauge*, a name which comes from the analogy to the condition with the same name in electrodynamics ($\partial_\mu A^\mu = 0$). In the Lorentz gauge, the linearized field equation (2.30) simplifies considerably as three of the terms on left-hand side vanish. Nothing happens on the right-hand side though, since the energy-momentum tensor never depends explicitly on partial derivatives of the metric (see for example the energy-momentum tensor for a perfect fluid (1.122)). Hence, in the Lorentz gauge, the linearized field equation (2.30) is simplified to an inhomogeneous wave equation:

$$\Box\overline{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}. \tag{2.32}$$

This is a very common differential equation in physics. The factor $1/c^2$ in the operator $\Box$ tells that the speed of gravity equals the speed of light c. The special retarded solution of (2.32) is

$$\overline{h}_{\alpha\beta} = 4\frac{G}{c^4}\int\frac{T_{\alpha\beta}(t_-, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3x', \tag{2.33}$$

where t_- is short-hand notation for the retarded time

$$t_- = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}. \tag{2.34}$$

It should be noted that the special retarded solution is not the only mathematical solution to the wave equation (2.32). There is also an "advanced solution" where t_- is replaced by $t_+ = t + \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ in

(2.33). The full mathematical solution of the wave equation (2.32) is a linear combination of the special retarded solution (2.33) and the advanced solution. The latter is not acceptable as a physical solution though, since it tells that the geometry of spacetime depends on the matter distribution in the future! The special retarded solution on the other hand, is consistent with the principle of causality and is therefore the physical interesting part of the mathematical solution.

Equations (2.30), (2.31) and (2.32) summarizes the formalism of linearized theory, and will be much used in the following chapters. In my subsequent work I will also need the wave equation (2.32) rewritten in terms of the perturbation $h_{\mu\nu}$ rather than the trace-reversed tensor $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. We use a bar to imply a corresponding operation on any other tensor. Notice that

$$\overline{\overline{h}}_{\alpha\beta} = \overline{h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h} = \overline{h}_{\alpha\beta} - \frac{1}{2}\overline{\eta_{\alpha\beta}}h = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h - \frac{1}{2}(\eta_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}4)h = h_{\alpha\beta}. \tag{2.35}$$

The wave equation (2.32) can thus be rewritten[6]

$$\Box h_{\alpha\beta} = -\frac{16\pi G}{c^4}\overline{T}_{\alpha\beta}. \tag{2.36}$$

This form of the linearized field equation is sometimes convenient as it is a differential equation for $h_{\alpha\beta}$ rather then $\overline{h}_{\alpha\beta}$. We will make use of this later. Notice that (2.36) is the linearized theories' version of (1.130).

Then, let us prove that it is always possible to choose a coordinate system where the Lorentz gauge is satisfied, ie. a coordinate system where the linearized field equation takes the form of a simple wave equation. In section 2.2 we showed that the gauge transformation of $h_{\mu\nu}$ under the symmetry group of linearized theory is

$$h_{\mu\nu} \to h_{\mu'\nu'} = h_{\mu\nu} - \frac{\partial\epsilon_\nu}{\partial x^\mu} - \frac{\partial\epsilon_\mu}{\partial x^\nu}. \tag{2.37}$$

Acting on this relation with $\eta^{\mu\nu}$ we find that the trace of $h_{\mu\nu}$ transforms as

$$h \to h' = h - 2\partial_\rho\epsilon^\rho, \tag{2.38}$$

and therefore, in terms of $\overline{h}_{\mu\nu}$, the gauge transformation becomes

$$\overline{h}_{\mu\nu} \to \overline{h}_{\mu'\nu'} = \overline{h}_{\mu\nu} - \frac{\partial\epsilon_\nu}{\partial x^\mu} - \frac{\partial\epsilon_\mu}{\partial x^\nu} + \eta_{\mu\nu}\partial_\rho\epsilon^\rho. \tag{2.39}$$

We now assume that, in a given coordinate system, $\partial^\nu\overline{h}_{\mu\nu}$ does not vanish, ie. the Lorentz gauge condition is not satisfied. From (2.39) we see that an infinitesimal coordinate transformation $x^\mu \to x^{\mu'}$ changes this to

$$\partial^{\nu'}\overline{h}_{\mu'\nu'} = \partial^\nu\overline{h}_{\mu\nu} - \partial^\nu\frac{\partial\epsilon_\nu}{\partial x^\mu} - \Box\epsilon_\mu + \partial_\mu\partial_\rho\epsilon^\rho = \partial^\nu\overline{h}_{\mu\nu} - \Box\epsilon_\mu. \tag{2.40}$$

Assuming that the Lorentz gauge condition is satisfied in the new coordinate system (ie. $\partial^{\nu'}\overline{h}_{\mu'\nu'} = 0$), we get the condition

$$\Box\epsilon_\mu = \partial^\nu\overline{h}_{\mu\nu}. \tag{2.41}$$

This is just the inhomogeneous wave equation again! As already discussed, this equation always admits solutions. Thus we have showed that it is always possible to choose coordinates such that the linearized field equation takes the form of a simple (inhomogeneous) wave equation.

[6] The Minkowski metric $\eta_{\alpha\beta}$ commutes with $\Box$.

2.5 The Newtonian limit

As we have obtained the linearized field equation from the Einstein equation, we are ready to check that there exists a Newtonian limit of general relativity. In terms of the gravitational potential U and the mass density ρ of the source, Newton's law of gravitation can be summarized as follows:

1. Mass generates gravitational potential according to (Poisson's equation)

$$\nabla^2 U = 4\pi G\rho. \tag{2.42}$$

2. Gravitational potential generates motion according to

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla U. \tag{2.43}$$

We know that Newtonian gravity must be valid in the non-relativistic limit of general relativity; the limit where the fields are weak and the motion of test-particles and sources are neglectible (ie. small compared to the speed of light). Hence, the linearized field equations and the geodesic equation should agree with Newtonian gravity when we neglect all velocity dependent terms and assume a stationary metric. To identify the gravitational potential U in terms of the metric, we consider the spatial part of the geodesic equation:

$$\begin{aligned}\ddot{x}^i &= -\Gamma^i_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &\approx -\Gamma^i_{00}c^2.\end{aligned} \tag{2.44}$$

The last line is a lowest order approximation which follows from assuming weak fields and slow motion ($v \ll c$) such that we can use the limits $\frac{dt}{d\tau} \to 1$ and $\frac{v}{c} \to 0$. Inserting the linearized Christoffel-connection, see (2.19), and neglecting all time-derivatives we are left with

$$\ddot{x}^i = -\frac{1}{2}c^2\partial_i h_{00}, \tag{2.45}$$

or as a vector equation

$$\ddot{\mathbf{x}} = -\frac{1}{2}c^2\nabla h_{00}. \tag{2.46}$$

Correspondence with Newtonian gravity (2.43) requires

$$U = -\frac{1}{2}c^2 h_{00}. \tag{2.47}$$

As we have now identified the gravitational potential in terms of the metric, we need to check whether the Newtonian field equation are reproduced from the linearized field equation:

$$\Box\bar{h}_{\mu\nu} = -16\pi\frac{G}{c^4}T_{\mu\nu}. \tag{2.48}$$

A very realistic model for the energy-momentum tensor is the perfect fluid. However, as we will show in a later chapter, the contribution in the metric from the pressure p, is suppressed by a factor $\sim \frac{p}{\rho c^2}$ relative to the contribution from the energy ρ. Under "normal" astrophysical conditions the pressure can thus be neglected, and it is sufficient to use dust as a model for the energy-momentum tensor, ie.

$T^{\mu\nu} = \rho \dot{x}^\mu \dot{x}^\nu$. Inserted into (2.48) we see that $\overline{h}_{00}$ is the only non-zero component of $\overline{h}_{\mu\nu}$ when we neglect the speed of the source. The relevant part of (2.48) is therefore

$$\nabla^2 \overline{h}_{00} = -16\pi \frac{G}{c^2}\rho. \tag{2.49}$$

From (2.28) we see that $h_{00} = \overline{h}_{00} - \frac{1}{2}\eta_{00}\overline{h} \approx \overline{h}_{00} + \frac{1}{2}\overline{h}_0^0 = \overline{h}_{00} - \frac{1}{2}\overline{h}_{00} = \frac{1}{2}\overline{h}_{00}$, which means that $U = -\frac{1}{2}c^2 h_{00} = -\frac{1}{4}c^2\overline{h}_{00}$. Inserting this into the above equation we obtain Poisson's equation

$$\nabla^2 U = 4\pi G\rho, \tag{2.50}$$

and we have confirmed that Newtonian gravity is the non-relativistic limit of general relativity. Notice that if we did not know the (arbitrary) constant in the Einstein equation (and hence in the linearized field equation), the Newtonian limit would have determined it.

2.6 Perspectives on linearized gravity.

As we have worked out the basic of linearized theory, it is well worth to pause for a moment and reflect on the results. We have seen that the transformation of $h_{\mu\nu}$ under the coordinate transformations of linearized theory, can be viewed as a gauge transformation where the associated invariant quantity is the linearized Riemann tensor. The perturbation $h_{\mu\nu}$ is then viewed as a basic potential which can be "gauged", and the linearized Riemann tensor is the associated invariant field quantity. There is a clear analogy to electrodynamics where the basic potential is the four-vector A^μ and the conserved field quantity is the electromagnetic field tensor $F^{\mu\nu}$. Recall that the components of $F^{\mu\nu}$ are the standard electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. The analogy is spelled out in table 2.1. Allthough this seems like an obvious analogy there are some important differences which should not be overlooked. In particular it should be stressed that in gravitational theory, there is no *unique* field quantity corresponding to the electric and magnetic fields in electrodynamics[7]. To justify this I will quote from chapter 16.5 of the celebrated textbook Gravitation (Misner, Thorne, Wheeler (1972), see [1]):

> *Many different mathematical entities are associated with gravitation: the metric, the Riemann curvature tensor, the Ricci curvature tensor, the curvature scalar, the covariant derivative, the connection coefficients, etc. Each of these play an important role in gravitation theory, and none is so much more central than others that it deserves the name "gravitational field". Thus it is that (...) the terms "gravitational field" and "gravity" refer in a vague, collective sort of way to all of these entities.*

Why then is the linearized Riemann tensor chosen as the field quantity in linearized theory? The simple answer is of course that the linearized Riemann tensor turns out to be the invariant quantity[8]. The analogy to the field theory of electromagnetism is therefore not a very deep one, but depends on the arbitrary choice of the linearized Riemann tensor as the "gravitational field". The reader should also be aware of an important difference in the equation of motion, see the last line of table 2.1. Notice that it is just in electrodynamics that the equation of motion is written in terms of the field

[7] It should be commented that a lot of authors, see for example [3], defines the metric as the gravitational field. There is no deep reason for this convention, and it is really just a matter of terminology what one chooses to call the gravitational field.

[8] Since the Ricci tensor, the Ricci scalar and the Einstein tensor are contractions of the Riemann tensor, the linearized versions of these tensors are also invariant under the gauge transformation. The linearized Christoffel connection (2.19) however is not invariant. Needless to say, the metric is not invariant as it is $h_{\mu\nu}$ that is being gauged.

quantities $\mathbf{E}$ and $\mathbf{B}$ (or $F^{\mu\nu}$ in the covariant formulation). This ensures physical invariance under gauge transformations, ie. the physical acceleration is unaffected by the transformation of potentials. In linearized theory the equation of motion is written in terms of the linearized Christoffel connection $\Gamma^{\mu(1)}_{\alpha\beta}$, see (2.19). This one is clearly not invariant under the gauge transformation of linearized theory[9]. The (coordinate) acceleration is therefore not invariant under the gauge transformations of linearized theory. This should not surprise you as the gauge transformation of linearized theory corresponds to a change of coordinate system! The *physical* acceleration of the particles though, is certainly not affected by the choice of coordinate system.

As a conclusion we may say that the analogy between linearized gravity and electrodynamics is only a formal one. There is clearly a very strong analogy between the basic equations of the theories, as summarized in table 2.1, but the foundations of the theories are very different. Electrodynamics is a standard field theory where the fields lives on a (fixed) spacetime background, while general relativity is the field theory of the dynamics of the background itself. Linearized gravity is just an approximation to general relativity, and $h_{\mu\nu}$ should, strictly speaking, not be interpreted as an ordinary field living on a flat spacetime background $\eta_{\mu\nu}$. Nevertheless, this view-point is actually still quite common, also in the main-stream literature, see for example box 18.2 (part D) in [1]. Of course, I agree that this view point make some sense since the coordinate-system is *almost* Minkowskian, but taking it too literary necessarily leads to self-contradictions. One of them is the fact that the coordinate acceleration, as given by the geodesic equation, is not, as we have seen, invariant under gauge transformations. As explained above this can be understood by recalling that the gauge transformations of linearized gravity corresponds to a change of coordinate systems, ie. from a coordinate system where the metric is $g_{\mu\nu} = \eta_{\mu\nu} + h^{\text{old}}_{\mu\nu}$ to a new coordinate system where it is $g_{\mu\nu} = \eta_{\mu\nu} + h^{\text{new}}_{\mu\nu}$. In the viewpoint where $h_{\mu\nu}$ is an ordinary field living on the flat spacetime background however, there is no associated change of coordinate systems under the transformation $h^{\text{old}}_{\mu\nu} \to h^{\text{new}}_{\mu\nu}$, ie. both $h^{\text{old}}_{\mu\nu}$ and $h^{\text{new}}_{\mu\nu}$ belongs to the same coordinate system with metric $g_{\mu\nu} = \eta_{\mu\nu}$.

2.7 Linearized theory and the equation of motion

In this section we shall study a consistency problem of the linearized theory. In the previous section we stated that the equation of motion for a particle in linearized theory is:

$$\frac{du^\mu}{d\tau} = -\Gamma^{\mu(1)}_{\alpha\beta} u^\alpha u^\beta. \tag{2.51}$$

Recall from chapter 1.7.3 that in the full theory, the equation of motion, ie. the geodesic equation, follows from the Einstein equation. We shall now follow the same procedure for linearized theory and check whether (2.51) is reproduced. The result of these calculations are mentioned in most textbooks on general relativity, but they do not care about showing you the calculations. I will therefore show the calculations in full detail; the procedure is just like in chapter 1.7.3, but the details somewhat different. You may want to review that chapter again before reading further.

First we act on the linearized field equation (2.32) with the partial differential operator:

$$\partial_\nu \Box \bar{h}^{\mu\nu} = \Box \overbrace{\partial_\nu \bar{h}^{\mu\nu}}^{=0} = -\frac{16\pi G}{c^4} \partial_\nu T^{\mu\nu}, \tag{2.52}$$

[9] A similar exercise as the one used to obtain (2.15) shows that the transformation of the Christoffel connection under the coordinate transformation (2.13) becomes $\Gamma^{\gamma(1)}_{\alpha\beta}(x) \to \Gamma^{\gamma'(1)}_{\alpha'\beta'}(x') = \Gamma^{\gamma(1)}_{\alpha\beta}(x) - 2\partial_\alpha\partial_\beta\epsilon^\gamma(x)$.

	LINEARIZED GRAVITY	ELECTRODYNAMICS
POTENTIALS	$h_{\mu\nu}$	A^μ
FIELD EQUATION	$\Box\overline{h}_{\alpha\beta} + \eta_{\alpha\beta}\partial^\rho\partial^\sigma\overline{h}_{\rho\sigma} - \partial^\rho\partial_\beta\overline{h}_{\alpha\rho}$ $-\partial^\rho\partial_\mu\overline{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\alpha\beta}$	$\Box A^\mu - \partial^\mu\partial_\alpha A^\alpha = -\mu_0 j^\mu$
GAUGE TRANSFORMATION	$h_{\mu\nu} \to h_{\mu\nu} - \frac{\partial\epsilon_\nu}{\partial x^\mu} - \frac{\partial\epsilon_\mu}{\partial x^\nu}$	$A_\mu \to A_\mu + \partial_\mu\chi$
INVARIANT FIELD QUANTITIES	$R^{(1)}_{\mu\nu\alpha\beta}$	$F_{\mu\nu}$
PREFERRED GAUGE (LORENTZ GAUGE)	$\partial_\mu h^{\mu\nu} = 0$	$\partial_\mu A^\mu = 0$
FIELD EQUATION IN PREFERRED GAUGE	$\Box\overline{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}$	$\Box A^\mu = -\mu_0 j^\mu$
EQUATION OF MOTION FOR PARTICLE	$\frac{du^\mu}{d\tau} = -\Gamma^{\mu(1)}_{\alpha\beta}u^\alpha u^\beta$	$\frac{du^\mu}{d\tau} = \frac{q}{m}F^{\mu\nu}u_\nu$

Table 2.1: Analogy between linearized gravity and electrodynamics.

and hence

$$\partial_\nu T^{\mu\nu} = 0. \tag{2.53}$$

According to [2, ch.4.4a] this relation is not gauge dependent, and therefore valid to linearized accuracy in an arbitrary coordinate system admitting $h_{\mu\nu} \ll 1$. To check the consequences of (2.53) for a free particle we insert the energy-momentum tensor for dust, ie. $T^{\alpha\beta} = \rho u^\alpha u^\beta$, and get

$$u^\beta\partial_\alpha(\rho u^\alpha) + \rho u^\alpha\partial_\alpha u^\beta = 0. \tag{2.54}$$

Multiplying with u_β and using the four-velocity identity $u^\beta u_\beta = -c^2$ we get

$$-c^2\partial_\alpha(\rho u^\alpha) + \rho u^\alpha u_\beta\partial_\alpha u^\beta = 0. \tag{2.55}$$

Some effort is now needed to show that the second term on the left-hand side of (2.55) is zero. First note that from the four-velocity identity it follows that

$$u_\mu\partial_\nu u^\mu + u^\mu\partial_\nu u_\nu = 0. \tag{2.56}$$

Then we need to show that both terms on the left-hand side of this equation are identical. It turns out that they are identical also in the full theory, and I will show this by starting from the definition of the covariant derivative of a one-form, see (1.77), and using that the metric is covariant constant

$(D_\alpha g_{\mu\nu} = 0)$:

$$\begin{aligned} u^\mu \partial_\nu u_\mu &\equiv u^\mu D_\nu u_\mu + u^\mu \Gamma^\lambda_{\mu\nu} u_\lambda \\ &= u^\mu D_\nu (g_{\mu\sigma} u^\sigma) + u^\mu \Gamma^\lambda_{\mu\nu} u_\lambda \\ &= u^\mu g_{\mu\sigma} D_\nu u^\sigma + u^\mu \Gamma^\lambda_{\mu\nu} u_\lambda \\ &= u_\mu D_\nu u^\mu + u^\mu \Gamma^\lambda_{\mu\nu} u_\lambda \\ &= u_\mu \partial_\nu u^\mu - u_\mu \Gamma^\mu_{\lambda\nu} u^\lambda + u^\mu \Gamma^\lambda_{\mu\nu} u_\lambda \\ &= u_\mu \partial_\nu u^\mu. \end{aligned} \tag{2.57}$$

Thus we have showed that

$$u^\mu \partial_\nu u_\mu = u_\mu \partial_\nu u^\mu, \tag{2.58}$$

and (2.56) therefore implies

$$u_\beta \partial_\alpha u^\beta = 0. \tag{2.59}$$

It then follows that the second term on the left-hand side of (2.55) is zero and we get

$$\partial_\alpha (\rho u^\alpha) = 0. \tag{2.60}$$

Inserting this into (2.54) we get

$$u^\alpha \partial_\alpha u^\beta = 0. \tag{2.61}$$

This can be written

$$\frac{dx^\nu}{d\tau} \frac{\partial u^\mu}{\partial x^\nu}, \tag{2.62}$$

or

$$\frac{du^\mu}{d\tau} = 0. \tag{2.63}$$

Thus the linearized field equation predicts that particles will move on straight lines! This is a well-known consistency problem of linearized theory. It means that aspects of the full non-linear[10] theory must be involved to obtain a meaningful framework. In particular one needs to make use of (a linearized version of) the geodesic equation. This shows that linearized theory should not be thought of as a consistent theory of gravity in itself.

[10] The geodesic equation is here viewed as an aspect of the full non-linear theory (since it can be derived from the full theory).

Chapter 3

The gravito-electromagnetic analogy

In the history of gravitational physics there is a long tradition for drawing analogies to electromagnetism. The obvious similarities between Newton's law of gravity and Coulomb's law of electricity has led many authors into exploring further analogies between gravity and electromagnetism. After the foundation for relativity and electrodynamics were laid at the beginning of the 20th' century, it was clear that Newton's theory of gravity was inconsistent with causality, and in need of modifications. Maxwell himself [11] actually considered the possibility of formulating gravity analogous to electrodynamics already in 1865. In 1900 Hendrik Lorentz [12] tried to explain gravity as an electromagnetic effect by suggesting that the attraction of oppositely charged particles where slightly stronger than the repulsion of equally charged particles. On large scale the net effect would manifest itself as a phenomena of universal attraction between masses, which is the usual signature of gravity. However, after Einstein introduced his theory of general relativity in 1916, it became clear that the phenomena of gravity and electromagnetism was not as similar as anticipated. Both the physical concepts and the mathematical formalism where very different. Electrodynamics is the theory of how charged particles interacts via forces, while general relativity claims that gravity is no force at all, and describes how fields produce spacetime curvature and how objects (fields) move (propagate) in that curved spacetime background. Electrodynamics is described by linear field equations (Maxwell's equations) with a known general solution, while general relativity provides a set of complicated non-linear differential equations with, even today, only a few known exact solutions. Nevertheless, in 1918 Josef Lense and Hans Thirring [13] showed that general relativity predicts phenomena with behaveour analogous to magnetic effects. They considered the predictions of general relativity in the case when the gravitational field is generated by a rotating spherical symmetric object, and found an effect which suggested an interpretation where spacetime itself was dragged around by the rotating object. The effect is today known as the frame-dragging effect, or the Lense-Thirring effect. For weak gravitational fields, linearized theory can be used to derive a generalized framedragging effect. This formulation shows that frame-dragging effects are very similar to magnetic phenomena in electrodynamics, where charge flow produces a magnetic field with a velocity dependent influence on the motion of test-particles. For this reason, the frame-dragging effect is often referred to as *gravitomagnetism*. This effect is clearly a well-established and undisputed theoretical part of general relativity, and to some extent also experimentally verified [14].

In this chapter we will review an approximation-scheme to general relativity which suggests that in the weak-field slow-motion limit of the theory there exist a formal analogy to Maxwell's theory of electrodynamics. This approximation starts from the linearized field equation, and leads to a formulation where the Einstein equation is written on a form similar to Maxwell's equations and the geodesic equation takes the same form as the Lorentz force law. In this formulation the gravitational field is described in terms of a gravito-electric field $\mathbf{E}_g$ and a gravito-magnetic field $\mathbf{B}_g$ analogous to the vector fields in electrodynamics. The formalism is often referred to as the GEM framework, where GEM is short hand for gravito-electromagnetic (or gravito-electromagnetism).

3.1 The gravito-electromagnetic (GEM) framework

In this section we shall briefly review how the GEM framework is obtained from general relativity. The purpose here is to present the standard treatment in the literature and not to discuss the validity, completeness or the consistency of the derivation in great detail. Apart from being more detailed, we will follow the line of papers like [15], [16] and the textbook [6]. When I later in this chapter simply refer to 'the literature', these references are representative for what I have in mind.

In chapter 2 we studied a *formal* analogy between linearized gravity and electromagnetism, based on the gauge-theory-like structure of linearized theory spelled out in table 2.1. In this chapter the perspective is different, we shall specialize to the Lorentz gauge and study *physical* analogies between the theories. The advantage of choosing the Lorentz-gauge is that it turns out being possible under certain circumstances to describe the metric in terms of an entity with four components, much like the four-potential A^μ of electrodynamics. It should be stressed though that this necessarily breaks the gauge-like analogy summarized in table 2.1[1].

The starting point for the analogy is the obvious analogy, reviewed in section 2.6, between the linearized field equation in Lorentz gauge and the (covariant) Maxwell equations also in Lorentz gauge:

$$\Box\overline{h}_{\alpha\beta} = -16\pi\frac{G}{c^4}T_{\alpha\beta}, \qquad \partial_\beta\overline{h}^{\alpha\beta} = 0, \qquad \text{(linearized gravity)} \tag{3.1}$$

$$\Box A^\mu = -\mu_0 j^\mu, \qquad \partial_\mu A^\mu = 0, \qquad \text{(electromagnetism).} \tag{3.2}$$

Focusing on the components, the equations are on exactly the same form. The energy-momentum tensor mimics the electromagnetic four-current j^μ, while the tensor $\overline{h}_{\alpha\beta}$ plays the role of the electromagnetic four potential A^μ. In the weak-field limit, where the linearized field equation gives good accuracy, this suggests an analogy between the dynamics of gravitational and electromagnetic phenomenas.

Although there is a clear analogy between linearized gravity and electromagnetism, the formalism is quite different since the gravitational field is a rank 2 tensor field, while the electromagnetic field is of rank 1. However, as we shall see, in the interesting special case when the speed of the source (of the gravitational field) is small compared to the speed of light, we only need to consider four of the components in (3.1), and hence only four of the metric components. In that case, the four significant components of $\overline{h}_{\alpha\beta}$ can be organized in a four-potential just like in electrodynamics.

We assume that the source of the gravitational field can be described by a non-relativistic pressureless perfect fluid, ie. non-relativistic dust. By 'non-relativistic' we mean that the relative speed of the fluid is small enough to allow for a coordinate system where the coordinate speed is small compared to the speed of light, ie. $v^i \equiv \frac{dx^i}{dt} \ll c$. Since we have also assumed a weak gravitational field, this means that we can write $\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} \approx \frac{dx^\mu}{dt} = (c, v^i)$. Using this approximation, the

[1] Once we have written the metric in terms of a four-potential it is no longer possible to change gauge condition.

energy-momentum tensor takes the form

$$T^{\mu\nu} = \rho \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}, \tag{3.3}$$

where ρ is the mass-density measured by an observer in rest relative to the fluid. Notice that $T^{00} = \rho c^2$, $T^{0i} = \rho c v^i$ and $T^{ij} = \rho v^i v^j$, and hence

$$T^{00} \gg T^{0i} \gg T^{ij}. \tag{3.4}$$

From the retarded solution of (3.1)

$$\overline{h}_{\alpha\beta} = 4\frac{G}{c^4}\int \frac{T_{\alpha\beta}(t_-, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x', \tag{3.5}$$

we see that (3.4) implies

$$|\overline{h}_{00}| >> |\overline{h}_{0i}| >> |\overline{h}_{ij}|. \tag{3.6}$$

This means that for a non-relativistic source we can neglect the spatial components $\overline{h}_{ij}$. In this approximation the formalism can be described solely by the four components $\overline{h}_{0\mu}$ since the metric tensor is symmetric. From here the GEM framework follows easily by introducing GEM potentials $\widehat{\phi}$ and $\widehat{\mathbf{A}}$. The scalar potential is defined by

$$\widehat{\phi} \equiv -\frac{c^2\overline{h}_{00}}{4}, \tag{3.7}$$

and the components of the three vector potential are defined by

$$\widehat{A}_i \equiv \frac{c\overline{h}_{0i}}{4}. \tag{3.8}$$

The linearized field equation then takes the form

$$\Box\widehat{\phi} = 4\pi G\rho, \qquad \Box\widehat{\mathbf{A}} = \frac{4\pi G}{c^2}\mathbf{j}, \tag{3.9}$$

where $\mathbf{j} = \rho\mathbf{v}$ is the mass flow. The significant part of the Lorentz gauge condition $\partial^\mu \overline{h}_{\mu\nu} = 0$, which actually is four conditions (one for each ν), is the $\nu = 0$ component, which in terms of the potentials is written

$$\frac{1}{c^2}\frac{\partial\widehat{\phi}}{\partial t} + \nabla\cdot\widehat{\mathbf{A}} = 0. \tag{3.10}$$

The field equations (3.9) and the Lorentz gauge condition (3.10) should now be compared to the corresponding equations in electrodynamics:[2]

$$\begin{aligned} &\Box\phi = -\mu_0 c^2 \rho_q, \\ &\Box\mathbf{A} = -\mu_0 \mathbf{j_q}, \\ &\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\mathbf{A} = 0 \qquad \text{(Lorentz gauge condition)}, \end{aligned} \tag{3.11}$$

where ρ_q is the charge density, and $\mathbf{j_q}$ is the charge flow (charge current density)[3].

[2]The equations (3.2), which are written on covariant form, are rewritten by using that the four-potential has components $(A^\mu) = (\phi/c, \mathbf{A})$ and that the four-current has components $(j_q^\mu) = (c\rho_q, \mathbf{j_q})$.

[3]I have added a 'q' in ρ_q and j_q since I want to reserve ρ and $\mathbf{j}$ to the corresponding quantities in gravity (mass-density and mass-flow).

Observe that by imposing a restriction on the speed of the source, the analogy has become a perfect one. Mass density ρ mimics charge density ρ_q, mass flow $\mathbf{j}$ mimics current flow $\mathbf{j}_q$, and the magnetic permeability is replaced by the constant $4\pi G/c^2$. So far we have not showed that this formal analogy is a physical one though. To do this we must consider the kinematics, and check whether the potentials $\widehat{\phi}$ and $\widehat{\mathbf{A}}$ has the same kind of influence on the motion of a test-particle as in the electromagnetic case. In electrodynamics the force on a test-particle with charge q is given by the Lorentz force law:

$$\frac{\mathbf{F}}{q} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}). \tag{3.12}$$

Starting from the geodesic equation, we will now work out a similar expression for the gravitational case. We will then see that the vector-potential $\widehat{\mathbf{A}}$ has the same kind of velocity-dependent influence on a test-particle in a gravitational field as the magnetic potential $\mathbf{A}$ has on a charged test-particle in the electromagnetic case.

To linearized accuracy the geodesic equation (1.100) reads

$$\frac{du^\mu}{d\tau} = -\Gamma^{\mu(1)}_{\alpha\beta} u^\alpha u^\beta, \tag{3.13}$$

where $\Gamma^{\mu(1)}_{\alpha\beta}$ is defined in (2.19). We have already assumed slow-motion in the source of the gravitational field. Now we impose a further physical condition, and assume slow-motion also for the test-particle. The spatial part of the geodesic equation can then be written

$$\begin{aligned} \frac{dv^i}{dt} &= -\Gamma^{i(1)}_{\alpha\beta} v^\alpha v^\beta \\ &= -c^2\Gamma^{i(1)}_{00} - 2c\Gamma^{i(1)}_{j0} v^j - \Gamma^{i(1)}_{jk} v^j v^k, \end{aligned} \tag{3.14}$$

where $v^i \equiv \frac{dx^i}{dt}$. The literature claims that it will be sufficient to go only to first order in $\frac{v^i}{c}$, which means that the last term in (3.14) can be neglected. The justification of this choice is of course that second order terms will be suppressed for non-relativistic speeds. This sounds like fair reasoning at first, but in chapter 4 we will examine the consistency of this treatment[4]. For now we will just follow the literature, as promised in the introduction, and neglect terms which are second order in the speed of the test-particle. Inserting the linearized Christoffel symbols, see (2.19), we have to first order in $\frac{v^i}{c}$:

$$\frac{dv^i}{dt} = -c^2\partial_0 h^i_0 + \frac{1}{2}c^2\partial^i h_{00} + c\partial^i h_{j0} v^j - c\partial_j h^i_0 v^j. \tag{3.15}$$

If we, like in the literature, also assume the metric is stationary, then the first term on the right-hand side of (3.15) is zero. A stationary metric is not preferable when we want to study dynamics (although it allows sources with spherical mass distribution to rotate). I will therfor keep things more general, and check how things turn out.

Next we need to express the components $h_{\alpha\beta}$ in terms of the potentials. In chapter 2.3 we showed that

$$h_{\alpha\beta} = \overline{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\overline{h}. \tag{3.16}$$

[4]We will find that by neglecting terms which are second order in the velocity of the test particle, we will neglect corrections to the equation of motion of the same magnitude as the Lense-Thirring effect.

Inserting the definitions $\widehat{\phi} \equiv -\frac{c^2\bar{h}_{00}}{4}$ and $\widehat{A}_i \equiv \frac{c\bar{h}_{0i}}{4}$ we get

$$h_{\alpha\beta} = \begin{pmatrix} -\frac{2\widehat{\phi}}{c^2} & \frac{4\widehat{A}_1}{c} & \frac{4\widehat{A}_2}{c} & \frac{4\widehat{A}_3}{c} \\ \frac{4\widehat{A}_1}{c} & -\frac{2\widehat{\phi}}{c^2} & 0 & 0 \\ \frac{4\widehat{A}_2}{c} & 0 & -\frac{2\widehat{\phi}}{c^2} & 0 \\ \frac{4\widehat{A}_3}{c} & 0 & 0 & -\frac{2\widehat{\phi}}{c^2} \end{pmatrix}. \tag{3.17}$$

Equation (3.15) can then be reformulated

$$\frac{dv^i}{dt} = -\partial^i\widehat{\phi} - 4\frac{\partial \widehat{A}^i}{\partial t} + 4v^j\left(\partial^i\widehat{A}_j - \partial_j\widehat{A}^i\right). \tag{3.18}$$

This equation can be written as a vector equation. For arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ we have $(\mathbf{a}\times\mathbf{b})^i = \epsilon^i_{\ jk}a^jb^k$ and $(\nabla\times\mathbf{b})^k = \epsilon^k_{\ lm}\partial^l b^m$, where $\epsilon^i_{\ jk}$ is the Levi-Civita symbol defined by

$$\epsilon^i_{\ jk} = \begin{cases} +1 & \text{, if } (i,j,k) = (1,2,3) \text{ or } (2,3,1) \text{ or } 3,1,2 \\ -1 & \text{, if } (i,j,k) = (1,3,2) \text{ or } (2,1,3) \text{ or } 3,2,1 \\ 0 & \text{, if indices are repeated.} \end{cases} \tag{3.19}$$

It then follows that

$$(\mathbf{v}\times(\nabla\times\widehat{\mathbf{A}}))^i = \epsilon^k_{\ lm}\epsilon^i_{\ jk}v^j\partial^l\widehat{A}^m. \tag{3.20}$$

Inserting the identity[5]

$$\epsilon^k_{\ lm}\epsilon^i_{\ jk} = \delta^i_l\delta_{jm} - \delta^i_m\delta_{jl}, \tag{3.21}$$

and working out the summations we get

$$(\mathbf{v}\times(\nabla\times\widehat{\mathbf{A}}))^i = v^j(\partial^i\widehat{A}_j - \partial_j\widehat{A}^i). \tag{3.22}$$

Using this, we can rewrite (3.18) as a vector equation:

$$\frac{d\mathbf{v}}{dt} = -\nabla\widehat{\phi} - 4\frac{\partial\widehat{\mathbf{A}}}{\partial t} + 4(\mathbf{v}\times(\nabla\times\widehat{\mathbf{A}})). \tag{3.23}$$

We have now collected the equations of the GEM-framework. In table 3.1 the formalism is summarized together with the corresponding equations of electrodynamics[6]. Notice that the analogy is perfect only for the field equations. For the equation of motion the analogy is not perfect, as there appear a curious factor of 4 in the terms with the vector-potential $\widehat{\mathbf{A}}$. It should also be commented that in the gravitational case q/m is unity, since the gravitational charge in the GEM framework is the mass.

The usual treatment in the literature is not to summarize gravito-electromagnetism in terms of the GEM-potentials, like in table 3.1, but to introduce vector fields $\mathbf{E}_g$ and $\mathbf{B}_g$ analogous to electrodynamics. These vector fields are defined by

$$\mathbf{E_g} = -\nabla\widehat{\phi} - \frac{\partial\widehat{\mathbf{A}}}{\partial t}, \qquad \mathbf{B_g} = \nabla\times\widehat{\mathbf{A}}. \tag{3.24}$$

[5]See for example Mathematical methods in the physical sciences, ch. 10.

[6]Note that to first order in v/c, we have $\mathbf{F}/m = \mathbf{a}$, where $\mathbf{F} = \frac{d}{dt}(\gamma m\mathbf{v})$.

GEM FRAMEWORK	ELECTROMAGNETISM
$\Box\widehat{\phi} = 4\pi G\rho$	$\Box\phi = -\mu_0 c^2 \rho_q$
$\Box\widehat{\mathbf{A}} = \frac{4\pi G}{c^2}\mathbf{J}$	$\Box\mathbf{A} = -\mu_0 \mathbf{j_q}$
$\frac{d\mathbf{v}}{dt} = -\nabla\widehat{\phi} - 4\frac{\partial\widehat{\mathbf{A}}}{\partial t} + 4(\mathbf{v}\times(\nabla\times\widehat{\mathbf{A}}))$	$\frac{\mathbf{F}}{m} = -\frac{q}{m}\nabla\phi - \frac{q}{m}\frac{\partial\mathbf{A}}{\partial t} + \frac{q}{m}(\mathbf{v}\times(\nabla\times\mathbf{A}))$

Table 3.1: GEM-framework in terms of potentials compared to electrodynamics in the Lorentz-gauge.

	GEM FRAMEWORK	ELECTROMAGNETISM
GAUSS LAW	$\nabla\cdot\mathbf{E_g} = -4\pi G\rho$	$\nabla\cdot\mathbf{E} = \frac{\rho_q}{\epsilon_0}$
AMPERES LAW	$\nabla\times\mathbf{B_g} - \frac{1}{c^2}\frac{\partial\mathbf{E_g}}{\partial t} = -\frac{4}{c^2}\pi G\mathbf{J}$	$\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{j_q}$
SOURCE FREE GAUSS LAW	$\nabla\cdot\mathbf{B_g} = 0$	$\nabla\cdot\mathbf{B} = 0$
FARADAY'S LAW	$\nabla\times\mathbf{E_g} + \frac{\partial\mathbf{B_g}}{\partial t} = 0$	$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0$
LORENTZ FORCE LAW	$\frac{d\mathbf{v}}{dt} = \mathbf{E_g} + 4(\mathbf{v}\times\mathbf{B_g})$	$\frac{\mathbf{F}}{m} = \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

Table 3.2: GEM-framework in terms of vector fields $\mathbf{E}_G$ and $\mathbf{B}_G$ compared to electrodynamics.

Using these definitions together with the field equations (3.9) and the Lorentz gauge condition (3.10) we can reformulate all equations of table 3.1 in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$ fields. This involve some calculation though, and a more direct approach[7] is just to perform the substitutions $\mu_g = \frac{-4\pi G}{c^2}$, $\rho_q \to \rho$, $\mathbf{j_q} \to \mathbf{j}$, $\mathbf{E} \to \mathbf{E_g}$ and $\mathbf{B} \to \mathbf{B_g}$ into Maxwell's equations. The result is summarized in table 3.2 together with the equation of motion. Notice that to formulate the equation of motion in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$ one must assume that the magnetic field is stationary, ie. $\frac{\partial\widehat{A}}{\partial t} = 0$. In the next section I will show in a detailed manner that this is not due to our choice of definitions for $\widehat{\phi}$, $\widehat{\mathbf{A}}$, $\mathbf{E}_g$ and $\mathbf{B}_g$. It is not possible to define these variables such that both the field equations *and* the equation of motion can be formulated in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$ without assuming a stationary $\widehat{\mathbf{A}}$ field. This condition clearly places strict limitations on the choice of coordinate frame as well as on the physical setting which can be handled by the formulation. The only interesting special case which the $\mathbf{E}_g$-$\mathbf{B}_g$-formulation can deal with, if gravitomagnetic effects are present, is in fact the gravitational field set up by an object which is rotating with constant angular velocity. The formulation in terms of potentials is therefore much more general, and for this reason it is the equations of table 3.1 I will refer to as the GEM-framework in the following chapters[8].

The gravito-electromagnetic analogy reviewed here has influenced a lot of papers in the gravita-

[7] The more cumbersome way to do it is showed in section 3.2.

[8] This is contrary to usual approach in the literature, where the potentials $\widehat{\phi}$ and $\widehat{\mathbf{A}}$ are introduced just as an intermediate step on the way to the formulation in terms of $\mathbf{E_g}$ and $\mathbf{B_g}$.

tional literature. See for example [17] for a list of some hundred articles; a large part of them deals, in some way, with the analogy to Maxwell's theory reviewed here. It is interesting that there is an analogy between gravitational and electrical phenomenas which are deeper then the obvious analogy between Newton's gravitational law and Coulomb's law of electricity. It is however also important to figure out the limitations of such an analogy. In chapter 4 we will carefully analyze the slow-motion weak-field limit of general relativity and electrodynamics in a consistent way and sort out the scope and limitations of this analogy.

3.2 GEM-potentials with free parameters

In the previous section we noticed that it was not possible to formulate the GEM equations in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$ without assuming stationary potential $\widehat{\mathbf{A}}$. In this section I will show that this is not due to the arbitrary choice of constants in the definitions of $\widehat{\phi}$, $\widehat{\mathbf{A}}$, $\mathbf{E}_g$ and $\mathbf{B}_g$. This section is not needed in preparation for later chapters. If the reader trust the claims previously made, this section might be skipped without any problems.

We start by introducing free parameters ι, κ and λ into the definitions (3.7), (3.8) and (3.24):

$$\begin{aligned}
\overline{h}_{00} &= -\iota\frac{4\widehat{\phi}}{c^2}, \\
\overline{h}_{0i} &= \frac{\kappa\widehat{A}_i}{c^2}, \\
\mathbf{E_g} &= -\nabla\widehat{\phi} - \lambda\frac{\partial\widehat{\mathbf{A}}}{\partial t}, \\
\mathbf{B_g} &= \nabla\times\widehat{\mathbf{A}}.
\end{aligned} \tag{3.25}$$

I have not inserted any free parameter into the definition of $\mathbf{B}_g$, since this would just give rise to a not interesting over-all factor in the equation of motion. Notice that our definitions in the previous section corresponds to $\iota = 1$, $\kappa = 4c$ and $\lambda = 1$. This convention have the following benefits:

- ($\iota = 1$) ensures that we have the standard expression $\widehat{\phi} = -\frac{GM}{r}$ outside a spherical symmetric mass distribution.
- ($\lambda = 1$) ensures that the definition of $\mathbf{E}_g$ is the same one as the corresponding field in electrodynamics.
- ($\kappa = 4c$) ensures that the Lorentz gauge condition is similar in both gravity and electrodynamics.

Allthough I had fairly good reasons to choose my definitions as I did, there are no standard conventions in the literature. For example [6] uses definitions corresponding to $(\iota,\kappa,\lambda) = (1, 2, \frac{1}{2c})$, while [16] uses $(\iota,\kappa,\lambda) = (-1,-2,\frac{1}{2c})$, [15] uses $(\iota,\kappa,\lambda) = (-1, 2, -\frac{1}{2c})$ and finally [18] uses $(\iota,\kappa,\lambda) = (1, 4c, 1)$. In my opinion the above mentioned benefits suggests that $(\iota,\kappa,\lambda) = (1, 4c, 1)$ should be the standard convention.

My intention by introducing these parameters however, is not to dwell on aesthetic aspects of the definitions, but to demonstrate that there exist no set of parameters such as the formulation in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$ is valid also in the non-stationary case. To do this we will need to redo the derivation in the previous section. To demonstrate how Maxwell's equations follow from the covariant formulation, I will not use the substitution-strategy of the the previous section.

The source-free Maxwell equations follows directly from the definitions of $\mathbf{E_G}$ and $\mathbf{B_G}$ by use of the vector identities $\nabla \cdot (\nabla \times \widehat{\mathbf{A}}) = 0$ and $\nabla \times \nabla\widehat{\phi} = 0$. The first one of these equations is

$$\nabla \cdot \mathbf{B_G} = \nabla \cdot (\nabla \times \widehat{\mathbf{A}}) = 0, \tag{3.26}$$

while the next one is

$$\begin{aligned} \nabla \times \mathbf{E_G} &= \nabla \times \left(-\nabla\widehat{\phi} - \lambda \frac{\partial \widehat{\mathbf{A}}}{\partial t} \right) \\ &= -\overbrace{\nabla \times \nabla\widehat{\phi}}^{=0} - \lambda \frac{\partial}{\partial t} \overbrace{(\nabla \times \widehat{\mathbf{A}})}^{\mathbf{B_G}} \\ &= -\lambda \frac{\partial \mathbf{B_G}}{\partial t}. \end{aligned} \tag{3.27}$$

Notice that the source-free Maxwell equations do not put any constraints on the parameters. The two remaining equations however, which follows from the Lorentz gauge condition and the field equations for $\widehat{\phi}$ and $\widehat{\mathbf{A}}$, do. In terms of the new definitions (3.25) these equations becomes

$$\Box\widehat{\phi} = \frac{1}{\iota} 4\pi G\rho, \qquad \Box\widehat{\mathbf{A}} = \frac{16\pi G}{\kappa c}\mathbf{j}, \qquad \frac{\iota}{c}\frac{\partial\widehat{\phi}}{\partial t} + \frac{\kappa}{4}\nabla \cdot \widehat{\mathbf{A}} = 0. \tag{3.28}$$

Working a little on the left hand side of the field equation for $\widehat{\phi}$, using the Lorentz gauge condition, we get:

$$\begin{aligned} \Box\widehat{\phi} &= -\frac{1}{c^2}\frac{d^2\widehat{\phi}}{dt^2} + \nabla^2\widehat{\phi} \\ &= \nabla^2\widehat{\phi} - \frac{1}{c^2}\frac{\partial}{\partial t} \overbrace{\left(\frac{\partial\widehat{\phi}}{\partial t}\right)}^{(-\frac{\kappa c}{4\iota}\nabla\cdot\widehat{\mathbf{A}})} \\ &= \nabla \cdot \left(\nabla\widehat{\phi} + \frac{\kappa}{4\iota c}\frac{\partial\widehat{\mathbf{A}}}{\partial t} \right) \\ &= -\nabla \cdot \mathbf{E_G}, \end{aligned} \tag{3.29}$$

where the last step give the following constraint on the parameters:

$$\lambda = \frac{\kappa}{4\iota c}. \tag{3.30}$$

Inserting this into the field equation we arrive at Gauss law:

$$\nabla \cdot \mathbf{E_G} = -\frac{1}{\iota} 4\pi G\rho. \tag{3.31}$$

The last equation is obtained from the field equation for $\widehat{\mathbf{A}}$ which reads

$$\Box\widehat{\mathbf{A}} = \frac{16\pi G}{\kappa c}\mathbf{j}, \tag{3.32}$$

or

$$\nabla^2\widehat{\mathbf{A}} = \frac{1}{c^2}\frac{\partial^2\widehat{\mathbf{A}}}{\partial t^2} + \frac{16\pi G}{\kappa c}\mathbf{j}. \tag{3.33}$$

To progress, we will need to take the gradient of the Lorentz gauge condition:

$$-\nabla(\nabla \cdot \widehat{\mathbf{A}}) = \frac{4\iota}{\kappa c}\frac{\partial}{\partial t}(\nabla\widehat{\phi}). \tag{3.34}$$

Adding $-\nabla(\nabla \cdot \widehat{\mathbf{A}})$ to the left hand side of (3.33), and $\frac{4\iota}{\kappa c}\frac{\partial}{\partial t}(\nabla\widehat{\phi})$ to the right hand side, we get

$$-\nabla(\nabla \cdot \widehat{\mathbf{A}}) + \nabla^2\widehat{\mathbf{A}} = \frac{4\iota}{\kappa c}\frac{\partial}{\partial t}(\nabla\widehat{\phi}) + \frac{1}{c^2}\frac{\partial^2\widehat{\mathbf{A}}}{\partial t^2} + \frac{16\pi G}{\kappa c}\mathbf{j}. \tag{3.35}$$

The left hand side is identified as $-\nabla \times (\nabla \times \widehat{\mathbf{A}})$ which equals $-\nabla \times \mathbf{B_g}$. Multiplying the equation by $-\lambda c^2$ it reads

$$\lambda c^2 \nabla \times \mathbf{B_g} = \frac{\partial}{\partial t}\left(-\frac{4\iota\lambda c}{\kappa}\nabla\widehat{\phi} - \lambda\frac{\partial^2\widehat{\mathbf{A}}}{\partial t^2}\right) - \frac{16\pi G\lambda c}{\kappa}\mathbf{j}. \tag{3.36}$$

We then obtain an equation corresponding to Amperes law in electromagnetism

$$\lambda c\nabla \times \mathbf{B_G} = \frac{1}{c}\frac{\partial \mathbf{E_G}}{\partial t} - \frac{16\lambda}{\kappa}\pi G\mathbf{j} \tag{3.37}$$

if we require $4\iota\lambda c/\kappa = 1$. But this constraint is the same as the one obtained from the field equation for ϕ, ie. (3.30). Re-deriving the equation of motion we find

$$\frac{d\mathbf{v}}{dt} = \iota\left(-\nabla\phi - \frac{\kappa}{\iota c}\frac{\partial \mathbf{A}}{\partial t}\right) + \frac{\kappa}{c}(\mathbf{v} \times (\nabla \times \mathbf{A})), \tag{3.38}$$

and hence, to rewrite it in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$, we must require $\lambda = \frac{\kappa}{\iota c}$. But this is not compatible with the constraint $\lambda = \frac{\kappa}{4\iota c}$ (eq. 3.30), which followed from the field equations. Thus we have proved that in the case of non-stationary fields, it is not possible to formulate gravity in terms of $\mathbf{E_g}$ and $\mathbf{B_g}$ (at least not without doing something radically more creative than changing the constants in the definitions).

3.3 The Lense-Thirring effect

In the previous section we reviewed an approximation-scheme to general relativity which lead to a Maxwell-like theory (the GEM framework). The gravito-electric part is the Newtonian contribution, while the gravito-magnetic part is the (generalized) frame-dragging effect. In the next chapter I will question the validity and completeness of this approximation-scheme. I will conclude that the GEM-framework is not valid as an approximation of general relativity in the weak-field slow-motion limit, in the sense that it does not give a *complete* description. This does however not affect the fact that gravitomagnetism is a well-established and theoretically undisputed part of general relativity. My criticism is directed against the framework of gravito-electromagnetism, *not* the phenomenon of gravitomagnetism.

In this section we will briefly review the physical significance of the gravito-magnetic effect. No detailed derivations will be shown, the purpose is only to give some sense of the physical significance of the effect, such as its order of magnitude and interpretation from a relativistic point of view. For derivations of formulas found in this section, see for example the textbook [19, page 234] or [9, ch. 6.2].

Gravitomagnetic field outside the earth First let us consider the impact of the magnetic field on a test particle in the gravitational field of the earth. It can be shown that the gravito-magnetic field outside a spherical symmetric rotating object is given by

$$\mathbf{B}_g = \frac{G}{2c^2r^3}(\mathbf{J} - 3\frac{\mathbf{J}\cdot\mathbf{r}}{r^2}\mathbf{r}), \tag{3.39}$$

where $\mathbf{J}$ is the angular momentum of the object. The earth is of course to good approximation a spherical symmetric rotating object. To find an estimate for the earth's angular momentum we can use that the moment of inertia of a solid sphere with mass M, radius R and constant mass density is $I = \frac{2}{5}MR^2$. For the earth this gives $|\mathbf{J}| = I|\mathbf{w}| \approx 10^{34}kgm^2/s$. Let us now consider a freely falling test-particle at equator moving in the radial direction (towards the center of mass of the earth) with speed $1000m/s$. The Newtonian acceleration is of course $|\mathbf{E}_g| = 9.8m/s^2$. Let us now estimate the impact of the gravito-magnetic field on the particle. At equator $\mathbf{J}\cdot\mathbf{r} = 0$ so $\mathbf{B}_g$ is pointing in the $\mathbf{J}$ direction with magnitude $B_g = \frac{G}{2c^2R^3}J \approx 10^{-14}s^{-1}$. Thus the gravitomagnetic field gives an acceleration in the easterly direction with magnitude $|4\mathbf{v}\times\mathbf{B}_g| \approx 10^{-10}m/s^2$. Hence the Lense-Thirring effect is suppressed by a factor 10^{-11} compared to the Newtonian acceleration in this example. This demonstrates how good the Newtonian approximation is in non-extreme gravitatational scenarios.

Galactic kinematics It would be interesting to estimate the gravito-magnetic correction to Newtonian gravity also on galaxy scales. One of the major challenges in modern cosmology is to explain the rotation-speed versus radius behavior for galaxies. The rotation speed of the outer regions are far larger than expected from the observed mass and standard gravitational theory. Galaxy kinematics is usually modeled using Newtonian gravity since there seems to be consensus that corrections from general relativity are negligible. There have been two main approaches to this problem. The main-stream approach is to postulate dark matter, an un-observed un-known kind of matter which must make up about 90% of the total mass of the galaxy to explain the velocity pattern. The other approach has been to search for alternative theories of gravity which can explain the kinematics without need of dark matter. Recently, however, the assumption that corrections from general relativity are neglectible has been challenged, see for example [20]. On this basis, let us make a simple order of magnitude estimate for the gravito-magnetic effect for a simpliefied modell of a galaxy. As modell for the galaxy we will use a rigid rotating sphere with the same mass ($M = 10^{12}M_{sun} = 10^{42}$kg) and radius ($R = 10^5$ly $= 10^{21}$m) as the Milkyway. This is of course not a realistic modell for a galaxy, but it is sufficient for an order of magnitude estimate like this. The Newtonian acceleration just outside the sphere is $E_g = \frac{GM}{R^2}$. Consider now a test-particle just outside the surface of the sphere. The test-particle has has speed $V = \sqrt{\frac{GM}{R}} = 10^5m/s$ in the tangential direction, and will therefore[9] follow a circular orbit around the sphere. We assume that the orbit of the test-particle coincides with the equator line of the rotating sphere. Let us assume that the sphere rotates with just the right angular velocity to ensure that the speed at the surface of the sphere is equal to the speed of the test particle: $\omega = \frac{V}{R}$. In this manner we ensure that the considered test-particle has the same motion as the (rest of the) galaxy. Calculating the angular momentum for the sphere in the same way as above, we get $|\mathbf{J}| = I|\omega| = \frac{2}{5}MRV = 10^{68}kgm^2/s$. At equator $\mathbf{B}_g$ points in the same direction as $\mathbf{J}$ and has magnitude $B_g = \frac{G}{2c^2R^3}J \approx 10^{-23}s^{-1}$ according to (3.39).

[9] Here, of course, I have just used Newton's law of gravity as well as his second law. Contributions from gravitomagnetism are neglected.

Inserting the numbers into (3.39) we get the following results:

$$\begin{aligned}|E_g| &\approx 10^{-10} m/s^2,\\ |4\mathbf{v}\times\mathbf{B}_g| &\approx 10^{-17} m/s^2.\end{aligned}$$

This shows that on galaxy scales the gravito-magnetic acceleration can be assumed to be suppressed approximately by a factor $1/10^7$. This is slightly more significant than in the above example, but it still supports the conventional assumption that contributions from general relativity can be neglected. Finally notice that the vector $\mathbf{v}\times\mathbf{B}_g$ points in the opposite direction of $\mathbf{E}_g$. As far as we can say from this simple analysis, this mean that you will need slightly *more* dark matter if you take account for the gravito-magnetic effect.

Spin precession The gravito-magnetic field causes precession of gyroscopes, and since it can be measured, this is perhaps the most interesting effect. The satellite experiment Gravito-Probe B has measured the precession of gyropsopes orbiting the earth. The data from the experiment are still being analyzed, but the effect is now verified to an accuracy of $\pm 10\%$[10]. We shall now pay attention to the effect of spin precession.

The spin precession formula for a gyroscope with spin $\mathbf{S}$ is

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\Omega}\times\mathbf{S}. \tag{3.40}$$

This equation tells that the spin $\mathbf{S}$ is constant in magnitude and precesses at a rate $|\boldsymbol{\Omega}|$ around the direction of $\boldsymbol{\Omega}$. The spin formula accounts for more than the frame-dragging effect. It also accounts for the so-called geodesic precession and Thomas precession (the last one familiar from special relativity). This section is about gravito-magnetic effects, but I will also briefly discuss the two other contributions to the spin precession. The reason for this is that precession of gyroscopes are of conceptual interest in Einstein's theory of gravity. In general relativity gyroscopes plays the role of defining the spatial directions of non-rotating reference frames. Consider an observer which is transporting three gyroscopes, spinning in mutually orthogonal directions, along his world-line. If the observer is accelerating (in the relativistic sense), he must act on the gyroscope with forces applied to the center of mass (no torque!). We will refer to gyroscopes transported in this way as *inertial-guidance gyroscopes*. It can be shown that three mutually orthogonal gyroscopes transported like this will remain mutually orthogonal all along the world line of the observer. This is the reason why it is natural to let inertial-guidance gyroscopes define the spatial directions of an accelerating, but non-rotating, reference frame. So in the relativistic sense, we can say that a non-rotating reference frame is one where the spatial axis precesses like the axis of an inertial-guidance gyroscope. In a later chapter on physical reference frames we will need the complete spin precession formula. It is therefore convenient to associate one spin-vector for each effect:

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\text{frame-dragging}} + \boldsymbol{\Omega}_{\text{geodesic}} + \boldsymbol{\Omega}_{\text{Thomas}}. \tag{3.41}$$

The formulas for the precession vectors are

$$\begin{aligned}\boldsymbol{\Omega}_{\text{frame-dragging}} &= -2\nabla\times\widehat{\mathbf{A}} = -2\mathbf{B}_g,\\ \boldsymbol{\Omega}_{\text{geodesic}} &= -\frac{3}{2}\mathbf{v}\times\nabla\frac{\widehat{\phi}}{c^2},\\ \boldsymbol{\Omega}_{\text{Thomas}} &= -\frac{1}{2c^2}\frac{\mathbf{F}}{m}\times\mathbf{v}.\end{aligned} \tag{3.42}$$

[10]See the web-page for the experiment: http://einstein.stanford.edu/

Here $\mathbf{v}$ is the center of mass velocity of the gyroscope and m is its mass. $\mathbf{F}$ denotes non-gravitational forces which acts against the center of mass of the gyroscope, ie. no torque. This means that for freely falling gyroscopes, as those in the Gravito-Probe B experiment, there is no Thomas precession. For the Gravito Probe B gyroscopes the geodesic precession dominates over the frame-dragging precession, $< |\Omega_{\text{precession}}| >= 8.4$arc seconds/year while $< |\Omega_{\text{frame-dragging}}| >= 0.048$arc seconds/year. Notice that the geodesic-precession and the Thomas-precession both depends on the motion (ie. $\mathbf{v}$) of the gyroscope, while the framedragging effect does not.

Physical interpretation The effect discussed in this section has three different names: 'gravitomagnetism', 'the Lense-Thirring effect' or finally 'the frame-dragging effect'. The first name is of course due to the obvious analogy to magnetic phenomenas in Maxwell's theory, while the second is in honor of the pioneers of the theoretical discovery. The name frame-dragging on the other hand, needs more explanation. The name originates from Schiff (1960), see [21], who introduced a theoretical fluid which is dragged along by rotating bodies (like the earth), and which in the next turn "dragges" gyroscopes. Gyroscopes defines the precession of non-rotating reference frames, and thereby the name 'frame-dragging'. This idea has influenced much of the literature, since it gives a simple pictorial explanation of the phenomenon. Consider for example the gravito-magnetic field $\mathbf{B}_g$ around the earth, see (3.39). From (3.41) and (3.42) we see that gyroscopes at the north-pole will precess in the same direction as the rotation of the earth (since $\Omega_{\text{frame-dragging}}$ is parallel to $\mathbf{J}$), while a gyroscope at equator will precess in the opposite direction (since there $\Omega_{\text{frame-dragging}}$ is anti-parallel to $\mathbf{J}$). Schiff's analogy to fluid mechanics gives a nice pictorial explanation of this behavior. The earth drags the fluid in the same direction as it rotates. Above the north-pole this causes the spin to precess in the same direction as the rotation of the earth, but at equator the fluid is dragged faster closer to the earth since the gravitational field falls off with increasing radius, and therefore the gyroscope precesses in the opposite direction of the rotation of the earth/fluid.

Part II

Approaching the gravito-electromagnetic analogy using post-Newtonian methods

Chapter 4

Post-Newtonian methods

In chapter 2 we noted a formal analogy between the linearized theory for gravitation and electrodynamics based on the view of the former as a "gauge theory". In chapter 3 we saw that, in the weak-field slow-motion approximation of general relativity, there is also a deep *physical* analogy to phenomena in electromagnetism. In particular we noticed that the frame-dragging effect can be viewed as a magnetic kind of phenomenon. As stressed before, this effect, often referred to as gravitomagnetism, is a well-established and undisputed theoretical part of general relativity. As reviewed in section 3.1, this analogy has led several authors to consider an approximation-scheme to general relativity which starts from the linearized field equation and leads to a Maxwell-like theory for gravitation, often referred to as the framework of gravito-electromagnetism (the GEM framework).

It is of course very interesting that there is an analogy between gravitational and electrical phenomenas which is deeper than the obvious analogy between Newton's gravitational law and Coulomb's law of electricity. It is however also important to figure out the limitations of such an analogy. Is it really correct that Einstein's field equation takes an Maxwell-like form in the weak-field slow-motion approximation, and that the geodesic equation becomes similar to the Lorentz force law? Does this framework provide a complete description of gravitational phenomenas in the weak-field slow-motion regime? In this chapter we will answer such questions. The entire chapter is reserved for a systematic study of the weak-field slow-motion approximation of general relativity and how it relates to the field theory of electrodynamics. Hopefully this can provide the needed clarifications and give some new insights.

In section 4.1 I will start with the full theory and explore how things turn out when following a systematic method to the weak-field slow-motion approximation of general relativity. Then, in 4.2, I will employ the same systematic method to electrodynamics. This consistent approach, employed to electrodynamics as well as general relativity, will enable us to see the limitations of the gravito-electromagnetic analogy. Finally, in section 4.3 I will, based on the work in 4.1 and 4.2, show that general relativity, in the weak-field slow-motion approximation, can be formulated in a framework very similar to electrodynamics. This demands that we define the gravitational charge to be a particular combination of rest-mass, kinetic energy, pressure and gravitational binding energy.

Before getting to all this, let me motivate why there are several good reasons to question the validity and completeness of the GEM framework. As we saw many examples of in section 3.3, the physical significance of the Lense-Thirring effect is, under non-extreme conditions, extremely tiny compared to the Newtonian part. Recall that in the GEM framework, dust is used as model for the energy-momentum tensor. Why is pressure neglected? If the GEM framework follows from a systematic method, it should be neglected only if the physical effect of the pressure of the source on a (free) test particle is suppressed compared to the (not neglected) Lense-Thirring effect. Furthermore, recall that the derivation of the GEM framework starts from the linear field equation. Are non-linear contributions suppressed compared to the tiny Lense-Thirring effect? A good derivation should definitively answer questions like this. In the next section we will address all these questions, and several others, and we will see that the phenomenon of gravitation is far richer than suggested by the GEM framework once we choose to go beyond the Newtonian limit.

4.1 The post-Newtonian approximation of general relativity

The purpose of this section is to apply perturbation theory to general relativity and study the weak-field slow-motion approximation. Let me start by discussing the methods and some of the terminology used in this chapter. First of all, the phrase *weak-field slow-motion approximation of general relativity*, refers to any (consistent) approximation of the theory which gives good accuracy when the gravitational field is weak and the relative speed of the source is small compared to the speed of light. Thus, as we have seen, the *Newtonian limit* is an example of such an approximation. It is however not the only example, but rather, as we shall see, the lowest order solution of a perturbative method to be introduced in this chapter. This method has long traditions in gravitational theory, and is usually referred to as the *post-Newtonian approximation-scheme* to general relativity. This scheme leads to approximate solutions which encompasses all physical effects up to a given level of accuracy. As we shall see, it can be viewed as an iteration process where the solutions converges towards the exact solution as the number of iterations increase. The first iteration gives back the Newtonian limit, while the next one gives the so called *post-Newtonian approximation*, sometimes also called the *post-Newtonian limit* (I will use both phrases interchangeably). The method is consistent in the sense that each iteration gives (approximate) solutions which takes account for all physical effects up to a given level of accuracy which is characteristic for that solution/iteration. The iteration can therefore be stopped at any desired level. We need a minimum of formalism before we can specify the characteristic accuracy of the post-Newtonian limit in a precise way, but the reader may appreciate to know that, according to [22], it is "sufficiently accurate to encompass most solar-system tests that can be performed in the foreseeable future".

This approximation scheme lays the foundations for the parametrized post-Newtonian (PPN) formalism, which is the standard framework for calculating experimental consequences of, and distinguishing between, metric theories of gravity. I will follow methods of the standard reference [3]. However, since our perspective is gravito-electromagnetism, and not experiments, I will make the necessary adjustments. I will keep using the Lorentz gauge condition, which is slightly different from the coordinates used in [3]. This is required for formulating the generalized frame-dragging effect analogous to magnetism. I will also use different conventions and define different variables whenever it is appropriate for the discussion of the gravito-electromagnetic analogy. Finally I will formulate the equation of motion in a completely different way. In [3] the equation of motion is formulated in terms of a defined effective mass, while our formulation is inspired from electrodynamics and written in terms of potentials.

Some of the calculations in this chapter demands a lot of effort. Since the standard reference [3], at least from the perspective of a student, is written in an extremely compact style, I will provide a lot of details. Allthough I use different coordinates, define my own variables and so on, I hope this can be useful for students and others who wants to gain skills in doing calculations in the PPN-formalism or gravitational perturbation theory. Some of the calculations are placed in the appendix. This is not because they are less important, but is done in order to secure a minimum of “flow” in my text.

Before starting let me discuss the coordinate system and summarize the basic equations needed in this chapter. As in chapter 3 we will use assymtotical Minkowskian coordinates, such that we have Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ far from gravitational systems. We will also expand the metric arround the Minkowski metric in the usual way, ie. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Even though we will not linearize our equations, this expansion will be very useful also in this chapter.

To post-Newtonian accuracy (see [3] for a complete discussion) it will be sufficient to use the perfect fluid as model for the energy-momentum tensor:

$$T^{\mu\nu} = (\rho + \frac{p}{c^2})u^\mu u^\nu + pg^{\mu\nu}. \tag{4.1}$$

It should be stressed that ρ is the *total* energy density as measured in the rest frame of the fluid. This is different from the convention in [3] where the total energy is $\rho(1 + \Pi)$, where ρ is the *rest mass* energy of the atoms in the fluid and $\rho\Pi$ is the internal kinetic and thermal energy.

We will work with the Einstein equation on the form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\overline{T}_{\mu\nu}, \tag{4.2}$$

where $\overline{T}_{\mu\nu} = (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$ and $T = g_{\mu\nu}T^{\mu\nu}$. In section 1.7.2 I showed that this form of the field equation is equivalent to the usual one.

Allthough, as it turns out, the full theory is needed to find the post-Newtonian approximation, the linearized field equation may still be used to simplify some calculations. We use it on the form of (2.36)[1]:

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\overline{T}_{\mu\nu}. \tag{4.3}$$

4.1.1 The post-Newtonian book-keeping system

Our goal, as explained above, is to find approximate solutions which takes account for all physical effects up to a given level of accuracy. In order to know which terms to neglect in the expansion of the field equation, it is therefore necessary to keep track on the relative size of quantities like the pressure, mass density, Newtonian potential and so on[2]. Order of magnitude estimates for the relative size of these quantities can be found from Newtonian considerations. In the literature these estimates are referred to as the *post-Newtonian book-keeping system*.

Consider a self-gravitating system with a typical mass density ρ and a typical pressure p. The system produces a Newtonian gravitational potential U which satisfy the Poisson's equation

$$\nabla^2 U = 4\pi G\rho. \tag{4.4}$$

We will use $U \approx -\frac{GM}{r}$ as an estimate[3] for U, where M is the total mass of the gravitational system and r is the distance to the center of mass of the system. Non-extreme gravitational systems tends to be stable (neither collapsing nor exploding). This means either that the gravitational system constitutes of bodies in bound trajectories, or that the gravitational force is canceled by pressure. Galaxies are

[1] In the linearized field equation $g_{\mu\nu} \to \eta_{\mu\nu}$ in the definitions of $T_{\mu\nu}$ and T.

[2] The Newtonian potential enters the scene because, as we saw in section 2.5, it gives an approximate solution for the component h_{00}.

[3] This is the monopole term in the multipole expansion of U. This term gives a good approximation to the true potential far away from the source.

examples of the former, while planets are examples of the latter. In the former case Newton's second law gives us the estimate $\frac{GM}{r^2} \sim \frac{v^2}{r}$ or by using the estimate for U:

$$|U| \sim v^2. \tag{4.5}$$

In the latter case we consider a spherical object with constant mass density ρ, radius R and total mass M. It is a straight forward exercise to show that in order to cancel the gravitational attraction the body must have pressure $p = \frac{2}{3}\pi G\rho^2(R^2 - r^2)$ which means that

$$p \sim G\rho^2 R^2. \tag{4.6}$$

Using that $|U| \sim \frac{GM}{R} \sim G\rho R^2$ we see that (4.6) implies

$$p \sim \rho|U|. \tag{4.7}$$

Equations (4.5) and (4.7) relates the pressure and velocity to the Newtonian potential:

$$(\frac{v}{c})^2 \sim \frac{U}{c^2} \sim \frac{p}{\rho c^2} \sim O(2). \tag{4.8}$$

The symbol $O(2)$ denotes the size of the above small quantities. A quantity of smallness $O(m)$ multiplied with a quantity with smallness $O(n)$ is $O(m+n)$. As $(\frac{v}{c})^2$ is O(2), we also have

$$\frac{v}{c} \sim O(1). \tag{4.9}$$

The book-keeping system also needs to tell us what happens when we differentiate with respect to coordinate time t. Time evolutions are related to the motion of the sources $\mathbf{v}$ by $\frac{\partial}{\partial t} \sim \mathbf{v} \cdot \nabla$, which give us our final relation

$$\frac{|\partial_0|}{|\partial_i|} \sim O(1). \tag{4.10}$$

This relation follows from the assumption that time variations in the gravitational field are due to the motion of its source. Thus we have neglected gravitational waves which propagates with the speed of light and where $\partial_0/\partial_i \sim O(0)$[4]. It turns out though that wave phenomena do not enter the scene in the post-Newtonian approximation[5] so (4.10) will be sufficient for our business.

Equations (4.8), (4.9) and (4.10) constitute the post-Newtonian book-keeping system. It turns out that the post-Newtonian approximation-scheme will involve expansion in small quantities like v/c, $\frac{p}{\rho c^2}$ and U/c^2. The book-keeping relates these expansions to each others and will be invaluable help in this chapter. Note that since v^2, U and p/ρ are small compared to c^2 (in the weak-field small-motion approximation), we essentially have an expansion in the *single* parameter c.

Our freedom in choice of reference frame is restricted by the small-motion condition, ie. $v << c$. This means that the book-keeping system also can be used in coordinate systems where the source of the potential U is moving, and that the relations holds also when the speed v denotes the speed of an external test-particle rather than the constituents of a self-gravitating body.

[4] Consider for example the wave function $f(t,x) = cos(kx - \omega t)$, where $k = \omega/u$ and u is the speed of the wave form (ie. $u = c$ for a gravitational wave). We have $\frac{\partial f}{\partial t} = -A\omega sin(kx - \omega t)$ and $\frac{\partial f}{\partial t} = -Aksin(kx - \omega t)$ and hence $\frac{\partial_0}{\partial_i} = \frac{\partial_t}{c\partial_i} \sim \frac{u}{c} = 1 \sim O(0)$.

[5] Wave phenomena are suppressed by a factor $O(3)$ relative to post-Newtonian effects, see [3, p.90].

4.1.2 Notation and conventions

We will use the convention that $h_{\mu\nu}[O(P)]$ and $T^{\mu\nu}[\rho c^2 O(P)]$ are estimates for $h^{\mu\nu}$ and $T^{\mu\nu}$ accurate to smallness $O(P)$ and $\rho c^2 O(P)$ respectly. In general the next order corrections to the quantities are of (at least) one order of magnitude smaller, so we have $h_{\mu\nu} = h_{\mu\nu}[O(P)] + O(P+1)$ and $T^{\mu\nu} = T^{\mu\nu}[\rho c^2 O(P)] + \rho c^2 O(P+1)$. The exact metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is given by the limit

$$h_{\mu\nu} = \lim_{P\to\infty} h_{\mu\nu}[O(P)], \tag{4.11}$$

and can in principle be found after an infinite number of iterations.

We will denote the accuracy in the estimates for the coordinate acceleration $\mathbf{a}$, a three vector with components $a^i = \frac{d^2x^i}{dt^2}$, in a similar way. For example $\mathbf{a}[O(2)\nabla U]$ denotes an estimate for $\mathbf{a}$ where all effects suppressed by a factor $O(2)$ compared to the Newtonian acceleration are taken account for. Effects of smallness $O(P)\nabla U$, where $P > 2$, however, are neglected.

We will also keep using the notation introduced in section 2.3, for example $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu(1)}_{\alpha\beta} + \Gamma^{\mu(2)}_{\alpha\beta}$, where $\Gamma^{\mu(1)}_{\alpha\beta}$ is linear in $h_{\alpha\beta}$ and $\Gamma^{\mu(2)}_{\alpha\beta}$ is quadratic. As always, Greek letters run from 0 to 3, while Latin letters run from 1 to 3. Repeated indices are summed over (Einstein summation convention). For Greek repeated letters one is always upper (contravariant), while the other one is lower(covariant). For Latin letters we also sum over repeated indices regardless of position, for example $h_{ii} \equiv h_{11} + h_{22} + h_{33}$. We use the notation of field theory in flat spacetime where spatial indices are placed up or down equivalently. This notation must be used very carefully though, since the raising/lowering operator in curved spacetime is the full metric $g_{\mu\nu}$ and *not* $\eta_{\mu\nu}$ (this will be discussed more thoroughly in a moment). For convenience we also introduce a somewhat unusual notation with $=$ or $\neq$ written between indices. For example $h_{i=j}$ refers to any spatial diagonal element of $h_{\mu\nu}$ (that is h_{11},h_{22} OR h_{33}), while $h_{i\neq j}$ refers to any spatial off-diagonal element.

We should also comment on the raising and lowering of indices. In linearized theory we defined $\eta_{\mu\nu}$ ($\eta^{\mu\nu}$) to be the lowering (raising) operator for the perturbation $h_{\mu\nu}$. In this chapter we want to describe all post-Newtonian effects regardless of whether they may be linear or not in $h_{\mu\nu}$. We will therefore always start out with the full metric $g_{\mu\nu}$ as raising/lowering operator, but sometimes, after a careful analysis of the loss of precision, it will be possible to replace $g_{\mu\nu}$ with $\eta_{\mu\nu}$.

4.1.3 The method

As we have introduced the post-Newtonian book-keeping system and defined our notation, we can now give a more detailed account on the method. The main idea of the approximative-scheme is to determine the coordinate acceleration $\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2}$ consistently to a given accuracy, say $\mathbf{a}[O(P)\nabla U]$. The equation of motion is therefore used to define the accuracy needed in $h_{\mu\nu}$ (and $T_{\mu\nu}$) to obtain the desired accuracy in $\mathbf{a}$. This is necessary to obtain a consistent description at each iteration, ie. predicting all physical effects at the considered precision.

The spatial components of the geodesic equation we can be written out as

$$\begin{aligned}\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2 \Big[&\partial_i h_{00} - 2\partial_0 h_{i0} - 2\frac{v^k}{c}\partial_k h_{i0} - 2\frac{v^k}{c}\partial_0 h_{ik} + 2\frac{v^k}{c}\partial_i h_{0k} \\ &- \frac{v^m}{c}\frac{v^n}{c}\partial_n h_{im} - \frac{v^m}{c}\frac{v^n}{c}\partial_m h_{in} + \frac{v^m}{c}\frac{v^n}{c}\partial_i h_{mn} + O(h^2) \Big].\end{aligned} \tag{4.12}$$

All terms contributes to the motion of the particle of course, but some terms are suppressed since they include time derivatives instead of spatial derivatives or include factors v/c. To get a consistent description, it is therefore *not* necessary to operate with the same accuracy in all the components of $h_{\mu\nu}$. Now, recall from section section 2.5 that to lowest order we have $h_{00} = -\frac{2U}{c^2} \sim O(2)$. Using the post-Newtonian book-keeping system, it is then easy to verify that determining $\mathbf{a}$ to accuracy $O(P)\nabla U$, requires knowledge of

$$\begin{array}{lll} h_{00} & \text{to accuracy} & O(P+2), \\ h_{0i} & \text{to accuracy} & O(P+1), \\ h_{ij} & \text{to accuracy} & O(P). \end{array} \tag{4.13}$$

For example if we demand Newtonian accuracy ($P = 0$), it is easy to verify that all terms in 4.12 will be accurate to (at least) order $\nabla U O(0)$ if we know h_{00} to $O(2)$, h_{0i} to $O(1)$ and h_{ij} to $O(0)$.

As we shall soon verify there are no corrections to $\mathbf{a}$ of order $O(1)\nabla U$ in general relativity. The post-Newtonian limit therefore requires knowledge of

$$\begin{array}{lll} h_{00} & \text{to accuracy} & O(4), \\ h_{0i} & \text{to accuracy} & O(3), \\ h_{ij} & \text{to accuracy} & O(2). \end{array} \tag{4.14}$$

We will refer to (4.14) as the *post-Newtonian accuracy* of the metric. It determines the acceleration to accuracy $O(2)\nabla U$, which means that the post-Newtonian limit properly accounts for all effects suppressed by a factor $O(2)$ compared to the Newtonian acceleration. If one demands an even better precision than that, one would have to calculate the post-post-Newtonian limit of general relativity, which determines $\mathbf{a}[O(4)\nabla U]$. Such accuracy is not necessary in our discussion of the gravito-electromagnetic analogy, but are important for example in the strong field around neutron stars (see [23] for a discussion of calculations beyond the post-Newtonian approximation).

We have stressed that the approximation scheme can be viewed as an iteration process starting from Newtonian theory and giving the post-Newtonian limit at the second step. Each iteration starts with the Einstein equation and gives improved estimates for the metric. The new estimate for $g_{\mu\nu}$ is then inserted into the energy-momentum tensor which gives an improved estimate for $T^{\mu\nu}$. Then one starts over again with the Einstein equation, inserts the improved estimate for $T^{\mu\nu}$, and get a further improved estimate for the metric. The first iteration starts with the lowest order solution for the metric ($g_{\mu\nu} \approx \eta_{\mu\nu}$) and the linearized field equation. However, already at the second iteration, giving the post-Newtonian approximation, the full non-linear field equation is needed. That is basically the systematics of the method.

4.1.4 The first iteration

The starting point for the first iteration is the lowest order solution for the metric, ie. $g_{\mu\nu} \approx \eta_{\mu\nu}$. To lowest order (4.1) give $T^{00} \approx \rho c^2$, $T^{0i} \approx \rho c v^i$, $T^{ij} \approx \rho v^i v^j + p\delta^{ij}$ and hence $T \equiv T^{\mu}_{\mu} \approx \eta_{\mu\nu} T^{\mu\nu} \approx -\rho c^2$. After lowering the indices with $\eta_{\mu\nu}$ -which is sufficient to find the lowest order estimates- we get:

$$\overline{T}_{\mu\nu} \approx \begin{cases} \frac{1}{2}\rho c^2 & \text{, if } (\mu = \nu) \\ -\rho c v^i \sim -\rho c^2 O(1) & \text{, if } (\mu, \nu) = (i, 0) \\ \rho v^i v^j \sim \rho c^2 O(2) & \text{, if } (\mu, \nu) = (i, j) \text{ and } i \neq j. \end{cases} \tag{4.15}$$

We observe that $\overline{T}_{i\neq j} \sim O(1)\overline{T}_{0k} \sim O(2)\overline{T}_{00}$ and that $\overline{T}_{i=j} \sim \overline{T}_{00}$. Hence (4.3) implies that $h_{i=j} \sim O(2)$, $h_{0i} \sim O(3)$ and $h_{i\neq j} \sim O(4)$, where we have used $h_{00} \sim O(2)$ (which follows from the Newtonian limit). Hence, linearized theory gives us a very useful estimate of the smallness of the components $h_{\mu\nu}$, which is well worth summarizing:

$$h_{\mu\nu} \sim \begin{cases} O(2) & \text{, if } \mu = \nu \\ O(3) & \text{, if } (\mu,\nu) = (0,i) \\ O(4) & \text{, if } (\mu,\nu) = (i,j) \text{ and } i \neq j. \end{cases} \tag{4.16}$$

Inserting (4.15) into (4.3) and using (4.10) to neglect small terms in the differential equations, we get

$$\begin{aligned} h_{00}[O(2)] &= -2\frac{U}{c^2} \sim O(2), \\ h_{0i}[O(3)] &= 4\tilde{A}_i/c \sim O(3), \\ h_{ij}[O(2)] &= -2\frac{U}{c^2}\delta_{ij} \sim O(2), \end{aligned} \tag{4.17}$$

where U is defined by Poisson's equation (4.4) and $\tilde{A}_i$ is defined by the equation

$$\nabla^2 \tilde{A}_i = \frac{4\pi G}{c^2}\rho v^i. \tag{4.18}$$

The first iteration has thus given us estimates for h_{00} to accuracy $O(2)$, h_{0i} to $O(3)$ and h_{ij} to $O(2)$. Actually, the components $h_{i\neq j}$ could have been determined to $O(4)$ (according to (4.16)), but this is not necessary neither in the Newtonian nor the post-Newtonian limit. From (4.13) and (4.14) we see that the first iteration has determined all components of the metric to Newtonian accuracy, and the components h_{0i} and h_{ij} to post-Newtonian accuracy.

From (4.17) we have $h_{00}[O(2)] = -\frac{2U}{c^2}$, $h_{0i}[O(1)] = 0$ and $h_{ij}[O(0)] = 0$, which is the accuracy needed for the Newtonian limit (see (4.13)). Inserting this back into (4.12) we get $\mathbf{a}[O(0)\nabla U] = -\nabla U$. Thus we have checked that the method gives back Newtonian theory in the first iteration.

4.1.5 The second iteration

The starting points for the second iteration are the results from the first iteration, ie. (4.17), which are used to find improved estimates for the energy-momentum tensor. The new estimate for $T^{\mu\nu}$ is then inserted into the full non-linear field equation to find improved estimates for the metric. However, as already observed, the only component of the metric we need an improved estimate for is h_{00}, which must be determined to accuracy $O(4)$. Hence it is only the zero-zero component of the field equation which needs to be considered:

$$R_{00} = \frac{8\pi G}{c^4}\overline{T}_{00}. \tag{4.19}$$

When we where calculating h_{00} to $O(2)$ we needed knowledge of $\overline{T}_{00}$ to accuracy $\rho c^2 O(0)$. This time we shall calculate h_{00} to $O(4)$ and we will need $\overline{T}_{00}$ to $\rho c^2 O(2)$. Our strategy is first to calculate the contra-variant components $T^{\mu\nu}[\rho c^2 O(2)]$, and then find $T_{\mu\nu}$ and $T = g_{\mu\nu}T^{\mu\nu}$ to accuracy $\rho c^2 O(2)$ by lowering indices. Finally we use these results to calculate

$$\overline{T}_{00} = T_{00} - \frac{1}{2}g_{00}T \tag{4.20}$$

to accuracy $\rho c^2 O(2)$. All calculations make extensively use of the book-keeping system, (4.16) and the relation

$$\frac{u^\mu}{c} \sim \begin{cases} O(0) & \text{, if } \mu = 0 \\ O(1) & \text{, if } \mu = i \end{cases} \tag{4.21}$$

to determine which terms can be neglected. After neglecting terms, lower order solutions for the metric (4.17) is used to replace the $h_{\mu\nu}$'s.

Then, let's get started. From the line-element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2 d\tau^2$ we find (see appendix A.1)

$$\frac{u^0}{c}[O(2)] = 1 - \frac{U}{c^2} + \frac{1}{2}(\frac{\mathbf{v}}{c})^2, \tag{4.22}$$

and

$$\frac{u^i}{c}[O(2)] = \frac{v^i}{c}. \tag{4.23}$$

From this follows

$$\frac{u^\mu}{c} \sim \begin{cases} 1 + O(2) & \text{, for } \mu = 0 \\ \frac{v^i}{c} + O(3) & \text{, for } \mu = i, \end{cases} \tag{4.24}$$

which is an improvement of (4.21) that will be useful in the following. Inserting (4.22) and (4.23) into (4.1) and neglecting terms smaller than $\rho c^2 O(2)$ we get (see appendix A.2):

$$T^{\mu\nu}[\rho c^2 O(2)] = \begin{cases} \rho c^2 \left(1 - 2\frac{U}{c^2} + (\frac{\mathbf{v}}{c})^2\right) & \text{, for } (\mu,\nu) = (0,0) \\ \rho c^2 \left(\frac{v^i}{c}\right) & \text{, for } (\mu,\nu) = (i,0) \\ \rho c^2 \left(\frac{v^i}{c}\frac{v^j}{c} + \frac{p}{\rho c^2}\delta^{ij}\right) & \text{, for } (\mu,\nu) = (i,j). \end{cases} \tag{4.25}$$

When calculating $T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}$ and $T = g_{\mu\nu}T^{\mu\nu}$ we must start with the full metric $g_{\mu\nu}$ as lowering operator and then neglect all terms smaller than the desired accuracy. It is simply not consistent to use $\eta_{\mu\nu}$ as a lowering operator at this level of accuracy. When finding which terms to neglect, the estimate

$$T^{\mu\nu} \sim \begin{cases} \rho c^2 O(0) & \text{, for } (\mu,\nu) = (0,0) \\ \rho c^2 O(1) & \text{, for } (\mu,\nu) = (0,i) \\ \rho c^2 O(2) & \text{, for } (\mu,\nu) = (i,j), \end{cases} \tag{4.26}$$

which follows from (4.25), will be useful. The result (see appendix A.3) is

$$T_{\mu\nu}[\rho c^2 O(2)] = \begin{cases} \rho c^2 \left(1 + 2\frac{U}{c^2} + (\frac{\mathbf{v}}{c})^2\right) & \text{, for } (\mu,\nu) = (0,0) \\ -\rho c^2 \left(\frac{v^i}{c})\right) & \text{, for } (\mu,\nu) = (i,0) \\ \rho c^2 \left(\frac{v^i}{c}\frac{v^j}{c} + \frac{p}{\rho c^2}\delta^{ij}\right) & \text{, for } (\mu,\nu) = (i,j) \end{cases} \tag{4.27}$$

and

$$T[\rho c^2 O(2)] = -\rho c^2 \left(1 - 3\frac{p}{\rho c^2}\right). \tag{4.28}$$

Inserting (4.27) and (4.28) into (4.20) we finally get what we have been working hard for:

$$\overline{T}_{00}[\rho c^2 O(2)] = \frac{1}{2}\rho c^2 \left(1 + 2\frac{v^2}{c^2} + 2\frac{U}{c^2} + 3\frac{p}{\rho c^2}\right). \tag{4.29}$$

Then we compute R_{00} to $O(4)$. To order $O(4)$ we do not need the terms which are cubic and quartic in $h_{\mu\nu}$, and we write

$$R_{\mu\nu}[O(4)] = R^{(1)}_{\mu\nu}[O(4)] + R^{(2)}_{\mu\nu}[O(4)], \tag{4.30}$$

where $R^{(1)}_{\mu\nu}$ (see (2.24)) is linear in $h_{\mu\nu}$:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\partial^\alpha\partial_\mu h_{\nu\alpha} + \partial^\alpha\partial_\nu h_{\mu\alpha} - \partial^\alpha\partial_\alpha h_{\mu\nu} - \partial_\nu\partial_\mu h\right), \tag{4.31}$$

and $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$:

$$
\begin{aligned}
R^{(2)}_{\mu\nu} = \;& \frac{1}{2}h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - \frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\nu\beta} - \frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\nu h_{\mu\beta} + \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} \\
&+\frac{1}{2}\partial^\beta h^\alpha_\nu\partial_\beta h_{\alpha\mu} - \frac{1}{2}\partial^\beta h^\alpha_\nu\partial_\alpha h_{\beta\mu} + \frac{1}{2}\partial_\beta h^{\beta\alpha}\partial_\alpha h_{\mu\nu} \\
&+\frac{1}{2}h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - \frac{1}{4}\partial_\alpha h_{\mu\nu}\partial^\alpha h - \frac{1}{2}\partial_\beta h^{\alpha\beta}\partial_\mu h_{\nu\alpha} \\
&-\frac{1}{2}\partial_\beta h^{\alpha\beta}\partial_\nu h_{\mu\alpha} + \frac{1}{4}\partial^\alpha h\partial_\mu h_{\nu\alpha} + \frac{1}{4}\partial^\alpha h\partial_\nu h_{\mu\alpha}.
\end{aligned}
\tag{4.32}
$$

We showed how to find $R^{(1)}_{\mu\nu}$ from the definition of the Riemann tensor in chapter 2.3. The components of $R^{(2)}_{\mu\nu}$ are found by following a similar procedure.

Inserting $\mu = 0$ and $\nu = 0$ we easily find

$$
R^{(1)}_{00}[O(4)] = -\frac{1}{2}\nabla^2 h_{00}[O(4)] + \partial_0\partial_k h_{0k}[O(3)] - \frac{1}{2}\partial_0\partial_0 h_{kk}[O(2)] - \frac{1}{2}\partial_0\partial_0 h_{00}[O(2)], \tag{4.33}
$$

and after a lot of work we also get

$$
\begin{aligned}
R^{(2)}_{00}[O(4)] =& \frac{1}{2}h_{kj}[O(2)]\partial_k\partial_j h_{00}[O(2)] - \frac{1}{4}\left|\nabla h_{00}[O(2)]\right|^2 \\
&+ \frac{1}{2}\partial_j h_{00}[O(2)]\left(\partial_k h_{jk}[O(2)] - \frac{1}{2}\partial_j h_{kk}[O(2)]\right).
\end{aligned}
\tag{4.34}
$$

In these expressions we have used the notation $\nabla^2 h_{00} \equiv \partial_k\partial^k h_{00}$ and $|\nabla h_{00}|^2 = \partial_k h_{00}\partial^k h_{00}$. Obtaining the expression for $R^{(1)}_{00}$ is easy algebra starting from $R^{(1)}_{\mu\nu}$. We have used the fact that to accuracy $O(4)$ we can write $\partial_0\partial_0 h^k_k = \partial_0\partial_0 h_{kk}$, $\partial_0\partial^k h_{0k} = \partial_0\partial_k h_{0k}$ and $\partial_0\partial^0 h_{00} = -\partial_0\partial_0 h_{00}$[6]. The expression for $R^{(2)}_{00}[O(4)]$ is found after a long, but straight forward, calculation where all terms smaller than $O(4)$ is neglected.

The results found in section 4.1.4 satisfy the Lorentz gauge condition since they where obtained from the linearized field equation in that gauge. For consistency we must therefore express $R^{(1)}_{00}[O(4)]$ and $R^{(2)}_{00}[O(4)]$ in the same gauge: $\partial^\mu\overline{h}_{\mu\nu} = 0$. The $\nu = 0$ component of this condition can be rewritten $\partial^k h_{0k} = -\frac{1}{2}\partial^0(h + 2h_{00})$, where we have used the definition $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. Inserting this into (4.33) we get an expression for $R^{(1)}_{00}$ to $O(4)$ in the Lorentz gauge:

$$
R^{(1)}_{00}[O(4)] = -\frac{1}{2}\Box h_{00}[O(4)]. \tag{4.35}
$$

$R^{(2)}_{00}[O(4)]$ will automatically fulfill the Lorentz gauge condition if we insert the lower order solutions for $h_{\mu\nu}$ given by (4.17), since these solutions follows from the linearized field equation in this gauge. The result (see appendix A.4) is

$$
R^{(2)}_{00}[O(4)] = -\nabla^2\left(\frac{U}{c^2}\right)^2 + \frac{16\pi G}{c^4}\rho U. \tag{4.36}
$$

[6]To avoid confusion: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is the lowering operator, but the contribution from $h_{\mu\nu}$ is neglected as the terms are evaluated only to accuracy $O(4)$.

Adding $R_{00}^{(1)}$ and $R_{00}^{(2)}$ we get

$$R_{00}[O(4)] = -\frac{1}{2}\Box h_{00}[O(4)] - \nabla^2\left(\frac{U}{c^2}\right)^2 + \frac{16\pi G}{c^4}\rho U. \tag{4.37}$$

Inserting this together with (4.29) into (4.19) we get the equation

$$-\Box h_{00}[O(4)] - 2\nabla^2\left(\frac{U}{c^2}\right)^2 = \frac{8\pi G}{c^2}\rho\left(1 + 2\frac{v^2}{c^2} - 2\frac{U}{c^2} + 3\frac{p}{\rho c^2}\right). \tag{4.38}$$

This differential equation gives us h_{00} to accuracy $O(4)$ using standard techniques for solving equations perturbatively, see appendix A.5. Combined with the results in (4.17), we have finally determined the metric to post-Newtonian accuracy:

$$\begin{aligned}
h_{00}[O(2)] &= -2\frac{U}{c^2}, \\
h_{00}[O(4)] &= -2\frac{U}{c^2} - 8\frac{U_k}{c^4} - 8\frac{U_g}{c^4} - 2\frac{U_p}{c^4} - 2\frac{U_t}{c^4} - 2(\frac{U}{c^2})^2, \\
h_{0i}[O(3)] &= 4\frac{\tilde{A}_{gi}}{c}, \\
h_{ij}[O(2)] &= -2\frac{U}{c^2}\delta_{ij},
\end{aligned} \tag{4.39}$$

where the potentials are defined by

$$\begin{aligned}
U &= -G\int d^3x' \frac{\rho'}{|\mathbf{x}-\mathbf{x}'|} \quad , \quad U_k = -G\int d^3x' \frac{\frac{1}{2}\rho' v'^2}{|\mathbf{x}-\mathbf{x}'|}, \\
U_g &= -G\int d^3x' \frac{-\frac{1}{2}\rho' U'}{|\mathbf{x}-\mathbf{x}'|} \quad , \quad U_p = -G\int d^3x' \frac{3p'}{|\mathbf{x}-\mathbf{x}'|}, \\
U_t &= -\frac{1}{4\pi}\int d^3x' \frac{\frac{\partial^2 U'}{\partial t^2}}{|\mathbf{x}-\mathbf{x}'|} \quad , \quad \tilde{A}_{gi} = -\frac{G}{c^2}\int d^3x' \frac{\rho' v'^i}{|\mathbf{x}-\mathbf{x}'|}.
\end{aligned} \tag{4.40}$$

The corresponding field equation are stated in table 4.1. The potentials are defined such that they all have a special physical significance. For example, the Newtonian kinetic energy density $\frac{1}{2}\rho v^2$ plays the same role for the potential U_k as the energy density ρ for U. In the special case of a static spherical mass distribution it can be shown that $-\frac{1}{2}\rho U = 3p$ can be interpreted as the gravitational potential energy density (see exercise 23.7 in [1]). Therefore, in this special case, the gravitational potential energy density plays the same role for U_g and U_p as the energy density ρ for U. The potential U_t has significance as the retardation effect associated with U. To see this note that[7] $U + \frac{U_t}{c^2} = \tilde{U} + O(6)$, where $\tilde{U}$ satisfy the wave equation $\Box\tilde{U} = 4\pi G\rho$, while U satisfy the causality violating Poisson equation $\nabla^2 U = 4\pi G\rho$. Note that the retardation effect is a $O(4)$ effect, ie. the retardation effect associated with U is suppressed by a factor $O(2)$. This is a general property of the slow-motion approximation where $|\partial_0^2|/|\partial_i^2| \sim O(2)$, which means that the retardation effects associated with the potentials U_k, U_g, U_p and $\tilde{A}_{gi}$ is of smallness $O(6)$ and therefore neglected.

[7] Use the technique of appendix A.5.

4.1.6 The equation of motion

We have calculated the metric to the desired accuracy, and we will now find an expression for the coordinate acceleration $a^i = \frac{d^2x^i}{dt^2}$ in terms of the introduced potentials. Allthough we have used different coordinates/gauges and defined our own potentials, we have so far followed the methods of the standard reference [3] fairly tight. In this section however, we will no longer follow the standard methods of the the PPN-formalism. The reason for this is that (in the PPN formalism) the source of the gravitational field is modeled as an discrete n-body system and the equation of motion is written as a sum over effective masses. Our motivation for dealing with post-Newtonian methods however, is a discussion of the gravito-electromagnetic analogy, and therefore we will formulate the equation of motion in the same fashion as in chapter 3, namely in terms of our defined potentials.

In chapter 3.1 we used the approximations $\frac{dx^i}{d\tau} \approx \frac{dx^i}{dt}$ and $\frac{d^2x^i}{d\tau^2} \approx \frac{d^2x^i}{dt^2}$ when writing out the equation of motion. In this chapter we start by finding an *exact* relativistic expression for the coordinate acceleration, and determine which terms to be neglected after a careful analysis which makes use of the book-keeping system. The coordinate acceleration can be rewritten

$$a^i = \frac{d^2x^i}{dt^2} = \left(\frac{dt}{d\tau}\right)^{-1} \frac{d}{d\tau}\left[\left(\frac{dt}{d\tau}\right)^{-1} \frac{dx^i}{d\tau}\right] = \left(\frac{dt}{d\tau}\right)^{-2} \frac{d^2x^i}{d\tau^2} - \left(\frac{dt}{d\tau}\right)^{-3} \frac{d^2t}{d\tau^2}\frac{dx^i}{d\tau}. \tag{4.41}$$

From the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}, \tag{4.42}$$

we find expressions for $\frac{d^2x^i}{d\tau^2}$ and $\frac{d^2t}{d\tau^2}$ which can be inserted into (4.41) to give

$$a^i = -c^2\Gamma^i_{00} - 2c\Gamma^i_{0n}v^n - \Gamma^i_{mn}v^m v^n + cv^i\Gamma^0_{00} + 2v^i\Gamma^0_{0n}v^n + \frac{v^i}{c}\Gamma^0_{mn}v^m v^n. \tag{4.43}$$

This is an exact relativistic expression for the (spatial) coordinate acceleration $\frac{d^2x^i}{dt^2}$ in terms of the coordinate velocity $\frac{dx^i}{dt}$ and the Christoffel connection. We will use the notation from section 2.3 and write

$$\Gamma^\mu_{\alpha\beta} = \Gamma^{\mu(1)}_{\alpha\beta} + \Gamma^{\mu(2)}_{\alpha\beta}, \tag{4.44}$$

where $\Gamma^{\mu(1)}_{\alpha\beta}$ was defined

$$\Gamma^{\gamma(1)}_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left[\partial_\beta h_{\delta\alpha}(x) + \partial_\alpha h_{\delta\beta}(x) - \partial_\delta h_{\alpha\beta}(x)\right], \tag{4.45}$$

and

$$\Gamma^{\mu(2)}_{\alpha\beta}[O(4)] = -\frac{1}{2}h^{\mu\delta}\left[\partial_\beta h_{\delta\alpha} + \partial_\alpha h_{\delta\beta} - \partial_\delta h_{\alpha\beta}\right]. \tag{4.46}$$

The first expression (4.45) is exact while the second (4.46) is only accurate to $O(4)$ since $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ is only accurate to linear order in $h^{\mu\nu}$ as we saw in section 2.3. Then we can write the coordinate acceleration in a corresponding way

$$a^i = a^i_{(1)} + a^i_{(2)}, \tag{4.47}$$

where $a^i_{(1)}$ is defined as (4.43) with $\Gamma^\mu_{\alpha\beta}$ replaced by $\Gamma^{\mu(1)}_{\alpha\beta}$ and similarly for $a^i_{(2)}$. Inserting the Christoffel symbols into $a^i_{(1)}$, and then replacing the $h_{\mu\nu}$'s with ϕ_g and A_{gi} according to (4.39) we get

$$\begin{aligned} a^i_{(1)}[O(2)\nabla U] = &- \partial^i U - 4\partial^i \frac{U_k}{c^2} - 4\partial^i \frac{U_g}{c^2} - \partial^i \frac{U_p}{c^2} - \partial^i \frac{U_t}{c^2} - \partial^i \frac{U^2}{c^2} \\ &+ 4v^j \left(\partial^i \tilde{A}_{gj} - \partial_j \tilde{A}^i_g\right) + 4\frac{v^i}{c}\frac{v^k}{c}\partial_k U - \frac{v^2}{c^2}\partial^i U \\ &+ \frac{3}{c^2} v^i \frac{\partial U}{\partial t} - 4\frac{\partial \tilde{A}^i_g}{\partial t}, \end{aligned} \tag{4.48}$$

(see appendix A.6 for details). A similar calculation for $a^i_{(2)}$ gives

$$a^i_{(2)}[O(2)\nabla U] = -\partial^i \frac{U^2}{c^2}, \tag{4.49}$$

(see appendix A.7). Adding $a^i_{(1)}$ and $a^i_{(2)}$, and rewriting the sum as a vector equation we get an expression accurate to order $O(2)\nabla U$:

$$\begin{aligned} \mathbf{a} = &- \nabla U - \nabla \frac{U_t}{c^2} - 4\frac{\partial \tilde{\mathbf{A}}_\mathbf{g}}{\partial t} + 4\mathbf{v} \times \left(\nabla \times \tilde{\mathbf{A}}_\mathbf{g}\right) \\ &+ 4\mathbf{v}(\mathbf{v} \cdot \nabla \frac{U}{c^2}) - v^2 \nabla \frac{U}{c^2} \\ &+ \mathbf{G}, \end{aligned} \tag{4.50}$$

where

$$\mathbf{G} = -2\nabla \frac{U^2}{c^2} + 3\frac{\mathbf{v}}{c}\frac{\partial U}{\partial t} - 4\nabla \frac{U_k}{c^2} - 4\nabla \frac{U_g}{c^2} - \nabla \frac{U_p}{c^2}. \tag{4.51}$$

Equation (4.50), together with the field equations for the potentials, forms a framework for predicting the post-Newtonian trajectory of a free particle in curved space-time. The results are summarized in table 4.1. The reason for introducing the vector $\mathbf{G}$, defined in (4.51), will become clear when we concider the electromagnetic case in section 4.2. To estimate the error of the post-Newtonian framework, one must calculate the post-post-Newtonian framework and consider the size of the new terms that appear in the acceleration. It turns out that there are no corrections of smallness $O(3)\nabla U$ to $\frac{d\mathbf{v}}{dt}$ in general relativity (see chapter 4.1 of [3]), which means that the error is of smallness $\sim O(4)\nabla U \sim (\frac{v}{c})^4 \nabla U \sim (\frac{U}{c^2})^2 \nabla U$.

FIELD EQUATIONS	SOURCE
$\nabla^2 U = 4\pi G\rho$	ENERGY DENSITY
$\nabla^2 U_k = 4\pi G(\frac{1}{2}\rho v^2)$	KINETIC ENERGY
$\nabla^2 U_g = 4\pi G(-\frac{1}{2}\rho U)$	GRAVITATIONAL POTENTIAL ENERGY
$\nabla^2 U_p = 4\pi G(3p)$	PRESSURE
$\nabla^2 U_t = \frac{\partial^2 U}{\partial t^2}$	EXPLICIT TIME-DEPENDENCE
$\nabla^2 \tilde{\mathbf{A}}_{\mathbf{g}} = \frac{4\pi G}{c^2}\rho\mathbf{v}$	ENERGY FLOW

Condition satisfied by solutions:

$$\frac{\partial}{\partial t}\frac{U}{c^2} + \nabla\cdot\tilde{\mathbf{A}}_{\mathbf{g}} = 0$$

Equation of motion:

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= -\nabla U - 4\nabla\frac{U_k}{c^2} - 4\nabla\frac{U_g}{c^2} - \nabla\frac{U_p}{c^2} - \nabla\frac{U_t}{c^2} - 2\nabla\frac{U^2}{c^2} \\ &\quad +4\mathbf{v}\times\left(\nabla\times\tilde{\mathbf{A}}_{\mathbf{g}}\right) + 4\mathbf{v}(\mathbf{v}\cdot\nabla\frac{U}{c^2}) - v^2\nabla\frac{U}{c^2} \\ &\quad +3\frac{\mathbf{v}}{c}\frac{\partial U}{\partial t} - 4\frac{\partial\tilde{\mathbf{A}}_{\mathbf{g}}}{\partial t}\end{aligned}$$

Error-estimate: $\sim \frac{v^4}{c^4}\nabla U$ or $\sim \frac{U^2}{c^4}\nabla U$

Table 4.1: Post-Newtonian limit in Lorentz-gauge.

4.1.7 Order of magnitude estimates

Allthough we know that the corrections to Newtonian gravity in the equation of motion (4.50) all are of the same (maximum) smallness ($O(2)\nabla U$), it would be interesting to sort out how the magnitude relates to physical characteristics of the system. In this section we shall characterize the magnitude of post-Newtonian effects in terms of physical properties of the source such as its speed, pressure, mass density and interior Newtonian potential (read: U). Exact calculations are possible in special cases, but we are more interested in the general case which means that we will make order of magnitude estimates (loss of precision is the cost for generality).

We assume that the source (of the gravitational field) can be characterized by a fluid with a *typical* speed v_s, a *typical* internal potential U_s, a *typical* mass density ρ_s and a *typical* pressure p_s. We let v_p denote the speed of the test particle. In terms of these typical quantities we give a rough estimate for the smallness of each effect (read: each term in (4.50)). The results are summarized in the two leftmost columns in table 4.2 (the two rightmost columns are discussed in the next section). The leftmost column indicates what effect (which term in (4.50)), while the second column indicates how much that effect is suppressed compared to the Newtonian gravitational acceleration.

Let me give an example on how these estimates are calculated. Consider the term $+4\mathbf{v}\times(\nabla\times\tilde{\mathbf{A}}_{\mathbf{g}})$ in (4.50), which you will find in row 8 of the table. The velocity of the source $v^i(x)$ can be related to the typical speed of the source v_s by writing $v^i(x) = f^i(x)v_s$, where $f^i(x)$ is a dimensionless vector field to which the norm ($\sqrt{(f^1)^2+(f^2)^2+(f^3)^2}$) varies around the value 1. Inserting this into the definition (4.40) of the potentials we get

$$\tilde{A}^i_g = -\frac{v_s}{c^2}G\int d^3x'\,\frac{\rho' f'^i}{|\mathbf{x}-\mathbf{x}'|}, \tag{4.52}$$

while the Newtonian potential is

$$U = -G\int d^3x'\,\frac{\rho'}{|\mathbf{x}-\mathbf{x}'|}. \tag{4.53}$$

Putting $f^i(x) = 1$, ie. neglecting the variation in the speed of the source, we get $\tilde{A}_{gi} \sim \frac{v_s}{c^2}U$ which means that

$$\frac{4\mathbf{v_p}\times(\nabla\times\tilde{\mathbf{A}}_{\mathbf{g}})}{\nabla U} \sim \frac{v_p}{c}\frac{v_s}{c}. \tag{4.54}$$

Thus we have showed that the effect $4\mathbf{v_p}\times(\nabla\times\tilde{\mathbf{A}}_{\mathbf{g}})$ is suppressed by a factor $\frac{v_p}{c}\frac{v_s}{c}$ compared to the Newtonian term. This shows explicitly that the frame-dragging effect is a *second* order effect in velocity; first order in the test particle velocity *and* first order in the velocity of the source. Hence there was no good reason to neglect second order effects in the particle velocity in section 3.1, see the discussion following equation (3.14).

The other estimates in table 4.2 are worked out in a similar manner, and there is no need for further details. The exceptions however, are the effects depending on (partial) time derivatives. These are the terms $-\nabla\frac{U_t}{c^2}$, $+3\frac{\mathbf{v}}{c}\frac{\partial U}{\partial t}$ and $-4\frac{\partial\tilde{\mathbf{A}}_{\mathbf{g}}}{\partial t}$ in (4.50). Concerning the first one, there is a time derivative involved in the definition of U_t, see (4.40). Time evolutions are related to the motion of the sources $\mathbf{v}$ by $\frac{\partial}{\partial t} \sim \mathbf{v}_s\cdot\nabla$. We made use of this in section 4.1.1 to introduce the post-Newtonian book-keeping system. This is a bit too simple though, since we can have a stationary system where $v_s \neq 0$ if the source is rotating, but with no translational velocity. Hence we write $\partial_t \lesssim v_s\partial_x$, where $\lesssim$ can be replaced by $\sim$ or $<$ depending on whether v_s is mainly due to center of mass motion or rotation. This explains the use of the symbol '$\lesssim$' in table 4.2.

It should be commented on that although the post-Newtonian effects all have roughly the same magnitude ($O(2)\nabla U$), they do not have the same chance to be measured by actual experiments. The potentials U_k, U_g and U_p all have the same $1/r$ dependence as the Newtonian potential U. Hence, from measurements of satellite trajectories one can only deduce an effective gravitational potential which goes like $1/r$. The contribution from kinetic energy, gravitational binding energy, pressure and mass density to this effective potential is model dependent and cannot be discovered by studying satellite orbits in the external field. The Lense-Thirring effect however, has qualitatively different behavior and can therefore be measured (the experiment is mentioned in chapter 3.3). The non-linear effect ∇U^2 also has great experimental significance since it falls off faster than the other potentials with increasing distance ($U^2 \propto 1/r^2$). This term is the reason why linearized theory fails to predict the perihelion-shift of Mercury (see box 7.1 in [1]).

EFFECT	SURPRESSED BY A FACTOR	DERIVED FROM	COMMENT
∇U	1	LINEARIZED THEORY	SOURCED BY (REST) MASS
∇U_k	$\sim (v_s/c)^2$	LINEARIZED THEORY	SOURCED BY KINETIC ENERGY
∇U_g	$\sim U_s/c^2$	FULL THEORY	SOURCED BY GRAVITATIONAL POTENTIAL ENERGY
∇U_p	$\sim p_s/(\rho_s c^2)$	LINEARIZED THEORY	SOURCED BY PRESSURE
∇U_t	$\lesssim (v_s/c)^2$	LINEARIZED THEORY	RETARDATION EFFECT
∇U^2	$\sim U_s/c^2$	FULL THEORY	NON-LINEAR EFFECT
$\mathbf{v} \times \left(\nabla \times \tilde{\mathbf{A}}_\mathbf{g}\right)$	$\sim (v_s/c)(v_p/c)$	LINEARIZED THEORY	LENSE-THIRRING EFFECT
$\mathbf{v}(\mathbf{v} \cdot \nabla U)$	$\sim (v_p/c)^2$	LINEARIZED THEORY	
$v^2 \nabla U$	$\sim (v_p/c)^2$	LINEARIZED THEORY	
$\frac{\mathbf{v}}{c}\frac{\partial U}{\partial t}$	$\lesssim (v_s/c)(v_p/c)$	LINEARIZED THEORY	EXPLICIT TIME DEPENDENCE
$\frac{\partial \tilde{\mathbf{A}}_\mathbf{g}}{\partial t}$	$\lesssim (v_s/c)^2$	LINEARIZED THEORY	EXPLICIT TIME DEPENDENCE

Table 4.2: Overview of physical effects in the post-Newtonian limit. The leftmost coloumn indicates which effect in the equation of motion (4.50) which is being considered. The next coloumn indicates how much that effect is suppressed compared to the Newtonian acceleration ∇U. The third coloumn tells whether elements from the full non-linear theory is needed in order to derive that effect ('linearized theory' means that the effect could have been derived starting with the linear field equation). The last coloumn gives a short comment on the origin or the characteristics of the effect.

4.1.8 Discussion

Comparing the post-Newtonian framework (table 4.1) to the GEM-framework , it is clear that the latter is far from a complete description regardless of whether we consider the time-independent case given by table 3.2 or the potential formulation given by table 3.1. The GEM-framework lacks several terms of the same smallness as the Lense-Thirring effect. This is obviously partly due to the fact that the GEM-framework is derived from linearized theory assuming a pressureless fluid. However, observe from equations (4.2), (2.32) and (4.37) that the only difference if starting from linearized theory, is that it is not possible to deal properly with the term U^2 and the potential U_g. Accordingly, most effects in the post-Newtonian limit can actually be derived from linearized theory. In the third column of table 4.2 we have indicated which effects that *could*[8] have been derived from linearized theory alone, and which one that demands the full non-linear field equation.

Why does the GEM-framework lack so many terms that can be derived from the linearized field equation with dust as model for the energy-momentum tensor? The missing terms is due to several inconsistencies in the approximations performed to obtain the GEM-framework. Firstly there are no good reasons for neglecting terms which are second order in v/c in the geodesic equation. As observed in the previous section, the Lense-Thirring effect is also a second order effect in velocity (see the discussion following equation (4.54)). Secondly, it is not consistent to naively use $\frac{d^2x^i}{d\tau^2} \approx \frac{d^2x^i}{dt^2}$. The correct approach is first to perform an exact relativistic calculation, like (4.41), and then subsequently neglect terms which are at least one order of magnitude smaller then the desired accuracy. Thirdly, the significance of the kinetic energy of the source is missed because one naively use $T^{00} \approx \rho c^2$, $T^{0i} \approx \rho c v^i$ and $T^{ij} \approx \rho v^i v^j \approx 0$ rather than developing an expression to the needed accuracy, as we did in (4.25).

Hence, we have showed that the GEM-framework is not even a proper approximation of the linearized field equation. The derivation of the GEM-framework has no foundations in a systematic method, but is motivated by a strong desire to write the field equations analogous to Maxwell's equations and the geodesic equation analogous to the Lorentz force law.

We have found the post-Newtonian limit of general relativity, and formulated it in a way that fits our discussion of the gravito-electromagnetic analogy (used appropriate coordinates/gauge, defined our own potentials and formulated the equation of motion in terms of the potentials). The next logical step in our project is to apply the same systematic method to electrodynamics, which is the subject of the next section.

4.2 The post-electrostatic limit of electrodynamics

In the previous chapter we studied general relativity in the weak-field slow-motion approximation. For the equation of motion we found that the post-Newtonian limit includes two terms which are second order in v/c. Here we will show that the same sort of effects also exist in electrodynamics.

Our approach will be to apply the same systematic method of section 4.1 to electrodynamics. This will enable us to compare the field theories for gravitation and electromagnetism in a consistent way beyond their lowest order approximations. In 4.1 we explored the post-Newtonian limit of general relativity. Now we shall find the corresponding limit of electrodynamics. For reasons that will become

[8] We have used the full non-linear field equation, of course, as the starting point for going to the post-Newtonian limit. It should therefore be stressed that also the effects labeled 'linearized theory' in the 'derived from'-column in table 4.2, are clearly, in our approach, derived from the full theory. The point though, is that the result would have become similar if we actually had started from the linearized field equation. The effects labeled 'full theory' however, can not be derived from the linearized field equation.

clear we shall name this limit the 'post-electrostatic limit' of electrodynamics (in flat space-time). We will find that in this limit the Lorentz force law leads to an expression for the acceleration which in addition to the magnetic-effect also includes the above mentioned second order effects in v/c.

In the case of gravity we introduced a book-keeping system to keep track on the size of terms in the expansions. This system was founded on the assumptions of slow motion and weak fields. We shall now introduce an analogous book-keeping system for electrodynamics. We will obviously not need to assume weak fields since Maxwell's equation and the Lorentz force-law are linear and exact laws. Instead, we will make another assumption about the fields:

$$c|\nabla \times \mathbf{A}| \lesssim |\nabla \phi|. \tag{4.55}$$

Subsequently, we will refer to this as the weak-magnetic-field assumption. In the gravitational case, the slow-motion assumption guarantees that an analogous relation is always satisfied[9]. However, in electromagnetism slow motion does generally not imply that magnetic effects are suppressed[10], which is the reason why we need to assume this explicitly. Notice however, that if there are no electric cancellation effects (all charges of same sign), then $\frac{c|\nabla\times\mathbf{A}|}{|\nabla\phi|} \sim O(1)$, ie. (4.55) is satisfied to good margin. Like in gravity, we will assume the speed of the source as well as the test-particle to be small:

$$\frac{v}{c} \sim O(1). \tag{4.56}$$

Once again, this implies the relation

$$\frac{|\partial_0|}{|\partial_i|} \sim O(1), \tag{4.57}$$

where we have neglected wave phenomena just like in the gravitational case (see the discussion below (4.10)). Equations (4.55), (4.56) and (4.57) constitute the book-keeping system for electrodynamics in the slow-motion weak-magnetic-field approximation. Applying the system to Maxwell's equations in Lorentz-gauge and the Lorentz force law (see table 3.1 on page 56), we find that the lowest order solution is

$$\frac{d\mathbf{v}}{dt} = -\frac{q}{m}\nabla\phi, \tag{4.58}$$

where ϕ satisfy the field equation

$$\nabla^2\phi = -\mu_0 c^2 \rho_q. \tag{4.59}$$

This is the basic relations of electrostatics. Accordingly, we will refer to the lowest order solution as the *electrostatic limit*, while the next iteration give us the *post-electrostatic limit*. Note that the electrostatic limit is a perfect formal analogy to the Newtonian limit of gravity, as all equations are on the same form and since the gravitational field couples to the invariant quantity ρ in the same manner as the electrical field couples to the invariant quantity ρ_q.

In the post-electrostatic limit Maxwell's equations can be written

$$\Box\phi = -\mu_0 c^2 \rho_q \tag{4.60}$$

and

$$\nabla^2\mathbf{A} = -\mu_0 \mathbf{j_q}. \tag{4.61}$$

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[9]From the book-keeping system we find $c|\nabla \times \mathbf{A}_g|/|\nabla\phi_g| \sim O(1)$ which implies that the analogous relation to (4.55) -that is $c|\nabla \times \mathbf{A}_g| \lesssim |\nabla\phi_g|$- is satisfied to good margin!

[10]For example the magnetic field around a current carrying wire is usually not suppressed compared to the electric field even if the speed of the electrons are slow. The reason is of course the fact that electric charge can be both positive and negative, which usually implies that the electric field cancels out.

The expression for the acceleration however, demands a little more effort. It is found by applying (4.55) and (4.56) to the Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{4.62}$$

If we naively use the Newtonian approximation $F \approx ma$, we will miss the second order effects in the test-particle velocity v/c. However, in situations where (4.55) and (4.56) holds, the second order effects in v/c are generally not suppressed compared to magnetic phenomenas and cannot be neglected. Therefore, like in the gravitational case, we will need to make an exact relativistic calculation before determining which terms to neglect. According to special relativity we have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma \mathbf{m} \mathbf{v}). \tag{4.63}$$

Inserting this into the Lorentz force law we get

$$\dot{\gamma}\mathbf{v} + \gamma\mathbf{a} = \mathbf{U}, \tag{4.64}$$

where **U** is the vector

$$\mathbf{U} = \frac{q\mathbf{E}}{m} + \frac{q}{m}\mathbf{v} \times \mathbf{B}. \tag{4.65}$$

Inserting

$$\dot{\gamma} = \frac{d}{dt}(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{c^2})^{-1/2} = \frac{1}{2c^2}\gamma^3\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{c^2}\gamma^3\mathbf{a}\cdot\mathbf{v} \tag{4.66}$$

into (4.64) and defining $\boldsymbol{\beta} = \mathbf{v}/c$ we get:

$$\gamma^3(\boldsymbol{\beta}\cdot\mathbf{a})\boldsymbol{\beta} + \gamma\mathbf{a} = \mathbf{U}. \tag{4.67}$$

This is the equation we want to solve for $\mathbf{a}$ to post-electrostatic accuracy, which turns out to be $\sim O(2)\nabla\phi$. To second order in $\boldsymbol{\beta}$ we get the solution

$$\frac{d\mathbf{v}}{dt} = \mathbf{U} - \frac{1}{2}(\boldsymbol{\beta}\cdot\boldsymbol{\beta})\mathbf{U} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{U}), \tag{4.68}$$

see appendix A.8 for details. Writing out the vectors $\boldsymbol{\beta}$ and $\mathbf{U}$ and then replacing $\mathbf{E}$ and $\mathbf{B}$ with potentials ϕ and $\mathbf{A}$ using (4.55) we get:

$$\begin{aligned}\frac{d\mathbf{v}}{dt} = \frac{q}{m}\Bigg[&-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{A}) \\ &+\frac{1}{2}v^2\nabla\frac{\phi}{c^2} + \mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi}{c^2})\Bigg].\end{aligned} \tag{4.69}$$

We have obtained the post-electrostatic limit of electrodynamics and the results are summarized in table 4.3. Discussion and comparison with the post-Newtonian limit of general relativity follows below.

4.2.1 The post-electrostatic limit of electrodynamics compared to the post-Newtonian limit of general relativity

As we have calculated the post-electrostatic limit of electrodynamics as well as the post-Newtonian limit of general relativity, we are enabled to compare the theories beyond their lowest order approximations in a consistent way. Let us compare the equation of motion in the electromagnetic case (4.69)

FIELD EQUATIONS	SOURCE
$\Box\phi = -\mu_0 c^2 \rho_q$	CHARGE
$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j_q}$	CHARGE FLOW
Condition satisfied by solutions: $\frac{\partial}{\partial t}\frac{\phi}{c^2} + \nabla \cdot \mathbf{A} = 0$	
Equation of motion: $\frac{d\mathbf{v}}{dt} = \frac{q}{m}\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) + \frac{1}{2}v^2 \nabla \frac{\phi}{c^2} + \mathbf{v}(\mathbf{v} \cdot \nabla \frac{\phi}{c^2})\right]$	
Error-estimate: $\sim (\frac{v}{c})^4 \lvert\nabla\phi\rvert$ or $\sim (\frac{v}{c})^3 c \lvert\nabla \times \mathbf{A}\rvert$	

Table 4.3: The post-electrostatic limit of electrodynamics summarized.

to the gravitational case (4.50). Notice that for each term in (4.69) there is a corresponding term in (4.50). We see that the second order effects in v/c exists in both gravity and electrodynamics, and apart from the coefficients the behavior is identical. We also observe that there are effects in both the gravitational and the electromagnetic case which depends on (partial) time derivatives of the vector potential. Even though there are several common effects/terms, there are also important differences. In particular note that there are no counter parts to any of the five terms in $\mathbf{G}$ in (4.50) (this was the reason for introducing the vector $\mathbf{G}$ in (4.50)). The first one of these terms is quadratic in the Newtonian potential U. This effect is a consequence of the fact that general relativity is a non-linear theory. The second term depends on (partial) time derivative of the scalar potential U. Something similar does not exist in the electromagnetic case. The three last terms in $\mathbf{G}$ are effects associated with the kinetic energy, gravitational binding energy and pressure of the source of the gravitational field. Hence, already at the second order approximations, the physics is considerably richer in the gravitational case than in the electromagnetic. These important differences where not captured by the formalism we reviewed in chapter 3.

We can conclude that there are important differences between gravity and electromagnetism once we choose to go beyond the lowest order solutions. However, we have also showed that there are important similarities. We have seen that physical analogies between gravity and electromagnetism goes further than the much considered analogies between Newtonian gravity and Coulomb's law, and the Lense-Thirring effect versus magnetism.

4.3 A new framework

In chapter 4.1 we showed that the GEM-framwork is actually not a valid approximation to general relativity in the weak-field slow-motion approximation. The GEM-framework includes only effects with obvious counter-parts in electrodynamics, which is the Newtonian acceleration, frame-dragging and retardation effects, but lacks several of the effects summarized in table 4.2. In the previous chapter we applied the same systematic methods to electrodynamics, and showed that the analogy between gravity and electromagnetism is not limited to the above mentioned effects. In this chapter we will reformulate the post-Newtonian limit from section 4.1 to a framework as similar as possible to the post-electrostatic limit of electrodynamics. We demand from our framework that it includes all post-Newtonian effects summarized in table 4.2. Within this restriction we formulate gravity in a language which mimics the post-electrostatic limit of electrodynamics as much as possible.

The greatest part of the job is already done in chapter 4.1, where we formulated the post-Newtonian limit of general relativity in the Lorentz gauge, as summarized in table 4.1. However, the obtained framework includes several scalar potentials U, U_k, U_g, U_p, U_t, contrary to electrodynamics where there are only one scalar potential ϕ. We will now show that also gravity -to post-Newtonian accuracy- can be formulated in terms of only one scalar potential ϕ_g and one vector potential $\mathbf{A}_g$, which obeys field equations similar to Mawells equations in Lorentz-gauge. This requires (as we will soon verify) that we define the 'gravitational charge' ρ_g as a special combination of rest-mass energy, kinetic-energy, gravitational binding-energy and pressure:

$$\rho_g \equiv \rho + 2\rho\frac{v^2}{c^2} - 2\rho\frac{U}{c^2} + 3\frac{p}{c^2}. \tag{4.70}$$

We define the gravitational scalar potential ϕ_g by the wave equation

$$\Box\phi_g = 4\pi G\rho_g, \tag{4.71}$$

and the vector potential $\mathbf{A}_g$ by Poisson's equation

$$\nabla^2\mathbf{A}_g = \frac{4\pi G}{c^2}\mathbf{j}_g, \tag{4.72}$$

where $\mathbf{j}_g = \rho_g\mathbf{v}$.

The plan is now basically to follow the procedure from chapter 4.1: first formulate $h_{00}[O(4)]$, $h_{0i}[O(3)]$ and $h_{ij}[O(2)]$ in terms of ϕ_g and $\mathbf{A_g}$ [11], and then rewrite the geodesic equation in terms of the same potentials. Remember that the notation $h_{\mu\nu}[O(p)] = f_{\mu\nu}$ means that $f_{\mu\nu}$ is an estimate for $h_{\mu\nu}$ of accuracy $O(p)$, that is $h_{\mu\nu} = f_{\mu\nu} + O(p+1)$. So if we have another function $\hat{f}_{\mu\nu}$ which are equal to $f_{\mu\nu}$ to accuracy $O(p)$, we have freedom to choose $\hat{f}_{\mu\nu}$ as an estimate for $h_{\mu\nu}$ instead of $f_{\mu\nu}$. Starting with (4.17) and (4.38), it is now possible to express the metric to the desired accuracy in terms of the potentials ϕ_g and $\mathbf{A}_g$ (see appendix A.9):

$$\begin{aligned} h_{00}[O(2)] &= -2\frac{\phi_g}{c^2}, \\ h_{00}[O(4)] &= -2\frac{\phi_g}{c^2} - 2\frac{\phi_g^2}{c^4}, \\ h_{0i}[O(3)] &= 4\frac{A_{gi}}{c}, \\ h_{ij}[O(2)] &= -2\frac{\phi_g}{c^2}\delta_{ij}. \end{aligned} \tag{4.73}$$

[11] The definitions of ρ_g, ϕ_g and $\mathbf{A_g}$ where chosen in order to make this possible. I found the definition of the gravitational charge after a little trial and error. However, for pedagogic reason it is better just to define them, and verify that they are appropriate by showing that the post-Newtonian metric (4.14) can be formulated in terms of them.

It will be useful to note that in terms of ϕ_g and $\mathbf{A}_g$ the Lorentz gauge condition $\partial_\mu \overline{h}^{\mu\nu} = 0$ reads [12]:

$$\frac{\partial}{\partial t}\frac{\phi_g}{c^2} + \nabla \cdot \mathbf{A_g} = 0. \tag{4.74}$$

Following the same procedure as in chapter 4.1 we get the following expression for the acceleration to post-Newtonian accuracy[13]:

$$\begin{aligned}\frac{d\mathbf{v}}{dt} =& - \nabla\phi_g - 2\nabla\frac{\phi_g^2}{c^2} \\ &+ 4\mathbf{v} \times (\nabla \times \mathbf{A_g}) + 4\mathbf{v}(\mathbf{v} \cdot \nabla\frac{\phi_g}{c^2}) - v^2\nabla\frac{\phi_g}{c^2} \\ &+ 3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2} - 4\frac{\partial \mathbf{A_g}}{\partial t}.\end{aligned} \tag{4.75}$$

The framework is summarized in table 4.4. Note the clear similarity to electrodynamics in the post-electrostatic limit (table 4.3). I have collected the most important relations from table 4.3 and 4.4 and summarized them in table 4.5. In the last mentioned table it is easy to compare the theories as I have written each term in the equation of motion for the gravitational case on the same line as the corresponding term in the electromagnetic case. Notice that ρ_g has exactly the same role in the field equations as the charge density ρ_q in Maxwell's equation, which is the reason why we have named it the gravitational charge. However, unlike electric charge, ρ_g is not an invariant quantity. The mass-density ρ and the pressure p are invariants by definition, ie. defined as the quantities measured by an observer comoving with the fluid. The kinetic energy density ρv^2 and the gravitational binding energy ρU are however not invariants since both v^2 and U depends on the gauge/coordinates.

The field equations are on exactly the same form due to the definition of the gravitational charge density ρ_g. It should be pointed out though, that there are no physical reason for the appearance of the gravitational charge flow $\mathbf{j_g} \equiv \rho_g \mathbf{v}$ rather than $\mathbf{j} = \text{(rest-mass flow)}$ in the field equation for $\mathbf{A_g}$, see (4.72). In the post-Newtonian limit, where h_{oi} must be accurate to $O(3)$, it does not matter whether $\mathbf{A_g}$ is sourced by $\mathbf{j}_g$ or $\mathbf{j}$. We have chosen to define it in terms of $\mathbf{j_g}$ because of our desire to write the post-Newtonian limit of general relativity as similar as possible to the post-electrostatic limit of electrodynamics. The potential ϕ_g however, *must* be sourced by the gravitational charge in order to determine h_{00} to $O(4)$ as required in the post-Newtonian approximation.

For the equation of motion however, it is not possible to write it on exactly the same form as the post-electrostatic limit. "Only" five out of seven terms are similar. The effects in gravity with no counterpart in electrodynamics, is the quadratic one in ϕ_g (which is $-2\nabla\frac{\phi_g^2}{c^2}$) and the term with explicit time-dependence on ϕ_g (which is $+3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2}$). The other effects exist in both gravity and electrodynamics. Quantitatively they are not identical due to different coefficients, but the behavior is similar. Also observe that each term in the equation of motion for electrodynamics has a factor $\frac{q}{m}$, where $q = \int d^3x\rho_q$ is the total charge of the test-particle. As expected, there is no similar term in the gravitational case which is due to the equivalence principle, ie. all objects fall with the same acceleration.

Thus we have showed that the post-Newtonian limit can be written in a language very similar to electrodynamics. However, the formal analogy to the post-electrostatic limit is not perfect. This is

[12]Notice that only the zero component of $\partial_\mu \overline{h}^{\mu\nu} = 0$ which can be rewritten in terms of ϕ_g and $\mathbf{A}_g$. The spatial components are suppressed, and not significant to post-Newtonian accuracy.

[13]Note that all terms in the first two lines of (4.50) is oriented along ϕ_g or $\mathbf{v}$. Different decompositions might be useful, see appendix A.10.

due to the fact that the gravitational charge is not invariant, and because there exist effects with no counterparts in electrodynamics. However, for the lowest order solutions of the approximation, which we found to be Newtonian gravity and electrostatics, the formal analogy is really perfect, since all equations are on the same form and since the gravitational field couples to the invariant quantity ρ in the same manner as the electrical field couples to the invariant quantity ρ_q[14]. This perfect analogy is summarized in table 4.6. As mentioned in the introduction, there exist a paper [18], where the standard GEM-framework is used to propose a coupling between gravitation and electromagnetism based on the fact that (most) particles carry both energy/mass and charge. This sort of coupling can however only exist for the lowest order solutions in the approximation, which is Newtonian gravity and electrostatics, since only at this level there is a perfect analogy.

In chapter 4.1 and 4.2 we studied approximations of general relativity and electrodynamics using perturbation theory. In this chapter we used the results to introduce a new framework where the post-Newtonian limit is formulated in a language as similar as possible to the post-electrostatic limit of electrodynamics. This has clarified the limitation and power of the analogy between gravitational and electromagnetic phenomenas. The analysis can however not yet be considered as complete. In the case of electrodynamics the acceleration is an observable, in the sense that the numerical values of $a^i = \frac{d^2x^i}{dt^2}$ equals the result of measurements performed by an observer at rest in the coordinate system. In the gravitational case however, the coordinate system is not Minkowskian[15] and it is not obvious that the acceleration is an observable in the same sense. This problem is the subject of chapter 5. First, however, we need to verify that my calculations in this chapter are correct.

[14]It should be mentioned that there is a difference since elecrical charge may have different signs, while mass is always positive.

[15]It is however very close to Mincowskian: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

FIELD EQUATIONS	SOURCE
$\Box\phi_g = 4\pi G\rho_g$	GRAVITATIONAL CHARGE
$\nabla^2 \mathbf{A}_g = \frac{4\pi G}{c^2}\mathbf{j}_g$	GRAVITATIONAL CHARGE FLOW

Condition satisfied by solutions:

$$\frac{\partial}{\partial t}\frac{\phi_g}{c^2} + \nabla \cdot \mathbf{A_g} = 0$$

Gravitational charge:

$$\rho_g \equiv \rho + 2\rho\frac{v^2}{c^2} - 2\rho\frac{U}{c^2} + 3\frac{p}{c^2}$$

Equation of motion:

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= -\nabla\phi_g - 2\nabla\frac{\phi_g^2}{c^2}\\ &\quad +4\mathbf{v}\times(\nabla\times\mathbf{A_g}) + 4\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) - v^2\nabla\frac{\phi_g}{c^2}\\ &\quad +3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2} - 4\frac{\partial\mathbf{A_g}}{\partial t}\end{aligned}$$

Error-estimate: $\sim \frac{v^4}{c^4}\nabla\phi_g \sim \frac{U^2}{c^4}\nabla\phi_g$

Table 4.4: Reformulation of the post-Newtonian limit of general relativity.

GRAVITY	ELECTRODYNAMICS
$\Box\phi_g = 4\pi G\rho_g$	$\Box\phi = -\mu_0 c^2\rho_q$
$\nabla^2\mathbf{A_g} = \frac{4\pi G}{c^2}\mathbf{j_g}$	$\nabla^2\mathbf{A} = -\mu_0\mathbf{j_q}$
$\rho_g \equiv \rho + 2\rho\frac{v^2}{c^2} - 2\rho\frac{U}{c^2} + 3\frac{p}{c^2}$	$\rho_q =$ CHARGE DENSITY
$\mathbf{j_g} \equiv \rho_g\mathbf{v}$	$\mathbf{j_q} \equiv \rho_q\mathbf{v}$
$\frac{d\mathbf{v}}{dt} = -\nabla(\phi_g + 2\frac{\phi_g^2}{c^2})$	$\frac{d\mathbf{v}}{dt} = \frac{q}{m}[-\nabla\phi$
$+4\mathbf{v}\times(\nabla\times\mathbf{A_g})$	$+\mathbf{v}\times(\nabla\times\mathbf{A})$
$+4\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2})$	$+\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi}{c^2})$
$-v^2\nabla\frac{\phi_g}{c^2}$	$+\frac{1}{2}v^2\nabla\frac{\phi}{c^2}$
$-4\frac{\partial\mathbf{A_g}}{\partial t}$	$-\frac{\partial\mathbf{A}}{\partial t}]$
$+3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2}$	

Table 4.5: The post-Newtonian limit of general relativity compared to electrodynamics in the post-electrostatic limit. Notice that each term in the gravitational acceleration is written on the same number of line as the corresponding term in electrodynamics.

NEWTONIAN LIMIT	ELECTRO-STATIC LIMIT
$\nabla^2\phi_g = 4\pi G\rho$	$\nabla^2\phi = -\mu_0 c^2\rho_q$
$\frac{d\mathbf{v}}{dt} = -\nabla\phi_g$	$\frac{d\mathbf{v}}{dt} = -\frac{q}{m}\nabla\phi$

Table 4.6: Newtonian limit of general relativity compared to the electro-static limit of electrodynamics. A perfect analogy.

4.4 Checking the calculations

In this chapter we have worked with the post-Newtonian limit of general relativity. Allthough we have used standard methods with a long tradition in gravitational theory, we need to verify that my calculations are correct. The reason for this is, as pointed out several times allready, that I have formulated the post-Newtonian limit in a way appropriate for our discussion of the gravito-electromagnetic analogy. In particular I have used different coordinates/gauge, introduced my own variables and potentials, and written the equation of motion in terms of potentials. It should be commented that there is a very rich litterature on the subject and that I am obviously not the first one to use the Lorentz gauge. However, I learned post-Newtonian methods from the textbook [3], which uses a different gauge. My approach was first to understand all steps in that textbook, before specializing to my own choice of coordinates and definitions.

The most obvious way to verify that my calculations are correct is to compare against exact solutions or to find relevant journal articles for comparison. I will start with the former alternative.

4.4.1 Comparison with the Scwarzschild solution

We shall compare the kinematics as predicted by the framework introduced in chapter 4.3 to the kinematics of the Schwartzschild spacetime. The Scwarzschild line element in the usual coordinates (r, θ, ϕ) (the socalled Schwarzschild coordinates) was given in (1.139). It turns out that these coordinates are not appropriate for comparison against my approximate solution which is isotropic in the spatial directions, ie. all the spatial diagonal components of the metric are equal. Therefore we will need to compare against the Scwarzschild solution in isotropic coordinates:

$$ds^2 = -\frac{\left(1-\frac{R_s}{4r}\right)^2}{\left(1+\frac{R_s}{4r}\right)^2}c^2dt^2 + \left(1+\frac{R_s}{4r}\right)^4(dx^2+dy^2+dz^2) = -c^2d\tau^2, \tag{4.76}$$

where $r^2 = x^2+y^2+z^2$. It was shown how this line element follows from the usual form in chapter 1.7.4. We will now apply the standard Lagrangian methods reviewed in chapter 1.6.4 to calculate the acceleration of a test particle with motion in the radial direction. From the line-element (4.76) we see that the Lagrangian becomes

$$\begin{aligned} L &\equiv \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &= -\frac{1}{2}\frac{(1-\frac{R_s}{4r})^2}{(1+\frac{R_s}{4r})^2}c^2\dot{t}^2 + \ldots, \end{aligned} \tag{4.77}$$

where the dots denotes terms which does not depend on $\dot{t} \equiv \frac{dt}{d\tau}$. Since the metric is stationary it follows from the Euler-Lagrange equations (1.105) that the time component of the conjugate momentum:

$$\begin{aligned} p_t &\equiv \frac{\partial L}{\partial \dot{t}} \\ &= -\frac{(1-\frac{R_s}{4r})^2}{(1+\frac{R_s}{4r})^2}c^2\dot{t}, \end{aligned} \tag{4.78}$$

is a constant of motion. From the line element (4.76) we see that

$$\dot{t} = \frac{1}{\sqrt{\frac{\left(1-\frac{R_s}{4r}\right)^2}{\left(1+\frac{R_s}{4r}\right)^2} - \left(1+\frac{R_s}{4r}\right)^4\frac{v^2}{c^2}}}, \tag{4.79}$$

where in general $v^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2$, but in our special case with radial motion $v^2 = (\frac{dr}{dt})^2$. Inserting (4.79) into (4.78) we get

$$p_t = -\frac{\frac{(1-\frac{R_s}{4r})^2}{(1+\frac{R_s}{4r})^4}c^2}{\sqrt{\frac{\left(1-\frac{R_s}{4r}\right)^2}{\left(1+\frac{R_s}{4r}\right)^6} - \frac{v^2}{c^2}}}. \tag{4.80}$$

Then we calculate the total time derivative $\frac{dp_t}{dt}$, insert it into $\frac{dp_t}{dt} = 0$, solves for the acceleration $a = \frac{d^2 r}{dt^2}$ and after tons of simple algebra we get:

$$\begin{aligned} a = &-\frac{1}{4}c^2\frac{(1-\frac{R_s}{4r})^2}{(1+\frac{R_s}{4r})^7}\frac{R_s}{r^2} - \frac{1}{4}c^2\frac{1-\frac{R_s}{4r}}{(1+\frac{R_s}{4r})^6}\frac{R_s}{r^2} \\ &+ \frac{1}{2}c^2\frac{1}{1-\frac{R_s}{4r}}\frac{R_s}{r^2}\frac{v^2}{c^2} + c^2\frac{1}{1+\frac{R_s}{4r}}\frac{R_s}{r^2}\frac{v^2}{c^2}. \end{aligned} \tag{4.81}$$

This expression can be expanded in power of the small quantities R_s/r and v/c. Going only to second order in these small quantities we are left with:

$$a = -c^2\frac{R_s}{r^2}\left[\frac{1}{2} - \frac{3}{2}\frac{v^2}{c^2} - \frac{R_s}{r}\right]. \tag{4.82}$$

Recall that $R_s \equiv 2\frac{GM}{c^2}$ where M is the total gravitatinal mass. To get (4.82) on a form more similar to our formulation of the post-Newtonian limit we introduce the potential $\phi = -\frac{GM}{r}$. Using that $\frac{R_s}{r} = -2\frac{\phi}{c^2}$ and $\frac{\phi}{r} = -\frac{GM}{r^2} = -|\nabla\phi|$, equation (4.82) can be rewritten as a vector equation:

$$\mathbf{a} = -\nabla\phi\left(1 - 3\frac{v^2}{c^2}\right) - 2\nabla\frac{\phi^2}{c^2}. \tag{4.83}$$

How does this fit with the equation of motion in chapter 4.3? The (exterior) Scwartzschild solution describes the spacetime outside a non-rotating stationary spherical mass distribution. Hence we have $\frac{\partial}{\partial t}\phi_g = 0$ and $\mathbf{A_g} = 0$ in the equation of motion (4.75). For radial motion we we have $\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) = v^2\nabla\frac{\phi_g}{c^2}$. Hence the equation of motion for the post-Newtonian limit (4.75) simplifies to

$$\mathbf{a} = -\nabla\phi_g\left(1 - 3\frac{v^2}{c^2}\right) - 2\nabla\frac{\phi_g^2}{c^2}, \tag{4.84}$$

exactly on the same form as (4.83)! Once again I stress the importance of using isotropic coordinates in the Scwartzschild solution. If we had used the ordinary Schwarzschild coordinates instead of isotropic coordinates the factor of 2 in front of $\nabla\frac{\phi^2}{c^2}$ in (4.83) would have been replaced by a factor 1, which does not fit with (4.84). This was very confusing to me the first time I tried to check my formalism in chapter 4.3 against the Scwarzschild spacetime. After a while I realized that I was dealing with two non-compatible coordinate systems (the radial coordinate in the usual Schwartzschild coordinates is not similar to the radial coordinate in the isotropic coordinates).

It is nice of course that our formalism reproduces a known solution. However, this is only a special case of a special case, namely *radial* motion in the Scwartzschild spacetime. It would of course be nice with a more general check than this one. In the next section I will compare against a journal paper which will provide a much more general check. The reason why I have included this section is that such calculations was important for me before I came aware of the PPN formalism and post-Newtonian methods. I compared against exact solutions to convince myself that the GEM-framework was not a complete description.

4.4.2 Comparison with Soffel et al.(1987)

We shall now compare the formalism of section 4.3 against the journal article [24]. This article deals with the two-body problem in the post-Newtonian approximation, ie. the kinematics of a two-body system. Equation (6) of this article gives an expression for the *relative* motion of two bodies:

$$\begin{aligned}\frac{d\mathbf{v}}{dt} = &-\frac{Gm}{r^2}\hat{\mathbf{n}} + \frac{Gm}{c^2r^2}\hat{\mathbf{n}}\left(\frac{Gm}{r}(2(\beta+\gamma)+2\nu) - \mathbf{v}^2(\gamma+3\nu) + \frac{3}{2}\nu(\hat{\mathbf{n}}\cdot v)^2\right) \\ &+ \frac{Gm}{c^2r^2}\mathbf{v}(\hat{\mathbf{n}}\cdot\mathbf{v})(2\gamma+2-2\nu),\end{aligned} \tag{4.85}$$

where m is the total mass ($m_1 + m_2$), $\mathbf{v}$ is the relative velocity, $\nu = \frac{m_1 m_2}{m^2}$ and γ and β are the so-called PPN parameters which takes the values $\gamma = \beta = 1$ in the case of general relativity. To compare against the framework in section 4.3, let us assume that body 2 is a test particle (ie. $m_2 = 0$) such that $m = m_1$ and $\nu = 0$. In that case the preferred (convenient) coordinate system is the one where body 1 (the source of the gravitational field) is at rest. In this coordinate system the metric will be stationary, and the test particle is moving in a stationary spacetime. Let us also introduce the potential $\phi = -\frac{Gm_1}{r}$ such that $-\frac{Gm_1}{r^2}\hat{\mathbf{n}} = -\nabla\phi$. In the case of general relativity ($\gamma = \beta = 1$), equation (4.85) can then be rewritten

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi - 2\nabla\frac{\phi^2}{c^2} + 4\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi}{c^2}) - v^2\nabla\frac{\phi}{c^2}. \tag{4.86}$$

In the preferred coordinate system this equation gives the acceleration of the test particle (which is similar to the relative acceleration). In the case of a static source the equation of motion (4.75) takes the form

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi_g - 2\nabla\frac{\phi_g^2}{c^2} + 4\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) - v^2\nabla\frac{\phi_g}{c^2}. \tag{4.87}$$

Thus our results are compatible with [24]. This gives a much more general test of the framework in 4.3 than in 4.4.1 since the test particle is no longer restricted to radial motion. The only terms in (4.75) not checked is then only those depending on the vector potential $\mathbf{A_g}$, and the one including a partial time derivative of ϕ_g. However, I am sure that the terms depending on $\mathbf{A}_g$ are correct, since this potential is associated with the Lense-Thirring effect, discussed in any paper of the gravito-electromagnetic analogy. I miss an independent check of the $\frac{\partial}{\partial t}\phi_g$ term though.

It should be commented that the gauge used in [24] is the same as the one used in [3]. This is not the same as the Lorentz gauge (which I have used). However, it turns out that in the stationary case the difference in gauge has no consequences for the scalar potential ϕ_g. The reason for this is that if we had chosen the gauge of [3], the operator $\Box$ in the field equation for ϕ_g (see equation (4.71)) would have been replaced by ∇^2. Hence, in the stationary case, the potential ϕ_g is invariant under the concidered change of gauge. You can verify all this by studying equation 5.28 in [3]. For the components g_{0i} though the change of gauge have significance, but we are not testing the Lense-Thirring effect here (as mention above I am convinced that the terms depending on $\mathbf{A}_g$ is correct).

Chapter 5

Proper reference frames

In chapter 4 we compared the post-Newtonian limit of general relativity with the corresponding limit of electrodynamics in a consistent way. We verified that there is a deep analogy between gravitational and electromagnetic phenomenas, but also showed that there are important differences. Our analysis can however not yet be regarded as a complete one, since what we have done is essentially just to compare the mathematical structure of two theories which are conseptually very different.

In this chapter we shall study the geometric significance of curvature in the post-Newtonian limit of general relativity. It was a very simple question concerning the interpretation of the equation of motion (for the gravitational case) which lead me to the work described in this chapter. We have worked out expressions for the coordinate acceleration $\frac{d^2\mathbf{x}}{dt^2}$ both for the electromagnetic and the gravitational case. In the former case we considered the phenomenon of electromagnetism in a flat spacetime background. Electrodynamics is then formulated in terms of the usual Lorentz coordinates, where the time coordinate has significance as proper time (as measured by an observer at rest in the coordinate system), and the spatial coordinates have significance has proper lengths/distances[1]. The coordinate acceleration therefore equals the physical acceleration measured by an observer equipped with a measuring rod and a clock at rest in the coordinate system. We therefore call the coordinate acceleration, in the electromagnetic case, an *observable*.

Throughout this chapter I will keep using the term observable in this very specific way. A quantity describing the state of a physical system is defined[2] as an observable only if the numerical value of its components equals the actual measured values. Hence, if T is a tensor describing a physical state with components $T^{\mu\nu}$ relative to a coordinate basis, and $T^{\hat{\mu}\hat{\nu}}$ relative to an orthonormal basis, it is $T^{\hat{\mu}\hat{\nu}}$ which we shall call the observable, *not* $T^{\mu\nu}$. In flat spacetime this is not an issue of course, since coordinates are chosen such that $T^{\mu\nu} = T^{\hat{\mu}\hat{\nu}}$ globally. Since the measured values also depend on the motion of the observer, it should be stressed that we always implicit assume an observer at rest in the considered coordinate system[3].

The gravitational case is far more subtle as spacetime is necessarily curved wherever gravitational phenomena exist. In curved spacetime it is not possible to introduce a globally Minkowskian coordinate system, and hence the coordinates has no immediate significance as proper time or proper

[1]Electrodynamics can also be formulated in terms of more general coordinates, but for practical reasons this is only done when it is really necessary, which is when spacetime is curved. We have only considered electromagnetism in a flat spacetime background however, and as alway, Lorentz coordinates are always implicit assumed.

[2]It turns out that the term observable is very rarely used in gravitational literature. I therefore needed to find an appropriate definition in line with traditions from other areas of physics.

[3]In the gravitational case there will also be a restriction on the position of the observer, since measurements are only well defined locally.

length. The coordinate acceleration $\frac{d^2\mathbf{x}}{dt^2}$ is therefore not equal to the physical measured acceleration, and hence not an observable according to our definition. However, in the weak-field slow-motion approximation, which we are considering, the coordinate system is *almost* Minkowskian. Furthermore, we have only operated with post-Newtonian accuracy, where the acceleration is accurate up to $O(2)\nabla U$. This suggests an interesting question: is the coordinate acceleration an observable to post-Newtonian accuracy? This question can of course be answered by calculating the physical measured acceleration, and then check whether there are corrections of order $O(2)\nabla U$ to the coordinate acceleration. However, we will first, in section 5.1, take a more direct approach, and show that the coordinate acceleration being an observable (to post-Newtonian accuracy) is incompatible with the equivalence principle. This allows us to conclude that the coordinate acceleration is not equal to the physical measured acceleration without having calculated the latter! Then, in section 5.2, we go on and find the actual expression for the measured acceleration in our post-Newtonian language. As the coordinate acceleration is not a tensor quantity[4], this expression can *not* be found in the usual way by calculating the components relative to an orthonormal basis. To find the measured acceleration we will introduce a special kind of coordinate system which is associated with the *proper reference frame* of the observer. To calculate the measured acceleration in our post-Newtonian language, we first make use of a result from the text-book [1]. Then I derive the same expression in a different way by taking a much geometric/visual approach. After having established the measured acceleration, we show, in section 5.3, that it can be used to gain insight into the kinematical differences between gravitation and electromagnetism.

In the following, in order to avoid too long sentences, we shall simply call the physical measured acceleration the *measured acceleration*. This is the physical measured acceleration of a particle with an arbitrary velocity. It should not be confused with the *proper acceleration*, which is defined as the physical acceleration measured in the instantaneous rest frame of the particle.

5.1 An accelerated observer in flat spacetime

In this section we shall make use of the equivalence principle, see section 1.2, to show that the coordinate acceleration cannot be an observable in the specific sense explained above. Recall that the equivalence principle can be formulated in two different ways. The first one states that a freely falling observer in curved spacetime cannot use local experiments to figure out whether he actually is freely falling in a gravitational field, or in an inertial frame in flat spacetime. The second formulation says that an observer at rest in a gravitational field doing local experiments will get similar results as an observer in an accelerated reference frame in a flat spacetime[5]. Here we will make use of the latter formulation.

Let us start by deciding what kind of experiment to consider. Imagine two observers, A and B. For simplicity we assume that A is at rest in a *static* gravitational field, while B is in an accelerated, but *non-rotating* reference frame in flat spacetime. In the gravitational case, since the field is static,

[4]Since it depends on the Christoffel connection. The tensor quantity associated with acceleration is not $\frac{d^2x^\mu}{dt^2}$, but the covariant derivative of the four-velocity $\frac{D}{d\tau}\frac{dx^\mu}{d\tau}$.

[5]In the relativistic sense both observers are accelerated as they both have a non-vanishing four-acceleration $D_{\mathbf{u}}\mathbf{u}$.

we have $\mathbf{A}_g = \frac{\partial}{\partial t}\phi_g = 0$[6], and the coordinate acceleration, see (4.75), is simplified to

$$\frac{d\mathbf{v}}{dt} = -\nabla\phi_g - 2\nabla\frac{\phi_g^2}{c^2} + 4\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) - v^2\nabla\frac{\phi_g}{c^2}. \tag{5.1}$$

To post-Newtonian accuracy this is equivalent to

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}_0 - 4\frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{a}_0\right) + \frac{v^2}{c^2}\mathbf{a}_0, \tag{5.2}$$

where

$$\mathbf{a}_0 = -\nabla\phi_g - 2\nabla\frac{\phi_g^2}{c^2}. \tag{5.3}$$

Thus we have written the coordinate acceleration $\mathbf{a}$ as a function of the coordinate velocity $\mathbf{v}$ and $\mathbf{a}_0$, the coordinate acceleration of a particle at rest.

We assume that our observers perform a very simple experiment of relativistic mechanics. They simply measure the acceleration of the test particles at a given point, and plot the measured acceleration $\overline{\mathbf{a}}$ as a function of $\overline{\mathbf{v}}$ and $\overline{\mathbf{a}_0}$, where $\overline{\mathbf{v}}$ is the measured velocity and $\overline{\mathbf{a}_0}$ is the measured acceleration of a particle at rest (ie. $\overline{\mathbf{v}} = 0$). From the plot they write down an analytic best-fit function for the acceleration as a function of $\overline{\mathbf{v}}$ and $\overline{\mathbf{a}_0}$:

$$\overline{\mathbf{a}} = f(\overline{\mathbf{v}}, \overline{\mathbf{a}_0}). \tag{5.4}$$

The equivalence principle then states that A and B will evaluate the *same* function $f(\overline{\mathbf{v}}, \overline{\mathbf{a}_0})$. If not, our suggestion would imply that it is (principial) possible to distinguish between the effects of a gravitational field and an accelerated reference frame by means of a local experiment, ie. a violation of the equivalence principle, the corner stone of gravitational physics!

We assume that the observers can measure all post-Newtonian effects, but no post-post-Newtonian effects[7]. If the coordinate acceleration is an observable, it then follows from 5.2 that A will find the following functional dependence from his plot:

$$f(\overline{\mathbf{v}}, \overline{\mathbf{a}_0}) = \overline{\mathbf{a}_0} - 4\frac{\overline{\mathbf{v}}}{c}\left(\frac{\overline{\mathbf{v}}}{c}\cdot\overline{\mathbf{a}_0}\right) + \frac{\overline{v}^2}{c^2}\overline{\mathbf{a}_0}. \tag{5.5}$$

It is then easy to check whether the coordinate acceleration is an observable to post-Newtonian accuracy or not. We simply compute the acceleration of a freely falling test-particle relative to an accelerated, but non-rotating, reference frame in a flat spacetime background and check whether the functional dependence on $\overline{\mathbf{v}}$ and $\overline{\mathbf{g}}$ is similar to (5.5).

It should be stressed that it is only due to simplicity that we have chosen a *static* gravitational field for A and a *non-rotating* accelerated reference frame for B. We could have done the analysis in full generality, but this is not necessary to solve our problem. To show that the coordinate acceleration (4.75) is not an observable, it is indeed enough to find a *single* example showing that this is not compatible with the equivalence principle!

[6]Static gravitational fields are those produced by non-moving sources. For example a stationary non-rotating mass distribution produces such a field. The field outside a rotating spherical mass-distribution is an example of a stationary field which is *not* static.

[7]Recall that post-Newtonian effects are of smallness $O(2)\nabla U$, while post-post-Newtonian effects are of smallness $O(4)\nabla U$

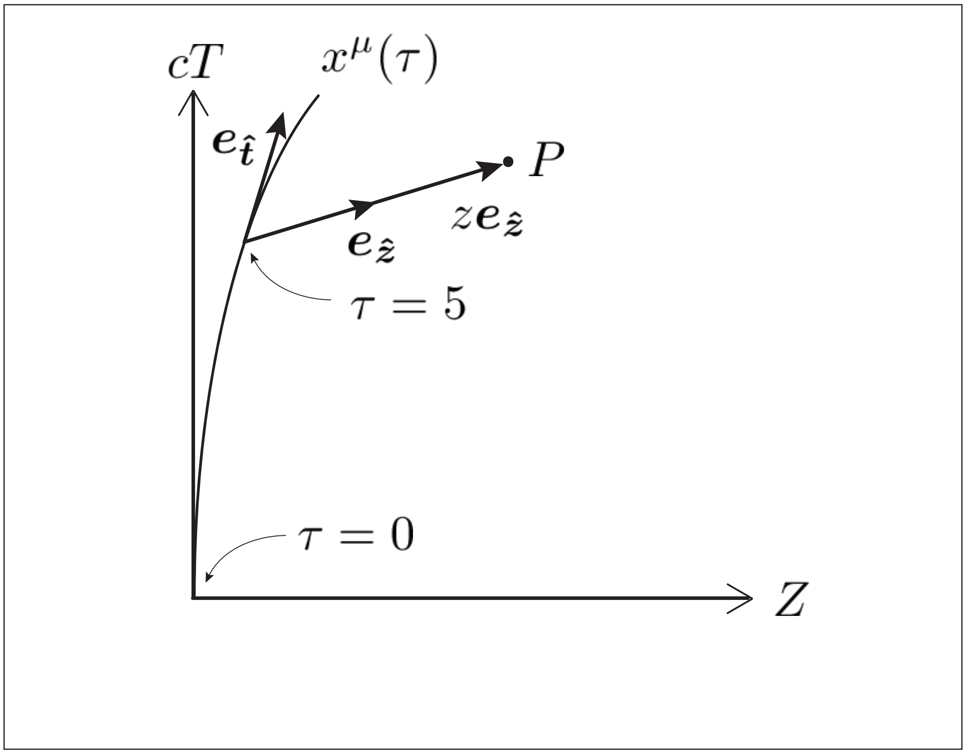

Figure 5.1: The world line $x^\mu(\tau)$ of the origin of accelerated reference frame traces out a hyperbola in the inertial system (T, X). The event P has time coordinate $t = 5$ since it lays in the simultaneity space of the event $(t, z) = (5, 0)$, ie. along the direction defined by the basis vector $\mathbf{e}_{\hat{\mathbf{z}}}$.

5.1.1 Hyperbolically accelerated reference frame

The accelerated, non-rotating reference frame in flat spacetime is a well-studied case in the literature. It is called a hyperbolically accelerated reference frame when the proper acceleration[8] of the origin of the accelerated reference frame is constant because the path (of the origin of the reference frame) then traces out a hyperbola in the inertial system. Such reference frames often serves as introductory material in first courses of general relativity, and will not be derived here. We shall rather give a brief review of the basic results.

Consider an inertial frame S with (Lorentz) coordinates T, X, Y, Z and Minkowski metric

$$ds^2 = -c^2 dT^2 + dX^2 + dY^2 + dZ^2. \tag{5.6}$$

We denote the coordinates of the hyperbolically accelerated reference frame t, x, y, z. One assumes that the origin of the coordinate systems ($X = Y = Z = 0$ and $x = y = z = 0$) coincide at time $T = t = 0$, and let the accelerated reference frame move along the Z axis of S, see figure 5.1. The proper acceleration $\mathbf{g}$ of a particle at rest at the origin of the accelerated reference frame is constant.

[8] Recall from special relativity that for linear acceleration the proper acceleration $\hat{a}$ is related to the coordinate acceleration a by $\hat{a} = \gamma^3 a$.

The coordinates t, x, y, z are then related to the Lorentz coordinates by the coordinate transformations:

$$\begin{aligned} T &= \frac{c}{g}\sinh\frac{gt}{c} + \frac{z}{c}\sinh\frac{gt}{c}, \\ X &= x, \\ Y &= y, \\ Z &= \frac{c^2}{g}(\cosh\frac{gt}{c} - 1) + z\cosh\frac{gt}{c}, \end{aligned} \tag{5.7}$$

see chapter [7, ch.4.2] for a detailed derivation. Let us now discuss the significance of the coordinates. The time coordinate t is the proper time as measured by a standard clock at rest at the origin of the accelerated reference frame. The spatial coordinates (x, y, z) have significance as proper lengths. This means for example that the point $(0, 0, z_0)$ lays a proper distance z_0 from the origin of the accelerated reference frame as measured by a measuring rod at rest relative to this frame. Accordingly, at the origin of the accelerated reference frame, $\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2}$ is the *measured acceleration*, ie. the acceleration of a freely falling test particle as measured by an observer at rest at the origin equipped with a standard clock and a measuring rod. We shall therefore assume that the observer B in the discussion above performs his experiment at the origin of the hyperbolically accelerated reference frame. What we need to do, is finding an expression for $\mathbf{a}$ in terms of $\mathbf{g}$ and the velocity $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ of the test particle. My approach is to calculate the metric associated with the accelerated reference frame and then compute the acceleration from the geodesic equation in the usual way.

Applying the standard transformation law for the metric to the coordinate transformation (5.7) we find

$$ds^2 = -\left(1 + \frac{gz}{c^2}\right)^2 c^2dt^2 + dx^2 + dy^2 + dz^2, \tag{5.8}$$

and accordingly

$$\begin{aligned} g_{00} &= -\left(1 + \frac{gz}{c^2}\right)^2, \\ g_{11} &= g_{22} = g_{33} = 1. \end{aligned} \tag{5.9}$$

The contravariant components defined by $g^{\mu\alpha}g_{\alpha_\nu} = \delta^\mu_\nu$ becomes

$$\begin{aligned} g^{00} &= -\frac{1}{\left(1 + \frac{gz}{c^2}\right)^2}, \\ g^{11} &= g^{22} = g^{33} = 1. \end{aligned} \tag{5.10}$$

The non-zero components of the Christoffel connection (see definition (1.87)) become

$$\Gamma^0_{03} = \Gamma^0_{30} = \frac{g}{c^2}\frac{1}{1 + \frac{gz}{c^2}} \tag{5.11}$$

and

$$\Gamma^3_{00} = \frac{g}{c^2}\left(1 + \frac{gz}{c^2}\right). \tag{5.12}$$

From the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{5.13}$$

we find

$$\mathbf{a} = \mathbf{g}\left(1 + \frac{gz}{c^2}\right) - 2\frac{1}{1 + \frac{gz}{c^2}}\frac{\mathbf{g}\cdot\mathbf{v}}{c^2}\mathbf{v}, \tag{5.14}$$

where $\mathbf{v}$ and $\mathbf{a}$ are the vectors with components $\frac{dx^i}{dt}$ and $\frac{d^2x^i}{dt^2}$ respectively. For details see appendix A.12. At the origin this simplifies to

$$\mathbf{a} = \mathbf{g} - 2\frac{\mathbf{g}\cdot\mathbf{v}}{c^2}\mathbf{v}. \tag{5.15}$$

5.1.2 Discussion

We have calculated that the (physical) acceleration of a freely falling test-particle as measured in an accelerated, but non-rotating reference frame in flat spacetime is

$$\mathbf{a} = \mathbf{g} - 2\frac{\mathbf{g}\cdot\mathbf{v}}{c^2}\mathbf{v}, \tag{5.16}$$

while the coordinate acceleration of a particle in a static gravitational field, ie. (5.2), is

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}_0 - 4\frac{\mathbf{a}_0\cdot\mathbf{v}}{c^2}\mathbf{v} + \frac{v^2}{c^2}\mathbf{a}_0 \tag{5.17}$$

in the post-Newtonian approximation. An observer able to perform measurements of the acceleration to post-Newtonian accuracy would therefore be able to distinguish between a real gravitational field and an accelerated reference frame, if the coordinate acceleration where an observable. This clearly shows that the coordinate acceleration being an observable is incompatible with the equivalence principle. Thus we have showed that the coordinate acceleration is *not* an observable. It is quite fascinating that it is possible to show this without calculating the actual expression for the measured acceleration.

For later reference let us briefly review the general case. If the accelerated reference frame in flat spacetime is also allowed to rotate with an angular velocity ω, an observer at the origin of the reference frame will measure an acceleration

$$\mathbf{a} = \mathbf{g} - 2\frac{\mathbf{g}\cdot\mathbf{v}}{c^2}\mathbf{v} - 2\omega\times\mathbf{v}, \tag{5.18}$$

see [1, ch 6.6] for a derivation.

5.2 Proper reference frames in curved spacetime

By making use of the equivalence principle, we have showed that the coordinate acceleration in the post-Newtonian approximation is not an observable; not the acceleration measured by an observer at rest in the coordinate system. In this section we shall find an expression for the the measured acceleration of a (freely falling) test particle in terms of our post-Newtonian language. First however, we will need to introduce the concept of a *proper reference frame* in curved spacetime.

5.2.1 The proper coordinate system

We will follow the treatment of [1, ch.13.6] to introduce the proper reference frame of an accelerated observer in curved spacetime.

We let $\overline{x}^\mu$ denote the coordinates associated with the proper reference frame, while x^μ, as usual, denotes the original coordinate system. We let τ be the proper time of the observer, ie. the time

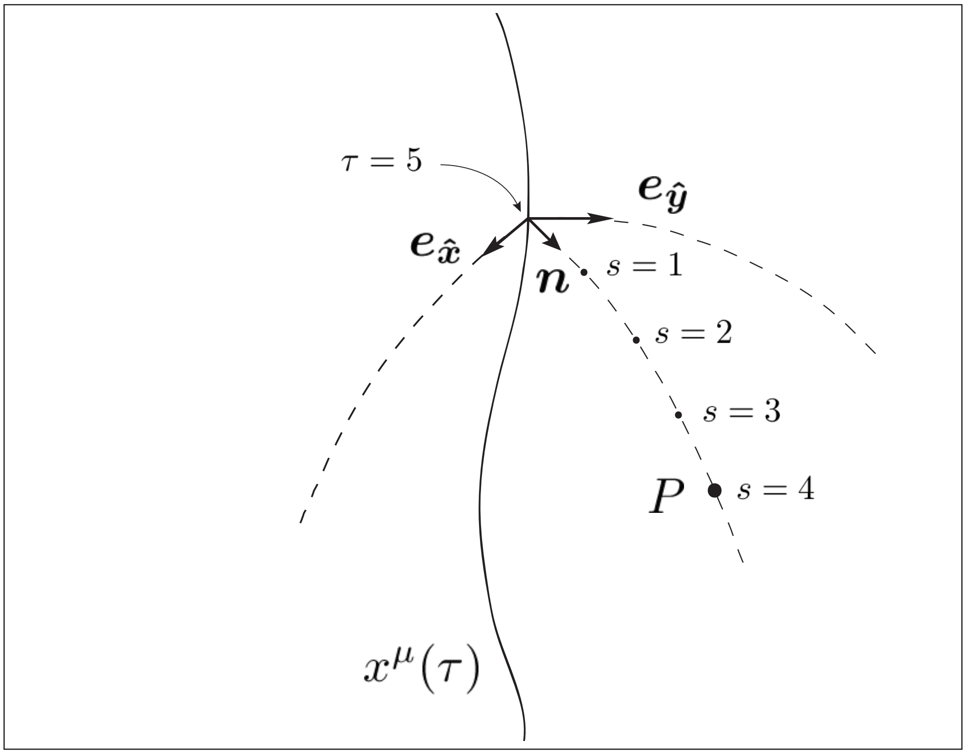

Figure 5.2: The origin of the proper reference frame moves along the world line $x^\mu(\tau)$. The point P is intersected by the geodesic starting out from the origin at proper time $\tau = 5$ with the (initial) direction $\mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{e}_{\hat{\mathbf{x}}} + \mathbf{e}_{\hat{\mathbf{y}}})$ at proper distance $s = 4$ from the origin. Thus the point P has proper coordinates $\bar{t} = 5$ and $\overline{x} = \overline{y} = 4/\sqrt{2}$.

measured on a comoving standard clock, and $u^\mu \equiv \frac{dx^\mu}{d\tau}$ the tangent vector to the observer's world line. As we are considering an observer at rest in the coordinate system, all spatial components of u^μ are zero. Remember from section 1.5.1 that associated with an observer there is a physical basis $\mathbf{e}_{\hat{\mu}}$ which satisfy $\mathbf{e}_{\hat{0}} = \frac{\mathbf{u}}{c}$ and $\mathbf{e}_{\hat{\mu}} \cdot \mathbf{e}_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$. These orthonormal vectors, from now on referred to as a *tetrad*, defines the time direction and the spatial directions of the laboratory which the observer is carrying with himself along his world line. This tetrad is the starting point for the construction of the proper reference frame and the associated coordinates $\overline{x}^\mu$.

The proper reference frame of an accelerated observer in curved spacetime is essentially a straight forward generalization of the accelerated reference frame in flat spacetime. The construction of the coordinates $\overline{x}^\mu$ are visualized in figure 5.2 and can be summarized in three steps:

1. From each point on his world line the observer sends out a continuum of purely *spatial* geodesics which covers the space around him. Each geodesic can be specified uniquely by the proper time τ, which denotes the starting point of the geodesic, and a pure spatial unit vector $\mathbf{n} = n^1\mathbf{e}_{\hat{1}} + n^2\mathbf{e}_{\hat{2}} + n^3\mathbf{e}_{\hat{3}}$ which specifies the (initial) direction of that geodesic relative to the tetrad vectors.

2. Each event near the observer's world line is intersected by precisely one of the geodesics. We let the parameter s be the proper length measured along the geodesic from the starting point, see figure 5.2. Each event near the observer is then uniquely specified by τ, $\mathbf{n}$ and s. The first two

parameters specifies which geodesic that is intersecting the event, while the last one specifies where on that geodesic the event is located.

3. An event P specified by the the the entities τ, $\mathbf{n}$ and s is then assigned the following coordinates in the proper reference frame:

$$(\overline{x}^0, \overline{x}^1, \overline{x}^2, \overline{x}^3) = (c\tau, sn^1, sn^2, sn^3). \tag{5.19}$$

Thus we have introduced the coordinates associated with the proper reference frame. The acceleration $\overline{a}^i = \frac{d^2\overline{x}^i}{d\bar{t}^2}$ of a particle at the origin of this coordinate system is the measured acceleration[9]. There are several things to comment concerning the points above. A geodesic is a curve whose tangent vectors are related by parallel transport. A spatial geodesic will therefore in general not follow a straight line in the original coordinate system x^μ, ie. $\frac{d^2x^\mu}{ds^2} \neq 0$ along the geodesic. The geodesic equation may be used for a space like geodesic if the parameter τ is replaced by the parameter s:

$$\frac{d^2x^\mu}{ds^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}. \tag{5.20}$$

Concerning point 2 above, it must be stressed that it is only near the world line of the observer that there is a unique geodesic which intersects the event. Far away from the observer the geodesics may cross due to the curvature of spactime or because of the (time varying) acceleration of the observer (concerning the latter, see [1] ch. 6.3 for more details). Accordingly the coordinate system $\overline{x}^\mu$ breaks down far away and can only be used close to the observer.

A reader familiar with the treatment in [1], may have noticed that we have not mentioned the 'transport law for the observer's tetrad'[10]. This is a relation describing the change of the observer's tetrad along his world line in terms of his four velocity, four acceleration and rotation relative to inertial-guidance gyroscopes. From this relation one can find expressions for the Christoffel coefficients, which can be used to find an expression for the measured acceleration of a freely falling particle. These calculations gives an exact expression for the measured acceleration in terms of the acceleration and rotation of the observer and the measured particle's velocity. The approach in [1] is formally correct of course, but does not provide any insight into why the expression for the measured acceleration is on a different form than the coordinate acceleration. I will try to derive the measured acceleration in a very different way which will provide much more insight into this. As I only consider an observer at rest in the original coordinate system, I will not need to introduce the transportation law for the tetrad. First however, in section 5.2.2, I will start with the result in [1], and reformulate it in terms of our post-Newtonian language. Then, in section 5.2.3, I derive the same result in a very different, and possibly new, way.

5.2.2 The measured acceleration

The result for the measured acceleration ($\frac{d^2\overline{x}^j}{d\bar{t}^2}$) given by equation 13.75 in [1] is:

$$\frac{d^2\overline{x}^j}{d\bar{t}^2}\mathbf{e}_{\hat{j}} = -\mathbf{A} + 2(\mathbf{A}\cdot\frac{\mathbf{v}}{c})\frac{\mathbf{v}}{c} - 2\omega\times\mathbf{v}, \tag{5.21}$$

where $\mathbf{A} \equiv D_{\mathbf{u}}\mathbf{u}$ is the four acceleration of the observer, ω is the rotation of the spatial part of the observer's tetrad relative to comoving inertial-guidance gyroscopes, and $\mathbf{v} = \frac{d\overline{x}^i}{d\bar{t}}\mathbf{e}_{\hat{j}}$ is the measured

[9] Outside the origin this does not hold, since the standard clock defining the $\bar{t}$ coordinate is attached to the origin of the proper reference frame.

[10] See eq. 13.60 in [1].

velocity of the freely falling test particle. The four acceleration $\mathbf{A}$ of the observer will in general have a non-vanishing zero component. The zero component of $\mathbf{A}$ relative to the observer's orthonormal basis however, *is* vanishing, ie. $A^{\hat{0}} = 0$, since the four acceleration is orthogonal to the four velocity (recall that $\mathbf{e}_{\hat{\mathbf{0}}} = \mathbf{u}/c$, such that $u^{\hat{i}} = 0$ and hence it follows from $u^\mu A_\mu = u^{\hat{\mu}} A_{\hat{\mu}} = 0$ that $A^{\hat{0}} = 0$). This explains how it can be that there are only spatial vectors (orthonormal basis) on the left-hand side of (5.21) while there are (in general) also time vectors (coordinate basis) on the right-hand side. Notice that (5.21) is on the same form as (5.18), which shows that the expression for the measured acceleration in curved spacetime is compatible with the equivalence principle.

What we are interested in, are the components $\frac{d^2\overline{x}^j}{d\overline{t}^2}$ which has significance as the measured acceleration. Therefore we introduce the (ordinary) three vector $\overline{\mathbf{a}}$, whose components are $\frac{d^2\overline{x}^i}{d\overline{t}^2}$, and write (5.21) as

$$\overline{a}^i = -A^{\hat{i}} + 2(\mathbf{A} \cdot \frac{\mathbf{v}}{c})\frac{v^{\hat{i}}}{c} - 2(\omega \times \mathbf{v})^{\hat{i}}. \tag{5.22}$$

Our task is to find the measured acceleration $\overline{\mathbf{a}}$ associated with an observer at *rest* in the original coordinate system and with *no rotation* relative to the grid lines of the original coordinate system. Hence we must find the four acceleration and the rotation relative to inertial-guidance gyroscopes of such an observer. We start with the former.

An observer at rest in the original coordinate system has four velocity

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = (c\frac{dt}{d\tau}, 0, 0, 0). \tag{5.23}$$

Thus the components of the four acceleration relative to the coordinate basis of the original coordinate system becomes:

$$\begin{aligned} A^\alpha &= \frac{D}{d\tau}u^\alpha \\ &= u^\beta D_\beta u^\alpha \\ &= u^0 D_0 u^\alpha \\ &= u^0(\partial_0 u^\alpha + \Gamma^\alpha_{0\beta} u^\beta) \\ &= u^0(\partial_0 u^\alpha + \Gamma^\alpha_{00} u^0), \end{aligned} \tag{5.24}$$

where I have used the definition of the covariant directional derivative (1.78) in the second line, (5.23) in the third line, the definition of the covariant derivative (1.72) in the fourth line and (5.23) once again in the last line. The space components of $\mathbf{A}$ becomes

$$A^i = (u^0)^2 \Gamma^i_{00}. \tag{5.25}$$

Instead of calculating the zero-component from (5.24), which is also possible, we use the fact that the four-acceleration A^μ is orthogonal to the four velocity u^μ, ie. $g_{\mu\nu} u^\mu A^\nu = 0$. To lowest order[11] this gives

$$A^0 = g_{i0} A^i, \tag{5.26}$$

where we have used that $u^i = 0$. Our goal is to formulate the measured acceleration in terms of our post-Newtonian language. Hence we must formulate u^0 and Γ^i_{00} in terms of the potentials ϕ_g and $\mathbf{A}_g$

[11] We have used that to lowest order $g_{00} = -1$.

introduced in section 4.3. By using results from appendix A.6 and the definition of ϕ_g and $\mathbf{A}_g$, see (4.71) and (4.72), we get

$$\Gamma^i_{00} = \partial^i \frac{\phi_g}{c^2} + 4\frac{\phi_g}{c^2}\partial^i \frac{\phi_g}{c^2} + \frac{4}{c^2}\frac{\partial}{\partial t}A^i_g. \tag{5.27}$$

As our observer is at rest in the (original) coordinate system ($dx = dy = dz = 0$), the line element becomes

$$\begin{aligned} g_{00}c^2dt^2 &= -c^2d\tau^2, \\ (-1 - 2\frac{\phi_g}{c^2} - 2\frac{\phi_g^2}{c^4})c^2dt^2 &= -c^2d\tau^2, \end{aligned} \tag{5.28}$$

which means that

$$(u^0)^2 = (c\frac{dt}{d\tau})^2 = \frac{c^2}{1 + 2\frac{\phi_g}{c^2} + 2\frac{\phi_g^2}{c^4}}. \tag{5.29}$$

Inserting (5.27) and (5.29) into (5.25) and evaluating to post-Newtonian accuracy we get

$$A^i = \partial^i\phi_g + 2\frac{\phi_g}{c^2}\partial^i\phi_g + 4\frac{\partial}{\partial t}A^i_g. \tag{5.30}$$

Inserting (5.30) into (5.26) and rewriting as a vector equation we find the lowest order solution for the zero component:

$$A^0 = \frac{4}{c}\mathbf{A_g}\cdot\nabla\phi_g. \tag{5.31}$$

What we need, according to (5.22), is to find the spatial components of $\mathbf{A}$[12] relative to the orthonormal basis $\mathbf{e}_{\hat{\mu}}$. This can be done by expressing the coordinate basis vectors e_μ as a linear combination of the orthonormal basis vectors $\mathbf{e}_{\hat{\mu}}$. The orthonormal basis $\mathbf{e}_{\hat{\mu}}$ is the physical basis associated with the observer, which means that $\mathbf{e}_{\hat{0}} = \mathbf{u}/c$. As the observer is at rest in the coordinate system ($u^i = 0$), we also know that $\mathbf{u}$ is parallel to $\mathbf{e_0}$. Accordingly, the vector $\mathbf{e_{\hat{0}}}$ is parallel to $\mathbf{e_0}$, and it follows that

$$\mathbf{e}_{\hat{0}} = \frac{\mathbf{e}_0}{\sqrt{|\mathbf{e}_0\cdot\mathbf{e}_0|}} = \frac{\mathbf{e}_0}{\sqrt{|g_{00}|}} = \mathbf{e}_0(1 - \frac{\phi_g}{c^2}), \tag{5.32}$$

or

$$\mathbf{e}_0 = \mathbf{e}_{\hat{0}}\left(1 + \frac{\phi_g}{c^2}\right). \tag{5.33}$$

A little more effort is needed for the spatial basis vectors. We start by writing

$$\mathbf{e}_i = \mathbf{e}_{\perp i} + \mathbf{e}_{\parallel i}, \tag{5.34}$$

where $\mathbf{e}_{\perp i}$ is the component of $\mathbf{e}_i$ orthogonal to $\mathbf{e}_0$, ie. $\mathbf{e}_{\perp i}\cdot\mathbf{e}_0 = 0$, and $\mathbf{e}_{\parallel i}$ is the component of $\mathbf{e}_i$ parallel to $\mathbf{e}_0$:

$$\mathbf{e_{\parallel i}} = \frac{\mathbf{e}_i\cdot\mathbf{e}_0}{\mathbf{e}_0\cdot\mathbf{e}_0}\mathbf{e}_0 = \frac{g_{i0}}{g_{00}}\mathbf{e}_0. \tag{5.35}$$

Thus we have

$$\mathbf{e}_{\perp i} = \mathbf{e}_i - \mathbf{e}_{\parallel i} = \mathbf{e}_i - \frac{g_{i0}}{g_{00}}\mathbf{e}_0. \tag{5.36}$$

[12]WARNING: Do not confuse the four acceleration $\mathbf{A}$ with the gravito-magnetic potential $\mathbf{A_g}$. I will promise to write the correct one!

The spatial vectors of the observer's tetrad ($\mathbf{e}_{\hat{i}}$) are found by normalizing the vectors $\mathbf{e}_{\perp i}$[13]:

$$\mathbf{e}_{\hat{i}} = \frac{\mathbf{e}_{\perp i}}{\sqrt{|\mathbf{e}_{\perp i} \cdot \mathbf{e}_{\perp i}|}} = \frac{\mathbf{e}_i - \frac{g_{i0}}{g_{00}}\mathbf{e}_0}{\sqrt{|(\mathbf{e}_i - \frac{g_{i0}}{g_{00}}\mathbf{e}_0) \cdot (\mathbf{e}_i - \frac{g_{i0}}{g_{00}}\mathbf{e}_0)|}} = \frac{\mathbf{e}_i - \frac{g_{i0}}{g_{00}}\mathbf{e}_0}{\sqrt{|g_{ii} - \frac{g_{i0}g_{i0}}{g_{00}}|}}. \tag{5.37}$$

Rewriting in terms of our potentials, neglecting contributions suppressed by $O(4)$, and solving for $\mathbf{e}_i$ yields:

$$\mathbf{e}_i = (1 - \frac{\phi_g}{c^2})\mathbf{e}_{\hat{i}} - \frac{4}{c}A_{gi}\mathbf{e}_0. \tag{5.38}$$

Combining (5.30), (5.31), (5.33) and (5.38) we get

$$\mathbf{A} = A^0\mathbf{e}_0 + A^i\mathbf{e}_i = \left(\partial^i\phi_g + \frac{\phi_g}{c^2}\partial^i\phi_g + 4\frac{\partial}{\partial t}A_g^i\right)\mathbf{e}_{\hat{\mathbf{i}}}. \tag{5.39}$$

Notice that the time-component of $\mathbf{A}$ vanish, just as it should in the orthonormal basis! From (5.39) we see that the components of $\mathbf{A}$ relative to the observer's tetrad are:

$$A^{\hat{i}} = \partial^i\phi_g + \frac{\phi_g}{c^2}\partial^i\phi_g + 4\frac{\partial}{\partial t}A_g^i. \tag{5.40}$$

We also need an expression for the rotation frequency ω of the observer's proper reference frame relative to inertial-guidance gyroscopes. At this point the reader may want to review the discussion following equation (3.40), where it is explained why it is natural, in general relativity, to let inertial-guidance gyroscopes define the axis of a non-rotating reference frame. As already mentioned, we consider an observer with no rotation relative to the grid lines of the original coordinate system. The total precession is in general, see (3.41) and (3.42 on page 61), a combination of the Lense-Thirring effect, geodesic precession and Thomas precession. In our case however, there is neither geodesic- nor Thomas precession since we consider an observer at rest in the coordinate system, ie. $\mathbf{v} = 0$ in (3.42). Thus, the precession of an inertial-guidance gyroscope carried by our observer becomes:

$$\mathbf{\Omega} = -2\nabla \times \mathbf{A_g}. \tag{5.41}$$

Hence the rotation of the observer's tetrad relative to an inertial-gyroscope becomes

$$\omega = 2\nabla \times \mathbf{A_g}. \tag{5.42}$$

According to (5.22), we need an expression for $(\omega \times \mathbf{v})^{\hat{i}}$. To post-Newtonian accuracy the orthonormal components are equal to the coordinate basis components:$(\omega \times \mathbf{v})^{\hat{i}} = (\omega \times \mathbf{v})^i$. Then we insert (5.40) and (5.42) into (5.22), neglecting terms in the usual way, and get our sought after expression, valid to post-Newtonian accuracy, for the measured acceleration:

$$\overline{a}^i = -\partial^i\phi_g - \frac{\phi_g}{c^2}\partial^i\phi_g + 4\mathbf{v} \times (\nabla \times \mathbf{A_g})^i + 2(\partial_j\phi_g\frac{v^j}{c})\frac{v^i}{c} - 4\frac{\partial}{\partial t}A_g^i, \tag{5.43}$$

or as a vector equation

$$\begin{aligned}\overline{\mathbf{a}} = & -\nabla\phi_g - \frac{\phi_g}{c^2}\nabla\phi_g \\ & + 4\mathbf{v} \times (\nabla \times \mathbf{A_g}) + 2\mathbf{v}(\mathbf{v} \cdot \nabla\frac{\phi_g}{c^2}) \\ & - 4\frac{\partial}{\partial t}\mathbf{A_g}.\end{aligned} \tag{5.44}$$

[13] There is *no* summation over repeated indices in this expression!

Thus we have reformulated equation (5.21) in terms of our post-Newtonian potentials ϕ_g and $\mathbf{A_g}$. Note that the term $4\mathbf{v} \times (\nabla \times \mathbf{A_g})$ in (5.44) corresponds to the term $-2\omega \times \mathbf{v}$ in (5.18). This shows that locally, the gravito-magnetic effect is equivalent to a rotating reference frame, just like the gravito-electric effect locally is equivalent to an accelerated reference frame (in flat spacetime). It should be mentioned that, to post-Newtonian accuracy, the coordinate velocity $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ in (5.44) can be replaced by the measured velocity $\overline{\mathbf{v}} = \frac{d\overline{\mathbf{x}}}{d\overline{t}}$. The measured acceleration (5.44) should be compared to the coordinate acceleration (4.75):

$$\begin{aligned}\frac{d\mathbf{v}}{dt} =& - \nabla\phi_g - 4\frac{\phi_g}{c^2}\nabla\phi_g \\ &+ 4\mathbf{v} \times (\nabla \times \mathbf{A_g}) + 4\mathbf{v}(\mathbf{v} \cdot \nabla\frac{\phi_g}{c^2}) - v^2\nabla\frac{\phi_g}{c^2} \\ &+ 3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2} - 4\frac{\partial \mathbf{A_g}}{\partial t}.\end{aligned} \tag{5.45}$$

Note that there are corrections to several of the coefficients in the measured acceleration relative to the coordinate acceleration. In the following section, we shall look at a special case and derive the measured acceleration from first principle. This will provide insight into the differences between the measured acceleration (5.44) and the coordinate acceleration (5.45).

5.2.3 Geometric derivation of the proper acceleration

In the previous section we did not derive the proper acceleration, but only reformulated a result in [1], in terms of our post-Newtonian language. In this section we shall look at a special case and derive the measured acceleration from the expression for the coordinate acceleration, and the knowledge of how the coordinates $\overline{x}^\mu$ relates to x^μ. This will provide geometric insight into the differences between the measured acceleration and the coordinate acceleration. The formal derivation of the measured acceleration in [1] is mathematically rigorous, but does not provide this kind of insight.

The difference between the proper acceleration and the coordinate acceleration is of course due to the different nature of the coordinate systems. I will now try to spell out more concretely which aspects of the proper reference frame these differences are due to. I claim that the reasons for $\overline{\mathbf{a}}$ not being equal to $\mathbf{a}$ can be divided into the following parts:

1. The standard clock defining the proper time coordinate $\overline{t}$ is attached to the origin of the proper reference frame, while the coordinate time t ticks at the same rate as a standard clock at infinity (far away from the gravitational system). We shall refer to this as the *clock effect*.

2. The grid lines of the proper coordinate system are stretched and squeezed relative to the grid lines of the original coordinate system. We shall refer to this as the *stretching/squeezing effect*.

3. The grid lines of the proper coordinate system are bent relative to the grid lines of the original coordinate system. We shall refer to this as the *bending effect*.

I will soon define mathematically what I mean by stretching/squeezing and bending of grid lines, but let us first try to visualize the idea. In figure 5.3 I have drawn the grid lines $\overline{x}^1 = \overline{x}$ and $\overline{x}^2 = \overline{y}$ into the x-y plane of the original coordinate system. Notice that the grid lines of the proper coordinates are represented in the original coordinate system as bent lines[14]. This is so because the tangent vectors to

[14] In the relativistic sense, where a straight line is a line whose tangent vectors are related by parallel transport, it is the grid lines of the original coordinate system which are bent, and the grid lines of the proper coordinates which are straight.

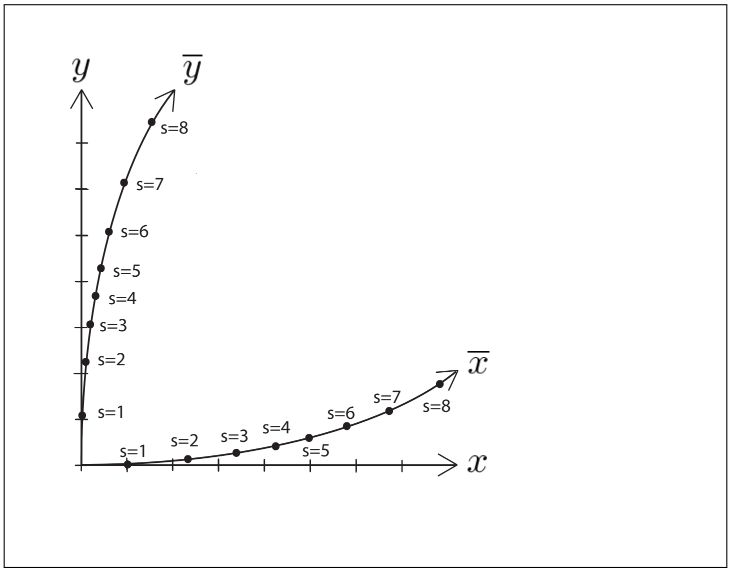

Figure 5.3: The grid lines of the proper coordinates $\overline{x}$ and $\overline{y}$ represented in the $x - y$ plane of the original coordinate system. The bending of the grid lines is what I call the bending effect, while the different spacing between the points $s = 1$,$s = 2$ and so on along the grid lines, is the stretching/squeezing effect.

the grid lines of the proper coordinates are related by parallel transport. I should stress that the figure only shows the *projection* of the proper grid lines $\overline{x}$ and $\overline{y}$ onto the $x - y$ plane. Allthough the tangent vector to the grid line has a vanishing time component at the origin[15], it will in general, as we shall see, have a non-vanishing time component other places. Thus it is not only the projection of the geodesic onto space-space planes which appear to be "bent", but also the projection onto space-time planes (for example the projection of the grid line $\overline{y}$ onto the (t, y) plane will be represented as a bent line). In the following, I shall refer to the bending of geodesics onto space-space planes simply as *spatial bending*. Thus, if I say that a given geodesic has no spatial bending, I mean that the projection of the geodesic onto *any* space-space plane is represented as a straight line (but the projection onto a time-space plane may not be straight). Notice also from the figure that the grid lines of the proper coordinates have a different scaling than the original coordinates. This is so because the proper coordinates represents proper distances, and this is what I refer to as the stretching/squeezing effect.

Point 3 in the list above is what make a derivation of $\overline{\mathbf{a}}$ from a geometric point of view tricky. This has so far prevented me from doing this derivation in full generality. We will therefore consider a special case where the particle moves in a direction where the grid lines are straight, ie. no spatial bending relative to the grid lines of the original coordinate system. Apart from that we shall be quite general. We will consider an example with a non-stationary metric, and we shall take into account for the stretching and squeezing of the grid lines.

We shall now make the above talk more mathematically precise. We start by studying the geometry of the geodesics radiating out from the observer, in a more mathematical, and hopefully less confusing[16], manner. We let $n^\mu(s) = \frac{dx^\mu}{ds}$ be the field of tangent vectors to a geodesic starting out in

[15] It is relative to the observer's tetrad that the geodesics starts out in a pure spatial direction. As we shall soon see, there will be a non-zero time component relative to the *coordinate basis*, also at the origin.

[16] The purpose of the above discussion was of course not to confuse you, but giving you a qualitative sense of the ideas before we start with the maths. If things above are unclear it should be read again after going through the maths.

the pure spatial direction $n^{\hat{\mu}}(0) = (0, \mathbf{n})$, where $\mathbf{n}$ is the pure space vector introduced in section 5.2.1. Since the the proper length s along the grid line is used as parameter, the tangent vector will have unit length all along the geodesic, which is easily demonstrated by considering the inner product:

$$g_{\mu\nu} n^{\mu} n^{\nu} = g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} = \frac{ds^2}{ds^2} = 1. \tag{5.46}$$

The first step in my derivation is to find the direction where a geodesic will have zero spatial bending (my plan is to align the coordinate system with this special direction). It is possible to realize from symmetry arguments that this direction must be parallel with $\nabla\phi_g$. However, as we shall now see, it is not hard, and quite instructive, to show how this follows from calculations. The total change of the tangent vector n^{μ} along the geodesic is given by the geodesic equation

$$\frac{dn^{\mu}}{ds} = -\Gamma^{\mu}_{\mu\nu} n^{\mu} n^{\nu}. \tag{5.47}$$

For the moment, we are only interested in the spatial part of the change of the tangent vector n^{μ}, ie. the space components of the above equation:

$$\begin{aligned} \frac{dn^{i}}{ds} &= -\Gamma^{i}_{\mu\nu} n^{\mu} n^{\nu} \\ &= -\Gamma^{i}_{kl} n^{k} n^{l}. \end{aligned} \tag{5.48}$$

In the second line we have used the approximation $n^0 = 0$, which is a good approximation in a region near the origin. It is convenient to work with 3-vector notation, and we introduce the ordinary three vector $\mathbf{N}$ defined by $n^{\mu} = (n^0, \mathbf{N})$[17]. We formulate the connection coefficients to lowest order in terms of our potentials ϕ_g and $\mathbf{A_g}$ (defined in (4.71) and (4.72)), insert it into (5.48) and get

$$\frac{d\mathbf{N}}{ds} = -\nabla\frac{\phi_g}{c^2} + 2\mathbf{N}\left(\mathbf{N}\cdot\nabla\frac{\phi_g}{c^2}\right), \tag{5.49}$$

see appendix (A.11) for details. This equation describes the *space* part of the change of the *total* tangent vector n^{μ} along the geodesic. Our interest though, is to find the direction where there is no (spatial) change/bending in the geodesic. We therefore need to find the part of $\frac{d\mathbf{N}}{ds}$ orthogonal to $\mathbf{N}$, which we shall denote $\left(\frac{d\mathbf{N}}{ds}\right)_{\perp}$. The part $\left(\frac{d\mathbf{N}}{ds}\right)_{\perp}$ represent what I called (spatial) "bending" above, while $\left(\frac{d\mathbf{N}}{ds}\right)_{\parallel} = \frac{d\mathbf{N}}{ds} - \left(\frac{d\mathbf{N}}{ds}\right)_{\perp}$ is the (spatial) stretching/squeezing component. Using that $\mathbf{N}$ is a unit vector (near the origin) we get:

$$\begin{aligned} \left(\frac{d\mathbf{N}}{ds}\right)_{\perp} &= \frac{d\mathbf{N}}{ds} - \mathbf{N}\left(\mathbf{N}\cdot\frac{d\mathbf{N}}{ds}\right) \\ &= -\nabla\frac{\phi_g}{c^2} + \mathbf{N}\left(\mathbf{N}\cdot\nabla\frac{\phi_g}{c^2}\right). \end{aligned} \tag{5.50}$$

Notice that when $\mathbf{N}$ is parallel to $\nabla\phi_g$ we have $\left(\frac{d\mathbf{N}}{ds}\right)_{\perp} = 0$. Thus we have formally showed that in the direction parallel to $\nabla\phi_g$ there is no spatial bending of the geodesic relative to the original coordinate system.

We shall now consider an example where the test particle's velocity *and* acceleration are limited to a single spatial direction, ie. one dimensional motion. We assume that the original coordinate

[17] Notice that $\mathbf{N}$ relates to the vector $\mathbf{n}$, introduced previously, in the following way: $\mathbf{N}(0) = \mathbf{n}$.

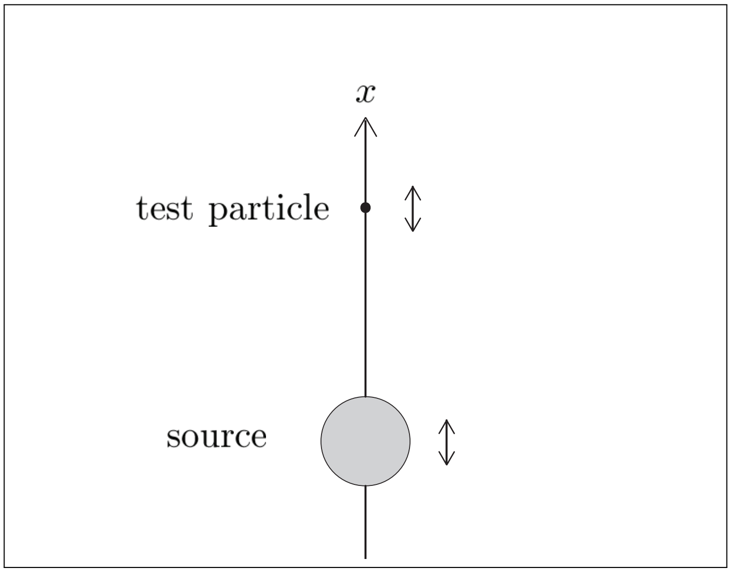

Figure 5.4: The source and the test particle are both restricted to motion along the x axis. The observer is at *rest* somewhere along the coordinate axis (at the moment of measurement the test particle and the observer are located at the same point of course). The source of the field is a non-rotating spherical symmetric mass distribution which moves with the center of mass always on the x axis. Along the x axis the gravito-magnetic vector potential $\mathbf{A}_g$ is aligned with the axis. The gradient of the potential ϕ_g is also parallel to the x-axis all along the axis.

system $x^i = (x, y, z)$ is aligned such that the motion is directed along the grid line $x^1 = x$. Since the acceleration is limited to the same line, we know that $\nabla\phi_g$ must point in the x-direction as well. Thus there are no spatial bending of the geodesic starting out in the direction $\overline{x}$, ie $\left(\frac{d\mathbf{N}}{ds}\right)_\perp = 0$. This assumptions ensures that also in the proper coordinates the motion is one-dimensional, ie. the velocity $\overline{\mathbf{v}}$ and the acceleration $\overline{\mathbf{a}}$ points along the grid line $\overline{x}$. We shall *not*, however, restrict ourselves to a stationary case, but allow for a time varying metric, ie $\phi_g = \phi_g(t, x)$ and $\mathbf{A_g} = \mathbf{A_g}(t, x)$. It may be instructive to visualize what kind of source which can produce such a field. In figure 5.4 we consider such an example, where the source of the field is a non-rotating spherical symmetric mass distribution which is allowed to move along the x axis (the center of mass always on the x axis.). The motion of the source produces a gravito-magnetic potential $\mathbf{A}_g$ as well. All along the x-axis the vector $\mathbf{A}_g$ is parallel to the velocity of the source, while $\nabla \times \mathbf{A_g} = 0$. It is therefore easy to realize from the equation of motion (5.45) that the acceleration induced by $\mathbf{A_g}$ is also directed along the x axis[18].

Then, it is time to start doing calculations. My idea is to write the (proper) coordinates $(\overline{t}, \overline{x})$ as a function of the original coordinates (t, x). By carrying out the differentiations $\overline{v} = \frac{d\overline{x}}{d\overline{t}}$ and $\overline{a} = \frac{d\overline{v}}{d\overline{t}}$, we shall reproduce the expression for the proper acceleration, see (5.44), in the special case when $\mathbf{v}$ is parallel to $\nabla\phi_g$. The coordinates $x^\mu = (t, x)$ represent the position and time of the test particle at a point P along it's world line, ie. an event P. To figure out the corresponding proper coordinates, we must consider the geodesic which is intersecting P, see figure 5.5 a). The event where the geodesic is sent out from the origin of the proper reference frame we denote P_0. We assign P_0 the (original) coordinates $x_0^\mu = (t_0, x_0)$. We define the curve $\mathcal{C}$ as the portion of the geodesic from P_0 to P (a

[18]The term $\frac{\partial \mathbf{A_g}}{\partial t}$ in (5.45) will be aligned with the x-axis since $\mathbf{A}_g$ *always* points along this axis. The term $4\mathbf{v} \times (\nabla \times \mathbf{A_g}) = 0$ since a moving particle produce a gravito-magnetic field of magnitude $|\nabla \times \mathbf{A_g}| \sim \sin\theta$, where θ is the angle between the velocity of the source and the position of the field ($\phi = 0$ in our example).

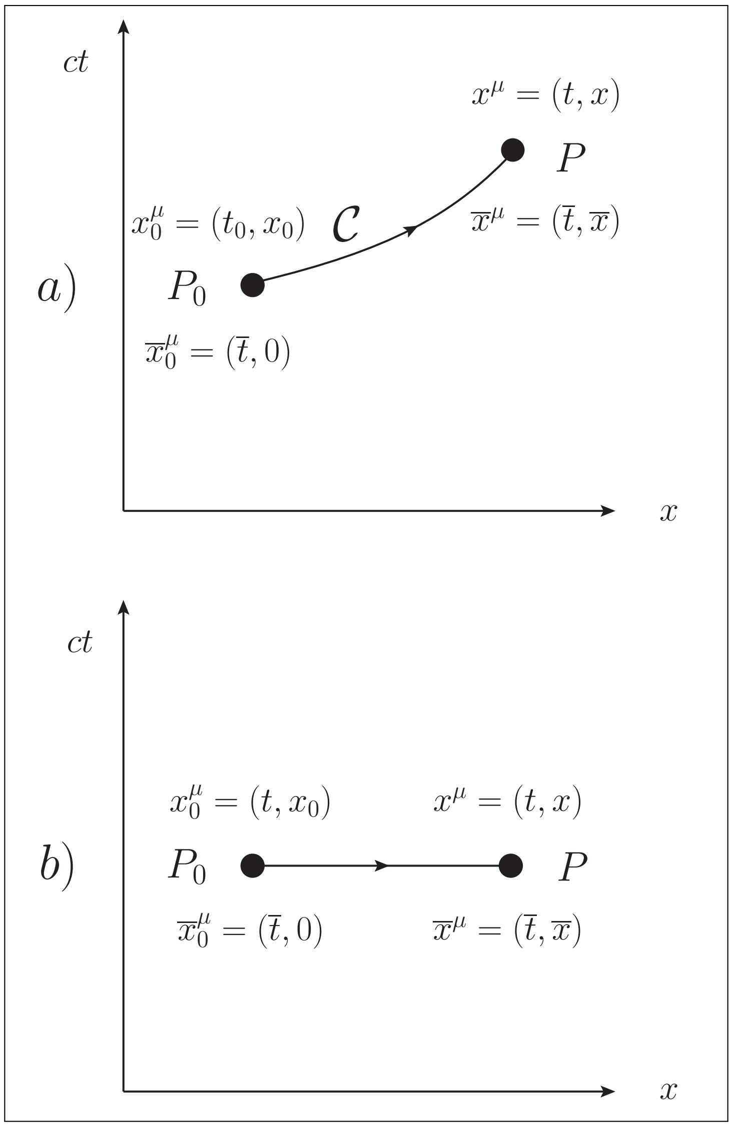

Figure 5.5: Figure a) shows an event P which is assigned the original coordinates (t, x) and proper coordinates $(\bar{t}, \overline{x})$. P is intersected by a geodesic which also intersects the world line of the observer at the event P_0. The curve $\mathcal{C}$ is the portion of the geodesic between P_0 and P (a closed interval). The tangent vector to the geodesic at P_0 coincide with the observer's orthonormal vector $\mathbf{e}_{\hat{x}}$ (not being displayed in the figure). Figure b) shows how things are simplified if we assume that the component of the tangent vectors to $\mathcal{C}$ in the $\mathbf{e}_0$ direction is neglectible. P_0 and P then has the same time coordinate also in the original coordinate system, ie. $t_0 = t$.

closed interval). The event P has proper coordinates $\overline{x}^\mu = (c\bar{t}, \overline{x})$. By definition (see section 5.2.1) the proper coordinates of P_0 becomes $\overline{x}_0^\mu = (c\bar{t}, 0)$.

Having clarified these conventions we set out to calculate the coordinates $\overline{x}^\mu$ as a function of the original coordinates x^μ. For simplicity we assume that the time coordinates $\bar{t}$ and t are synchronized at time zero, ie. the event $\overline{x}^\mu = (0, 0)$ has time coordinate $t = 0$. Since we consider an observer at rest in the original coordinate system (dx=dy=dz=0), it follows from the line element that the proper time coordinate is given by

$$\bar{t} = \int_0^{t_0} dt' \sqrt{|g_{00}(t', x_0)|} = \int_0^{t_0} dt'(1 + \frac{\phi_g(t', x_0)}{c^2}). \tag{5.51}$$

Note that the potential ϕ_g in the integrand has no position dependence, ie. $\phi_g = \phi_g(t, x_0)$. This is due to the facts that the observer's standard clock (which is defining the $\bar{t}$ coordinate) is attached to the origin of the proper reference frame and that the origin of the proper reference frame is at rest in the original coordinate system. The proper space component is the proper distance of the space-like curve $\mathcal{C}$ which is given by the curve integral

$$\overline{x} = \int_{\mathcal{C}} d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\mu}{d\lambda}}, \tag{5.52}$$

see (1.47). Equations (5.51) and (5.52) defines the proper coordinates $\bar{t}$ and $\overline{x}$. It is not easy however to find the functional dependence on t and x from these definitions. The problem is that in general $t_0 \neq t$. If we could use $t_0 \approx t$ however, things would simplify considerably. In the integral defining the $\bar{t}$ coordinate we could replace the t_0 in the upper integral limit with t, and the curve integral defining the $\overline{x}$ coordinate would become a simple one dimensional integral along the line t=constant, see figure 5.5 b). The magnitude of $t - t_0$ can be determined from the zero component of (5.47). We should therefore consider this equation to check whether it is possible to use the approximation $t_0 \approx t$ in the post-Newtonian approximation.

Near the origin of the proper reference frame the zero component of (5.47) yields

$$c\frac{d^2t}{ds^2} = -\Gamma^0_{kl} n^k n^l. \tag{5.53}$$

The Christoffel connection in terms of the metric is given in appendix A.6. Formulating it in terms of our potentials ϕ_g and $\mathbf{A}_g$ and rewriting (5.53) as a vector equation (in the notation introduced prior to (5.49)) we get

$$\frac{d^2t}{ds^2} = \frac{4}{c^2}\mathbf{N} \cdot (\mathbf{N} \cdot \nabla)\mathbf{A_g} + \frac{\mathbf{N}^2}{c^2}\partial_t \frac{\phi_g}{c^2}. \tag{5.54}$$

By comparing $\frac{d^2t}{ds^2}$ with the post-Newtonian effects summarized in table 4.2 we can make the following order of magnitude estimate:

$$\frac{d^2t}{ds^2} \sim \frac{1}{c^3} O(1)\nabla U, \tag{5.55}$$

where U is the Newtonian potential as usual. This shows that the acceleration of the curve $\mathcal{C}$ in the time direction is very small. We shall now show that it gives rise to a correction in the measured acceleration $\overline{\mathbf{a}}$ which is of order $O(4)\nabla U$, ie. four order of magnitudes smaller than the Newtonian acceleration, and therefore neglectible in the post-Newtonian approximation (where the precision is of order $O(2)\nabla U$). To make a lowest order estimate of the contribution we shall treat $\frac{d^2t}{ds^2}$ as a constant k along the curve $\mathcal{C}$, ie. $\frac{d^2t}{ds^2} = k \sim \frac{1}{c^3}O(1)\nabla U$. Integrating with respect to s we get

$$\frac{dt}{ds} = ks + k_2. \tag{5.56}$$

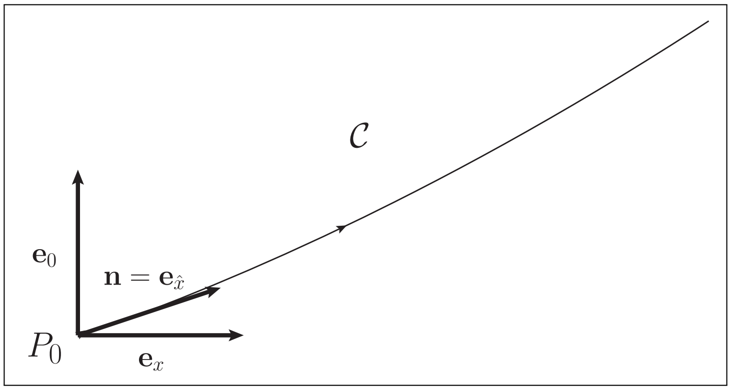

Figure 5.6: This figure is a "zoom-in" of figure 5.5 a) around P_0. Unlike figure 5.5a however, it also displays the vector $\mathbf{n} = \mathbf{e}_{\hat{x}}$ (which defines the initial direction of the curve $\mathcal{C}$) and the coordinate basis vectors $\mathbf{e}_x$ and $\mathbf{e}_0$ (at P_0).

The integration constant is found by evaluating $\frac{dt}{ds}$ at $s = 0$:

$$k_2 = \frac{dt}{ds}(s = 0). \tag{5.57}$$

Hence we must evaluate the difference coefficient of the curve $\mathcal{C}$ at the origin P_0 to determine k_2. Figure 5.6 shows the area around the origin and is basically a "zoom in" of figure 5.5a around P_0. Unlike figure 5.5a however, it also displays the vector $\mathbf{n} = \mathbf{e}_{\hat{x}}$ which defines the initial direction of the curve $\mathcal{C}$, and the coordinate basis vectors $\mathbf{e}_x$ and $\mathbf{e}_0$ at P_0[19]. We have already worked out an expression for how $\mathbf{e}_{\hat{x}}$ relates to the coordinate basis vectors $\mathbf{e}_0$ and $\mathbf{e}_x$ (see (5.38)):

$$\mathbf{e}_{\hat{x}} = (1 + \frac{\phi_g}{c^2})\mathbf{e}_i + \frac{4}{c}A_g\mathbf{e}_0, \tag{5.58}$$

where we have used that the x component of $\mathbf{A_g}$, ie. A_{gx}, is equal to the magnitude of $\mathbf{A_g}$, ie. $A_{gx} = |\mathbf{A_g}| = A_g$. From this equation together with figure 5.6 it is readily seen that to lowest order[20] the difference coefficient of $\mathcal{C}$ at P_0 becomes

$$\frac{d(ct)}{dx} = \frac{d(ct)}{ds} = \frac{4}{c}A_g(x_0), \tag{5.59}$$

and according to (5.57) we get

$$k_2 = \frac{4}{c^2}A_g(x_0) \sim \frac{1}{c}O(3). \tag{5.60}$$

Hence we have showed that k_2 is of the same order of magnitude as k. Integrating (5.56) yields

$$t = t_0 + \frac{1}{2}ks^2 + k_2 s. \tag{5.61}$$

Inserting this into (5.51) we see that

$$\bar{t} = \int_0^{t-\frac{1}{2}ks^2-k_2 s} dt'(1 + \frac{\phi_g(t', x_0)}{c^2}), \tag{5.62}$$

[19] It should be mentioned that displaying coordinate basis vectors in a coordinate system as we have done in figure 5.6 necessarily becomes misleading, as the coordinate vectors $\mathbf{e}_0$ and $\mathbf{e}_i$ appear to be orthogonal, but it is $\mathbf{e}_{\hat{x}}$ which really is orthogonal to $\mathbf{e}_0$, *not* $\mathbf{e}_x$.

[20] The first step is correct to lowest order.

which to lowest order yields

$$\bar{t} = \int_0^t dt'(1 + \frac{\phi_g(t', x_0)}{c^2}) - \frac{1}{2}ks^2 - k_2 s. \tag{5.63}$$

To show that the contribution from the extra terms on the right-hand side of (5.63) can be neglected, we need (in principal) to keep them in all our calculations and check what kind of contribution it gives rise to in the final expression for $\bar{\mathbf{a}}$. To avoid too long calculations however, I will rather use

$$\bar{t} = \int_0^t dt'(1 + \frac{\phi_g(t', x_0)}{c^2}), \tag{5.64}$$

in the following, and the contributions from the "extra" term $-(\frac{1}{2}ks^2 + k_2 s)$ will be commented in appendix A.13. The result turns out to be that the extra term gives rise to a contribution in the measured acceleration of smallness $O(4)\nabla U$. This is two order of magnitudes beyond the post-Newtonian accuracy, which means that 5.64 is a correct estimate for $\bar{t}$ in the post-Newtonian approximation.

Next we evaluate the curve integral (5.52) which is defining the $\overline{x}$ coordinate in the same manner. Writing out the components, (5.52) can be written:

$$\overline{x} = \int_{\mathcal{C}} d\lambda \sqrt{g_{00}\frac{d(ct)}{d\lambda}\frac{d(ct)}{d\lambda} + 2g_{0x}\frac{d(ct)}{d\lambda}\frac{dx}{d\lambda} + g_{xx}\frac{dx}{d\lambda}\frac{dx}{d\lambda}}. \tag{5.65}$$

Using the x coordinate as parameter[21], ie. putting $\lambda = x$, (5.65) becomes

$$\overline{x} = \int_{\mathcal{C}} dx \sqrt{g_{00}\frac{d(ct)}{dx}\frac{d(ct)}{dx} + 2g_{0x}\frac{d(ct)}{dx} + g_{xx}}. \tag{5.66}$$

To evaluate the curve integral we shall make use of the same simplification as above, namely that the acceleration $\frac{d^2t}{ds^2} = k$ is a constant. This will provide good accuracy around the origin. From (5.56) we get to lowest order

$$\frac{d(ct)}{dx} = c(kx + k_2), \tag{5.67}$$

where we have used the lowest order approximations $\frac{d(ct)}{dx} \approx \frac{d(ct)}{ds}$ and $s \approx x$. Inserting (5.67) into (5.66) and replacing the metric with our potentials in the usual way we get

$$\overline{x} = \int_{\mathcal{C}} dx \sqrt{1 - 2\frac{\phi_g}{c^2} - c^2(kx + k_2)^2 + 8A_g(kx + k_2)}, \tag{5.68}$$

or

$$\overline{x} = \int_{\mathcal{C}} dx \left(1 - \frac{\phi_g}{c^2} - \frac{1}{8}\left(\frac{\phi_g}{c^2}\right)^2 - \frac{1}{16}\left(\frac{\phi_g}{c^2}\right)^3 - c^2(kx + k_2)^2 + 8A_g(kx + k_2)\right). \tag{5.69}$$

The second term in the integrand is of smallness $O(2)$, the third is of smallness $O(4)$ and the remaining ones are roughly speaking of smallness $O(6)$ (around the origin). Notice from (5.69) that to lowest

[21] A parameter of a curve must satisfy two conditions; it must provide an unique mapping from the curve to $\mathbb{R}^1$ *and* it must be invariant. The coordinate x provides an *unique* mapping since the curve has monotonically increasing x coordinate. When we use the x coordinate as an invariant parameter we must remember that under a change of coordinate system the curve parameter does not change (ie. the parameter is still the x coordinate of the *old* coordinate system). We shall however not consider change of coordinate systems (we only operate with *one* set of coordinates x^μ, in addition to the proper coordinates $\overline{x}^\mu$ of course), so this is not going to be an issue.

order we have $\overline{x} = x - x_0$. It turns out that the post-Newtonian approximation requires knowledge of $\overline{x}$ to accuracy $O(2)(x - x_0)$. Hence we can drop all but the two first terms in (5.69) to find the measured acceleration to post-Newtonian accuracy:

$$\overline{x} = \int_{x_0}^{x} dx' \left(1 - \frac{\phi_g(t(x'), x')}{c^2}\right). \tag{5.70}$$

Notice that the time coordinate t in the argument of ϕ_g in the expression above depends on the parameter x', ie. $t = t(x')$. At P_0 and P we have $t(x'_0) = t_0$ and $t(x') = t$ respectively. From (5.61) we see that near the origin[22] $t - t_0$ is of smallness $O(4)$, and hence (5.70) may be replaced with

$$\overline{x} = \int_{x_0}^{x} dx' \left(1 - \frac{\phi_g(t, x')}{c^2}\right), \tag{5.71}$$

to great accuracy. Hence we have replaced the curve integral (5.69) with a simple integral along the x axis. The error generated in the measured acceleration by using (5.71) instead of (5.69) is of smallness $O(4)\nabla U$. We shall now start doing calculations using (5.64) and (5.71) as our definitions of $\bar{t}$ and $\overline{x}$ respectively. These definitions corresponds to neglecting the time-space "bending" of the geodesic from P_0 to P, see figure 5.5 b).

We are finally ready to calculate $\overline{v}^i = \frac{d\overline{x}^i}{d\bar{t}}$ and $\overline{a}^i = \frac{d\overline{v}^i}{d\bar{t}}$ from equations (5.64) and (5.71). It will be useful to note from (5.64) that

$$\frac{dt}{d\bar{t}} = 1 - \frac{\phi_g(t, x_0)}{c^2}. \tag{5.72}$$

We split the total time derivative into an implicit and explicit part:

$$\frac{d}{dt} = \frac{dx^j}{dt}\partial_j + \partial_t. \tag{5.73}$$

Using (5.71), (5.72) and (5.73) we calculate the measured velocity:

$$\begin{aligned} \frac{d\overline{x}}{d\bar{t}} &= \frac{dt}{d\bar{t}}\frac{d}{dt}\overline{x} \\ &= \left(1 - \frac{\phi_g(t, x_0)}{c^2}\right)\left(\frac{dx}{dt}(1 - \frac{\phi_g(t, x)}{c^2}) - \int_{x_0}^{x} dx' \frac{\partial}{\partial t}\frac{\phi_g(t, x')}{c^2}\right), \end{aligned} \tag{5.74}$$

or to post-Newtonian accuracy:

$$\frac{d\overline{x}}{d\bar{t}} = \frac{dx}{dt}(1 - \frac{\phi_g(t, x_0)}{c^2} - \frac{\phi_g(t, x)}{c^2}) - \int_{x_0}^{x} dx' \frac{\partial}{\partial t}\frac{\phi_g(t, x')}{c^2}. \tag{5.75}$$

For later use note that at the origin the measured velocity becomes particularly simple:

$$\frac{d\overline{x}}{d\bar{t}} = \frac{dx}{dt}(1 - 2\frac{\phi_g(t, x_0)}{c^2}). \tag{5.76}$$

Even though we are interested in the acceleration at the origin (where it has significance as the measured acceleration), it is of vital importance that we calculate the derivative of $\frac{d\overline{x}}{d\bar{t}}$ on the form given

[22] Near the origin we can use $s = O(0)$.

by (5.75). In the end however, when all calculations are performed, we shall evaluate the answer at the origin. Using (5.72) and (5.75) we see that:

$$\begin{aligned}\frac{d^2\overline{x}}{d\overline{t}^2} =& \frac{dt}{d\overline{t}}\frac{d}{dt}\frac{d\overline{x}}{d\overline{t}} \\ =& \left(1-\frac{\phi_g(t,x_0)}{c^2}\right)\left[\frac{d^2x}{dt^2}\left(1-\frac{\phi_g(t,x_0)}{c^2}-\frac{\phi_g(t,x)}{c^2}\right)\right. \\ & -\frac{dx}{dt}\frac{d}{dt}\left(\frac{\phi_g(t,x_0)}{c^2}+\frac{\phi_g(t,x)}{c^2}\right) \\ & \left. -\frac{d}{dt}\int_{x_0}^{x}dx'\frac{\partial}{\partial t}\frac{\phi_g(t,x')}{c^2}\right].\end{aligned} \tag{5.77}$$

This expression contains several total time derivatives which must be evaluated using (5.73):

$$\frac{d}{dt}\phi_g(t,x_0) = \frac{\partial}{\partial t}\phi_g(t,x_0), \tag{5.78}$$

$$\frac{d}{dt}\phi_g(t,x) = \frac{dx^j}{dt}\partial_j\phi_g(t,x) + \frac{\partial}{\partial t}\phi_g(t,x), \tag{5.79}$$

and

$$\frac{d}{dt}\int_{x_0}^{x}dx'\frac{\partial}{\partial t}\frac{\phi_g(t,x')}{c^2} = \frac{dx}{dt}\frac{\partial}{\partial t}\frac{\phi_g(t,x)}{c^2} + \int_{x_0}^{x}dx'\frac{\partial^2}{\partial t^2}\frac{\phi_g(t,x')}{c^2}. \tag{5.80}$$

Inserting (5.78),(5.79) and (5.80) into (5.77) we get, to post-Newtonian accuracy:

$$\begin{aligned}\frac{d^2\overline{x}}{d\overline{t}^2} =& \frac{d^2x}{dt^2}\left(1-2\frac{\phi_g(t,x_0)}{c^2}-\phi_g(t,x)\right) \\ & -\frac{dx}{dt}\left(\frac{\partial}{\partial t}\frac{\phi_g(t,x_0)}{c^2}+2\frac{\partial}{\partial t}\frac{\phi_g(t,x)}{c^2}+\frac{dx^j}{dt}\partial_j\frac{\phi_g(t,x)}{c^2}\right) \\ & -\left(1-\frac{\phi_g(t,x_0)}{c^2}\right)\int_{x_0}^{x}dx'\frac{\partial^2}{\partial t^2}\frac{\phi_g(t,x')}{c^2}.\end{aligned} \tag{5.81}$$

Evaluated at the origin ($x = x_0$) equation (5.81) is simplified to

$$\frac{d^2\overline{x}}{d\overline{t}^2} = \frac{d^2x}{dt^2}\left(1-3\frac{\phi_g(t,x_0)}{c^2}\right) - 3\frac{dx}{dt}\frac{\partial}{\partial t}\frac{\phi_g(t,x_0)}{c^2} - \frac{dx}{dt}\frac{dx^j}{dt}\partial_j\frac{\phi_g(t,x)}{c^2}, \tag{5.82}$$

or as a vector equation:

$$\overline{\mathbf{a}} = \mathbf{a}\left(1-3\frac{\phi_g(t,x_0)}{c^2}\right) - 3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g(t,x_0)}{c^2} - \mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}). \tag{5.83}$$

Inserting the coordinate acceleration $\mathbf{a}$ (see (4.75)) into (5.83), and as always evaluating to post-Newtonian accuracy, yields:

$$\begin{aligned}\overline{\mathbf{a}} =& -\nabla\phi_g - \frac{\phi_g}{c^2}\nabla\phi_g \\ & + 3\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) - v^2\nabla\frac{\phi_g}{c^2} \\ & - 4\frac{\partial\mathbf{A_g}}{\partial t} + 4\mathbf{v}\times(\nabla\times\mathbf{A_g}).\end{aligned} \tag{5.84}$$

In our one-dimensional case we have $\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) = v^2\frac{\phi_g}{c^2}$ and $\nabla\times\mathbf{A_g} = 0$, so (5.84) can be written

$$\begin{aligned}\overline{\mathbf{a}} =& -\nabla\phi_g - \frac{\phi_g}{c^2}\nabla\phi_g \\ &+ 2\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) - 4\frac{\partial\mathbf{A_g}}{\partial t}.\end{aligned} \tag{5.85}$$

This is in perfect agreement with (5.44)! Thus we have found a simple geometric way to derive the measured acceleration to post-Newtonian accuracy. My derivation is (fairly) rigorous, but, unfortunately, the derivation is restricted to a special case. However, if the reader allows me to use some loose and intuitive arguments, ie. to speculate, I will show how to obtain the most general expression. Let us consider the general case where the test particle pass through the origin of the proper reference frame with an arbitrary velocity $\mathbf{v}$ (which is not necessarily parallel with $\nabla\phi_g$). My hypothesis is that (5.84), also in that case, correctly accounts for the clock effect and the stretching/squeezing effect, but miss the bending effect. Let us consider a non freely falling particle following a world line with vanishing acceleration in the proper reference frame, ie. zero measured acceleration. For the moment, let us simplify the discussion and assume that the metric is stationary (which means that two geodesics radiating out from origo in the same direction, but at different times, have the same projection onto any space-space plane). If the measured acceleration is zero, it means that the projection of the particles world line onto the three space must coincide with the projection of a given geodesic onto the same space (if the world line of the particle intersects different geodesics, it means that the the vector $\mathbf{n}$ is changing, and hence the measured acceleration is not zero). In the original coordinates however, there will be spatial acceleration as the geodesic draws a curved/bent path in the coordinate system. We assume the particle passes through the origin (along the geodesic) in a direction $\mathbf{U}$ with a speed v, and hence velocity $\mathbf{v} = v\mathbf{U}$. From (5.50) we realize that the "bending-acceleration" of this particle must be

$$\frac{d\mathbf{v}}{dt} = -v^2\nabla\frac{\phi_g}{c^2} + \mathbf{v}\left(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}\right). \tag{5.86}$$

This term gives a contribution to the coordinate acceleration in the original coordinate system, but is not observed in the proper reference frame. Hence, if we want to take into account the bending of the proper grid lines, (5.86) must be *subtracted* from (5.84), which yields:

$$\begin{aligned}\frac{d\overline{\mathbf{v}}}{d\overline{t}} =& -\nabla\phi_g - \frac{\phi_g}{c^2}\nabla\phi_g \\ &+ 4\mathbf{v}\times(\nabla\times\mathbf{A_g}) + 2\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) \\ &- 4\frac{\partial}{\partial t}\mathbf{A_g}.\end{aligned} \tag{5.87}$$

This is in perfect agreement with the measured acceleration found in section 5.2.1, see (5.44). Of course, I don't consider this as a proper derivation for the general case. In the mathematical sense, it should be regarded as pure speculation, but hopefully the reeder agrees that the arguments are quite reasonable. A rigor geometric derivation, in the general case, would be a generalization of the procedure for the one dimensional case, ie. writing the coordinates $\overline{x}^\mu$ as a function of the original coordinates x^μ and then perform the derivations. I have not figured out yet how to do this in practice, but the special case considered here certainly provide a good starting point for further work. I think it would be very interesting to work it out for the general case because it would enable us to discuss the relation between coordinate effects and (locally) measurable effects in a more concrete and visual manner than I have seen anywhere else. In particular we would be able to point out exactly the origin of the difference between the coordinate acceleration and the measured acceleration, as I will demonstrate below.

The difference between the measured acceleration and the coordinate acceleration is

$$\bar{\mathbf{a}} - \mathbf{a} = 3\frac{\phi_g}{c^2}\nabla\phi_g - 2\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) + v^2\nabla\frac{\phi_g}{c^2} - 3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2}. \tag{5.88}$$

If it turns out that my speculation is correct, which is quite reasonable I think, we can specify more precisely what kind of effects these differences are due to. As I claimed in the beginning of this section the difference between $\bar{\mathbf{a}}$ and $\mathbf{a}$ is due to a combination of what I called the *clock effect*, the *stretching/squeezing* effect and the *bending effect*. We therefore write:

$$(\bar{\mathbf{a}} - \mathbf{a}) = (\bar{\mathbf{a}} - \mathbf{a})_{\text{clock}} + (\bar{\mathbf{a}} - \mathbf{a})_{\text{stretch/squeeze}} + (\bar{\mathbf{a}} - \mathbf{a})_{\text{bending}}. \tag{5.89}$$

Studying equations (5.72), (5.74), (5.77)-(5.83) and (5.86) we realize that

$$(\bar{\mathbf{a}} - \mathbf{a})_{\text{clock}} = 2\frac{\phi_g}{c^2}\nabla\phi_g - \mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2}, \tag{5.90}$$

$$(\bar{\mathbf{a}} - \mathbf{a})_{\text{stretch/squeeze}} = \frac{\phi_g}{c^2}\nabla\phi_g - 2\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2} - \mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}), \tag{5.91}$$

and

$$(\bar{\mathbf{a}} - \mathbf{a})_{\text{bending}} = v^2\nabla\frac{\phi_g}{c^2} - \mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}). \tag{5.92}$$

5.2.4 Discussion

It is now time to pause for a moment and reflect on the obtained results. In the previous section we provided a visual kind of insight into the difference between the coordinate acceleration and the locally measured acceleration. This was an unexpected additional result, a bit on the side of the main scope of this thesis perhaps, which followed from my attempt to derive the measured acceleration from a geometric point of view. Here we shall go back to our primary purpose and discuss the measured acceleration in the context of the gravito-electromagnetic analogy.

For convenience, let me start by repeating the equations of motion for the post-Newtonian limit of general relativity (4.75)

$$\begin{aligned}\frac{d\mathbf{v}}{dt} =& -\nabla\phi_g - 4\frac{\partial\mathbf{A_g}}{\partial t} + 4\mathbf{v}\times(\nabla\times\mathbf{A_g}) + 4\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) - v^2\nabla\frac{\phi_g}{c^2}\\ &- 4\frac{\phi_g}{c^2}\nabla\phi_g + 3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2},\end{aligned} \tag{5.93}$$

and the equation of motion in the post-electrostatic limit of electrodynamics (4.69)

$$\frac{d\mathbf{v}}{dt}\frac{m}{q} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times(\nabla\times\mathbf{A}) + \mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi}{c^2}) + \frac{1}{2}v^2\nabla\frac{\phi}{c^2}. \tag{5.94}$$

As discussed in section 4.3 there are two important qualitative kinds of differences between these equations. In particular there are no effects in electromagnetism corresponding to the terms on the second line in (5.93). These are the quadratic term in ϕ_g and the term with explicit time-dependence on ϕ_g (which is $+3\mathbf{v}\frac{\partial}{\partial t}\frac{\phi_g}{c^2}$). The former one is an important characteristic of general relativity and is due to the fact that it is a non-linear theory. It is harder, however, to explain the origin of the latter effect. It is interesting, however, to note that in the measured acceleration there is no term with explicit time-dependence on ϕ_g:

$$\begin{aligned}\overline{\mathbf{a}} = &-\nabla\phi_g - 4\frac{\partial}{\partial t}\mathbf{A_g} + 4\mathbf{v}\times(\nabla\times\mathbf{A_g}) + 2\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi_g}{c^2}) \\ &-\frac{\phi_g}{c^2}\nabla\phi_g.\end{aligned} \tag{5.95}$$

Thus the rate of change of the scalar potential is not locally observable, neither in the gravitational nor in the electromagnetic case. Hence we have discovered that the gravito-electromagnetic analogy becomes even stronger when one evaluates gravitational phenomena in a proper reference frame, which is quite interesting since in that case the theories are treated on a more equal footing conceptually (the equation of motion describes the measured acceleration in both cases). It is really interesting to clarify whether there is some kind of deep reason for this, or whether it should be viewed as a coincidence. This discussion will be saved for the conclusion, chapter 6.

Comparing (5.93) and (5.94) we note that there is a numerical difference of order $O(2)\nabla U$ between the coordinate acceleration and the locally measured acceleration. This shows that curvature has geometric significance also for weak fields when demanding post-Newtonian accuracy. For the first iteration in the approximation-scheme however, which predicts the acceleration to $O(0)\nabla U$ and give back Newtonian theory, this smallness has no significance. Conceptually this means that in Newtonian theory gravity can be interpreted as an ordinary force co-existing with electromagnetism and other forces in a flat spacetime background. This is of course just how Newton's theory of gravity usually is interpreted, but now we have justified the perspective by means of detailed calculations. The same calculations also show that beyond Newtonian theory there is no room for such an interpretation. The difference between the coordinate acceleration and the locally measured acceleration which appear already in the post-Newtonian approximation, is a characteristic of curved spacetime. Thus it is not completely satisfying just to compare the mathematical structure of the theories in a discussion of the gravito-electromagnetic analogy.

This chapter has extended and complemented the discussion of the gravito-electromagnetic analogy. In chapter 4 we only compared the mathematical structure of the theories (in a consistent way), and showed that a perfect formal analogy only exist between the lowest order approximations, ie. between Newtonian gravity and electrostatics. Our results from this chapter shows that the analogy is also perfect *conceptually* at this level since there is no difference between the measured acceleration and the coordinate acceleration (to Newtonian accuracy). Beyond the lowest order approximations however, ie. the post-Newtonian and the post-electrostatic approximations, we have showed that the perfect analogy breaks down both formally (the mathematical structure is not quite on the same form) *and* conceptually (curvature has geometric significance in the post-Newtonian approximation).

5.3 Energy considerations

In chapter 4 we compared the post-Newtonian limit of general relativity to the post-electrostatic limit of electrodynamics. Allthough we found several kind of similarities, there was also important differences, both qualitatively and quantitatively. In this section we shall see that the expression for the measured acceleration can provide insight into some of these differences. We shall study kinematics in the special case of radial motion outside a spherical static mass/charge distribution, since in that case energy considerations are particularly simple.

For radial motion outside a static mass/charge distribution we have[23] $\mathbf{v}(\mathbf{v}\cdot\nabla\frac{\phi}{c^2}) = v^2\nabla\frac{\phi}{c^2}$, $\mathbf{A} = 0$ and $\frac{\partial}{\partial t}\phi = 0$. Thus the equation of motion (4.69) becomes

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\left(-\nabla\phi + \frac{3}{2}\frac{v^2}{c^2}\nabla\phi\right) \tag{5.96}$$

in the electromagnetic case, and

$$\frac{d\mathbf{v}}{dt} = -\nabla(\phi_g + 2\frac{\phi_g^2}{c^2}) + 3\frac{v^2}{c^2}\nabla\phi_g \tag{5.97}$$

in the gravitational case (see (4.75)). Notice that the coefficient in front of the velocity dependent term is 3 in the gravitational case while it is $3/2$ in the electromagnetic case. Our project will be to show that in both cases, this coefficient is determined by energy conservation. This will not only provide insight into why the velocity dependence is different in the gravitational case, but also shed light on the huge conceptual differences between the theories. We therefore write equation (5.96) as

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\left(-\nabla\phi + k_1\frac{v^2}{c^2}\nabla\phi\right) \tag{5.98}$$

and (5.97) as

$$\frac{d\mathbf{v}}{dt} = -\nabla(\phi_g + 2\frac{\phi_g^2}{c^2}) + k_2\frac{v^2}{c^2}\nabla\phi_g. \tag{5.99}$$

Then I claim that k_1 and k_2 can be *derived* by demanding energy conservation. It is not hard to imagine how this can be done in the electromagnetic case. We consider two equally charged test particles with the same mass starting at rest from r_1 and traveling to r_0 along a radial path in the electric field. The first particle is only influenced by the electric field and the motion is determined by (5.98), see figure 5.7. In addition to the potential energy, this particle will have a non-vanishing kinetic energy E_0 at r_0. The second particle is, in addition to the electric field, influenced by another force $F(r)$ which has the same magnitude as the electric force all along the path, but acts in the opposite direction. We assume that $F(r)$ cancels the electric force *almostly* perfect such that the particle moves quasi statically from r_1 to r_0. It may be instructive to imagine a continuum of power plants along the path of the (second) particle, each with a turbine and a battery, see figure 5.8. The force $F(r)$ originates from the turbines of the local power plant at r. As $F(r)$ acts in the opposite direction compared to the motion of the particle, the turbine does a negative work on the particle: $dW = \mathbf{F}(r)\cdot\mathbf{dr} < 0$. The energy $dE = |dw|$ is stored in the local battery. Since the particle moves quasi-statically, it will have no kinetic energy at r_0, but the same potential energy as the first particle. As both particles started from r_1 with the same energy, there must be a total energy E_0 stored in the batteries.

Hence we have two different ways to calculate the energy E_0. The first one depends on the coefficient k_1 since the motion is determined by (5.98), while the second one does not since the motion is quasi static. Thus we have proposed an energy argument which will determine the coefficient k_1.

We will apply exactly the same idea for the gravitational case. We consider two test particles with the same mass starting at rest from r_1 and moving to r_0 along a radial path. The first particle is freely falling, and the motion is determined by (5.99). The second particle moves quasi-statically and gives energy to the local batteries. Allthough the basic idea is the same one as in the electromagnetic case, it will be much more challenging, at least conseptually, to apply the idea. The kinetic energy of the first particle at r_0, we denote $E_{\hat{0}}$, where the 'hat' reminds us that $E_{\hat{0}}$ is the *locally measured*

[23] In the gravitational case $\phi \rightarrow \phi_g$ and $\mathbf{A} \rightarrow \mathbf{A_g}$ of course.

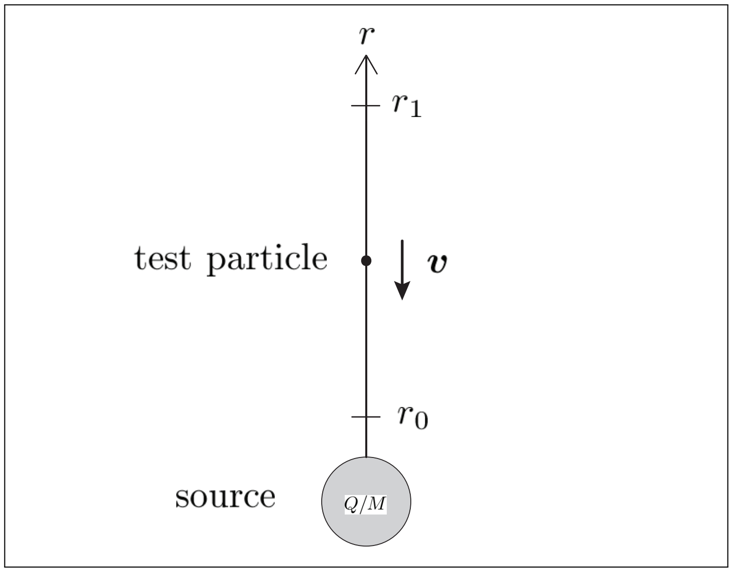

Figure 5.7: A freely falling test particle starts at rest from r_1 and is accelated towards the static source. The source is a spherical symmetric distribution of charge in the electromagnetic case, and a spherical symmetric mass distribution in the gravitational case. The total charge(mass) of the source is Q(M) (and q(m) for the test particle). The energy of the freely falling test particle is measured by an observer at r_0.

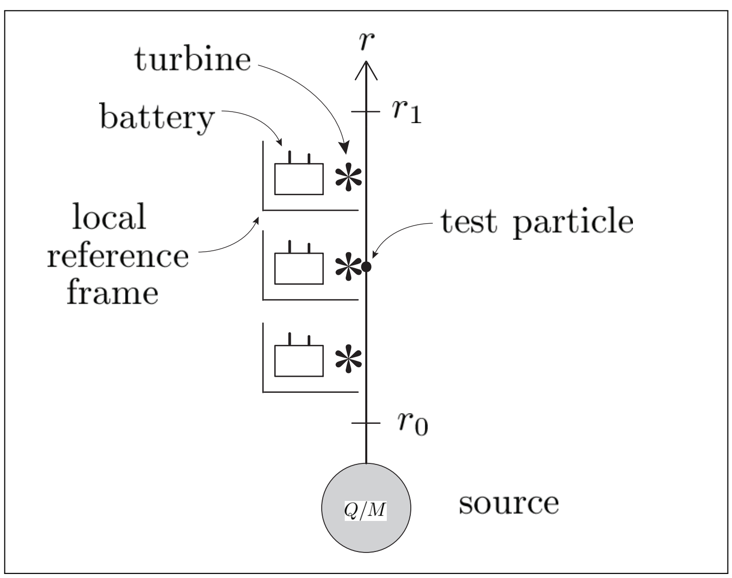

Figure 5.8: The test particle moves quasi statically from r_1 to r_0. There is a (infinite) continuum of local proper reference frames between r_1 and r_0 (three of them are represented explicitly in the figure). It is instructive to imagine a local power plant associated with each local proper reference frame, here represented by a turbine and a battery.

kinetic energy. To calculate the energy stored in a local battery, we must consider the associated local proper reference frame. Hence the term 'local power plant' gets a more literal meaning, and we will need the expression for the measured acceleration found in section 5.2 to determine the local force $\hat{F}$. Furthermore, the total energy stored in all the batteries can not be assumed to equal $E_{\hat{0}}$. The reason for this is that the energy stored in the batteries itself couples to the gravitational field. It is instructive to imagine that the energy in all the local batteries is inverted into light and propagated down to r_0 where it is stored as energy in a single battery. It is the total energy stored in this single battery at r_0 which should equal $E_{\hat{0}}$. Hence we cannot simply integrate over the continuum of local batteries, but must take into account the Doppler shift. This is not necessary in the electromagnetic case, since the energy stored in the batteries does not couple to the electric field (only charge couples to the electric field). We shall see that, by taking all this into account, it will be possible to derive the coefficient k_2. First however, as a warm-up exercise and in order to check that our idea works, we shall do the calculations for the electromagnetic case.

5.3.1 The electromagnetic case

We assume that the electric field ϕ is produced by (a static) point charge with charge Q:

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}. \tag{5.100}$$

We shall start by calculating E_0, the kinetic energy of the "freely" falling particle at r_0. We need to find the speed of the particle at r_0 by solving the differential equation (5.98) with initial condition $v(r_1) = 0$. It will be convenient to introduce the variable

$$R = \frac{-Qq}{4\pi\epsilon_0 mc^2}, \tag{5.101}$$

where q and m are the charge and mass of the test particle. Since the charge q is attracted towards the charge Q, we have $qQ < 0$ and hence $R > 0$. Also note that R has dimension length. A charged particle at distance R must have a speed comparable to the speed of light in order to escape from the field[24]. So, loosely speaking, we may say that R is to electrodynamics what the black-hole horizon R_s is to general relativity. In the following we shall assume that $\frac{R}{r_1} \ll 1$ and $\frac{R}{r_0} \ll 1$.

We reformulate the vector equation (5.98) to a differential equation in the variable $v = \frac{dr}{dt}$:

$$\frac{dv}{dt} + f(r)v^2 + g(r) = 0, \tag{5.102}$$

where $f(r) = -k_1\frac{R}{r^2}$ and $g(r) = c^2\frac{R}{r^2}$. This is a first-order, ordinary, non-linear differential equation. It is non-linear due to the quadratic term in v. Non-linear differential equations are in general notoriously hard, and often impossible, to solve. This one however, can be transformed to a linear differential equation by a change of variable[25]: $w(r) = v^2$. Notice that

$$\frac{d}{dr}w(r) = 2v\frac{d}{dr}v = 2v\frac{dt}{dr}\frac{dv}{dt} = 2\frac{dv}{dt}, \tag{5.103}$$

[24] The binding energy of a test particle with charge q at radius R is $q\phi(R) = -mc^2$. So in order to escape the particle must have kinetic energy mc^2, which means that the speed is comparable to the speed of light.

[25] I "found" this solution by reading trough a long list of exactly solve-able differential equations at the website: *http://eqworld.ipmnet.ru/*.

and hence (5.102) becomes

$$\frac{dw}{dr} + 2f(r)w(r) + 2g(r) = 0. \tag{5.104}$$

This is an ordinary, non-linear differential equation which can be solved by standard strategies, and the solution with initial condition $w(r_1) = 0$ is

$$w(r) = \frac{c^2}{k_1}\left[1 - e^{-2k_1(\frac{R}{r} - \frac{R}{r_1})}\right] \tag{5.105}$$

or

$$\frac{v^2}{c^2} = \frac{1}{k_1}\left[1 - e^{-2k_1(\frac{R}{r} - \frac{R}{r_1})}\right]. \tag{5.106}$$

We let $v_0 = v(r_0)$ and $\gamma_0 = 1/\sqrt{1 - \frac{v_0^2}{c^2}}$. The kinetic energy E_0 of the particle at r_0 becomes

$$\begin{aligned}\frac{E_0}{mc^2} &= \gamma_0 - 1 \\ &= \frac{R}{r_0} - \frac{R}{r_1} \\ &\quad + (\frac{3}{2} - k_1)(\frac{R}{r_0})^2 + (\frac{3}{2} - k_1)(\frac{R}{r_1})^2 + (2k_1 - 3)\frac{R}{r_0}\frac{R}{r_1} \\ &\quad + \mathcal{O}(\frac{R}{r})^3,\end{aligned} \tag{5.107}$$

where we have expanded the exponential function in (5.106) to second order in $\frac{R}{r_1}$ and $\frac{R}{r_0}$.

Then let us consider the second particle which is moving quasi statically. In order to cancel the electric force the turbines must act on the particle with a force $\mathbf{F}$ with magnitude

$$F = |q\mathbf{E}| = |q\nabla\phi| = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2} = \frac{R}{r^2}mc^2. \tag{5.108}$$

The total work performed by the turbines on the particle can be found by integrating the force from r_1 to r_0:

$$W = \int_{r_1}^{r_0} \mathbf{F}\cdot d\mathbf{r} = Rmc^2\int_{r_1}^{r_0} dr\frac{1}{r^2} = -mc^2(\frac{R}{r_0} - \frac{R}{r_1}). \tag{5.109}$$

Hence the amount of energy stored in the batteries is $-W$. As both particles started from r_1 with the same energy, it follows from energy conservation that

$$E_0 = -W = mc^2(\frac{R}{r_0} - \frac{R}{r_1}). \tag{5.110}$$

This is in agreement with (5.107) for the value $k_1 = \frac{3}{2}$. Thus we have successfully showed how the velocity dependent term in the equation of motion can be derived from an energy argument.

5.3.2 The gravitational case

We assume that the gravitational field ϕ_g is produced by spherical symmetric mass distribution with total mass M:

$$\phi_g = -\frac{GM}{r} = -\frac{1}{2}c^2\frac{R_s}{r}, \tag{5.111}$$

where $R_s = 2\frac{GM}{c^2}$ is the Schwarzschild radius. Needless to say, we shall assume that $\frac{R_s}{r_0} \ll 1$ and $\frac{R_s}{r_1} \ll 1$. As in the previous section we shall start by considering the kinetic energy of the freely falling particle. The differential equation (5.99) can be reformulated

$$\frac{dv}{dt} + f(r)v^2 + g(r) = 0, \tag{5.112}$$

where $v = \frac{dr}{dt}$, $f(r) = -\frac{1}{2}k_2\frac{R_s}{r^2}$ and $g(r) = \frac{1}{2}c^2\frac{R_s}{r^2} - c^2\frac{R_s^2}{r^3}$. This differential equation is on exactly the same form as (5.102), and can be solved in the same manner. The solution with initial condition $v(r_1) = 0$ is

$$\frac{v(r)^2}{c^2} = \frac{1}{k_2^2}\left[2 + k_2 - 2k_2\frac{R_s}{r} + e^{k_2\left(\frac{R_s}{r_1} - \frac{R_s}{r}\right)}\left(-2 - k_2 + 2k_2\frac{R_s}{r_1}\right)\right]. \tag{5.113}$$

Expanding this expression to second order in $\frac{R_s}{r}$ and $\frac{R_s}{r_1}$ we get

$$\begin{aligned}\frac{v(r)^2}{c^2} =& c^2\left(\frac{R_s}{r} - \frac{R_s}{r_1}\right)\\ &- c^2\left(\frac{R_s}{r}\right)^2(1 + \frac{1}{2}k_2) + c^2\left(\frac{R_s}{r_1}\right)^2(1 - \frac{1}{2}k_2) + c^2k_2\frac{R_s}{r}\frac{R_s}{r_1}\\ &+ \mathcal{O}(\frac{R_s}{r})^3.\end{aligned} \tag{5.114}$$

Thus we have found an expression for the coordinate velocity squared. From this it is possible to calculate the energy $E_{\hat{0}}$. At this point however, we must start being careful. We have defined $E_{\hat{0}}$ as the kinetic energy *measured* by an observer at rest at r_0. The coordinate velocity $v = \frac{dr}{dt}$ however, is not similar to the measured velocity $\overline{v} = \frac{d\overline{r}}{d\overline{t}}$. The idea is now to make use of the fact that in curved spacetime, all laws from special relativity are valid locally. Hence we can use the standard expression for the kinetic energy, $E_k = (\gamma-1)mc^2$, but the coordinate velocity must be replaced by the measured velocity:

$$\frac{E_{\hat{0}}}{mc^2} = \overline{\gamma}_0 - 1, \tag{5.115}$$

where

$$\overline{\gamma}_0 = \frac{1}{\sqrt{1 - (\frac{\overline{v}_0}{c})^2}} \tag{5.116}$$

and $\overline{v}_0$ is the measured velocity at r_0, ie. $\left.\frac{d\overline{r}}{d\overline{t}}\right|_{r=r_0}$. We calculated the measured velocity in section 5.2.3. From (5.76) we see that the measured velocity squared at r_0 becomes

$$\frac{\overline{v}_0^2}{c^2} = \frac{v_0^2}{c^2}(1 - 4\frac{\phi_{g0}}{c^2}) = \frac{v_0^2}{c^2}\left(1 + 2\frac{R_s}{r_0}\right). \tag{5.117}$$

Using (5.114) we rewrite $\frac{\overline{v}_0^2}{c^2}$ in terms of r_0 and r_1. Inserting this into (5.115) and expanding to second order in $\frac{R_s}{r_0}$ and $\frac{R_s}{r_1}$ we get:

$$\begin{aligned}\frac{E_{\hat{0}}}{mc^2} =& \frac{1}{2}\left(\frac{R_s}{r_0} - \frac{R_s}{r_1}\right)\\ &+ \frac{1}{4}(\frac{7}{2} - k_2)\left(\frac{R_s}{r_0}\right)^2 + \frac{1}{4}(\frac{7}{2} - k_2)\left(\frac{R_s}{r_1}\right)^2 + \frac{1}{2}(k_2 - \frac{7}{2})\frac{R_s}{r_0}\frac{R_s}{r_1}\\ &+ \mathcal{O}(\frac{R_s}{r})^3.\end{aligned} \tag{5.118}$$

Then, let us consider the quasi static particle. We start by considering the physics in a local proper reference frame. Since the local observer is considering a particle moving quasi statically, he can model the gravitational "force" $\overline{F}_g$ like an ordinary force[26], and apply Newton's second law

$$\overline{\mathbf{F}}_{\mathbf{g}} = m\overline{\mathbf{a}}, \tag{5.119}$$

to determine the effective gravitational force. Inserting the measured acceleration, see (5.44), with $\mathbf{v} = 0$ into (5.119) we get

$$\overline{\mathbf{F}}_{\mathbf{g}} = -m(\nabla\phi_g + \frac{\phi_g}{c^2}\nabla\phi_g), \tag{5.120}$$

or

$$\overline{F}_g = -\frac{1}{2}mc^2\frac{R_s}{r^2} + \frac{1}{4}mc^2\frac{R_s^2}{r^3}. \tag{5.121}$$

In order to cancel the gravitational force the turbines will have to apply a force $\overline{\mathbf{F}} = -\overline{\mathbf{F}}_{\mathbf{g}}$. It is instructive to imagine that for each line segment dr there is one battery storing an amount dE of energy:

$$dE = |\overline{\mathbf{F}} \cdot \mathbf{d}\overline{\mathbf{r}}| = \left| -\frac{1}{2}mc^2\frac{R_s}{r^2} + \frac{1}{4}mc^2\frac{R_s^2}{r^3} \right| \left(1 + \frac{1}{2}\frac{R_s}{r}\right) dr = \frac{1}{2}mc^2\frac{R_s}{r^2}dr. \tag{5.122}$$

As explained in the introduction, we can not simply integrate over all the batteries and expect it to add up to $E_{\hat{0}}$. Indeed, this would have been a violation of energy conservation, as the kinetic energy of the freely falling particle is measured at the end of the path (r_0). Thus we must take into account the Doppler shift of the energy. We let $dE_{\hat{0}}$ denote the value of the energy dE at r_0. Using the standard Doppler shift formula for light in a stationary gravitational field, we get

$$\begin{aligned} dE_{\hat{0}} &= dE\frac{|g_{00}(r)|}{|g_{00}(r_0)|} \\ &= dE(1 + \frac{1}{2}\frac{R_s}{r_0} - \frac{1}{2}\frac{R_s}{r}) \\ &= dr\left(\frac{1}{2}\frac{R_s}{r^2}(1 + \frac{1}{2}\frac{R_s}{r_0}) - \frac{1}{4}\frac{R_s^2}{r^3}\right)mc^2. \end{aligned} \tag{5.123}$$

The total energy stored in the single battery at r_0 becomes

$$\begin{aligned} \frac{E_{\hat{0}}}{mc^2} &= \int_{r_0}^{r_1} dE_{\hat{0}}/mc^2 \\ &= \frac{1}{2}\left(\frac{R_s}{r_0} - \frac{R_s}{r_1}\right) + \frac{1}{8}\left(\frac{R_s}{r_0}\right)^2 + \frac{1}{8}\left(\frac{R_s}{r_1}\right)^2 - \frac{1}{4}\frac{R_s}{r_0}\frac{R_s}{r_1}. \end{aligned} \tag{5.124}$$

Comparison with (5.118) gives $k_2 = 3$. Thus we have derived the velocity dependent term in the equation of motion from an energy argument by using insight into the conceptual foundations of general relativity.

[26] If you compare the measured acceleration for gravitation with the equation of motion in electrodynamics, you will see that the velocity dependent terms are different. For quasi-static motion however, we can neglect these terms, and gravity can be modeled as an ordinary force by a local observer.

5.3.3 Comments

There are a couple of things from this section which deserves comments. First of all, the calculations done here are done in a post-Newtonian like manner, ie. neglecting small terms in a consistent way. For stationary spacetimes however, energy conservation should hold exactly. I have done the same calculations by using exact methods with the full non-linear Schwarzschild metric. This calculation verifies, boringly enough, what any introduction book to general relativity teach us, namely that energy is conserved in Schwarzschild spacetime since the time coordinate is a cyclic coordinate.

I think it is very fascinating that it is possible to derive the velocity dependent term in the equation of motion, like we have done here, by considering the physics from a local special relativistic perspective. By integrating over the continuum of local reference frames, taking into account how they are related to each others (Doppler shift), we have derived something of a global character, namely the velocity dependence in the equation of motion. I think box 6.1 in [1], with the title 'General relativity is built on special relativity', shed some light onto our work:

> *A tourist in a powered interplanetary rocket feels "gravity". Can a physicist by local effects convince him that this "gravity" is bogus? Never, says Einstein's principle of the local equivalence of gravity and accelerations. But then the physicist will make no errors if he deludes himself into treating true gravity as a local illusion caused by acceleration. Under this delusion, he barges ahead an solves gravitational problems by using special relativity: if he is clever enough to divide every problem into a network of local questions, each solvable under such a delusion, then he can work out all influences of any gravitational field. Only three basic principles are invoked: special-relativity physics, the equivalence principle, and the local nature of physics. They are simple and clear. To apply them, however, imposes a double task: (1) take spacetime apart into locally flat pieces (where the principles are valid), and (2) put these pieces together again into a comprehensible picture. To undertake this dissection and reconstitution, to see curved dynamic spacetime inescapably take form, and to see the consequences for physics: that is general relativity.*

Chapter 6

Conclusion

The very simple starting point for this thesis was the so-called gravito-electromagnetic framework reviewed in chapter 3. As this framework strongly resembles Maxwell's equations and the Lorentz force law of electrodynamics, it has received some attention, and a lot is written about this and similar kinds of analogies. The reason for the interest has probably little to do with the physicist's dream about unifying gravity with the ordinary forces of nature. A more probable explanation is that the approximation is written on a form which facilitates physical intuition. The gravito-electromagnetic framework is immediately familiar to any person who has studied electrodynamics. Given the mathematically extremely complicated form of the full theory, it is not surprising that such a formulation has become popular. For a physicist with experience from serious approximation schemes to general relativity though, it should be clear that the framework is not based on a systematic method. It takes account for the leading term, ie. the Newtonian contribution, but for the second order effects it only includes frame-dragging and retardation effects. Nevertheless, in several papers, as those referred to in chapter 3, it is presented as a result of an approximation scheme where the only assumptions are weak fields and slow motion. For students and physicists not specializing in gravitational perturbation theory this may lead to misinterpretations and confusion. It may give rise to a myth that there exist a formal analogy between general relativity and electrodynamics which is much stronger than it deserves.

I have no doubt that physicists will continue to explore different kinds of analogies between the field theories for gravitation and electromagnetism. Such analogies may be interesting and sometimes they provide new insights into the phenomenon of gravitation. I hope however, that this thesis has demonstrated the need for a careful approach. In particular I hope that the authors will analyze, and share with us, the limitations of their proposed analogies. This is at least important for readers in lack of "hands on" experience with the methods being used.

My idea was to study the gravito-electromagnetic analogy in a systematic way using the framework referred to as the post-Newtonian approximation scheme. In chapter 4 we employ these methods to electrodynamics as well as gravitation, and compare the mathematical structure of the approximations in a consistent way. For electrodynamics the lowest order solution gives back the equations characterizing electrostatics. I therefore give the name 'the post-electrostatic limit of electrodynamics' to the approximation corresponding to the post-Newtonian limit of general relativity. Employing the same systematic method to both theories enabled me to compare the phenomenas in a consistent way beyond the lowest order approximations, see section 4.2.1. To put it short, we found that for each term/effect in the equation of motion for the electromagnetic case, there is a corresponding term/effect in the gravitational case. This can however not be put the other way around as there are several effects in the gravitational case, such as non-linearities and contributions from pressure, kinetic energy and gravitational binding energy, with no counter-part in electrodynamics. Despite interesting similarities, we must conclude that the physics is far richer in the gravitational case, which becomes clear when going beyond the lowest order approximations. I am not aware of any papers where this has been studied and clarified in a consistent way like here, and as far as I know it is a new approach.

In chapter 4.3 I reformulate the post-Newtonian limit of general relativity in a form which resembles Maxwell's equations. I show that the post-Newtonian limit can be formulated in terms of one scalar potential ϕ_g and one vector potential $\mathbf{A}_g$, just like electrodynamics. This demands that the gravitational "charge" is defined as a special combination of rest mass energy, pressure, kinetic energy and gravitational binding energy. Recently (April 2010) I have become aware that the idea of writing the post-Newtonian limit on a Maxwell-like form is not new. In 1991 T. Damour, M. Soffel and C. Xu wrote the post-Newtonian approximation in a Maxwell-like form, see [25]. Their results are extracted on a more readable form in [26]. Their formulation is actually in terms of two vector fields $\mathbf{g}$ and $\mathbf{H}$[1] which corresponds to the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in electrodynamics. This is not similar to my formulation which corresponds to the potentials ϕ and $\mathbf{A}$ of electrodynamics. It is, however, easy to show that the potentials which $\mathbf{g}$ and $\mathbf{H}$ are derived from in [26] are similar to my potentials ϕ_g and $\mathbf{A}_g$ to the required accuracy (compare (2.4a) and (2.4b) in [26] with (4.71) and (4.72)). This verifies that my formulation is viable.

The rest of my work (4.3 and all of chapter 5) is in terms of the above mentioned potentials ϕ_g and $\mathbf{A}_g$. In section 4.3 we notice that the equation of motion takes a form which is very similar to the equation of motion in the post-electrostatic limit of electrodynamics when it is written in terms of these potentials. The only qualitative differences are the appearance of two terms in the gravitational case with no counterpart in the electromagnetic case. One of these terms are quadratic in ϕ_g, while the other depends explicitly on the rate of change of ϕ_g (that is $\partial_t\phi_g$). The former term is due to the fact that general relativity is a non-linear theory. It is harder to explain why the latter term is absent in electrodynamics, but I will try to clarify this, in a moment, based on the results from chapter 5.

The work described in chapter 5 is motivated by a question concerning the interpretation of the equation of motion in the gravitational case. In particular I want to clarify whether the coordinate acceleration is numerically equal to the locally measured acceleration (to post-Newtonian accuracy). By taking a direct approach, making use of the equivalence principle, I show in 5.1 that the answer is definitely "no"; to post-Newtonian precision the coordinate acceleration is not the same as the locally observable acceleration. Then, in section 5.2, I make use of a result in [1] and find an actual expression for the measured acceleration in terms of our post-Newtonian language. The expression has several quantitative differences compared to the coordinate acceleration. This observation makes me conclude that curvature has geometric significance in the post-Newtonian approximation of general relativity. For the lowest order approximations however, there is no difference between the coordinate acceleration and the locally observable acceleration, so the curvature has no geometric significance. This observation gives a perturbation theory based justification for the view, in the Newtonian limit, of gravity as an ordinary force co-existing with ordinary forces in a fixed background spacetime. Taken together with the results from chapter 4, we conclude that there is an almost perfect analogy between the lowest order approximations of general relativity and electrodynamics both formally (the mathematical structure is on the same form) *and* conseptually (the background spacetime can be viewed as fixed and flat in both cases). Beyond the lowest order approximations however, my work shows that the perfect analogy breaks down both formally (the mathematical structure is not quite

[1]This formulation is not perfectly successful however since the equation of motion in addition to $\mathbf{g}$ and $\mathbf{H}$ also includes the scalar potential which $\mathbf{g}$ is defined in terms of.

on the same form) *and* conceptually (curvature has geometric significance in the post-Newtonian approximation). I have not seen any papers where the significance of curvature in the context of the gravito-electromagnetic analogy has been discussed and clarified in this way.

The single most interesting result from chapter 5, in my view, is that the analogy becomes stronger when the phenomenas are treated on a more equal footing conceptually. In particular, I showed that the term with explicit time dependence on ϕ_g in the coordinate acceleration does not show up in the expression for the locally measured acceleration. This means that the rate of change of the scalar potential is neither locally observable in the gravitational nor in the electromagnetic case. I am not sure there is a deep reason for this, but it seems reasonable that the appearance of the term $\partial_t \phi_g$ in the equation of motion is a manifestation of curvature. Since, according to the equivalence principle, curvature is not locally observable, it makes sense that the term is absent in the expression for the measured acceleration. Concerning the *numerical value* of the potential, we can almost say it is a general principle of field theory that it is not observable (it is its gradient which has significance, and the zero point for the potential can be chosen arbitrarily). One can wonder if it might be a general principle of field theory also that the *rate of change* of the scalar potential is not locally observable. Is it really just a coincidence that it is neither locally observable in the gravitational nor the electromagnetic case?

It should also be mentioned that the derivation of the measured acceleration showed in 5.2.3 gives a geometric/visual kind of insight into the difference between the coordinate acceleration and the measured acceleration. I think this approach is entirely new, but as explained in that section more effort is needed before I am completely satisfied. Another highlight from the same chapter was the energy considerations in section 5.3 where I gave an example on how local special relativistic physics can provide insight into some of the (quantitative) differences between the kinematics in the gravitational and the electromagnetic case.

Let me finally comment on the role of the gravito-electromagnetic analogy in the education of physicists. As pointed out in [27], most universities treat gravity and electromagnetism as two completely separated topics. The authors advocates the view that in general students should have a glimpse to the interplay between classical electromagnetism and general relativity. After one year of work on a related topic I do not completely agree. Lecturers should certainly comment on analogies to electrodynamics when teaching frame-dragging effects. They could also comment on analogies when teaching gravitational perturbation theory, although this is usually not part of the curriculum in a first course on general relativity. I think it would be wise however to stop there. For students being taught general relativity for the first time focusing too much on electrodynamics would probably rather lead to mis-interpretations than real insight. I am at least very happy that my lecturer in general relativity mainly focused on the essence of the theory and helped us appreciating the view of gravitation as a geometric phenomenon.

Bibliography

[1] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler (1973),
Gravitation, Freeman.

[2] Robert M. Wald (1984),
General Relativity, The University of Chicago Press.

[3] Clifford M. Will(1993),
Theory and experiment in gravitational physics, (rev.edn), Cambridge University Press.

[4] James B. Hartle(2003),
Gravity, Addison Wesley.

[5] Sean M. Carroll(1997),
Lecture notes on General Relativity, arXiv:gr-qc/9712019v1.

[6] Øyvind Grøn and Sigbjørn Hervik (2007),
Einstein's General Theory of Relativity: With Modern Applications in Cosmology, Springer.

[7] Øyvind Grøn (2009),
Lecture Notes on the General Theory of Relativity, Springer.

[8] Bernard F. Schutz (1985),
A first course in general relativity, Cambridge University Press.

[9] L. Ryder (2009),
Introduction to General Relativity, Cambridge University Press.

[10] Richard P. Feynman,
Feynman lectures on gravitation, Westview Press (2003).

[11] J.C. Maxwell(1865),
Phil. Trans., 155, 492.

[12] H.A. Lorentz,
Considerations on Gravitation, Proc. Acad. Amsterdam 2: 559Ű574.

[13] J. Lense and H. Thirring ,
Phys. Z. **19**, 156 (1918).

[14] C.W.F. Everitt and B.W. Parkinson (2009),
Gravity Probe B Science ResultsŮNASA Final Report, December 2008.

[15] M.L.Ruggiero and A. Tartaglia (2002),
Gravitomagnetic effects, Nuovo Cim. 117B (2002) 743-768 (arXiv:gr-qc/0207065v2).

[16] B. Mashoon (2002),
Gravitoelectromagnetism, Plenary talk given at EREs2000 (Valladolid, Spain, 6-9 September 2000); (arXiv:gr-qc/0011014v1).

[17] D. Bini, R. T. Jantzen,
A List of References on Spacetime Splitting and Gravitoelectromagnetism, arXiv:gr-qc/0010070v1.

[18] M. Tajmar, C. de Matos (2000)
Coupling of Gravitation and Electromagnetism in the Weak Field Approximation, arXiv:gr-qc/0003011v1.

[19] S. Weinberg (1972),
Gravitation and Cosmology, Wiley.

[20] A. Rakic, D.J.Schwarz,
General Relativistic Rotation Curves in a Post-Newtonian Light, arXiv:0811.1478v1.

[21] L. I. Schiff (1960),
Possible new experimental test of general relativity, Physical Review Letters 4, 215-217.

[22] Clifford M. Will(1993),
The Confrontation between General Relativity and Experiment, LivingRev.Rel.9:3,2005 (arXiv:gr-qc/0510072v2).

[23] G. Schaefer,
Post-Newtonian methods: Analytic results on the binary problem, arXiv:0910.2857v1.

[24] M. Soffel, H. Ruder, M. Schneider (1987)
The two-body problem in the (truncated) PPN-theory, Celestial Mechanics 40, 77-85.

[25] T. Damour, M. Soffel, C. Xu (1991)
Phys. Rev. D 43, 3273.

[26] Jeffrey D. Kaplan, David A. Nichols, Kip S. Thorne (2009)
Post-Newtonian Approximation in Maxwell-Like Form,Phys.Rev.D 80: 124014,2009.

[27] A. Tartaglia, M. L. Ruggiero,
Gravito-electromagnetism versus electromagnetism, Eur.J.Phys. 25 (2004) 203-210. arXiv:gr-qc/0311024v1

[28] M.H. SOFFEL and V.A. BRUMBERG (1991)
Relativistic reference frames including time scales: Questions and answers, Celest. Mech. 52(4), 355Ű373

A Appendix

A.1

Here we will compute u^0/c and u^i/c to $O(2)$. We start with the former. From the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2 d\tau^2, \tag{A.1}$$

or equivalently

$$g_{\mu\nu}u^\mu u^\nu = -c^2, \tag{A.2}$$

we can write

$$\begin{aligned}(\frac{u^0}{c})^2[O(2)] &= -\frac{1}{g_{00}} - 2\frac{g_{0i}}{g_{00}}\frac{u^0}{c}\frac{u^i}{c} - \frac{g_{i=j}}{g_{00}}\frac{u^i}{c}\frac{u^j}{c} - \frac{g_{i\neq j}}{g_{00}}\frac{u^i}{c}\frac{u^j}{c} \\ &= -\frac{1}{g_{00}} - \frac{g_{i=j}}{g_{00}}\frac{u^i}{c}\frac{u^j}{c} + O(4) \\ &= 1 - 2\frac{U}{c^2} + (\frac{\mathbf{v}}{c})^2.\end{aligned} \tag{A.3}$$

The first line is a simple re-formulation of the line-element (A.2). In the *second* line we have made use of the fact that the first term on the right hand side of the *first* line is $O(0)$, while the second term is $O(4)$, the third term $O(2)$ and finally the last term (still on the *first* line in the same equation) is $O(6)$. In the third line we have simply inserted lower order solutions for $g_{\mu\nu}$ and u^μ. Taking the square root of (A.3) we get our solution:

$$\frac{u^0}{c}[O(2)] = 1 - \frac{U}{c^2} + \frac{1}{2}(\frac{\mathbf{v}}{c})^2. \tag{A.4}$$

Using this result, it is straight forward to calculate the components u^i to $O(2)$:

$$\begin{aligned}\frac{u^i}{c} &= \frac{1}{c}\frac{dx^i}{d\tau} = \frac{1}{c}\frac{dt}{d\tau}\frac{dx^i}{dt} = \frac{v^i}{c}\frac{u^0}{c} = \frac{v^i}{c}(1 + O(2)) \\ &= \frac{v^i}{c} + O(3),\end{aligned} \tag{A.5}$$

and hence

$$\frac{u^i}{c}[O(2)] = \frac{v^i}{c}. \tag{A.6}$$

A.2

Here we shall compute the contravariant components of the energy-momentum tensor. We start with the time-time component:

$$\begin{aligned} T^{00} &= (\rho + \frac{p}{c^2})u^0u^0 + pg^{00} \\ &= \rho c^2 \left((\frac{u^0}{c})^2 + \frac{p}{\rho c^2}(\frac{u^0}{c})^2 + \frac{p}{\rho c^2} g^{00} \right) \\ &= \rho c^2 \left((\frac{u^0}{c})^2 + O(4) \right), \end{aligned} \tag{A.7}$$

which give

$$T^{00}[O(2)] = \rho c^2 \left(1 - 2\frac{U}{c^2} + (\frac{\mathbf{v}}{c})^2 \right). \tag{A.8}$$

Then we calculate the time space components:

$$\begin{aligned} T^{0i} &= (\rho + \frac{p}{c^2})u^0u^i + pg^{0i} \\ &= \rho c^2 \left(\frac{u^0}{c}\frac{u^i}{c} + \frac{p}{\rho c^2}\frac{u^0}{c}\frac{u^i}{c} + \frac{p}{\rho c^2} g^{0i} \right) \\ &= \rho c^2 \left(\frac{v^i}{c} + O(3) \right). \end{aligned} \tag{A.9}$$

In the last step we have used (4.24) and the post-Newtonian book-keeping system to evaluate the order of magnitude of the terms in the second line. From (A.9) we see that

$$T^{0i}[\rho c^2 O(2)] = \rho c^2 \frac{v^i}{c}. \tag{A.10}$$

Finally we calculate the space space components:

$$\begin{aligned} T^{ij} &= (\rho + \frac{p}{c^2})u^iu^j + pg^{ij} \\ &= \rho c^2 \left(\frac{u^i}{c}\frac{u^j}{c} + \frac{p}{\rho c^2}\frac{u^i}{c}\frac{u^j}{c} + \frac{p}{\rho c^2} g^{ij} \right) \\ &= \rho c^2 \left(\frac{v^i}{c}\frac{v^j}{c} + \frac{p}{\rho c^2}\delta^{ij} + O(4) \right), \end{aligned} \tag{A.11}$$

and we can write[1]:

$$T^{ij}[\rho c^2 O(2)] = \rho c^2 \left(\frac{v^i}{c}\frac{v^j}{c} + \frac{p}{\rho c^2}\delta^{ij} \right). \tag{A.12}$$

We have worked out all the contravariant components $T^{\mu\nu}$ to the desired accuracy, and can summarize our results:

$$T^{\mu\nu}[\rho c^2 O(2)] = \begin{cases} \rho c^2 \left(1 - 2\frac{U}{c^2} + (\frac{\mathbf{v}}{c})^2\right) & \text{, if } (\mu\nu) = (0,0) \\ \rho c^2 \left(\frac{v^i}{c}\right) & \text{, if } (\mu\nu) = (i,0) \\ \rho c^2 \left(\frac{v^i}{c}\frac{v^j}{c} + \frac{p}{\rho c^2}\delta^{ij}\right) & \text{, if } (\mu,\nu) = (i,j). \end{cases} \tag{A.13}$$

[1]In this case $T^{ij}[\rho c^2 O(2)] = T^{ij}[\rho c^2 O(3)]$, but terms of smallness $\rho c^2 O(3)$ are not relevant to us and we do not need this information.

A.3

We will here calculate $T_{\mu\nu}$ and T to accuracy $\rho c^2 O(2)$. We start with the former. The calculations of the components T_{00}, T_{0i} and T_{ij} have a repetive character and goes like this: a) write out the Einstein summation convention, b) replace $g_{\mu\nu}$ with $\eta_{\mu\nu} + h_{\mu\nu}$, c) analyze what terms can be neglected using (4.8), (4.16) and (4.26) , d) replace the $h_{\mu\nu}$'s with lower order solutions given in (4.16), e) insert the contravariant components $T^{\mu\nu}$. First we perform step a),b),c) and d) for each of the components

$$\begin{aligned} T_{00} &= g_{0\mu}g_{0\nu}T^{\mu\nu} \\ &= g_{00}g_{00}T^{00} + 2g_{00}g_{0i}T^{0i} + g_{0i}g_{0j}T^{ij} \\ &= (-1+h_{00})(-1+h_{00})T^{00} + 2(-1+h_{00})h_{0i}T^{0i} + h_{0i}h_{0j}T^{ij} \\ &= (1+4\frac{\phi}{c^2})T^{00} + O(4)\rho c^2, \end{aligned} \tag{A.14}$$

$$\begin{aligned} T_{0i} &= g_{0\mu}g_{i\nu}T^{\mu\nu} \\ &= g_{00}g_{i0}T^{00} + g_{0j}g_{i0}T^{j0} + g_{00}g_{ij}T^{0j} + g_{0k}g_{il}T^{kl} \\ &= (-1+h_{00})h_{i0}T^{00} + h_{0j}h_{i0}T^{j0} + (-1+h_{00})(\eta_{ij}+h_{ij})T^{0j} + h_{0k}(\eta_{il}+h_{il})T^{kl} \\ &= -\eta_{ij}T^{0j} + \rho c^2 O(3), \end{aligned} \tag{A.15}$$

and finally

$$\begin{aligned} T_{ij} &= g_{i\mu}g_{j\nu}T^{\mu\nu} \\ &= g_{i0}g_{j0}T^{00} + g_{ik}g_{j0}T^{k0} + g_{i0}g_{jk}T^{0k} + g_{ik}g_{jl}T^{kl} \\ &= h_{i0}h_{j0}T^{00} + (\eta_{ik}+h_{ik})h_{j0}T^{k0} + h_{i0}(\eta_{jk}+h_{jk})T^{0k} + (\eta_{ik}+h_{ik})(\eta_{jl}+h_{jl})T^{kl} \\ &= \eta_{ik}\eta_{jl}T^{kl} + \rho c^2 O(4). \end{aligned} \tag{A.16}$$

Then we insert the contravariant components (step e)) and get:

$$T_{\mu\nu} = \begin{cases} \rho c^2 \left(1 + 2\frac{U}{c^2} + (\frac{\mathbf{v}}{c})^2 + O(4)\right) & \text{, if } (\mu\nu) = (0,0) \\ -\rho c^2 \left(\frac{v^i}{c} + O(3)\right) & \text{, if } (\mu\nu) = (i,0) \\ \rho c^2 \left(\frac{v^i}{c}\frac{v^j}{c} + \frac{p}{\rho c^2}\delta^{ij} + O(4)\right) & \text{, if } (\mu,\nu) = (i,j). \end{cases} \tag{A.17}$$

Then we calculate T using the same strategy:

$$\begin{aligned} T &= g_{\mu\nu}T^{\mu\nu} \\ &= g_{00}T^{00} + 2g_{i0}T^{i0} + g_{ij}T^{ij} \\ &= (-1+h_{00})T^{00} + 2h_{i0}T^{i0} + (\eta_{ij}+h_{ij})T^{ij} \\ &= (-1+h_{00})T^{00} + \eta_{ij}T^{ij} + \rho c^2 O(4) \\ &= -\rho c^2 \left(1 + (\frac{\mathbf{v}}{c})^2 + O(4)\right) + \eta_{ij}\rho c^2 \left(\frac{v^i}{c}\frac{v^j}{c} + \frac{p}{\rho c^2}\delta^{ij} + O(4)\right) + \rho c^2 O(4) \\ &= -\rho c^2 \left(1 - 3\frac{p}{\rho c^2} + O(4)\right), \end{aligned} \tag{A.18}$$

thanks God for all the cancellations! So we get:

$$T[\rho c^2 O(2)] = -\rho c^2 \left(1 - 3\frac{p}{\rho c^2}\right). \tag{A.19}$$

A.4

We will show how (4.36) follows from (4.34) by inserting (4.17). Let us take it term by term:

$$\begin{aligned}\frac{1}{2}h_{kj}[O(2)]\partial_k\partial_j h_{00}[O(2)] &= 2\frac{U}{c^2}\delta_{kj}\partial_k\partial_j\frac{U}{c^2}\\ &= 2\frac{U}{c^4}\nabla^2 U\\ &= \frac{8\pi G}{c^4}\rho U,\end{aligned}\tag{A.20}$$

$$\begin{aligned}\frac{1}{2}\partial_j h_{00}[O(2)]\left(\partial_k h_{jk}[O(2)] - \frac{1}{2}\partial_j h_{kk}[O(2)]\right) &= 2\delta_{jk}\partial_j\frac{U}{c^2}\partial_k\frac{U}{c^2} - \partial_{kk}(\partial_j\frac{U}{c^2})(\partial_j\frac{U}{c^2})\\ &= -(\partial_j\frac{U}{c^2})(\partial_j\frac{U}{c^2})\\ &= -|\nabla\frac{U}{c^2}|^2,\end{aligned}\tag{A.21}$$

and finally

$$-\frac{1}{4}|\nabla h_{00}|^2 = -|\nabla\frac{U}{c^2}|^2.\tag{A.22}$$

Adding terms we get:

$$\begin{aligned}R^{(2)}_{00} &= \frac{U}{c^4}8\pi G\rho - 2|\nabla\frac{U}{c^2}|^2\\ &= -\nabla^2(\frac{U}{c^2})^2 + \frac{16\pi G}{c^4}\rho U.\end{aligned}\tag{A.23}$$

In the second line we have used

$$2|\nabla\frac{U}{c^2}|^2 = \nabla^2(\frac{U}{c^2})^2 - \frac{8\pi G}{c^4}\rho U,\tag{A.24}$$

which follows from the calculation

$$\begin{aligned}\nabla^2(\frac{U}{c^2})^2 &= \nabla\left(2\frac{U}{c^2}\nabla(\frac{U}{c^2})\right)\\ &= 2\frac{U}{c^4}\nabla^2 U + 2|\nabla\frac{U}{c^2}|^2\\ &= \frac{8\pi G}{c^4}\rho U + 2|\nabla\frac{U}{c^2}|^2.\end{aligned}\tag{A.25}$$

A.5

Here we will determine h_{00} to $O(4)$ from (4.38) using standard techniques from perturbation theory. Let us write $h_{00} = h_1 + h_2 + h_3 \ldots$, where h_1 is the solution of the lowest order part of (4.38), h_2 is the solution of the next lowest order part and so on. According to our notation this means $h_{00}[O(2)] = h_1$ and $h_{00}[O(4)] = h_1 + h_2$. Using the book-keeping system we find that to lowest order (4.38) yields

$$-\nabla^2 h_1 = \frac{8\pi G}{c^2}\rho,\tag{A.26}$$

or

$$\nabla^2(-\frac{1}{2}h_1 c^2) = 4\pi G\rho. \tag{A.27}$$

Accordingly $h_1 = h_{00}[O(2)] = -2\frac{U}{c^2}$ in agreement with (4.17). To next order (4.38) yields

$$-\nabla^2 h_2 + \partial_0^2 h_1 - 2\nabla^2 \frac{U^2}{c^4} = \frac{8\pi G}{c^2}\rho\left(2\frac{v^2}{c^2} - 2\frac{U}{c^2} + 3\frac{p}{\rho c^2}\right). \tag{A.28}$$

Inserting for h_1 and rewriting we get

$$\nabla^2\left(-\frac{1}{2}c^2 h_2 - \frac{U^2}{c^2}\right) = 4\pi G\rho\left(2\frac{v^2}{c^2} - 2\frac{U}{c^2} + 3\frac{p}{\rho c^2}\right) + \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}. \tag{A.29}$$

Integrating this equation we get

$$-\frac{1}{2}c^2 h_2 - \frac{U^2}{c^2} = -G\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\rho'\left(2\frac{v^2}{c^2} - 2\frac{U}{c^2} + 3\frac{p}{\rho c^2}\right) + \frac{1}{4\pi c^2}\frac{\partial^2 U}{\partial t^2}\right]. \tag{A.30}$$

Replacing the integral with the potentials given by (4.40) we get

$$h_2 = -8\frac{U_k}{c^4} - 8\frac{U_g}{c^4} - 2\frac{U_p}{c^4} - 2\frac{U_t}{c^4} - 2(\frac{U}{c^2})^2. \tag{A.31}$$

Using $h_{00}[O(4)] = h_1 + h_2$ we get the result in (4.39).

A.6

Here we will show how to obtain equation (4.48). The components $\Gamma^{\mu(1)}_{\alpha\beta}$ defined by (4.45) are

$$\Gamma^{0(1)}_{00} = -\frac{1}{2}\partial_0 h_{00}, \quad \Gamma^{0(1)}_{0k} = -\frac{1}{2}\partial_k h_{00}, \quad \Gamma^{0(1)}_{km} = -\frac{1}{2}\partial_m h_{0k} - \frac{1}{2}\partial_k h_{0m} + \frac{1}{2}\partial_0 h_{km},$$

$$\Gamma^{i(1)}_{00} = \partial_0 h^i_0 - \frac{1}{2}\partial^i h_{00}, \quad \Gamma^{i(1)}_{m0} = \frac{1}{2}\partial_0 h^i_m + \frac{1}{2}\partial_m h^i_0 - \frac{1}{2}\partial^i h_{m0},$$
$$\Gamma^{i(1)}_{mn} = \frac{1}{2}\partial_n h^i_m + \frac{1}{2}\partial_m h^i_n - \frac{1}{2}\partial^i h_{mn}.$$

Inserting this into $a^i_{(1)}$ we get

$$\begin{aligned} a^i_{(1)} = c^2[&+\frac{1}{2}\partial^i h_{00} - \partial_0 h^i_0 \\ &- \partial_0 h^i_j \frac{v^j}{c} - \partial_j h^i_0 \frac{v^j}{c} + \partial^i h_{j0}\frac{v^j}{c} \\ &- \frac{1}{2}\partial_k h^i_j \frac{v^j}{c}\frac{v^k}{c} - \frac{1}{2}\partial_j h^i_k \frac{v^j}{c}\frac{v^k}{c} + \frac{1}{2}\partial^i h_{jk}\frac{v^j}{c}\frac{v^k}{c} \\ &- \frac{1}{2}\frac{v^i}{c}\partial_0 h_{00} - \frac{v^i}{c}\frac{v^k}{c}\partial_k h_{00} - \frac{v^i}{c}\partial_k h_{0m}\frac{v^k}{c}\frac{v^m}{c} + \frac{1}{2}\frac{v^i}{c}\partial_0 h_{km}\frac{v^k}{c}\frac{v^m}{c}]. \end{aligned} \tag{A.32}$$

The first term is of size $\sim O(0)\nabla U$, the last two terms are $\sim O(4)\nabla U$ while the rest is $\sim O(2)\nabla U$. Replacing the $h_{\mu\nu}$'s with potentials according to (4.39) and neglecting terms of smallness $\sim O(4)\nabla U$ we get the expression for $a^i_{(1)}[O(2)\nabla U]$ given by (4.48).

A.7

Here we will show how to obtain equation (4.49). To determine $a^i_{(2)}$ to $O(2)\nabla U$, we need an expression for $\Gamma^{\mu(2)}_{\alpha\beta}$ to $O(4)$. Since $\Gamma^{\mu(2)}_{\alpha\beta}$ is quadratic in $h_{\mu\nu}$ we only need to consider $h_{\mu\nu}$'s of size $O(2)$. From (4.39) we see that

$$h_{\mu\nu}[O(2)] = -2\frac{U}{c^2}\delta_{\mu\nu}. \tag{A.33}$$

Inserting this into (4.46) we get

$$\Gamma^{\mu(2)}_{\alpha\beta}[O(4)] = -2\frac{U}{c^2}\left[\partial_\beta \frac{U}{c^2}\delta^\mu_\alpha + \partial_\alpha \frac{U}{c^2}\delta^\mu_\beta - \partial_\delta \frac{U}{c^2}\delta^{\mu\delta}\delta_{\alpha\beta}\right], \tag{A.34}$$

where $\delta_{\mu\nu}$, $\delta^{\mu\nu}$ and δ^μ_ν all are, by definition, components of the unit-matrix I_4 (and are therefor not tensors since they cannot be raised or lowered with the metric). From (4.43) (with $\Gamma^\mu_{\alpha\beta}$ replaced by $\Gamma^{\mu(2)}_{\alpha\beta}$) we see that to accuracy $O(2)\nabla U$ in a^i the only component we need to consider is $\Gamma^{i(2)}_{00}$:

$$\Gamma^{i(2)}_{00} = 2\frac{U}{c^2}\partial^i\frac{U}{c^2}, \tag{A.35}$$

and hence

$$a^i_{(2)}[O(4)] = -2U\partial^i\frac{U}{c^2} = -\partial^i\frac{U^2}{c^2}. \tag{A.36}$$

A.8

Here we will show that the solution of (4.67) to second order in β is given by (4.68). To second order in β (4.67) simplifies to

$$(\beta\cdot\mathbf{a})\beta + (1+\frac{\beta^2}{2})\mathbf{a} = \mathbf{U}. \tag{A.37}$$

In terms of components this vector equation can be written

$$(a_k\beta^k)\beta_i + (1+\frac{1}{2}\beta^k\beta_k)a_i = U_i. \tag{A.38}$$

This is a set of three equations (one for each i) with three unknowns a_i. In principle the set of equations can be written out explicitly ($i = 1, 2, 3$) and solved by standard high-school strategies. However, in order to avoid too much cumbersome algebra we take another approach. The above expression can be written

$$a_i(1+\beta_i^2+\frac{1}{2}\beta^k\beta_k) = U_i - \beta_i(a_k\beta^k - a_i\beta_i). \tag{A.39}$$

Notice that all terms on the left-hand side includes a_i, while the right-hand side does not include a_i at all (when the summations are written out explicitly the a_i-terms cancel). Hence, to second order in β, we get the following equation for a_i in terms of the other unknowns:

$$a_i = U_i(1-\beta_i^2-\frac{1}{2}\beta_k\beta^k) - \beta_i(a_k\beta^k - a_i\beta_i). \tag{A.40}$$

Notice that all terms on the right-hand side which includes components of $\mathbf{a}$, are second order in β. Since we only go to second order in β we must insert the zero-order expression for the acceleration, which is $a_i = U_i$. Inserting this we get:

$$\begin{aligned} a_i &= U_i(1 - \beta_i^2 - \frac{1}{2}\beta_k\beta^k) - \beta_i(U_k\beta^k - U_i\beta_i) \\ &= U_i - \frac{1}{2}U_i\beta_k\beta^k - \beta_i U_k \beta^k. \end{aligned} \tag{A.41}$$

Rewriting this as a vector equation we get (4.68).

A.9

Here we will justify that the post-Newtonian metric can be written in terms of ϕ_g and $\mathbf{A_g}$ as in 4.73. First observe that to lowest order (4.71) reads

$$\nabla^2 U = 4\pi G\rho, \tag{A.42}$$

which shows that $\phi_g = U$ to lowest order. This justifies the expressions for $h_{00}[O(2)]$ and $h_{ij}[O(2)]$ in (4.39). Then observe that to lowest order (4.72) reads

$$\nabla^2 \mathbf{A}_g = 4\pi G\mathbf{j}, \tag{A.43}$$

where $\mathbf{j} \equiv \rho\mathbf{v}$ is the mass-flow. Comparing with (4.18) we see that $A_{gi} = \tilde{A}_i$ to lowest order, which justifies the expression for $h_{0i}[O(3)]$. The expression for $h_{00}[O(4)]$ is derived from (4.38). Adding a term $2\partial_0\partial_0(\frac{U}{c^2})^2$ to the left hand side of the equation, it can be rewritten

$$\Box\left(-\frac{c^2}{2}h_{00}[O(4)] - \frac{U^2}{c^2}\right) = 4\pi G\rho_g. \tag{A.44}$$

This is also a valid differential equation for $h_{00}[O(4)]$ since the added term is of smallness $O(6)$. Comparing (4.71) and (A.44) we identify

$$-\frac{c^2}{2}h_{00}[O(4)] - \frac{U^2}{c^2} = \phi_g. \tag{A.45}$$

To lowest order we have $\frac{U^2}{c^4} = \frac{\phi_g^2}{c^4}$. Inserting this into the above equation we get the wanted expression for h_{00} to $O(4)$:

$$h_{00}[O(4)] = -2\frac{\phi_g}{c^2} - 2\frac{\phi_g^2}{c^4}. \tag{A.46}$$

A.10

All terms in 4.75 which includes ϕ_g is oriented either along the direction $\nabla\phi_g$ or $\mathbf{v}$. Depending on preferences it might be useful to decompose the vector $\mathbf{v}$ into one direction parallel with $\nabla\phi_g$ and one direction orthogonal to $\nabla\phi_g$:

$$\mathbf{v} = \nabla\phi_g\frac{(\mathbf{v}\cdot\nabla\phi_g)}{(\nabla\phi_g)^2} + \nabla\phi_g \times \frac{\mathbf{v}\times\nabla\phi_g}{(\nabla\phi_g)^2}, \tag{A.47}$$

or alternatively $\nabla\phi_g$ decomposed onto the directions $\mathbf{v}_{\|}$ and $\mathbf{v}_{\perp}$:

$$\nabla\phi_g = \frac{\mathbf{v}}{v^2}(\mathbf{v}\cdot\nabla\phi_g) + \frac{\mathbf{v}}{v^2}\times(\nabla\phi_g\times\mathbf{v}). \tag{A.48}$$

These relations follows directly from the vector identity $\mathbf{a}b^2 = \mathbf{b}(\mathbf{a}\cdot\mathbf{b}) + \mathbf{b}\times(\mathbf{a}\times\mathbf{b})$.

A.11

Here we shall show how (5.48) follows from (5.49). The connection coeffisients Γ^i_{mn} can be found in appendix A.6. Rewriting it to lowest order in terms of the potential ϕ_g defined in (4.71), we find that the only non-vanishing components are:

$$\begin{aligned}
-\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^1_{33} &= \partial_1 \frac{\phi_g}{c^2}, \\
\Gamma^2_{11} = -\Gamma^2_{22} = \Gamma^2_{33} &= \partial_2 \frac{\phi_g}{c^2}, \\
\Gamma^3_{11} = \Gamma^3_{22} = -\Gamma^3_{33} &= \partial_3 \frac{\phi_g}{c^2}, \\
\Gamma^2_{21} = \Gamma^3_{31} &= -\partial_1 \frac{\phi_g}{c^2}, \\
\Gamma^1_{12} = \Gamma^3_{32} &= -\partial_2 \frac{\phi_g}{c^2}, \\
\Gamma^1_{13} = \Gamma^2_{23} &= -\partial_3 \frac{\phi_g}{c^2}.
\end{aligned} \tag{A.49}$$

Inserting this into (5.48) we find for the component $i = 1$:

$$\frac{dU^1}{ds} = -\partial_1 \frac{\phi_g}{c^2} + 2U^1 \left(\partial_1 \frac{\phi_g}{c^2} U^1 + \partial_2 \frac{\phi_g}{c^2} U^2 + \partial_3 \frac{\phi_g}{c^2} U^3 \right), \tag{A.50}$$

and similar for the other components. As a vector equation this can be written on the form (5.49).

A.12

From the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \tag{A.51}$$

we find that the only non-zero components of $\frac{d^2 x^\mu}{d\tau^2}$ are

$$\frac{d^2 t}{d\tau^2} = -2 \frac{g}{c^2} \frac{1}{1 + \frac{gz}{c^2}} \frac{dt}{d\tau} \frac{dz}{d\tau} \tag{A.52}$$

and

$$\frac{d^2 z}{d\tau^2} = -g(1 + \frac{gz}{c^2}) \left(\frac{dt}{d\tau} \right)^2. \tag{A.53}$$

A test particle (initially) at rest in origo will have $\frac{dt}{d\tau} = 1$ and therefore $\frac{d^2 z}{dt^2} = -g$. The vector $\mathbf{g} = -g\mathbf{e}_z$ is therefore the 3-acceleration of a freely falling test particle initially at rest in origo, ie. the same vector as the $\mathbf{g}$ in (5.4).

We introduce the notation

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = (U^0, \mathbf{U}), \tag{A.54}$$

$$A^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} = (A^0, \mathbf{A}), \tag{A.55}$$

$$\mathbf{v} = \frac{dx^i}{dt}\mathbf{e}_i, \tag{A.56}$$

and

$$\mathbf{a} = \frac{d^2x^i}{dt^2}\mathbf{e}_i. \tag{A.57}$$

Our goal is to determine the functional dependence of $\mathbf{a}$ on $\mathbf{g}$ and $\mathbf{v}$, ie. determining $\mathbf{a}(\mathbf{g}, \mathbf{v})$. We start by calculating $A^0(\mathbf{g}, \mathbf{v})$ and $\mathbf{A}(\mathbf{g}, \mathbf{v})$ as an intermediate step. The line element (5.8) can be written

$$-\left((1+\frac{gz}{c^2})^2 - \frac{v^2}{c^2}\right)c^2dt^2 = -c^2d\tau^2, \tag{A.58}$$

which gives

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}}}. \tag{A.59}$$

Equations (A.52) and (A.53) can then be written

$$A^0 = 2c\frac{\mathbf{g}\cdot\mathbf{v}}{c^2}\frac{1}{\left(1+\frac{gz}{c^2}\right)\left(\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}\right)} \tag{A.60}$$

and

$$\mathbf{A} = \frac{(1+\frac{gz}{c^2})}{\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}}\mathbf{g}, \tag{A.61}$$

where we have used that $g\frac{dz}{dt} = -\mathbf{g}\cdot\mathbf{v}$. To find $\mathbf{a}(\mathbf{g}, \mathbf{v})$ from (A.60) and (A.61) we need two more equations to eliminate A^0 and $\mathbf{A}$. We find the needed relations from the definitions of U^μ and A^μ, ie. (A.54) and (A.55). First note that

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt}, \tag{A.62}$$

which means that

$$U^0 = \frac{c}{\sqrt{\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}}} \tag{A.63}$$

and

$$\mathbf{U} = \frac{1}{\sqrt{\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}}}\mathbf{v}. \tag{A.64}$$

Derivation of U^μ with respect to t gives:

$$\frac{dU^0}{dt} = c\frac{\left(1+\frac{gz}{c^2}\right)\frac{\mathbf{g}\cdot\mathbf{v}}{c^2} + \mathbf{v}\cdot\mathbf{a}}{\left(\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}\right)^{3/2}} \tag{A.65}$$

and

$$\frac{d\mathbf{U}}{dt} = \frac{1}{\sqrt{\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}}}\mathbf{a} + \frac{\left(1+\frac{gz}{c^2}\right)\frac{\mathbf{g}\cdot\mathbf{v}}{c^2} + \mathbf{v}\cdot\mathbf{a}}{\left(\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}\right)^{3/2}}\mathbf{v}, \tag{A.66}$$

where we have used that $\frac{d}{dt}\mathbf{v}^2 = 2\mathbf{v}\cdot\mathbf{a}$ and that $g\frac{dz}{dt} = -\mathbf{g}\cdot\mathbf{v}$. Accordingly we get

$$A^0 = \frac{dt}{d\tau}\frac{dU^0}{dt} = c\frac{\left(1+\frac{gz}{c^2}\right)\frac{\mathbf{g}\cdot\mathbf{v}}{c^2} + \mathbf{v}\cdot\mathbf{a}}{\left(\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}\right)^2} \tag{A.67}$$

and

$$\mathbf{A} = \frac{dt}{d\tau}\frac{d\mathbf{U}}{dt} = \frac{1}{\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}}\mathbf{a} + \frac{\left(1+\frac{gz}{c^2}\right)\frac{\mathbf{g}\cdot\mathbf{v}}{c^2} + \mathbf{v}\cdot\mathbf{a}}{\left(\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}\right)^2}\mathbf{v}. \tag{A.68}$$

Notice that the last term in (A.68) is almost the same as (A.67). Hence it follows (algebraically) from the two previous equations that:

$$\mathbf{a} = \left(\left(1+\frac{gz}{c^2}\right)^2 - \frac{v^2}{c^2}\right)\left(\mathbf{A} - \frac{A^0}{c}\mathbf{v}\right). \tag{A.69}$$

By inserting (A.60) and (A.61) into (A.69) we finally get the sought after expression:

$$\mathbf{a} = \mathbf{g}\left(1+\frac{gz}{c^2}\right) - 2\frac{1}{1+\frac{gz}{c^2}}\frac{\mathbf{g}\cdot\mathbf{v}}{c^2}\mathbf{v}. \tag{A.70}$$

A.13

Here we shall comment on the contributions from the terms $-\frac{1}{2}ks^2 - k_2 s$ on the right-hand side of (5.63) to the measured acceleration. Starting with (5.63) instead of (5.64) we get an additional contribution $ks\frac{ds}{dt} + k_2\frac{ds}{dt}$ in the expression for $\frac{dt}{dt}$ (see (5.72)). Hence we get an additional contribution $ks\frac{ds}{dt}\frac{dx}{dt} + k_2\frac{ds}{dt}\frac{dx}{dt}$ to the measured velocity (see (5.75)). This gives rise to an additional contribution

$$\frac{d}{dt}\left(ks\frac{ds}{dt}\frac{dx}{dt} + k_2\frac{ds}{dt}\frac{dx}{dt}\right) + k_2\frac{ds}{dt}\frac{d^2x}{dt^2}$$

to the measured acceleration. To lowest order we have $\frac{ds}{dt} = \frac{dx}{dt} = v$ and $\frac{d^2x}{dt^2} = -\nabla U$, where U is the Newtonian gravitational potential. Performing the derivation in the expression above and evaluating in the origin we find that the contributions from the negleced terms to the measured acceleration becomes

$$kv^3 + 3k_2 v\nabla U \sim O(4)\nabla U. \tag{A.71}$$

Notice that both terms above are of the same order of magnitude. Hence we have showed that the contribution is two orders of magnitude lower than the post-Newtonian precision, and therefore negligible.